J. DIFFERENTIAL GEOMETRY
16 (1981) 105–115

MINIMAL IMMERSIONS OF COMPACT IRREDUCIBLE HOMOGENEOUS RIEMANNIAN MANIFOLDS

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0. Introduction

The purpose of this paper is to study the space of isometric minimal immersions of a compact irreducible homogeneous Riemannian manifold M^m into a standard sphere S^n. By a theorem of Takahashi [6], any compact irreducible homogeneous Riemannian manifold can be isometrically minimally immersed into some $S^n(r)$ using its spaces of eigenfunctions satisfying the equation

$$\Delta\varphi = -\lambda\varphi \tag{0.1}$$

for some constant λ. The set λ such that (0.1) has nontrivial solution is called the spectrum of the Laplace operator Δ on M, denoted by Spec(M). It is also known that [4] the coordinate functions of any isometric minimal immersions of M into $S^n \subseteq \mathbf{R}^{n+1}$ are eigenfunctions of the Laplacian. In 1971, do Carmo and Wallach [2] consider the case when M is also a standard sphere. However, some of their results also hold when M is a compact irreducible homogeneous Riemannian manifold.

The main result which we have obtained in the paper is a classification theorem of all isometric minimal immersions. In fact, we show that if Φ: $M \to S^n(r)$ is an isometric minimal immersion, then $\Phi(M) = N$ is also a compact irreducible homogeneous Riemannian manifold which is embedded in $S^n(r)$. The map Φ: $M \to N$ is in fact a covering map, and N inherits the homogeneous structure of M.

As an application of the above theorem, we show that if N is a compact Riemannian manifold which is isometrically covered by M. Then N can be isometrically minimally immersed into some $S^n(r)$ iff N has the induced homogeneous structure of M. We also give necessary and sufficient conditions for an eigenspace E_λ of M to be invariant under the group of deck transformations $\Gamma(N)$ with respect to the covering map π: $M \to N$. An

Communicated by S. T. Yau, October 30, 1979.

interesting corollary of this is that if N is a lens space which is k-fold covered by S^{2m-1}, then N cannot be isometrically minimally immersed into any standard spheres unless $k = 1$ or 2.

In the last section, we consider the question whether a compact irreducible homogeneous Riemannian manifold can always be isometrically minimally embedded into some S^n. Using the Weyl formula, we show that if $M = G/H$, where G acts effectively on M, and if the center $Z(G)$ of G is a cyclic group, then there exists infinitely many eigenspaces of M which give isometric minimal embeddings of M into $S^n(r)$.

We will adopt the convention that any isometric minimal immersioin $\Phi: M \to S^n(r)$ is full, i.e., $\Phi(M)$ is not contained in any totally geodesic $S^p(r)$ of $S^n(r)$ with $p < r$.

The author would like to acknowledge his gratitude to R. Niles for pointing out the group theoretic observation in Proposition 11, and also thanks C. Croke for many helpful discussions during the preparation of the last section of this paper.

1. Spaces of isometric minimal immersions

Definition. A homogeneous manifold $M^m = G/H$ is said to be irreducible if its isometry group G is compact and its isotropy subgroup H acts irreducibly on the tangent space at $eH \in M$, where e is the identity element of G. In addition, we also assume that G acts effectively on M.

For the sake of completeness, we will outline the proof of do Carmo and Wallach for general compact irreducible homogeneous Riemannian manifolds.

Proposition 1. *Let $\Phi: M^m \to S^n(r)$ be an isometric minimal immersion of M into $S^n(r)$. Then $r^2 = m/\lambda$ for some $\lambda \in$ Spec(M). Moreover, for a fixed λ, the set of such isometric minimal immersions can be parametrized by a compact convex body in a finite dimensional vector space.*

Proof. If we consider $S^n(r) \subseteq \mathbf{R}^{n+1}$, then it is known that [4] the coordinate functions of $\Phi: M^m \to \mathbf{R}^{n+1}$ are eigenfunctions with eigenvalue m/r^2. Up to orthogonal transformation, we may assume that $\Phi = A\Psi$, where A is a semi-positive symmetric matrix and Ψ denotes the standard immersion given by $\Psi = (\alpha\varphi_1, \cdots, \alpha\varphi_{k+1})$ with $\{\varphi\}_{i=1}^{k+1}$ being an orthonormal basis of $E_\lambda = \{f | \Delta f = -\lambda f\}$, $\lambda = m/r^2$.

Let us denote $V_1 = d\Psi(T_x M) \subseteq T_{\Psi(x)} S^k(r)$ and $S^2(V_1) = \{$symmetric squares of $V_1\}$. Also let $W_0 = \{G \cdot S^2(V_1)\}_{\mathbf{R}}$-linear span of the orbit of $S^2(V_1)$ in $S^2(E_\lambda)$ where E_λ is identified to $T_{\Psi(x)}\mathbf{R}^{k+1}$.

One can identify the symmetric square $S^2(E_\lambda)$ with the space of symmetric linear maps of E_λ, where the linear map is defined by

$$uv(t) = \frac{1}{2}(\langle u, t\rangle v + \langle v, t\rangle u) \tag{1.1}$$

for $t \in E_\lambda$ and $uv \in S^2(E_\lambda)$. One obtains an induced inner product on $S^2(E_\lambda)$ given by $(A, B) = \operatorname{tr}(AB)$, for all $A, B \in S^2(E_\lambda)$, and the induced action of $g \in G$ on $S^2(E_\lambda)$ is given by $g \cdot A = gAg^{-1}$. Clearly $\langle Au, Av\rangle = (A, uv)$. We define $W = \{u \in S^2(E_\lambda) | (u, W_0) = 0\}$.

Now we claim that $\Phi = A\Psi$ is an isometric immersion iff $A^2 - I \in L$, where $L = \{c \in W | C + I \geqslant 0\}$. In fact, $A\Psi$ is an isometric immersion iff

$$\langle AgX^*, AgX^*\rangle = 1 \tag{1.2}$$

for all $g \in G$, $X \in S_xM$, $X^* = d\Psi(X)$. However, this is equivalent to

$$(A^2 - I, g \cdot (X^*)^2) = 0, \tag{1.3}$$

which means $A^2 - I \in W$, hence $A^2 - I \in L$ since $A \geqslant 0$ and is symmetric. The converse follows similarly. Of course, by Takahashi's theorem, if Φ is an isometric immersion then Φ is minimal.

Therefore the equivalent classes of isometric minimal immersions can be parametrized by the set $L \subset W$. Clearly L is a convex set with boundary. Moreover since $\operatorname{tr} A^2 = \dim E_\lambda$ for $A^2 - I \in L$, we conclude that if $c \in L$, then $\operatorname{tr} c = 0$. This implies that the eigenvalues of the elements in L are bounded, hence L is compact. In fact, the boundary points of L correspond to A being singular, i.e., $n < \dim E_\lambda - 1$.

2. A classification theorem

Definition. A function $f_0 \in E_\lambda$ is said to be the normalized zonal function at $x_0 \in M$ with respect to E_λ if it satisfies the following properties:

(i) f_0 is constant on the orbit of H_0 = isotropy subgroup of G which fixes x_0,

(ii) f_0 is perpendicular (in the L^2 sense) to the set of functions in E_λ which vanish at x_0,

(iii) $f_0(x_0) = \|f_0\|_\infty$,

(iv) $\|f_0\|_2 = 1$.

Proposition 2. *In each eigenspace E_λ of M and for a fixed $x_0 \in M$, there exists a unique normalized zonal function at x_0 with respect to E_λ.*

Proof. The proof of this proposition is contained in [3] and [5]. However, we will sketch the proof here.

Let us consider the space $E = \{f \in E_\lambda | \langle f, g\rangle = 0$ for all g such that $g(x_0) = 0\}$. It is easy to see that E is a 1-dimensional subspace of E_λ. Consider $f_0 \in E$ such that $\|f_0\|_2 = 1$, and $f_0(x_0) \neq 0$. Since E is invariant under the action of H_0, f_0 satisfies conditions (i), (ii) and (iv).

On the other hand, if we define the function

$$F(x) = \sum_{i=1}^{k+1} \varphi_i^2(x), \quad \text{for } x \in M, \tag{2.1}$$

where $\{\varphi_i\}_{i=1}^{k+1}$ is an orthonormal basis of E_λ, by the homogeneity assumption and the fact that $F(x)$ is well defined under an orthogonal change of basis of E_λ, $F(x) =$ constant. In particular,

$$F(x_0) = F(x). \tag{2.2}$$

If we pick an orthonormal basis such that $f_0 = \varphi_1$, then

$$F(x_0) = f_0^2(x_0). \tag{2.3}$$

Hence

$$\sum_{i=1}^{k+1} \varphi_i^2(x) = f_0^2(x_0). \tag{2.4}$$

Integrating both sides yields

$$k + 1 = V \cdot f_0^2(x_0), \tag{2.5}$$

where $V = V(M)$ is the volume of M. But

$$\frac{k+1}{V} = \sum_{i=1}^{k} \varphi_i^2(x)$$

implies that

$$\|\varphi\|_\infty^2 \leqslant \frac{k+1}{V}, \quad \text{for all } \varphi \in E_\lambda. \tag{2.6}$$

In particular,

$$\|f_0\|_\infty^2 \leqslant \frac{k+1}{V} = f_0^2(x_0),$$

which proves the proposition.

Lemma 3. *Let Φ: $M \to S^n(r)$ be an isometric minimal immersion. Suppose Φ corresponds to an interior point of L as discussed in Proposition* 1. *If N denotes the image of Φ in $S^n(r)$, then N is an isometrically minimally embedded submanifold of $S^n(r)$. Moreover Φ: $M \to N$ is a covering map.*

Proof. Clearly, we need only to show that the preimage set of each point $z \in N$ consists of exactly q points. By scaling, we may assume that

$$\dim E_\lambda = V(M). \tag{2.7}$$

By an orthonormal change of basis, if necessary, we may assume N contains p = north pole of $S^n(r)$. We claim that if $\Phi(x_0) = p$ then the preimage $\Phi^{-1}(p)$ of p consists of points in M which take on the maximum value of the normalized zonal function f_0 at x_0.

Indeed, if $\Phi(x) = (\varphi_1(x), \cdots, \varphi_{n+1}(x))$, then $\Phi(x_0) = p$ implies $\varphi_1(x_0) = r$ and $\varphi_\alpha(x_0) = 0$ for $\alpha \neq 1$. This means that $\varphi_\alpha \in E_0 = \{f \in E_\lambda | f(x_0) = 0\}$. Since by assumption $n + 1 = \dim E_\lambda = k + 1$, we conclude that $\langle \varphi_\alpha \rangle_{\alpha=2}^{n+1} = E_0$. Hence $\varphi_1 = af_0 + bg$ for some $a, b \in \mathbf{R}$ and $g \in E_0$. However, by $r = \varphi_1(x_0) = af_0(x_0)$, (2.5) and (2.7) we have

(2.8) $$r = af_0(x_0) = a.$$

Hence

(2.9) $$\varphi_1 = rf_0 + bg.$$

If $x \in$ {maximal points of f_0}, then $f_0(x) = 1$. From (2.6) we conclude that

(2.10) $$g(x) = 0, \quad \text{for } g \in E_0,$$

which means $E_0 = E_1 = \{f \in E_\lambda | f(x) = 0\}$ because $\dim E_0 = n = \dim E_1$. Therefore

$$\varphi_1(x) = rf_0(x) = r$$

and

$$\varphi_\alpha(x) = 0, \quad \alpha \neq 1,$$

which implies $\Phi(x) = p$.

Conversely, if $\Phi(x) = p$, then $\varphi_1(x) = r$ and $\varphi_\alpha(x) = 0$ for $\alpha \neq 1$. Thus

$$\varphi_\alpha \in E_1 = \{f \in E_\lambda | f(x) = 0\},$$

and $E_1 = E_0$. It follows that

(2.11) $$r = \varphi_1(x) = rf_0(x) + bg(x) = rf_0(x).$$

However $f_0(x) = 1$ implies that x takes on the maximum value of f_0. The lemma then follows directly.

Theorem 4. *Let Φ: $M \to S^n(r)$ be an isometric minimal immersion of M into $S^n(r)$. Then the image N of Φ is a compact homogeneous space which is isometrically minimally embedded in $S^n(r)$. Moreover, the homogeneous structure of N is the one induced from M, i.e., the group of deck transformations $\Gamma(N)$ with respect to the covering map Φ: $M \to N$ is contained in the center $Z(G)$ of G.*

Proof. We will first prove the theorem for those Φ which correspond to the interior points of L. We claim that for any $g \in G$, g commutes with the element of $\Gamma(N)$.

Observe that g preserves fibers over N. Indeed if $\bar{x}, \bar{y} \in N$, then $\Phi^{-1}(\bar{x})$ and $\Phi^{-1}(\bar{y})$ coincide with the sets $\{x \in M | f_1(x) = \|f_1\|_\infty\}$ and $\{x \in M | f_2(x) = \|f_2\|_\infty\}$ respectively, where f_1 and f_2 are normalized zonal functions at preimage points of $\bar{x}$ and $\bar{y}$. However if $g \in G$ and $g(x_1) = y_1$ with $x_1 \in \Phi^{-1}(\bar{x})$ and $y_1 \in \Phi^{-1}(\bar{y})$, then $g \cdot f_2 = f_2 \circ g$ is a zonal function at y_1. Hence by uniqueness $f_1 = g \cdot f_2$. This shows $\Phi^{-1}(\bar{x}) = \Phi^{-1}(\bar{y})$.

Since G is a Lie group, in order to show the claim, it suffices to show that g commutes with $\Gamma(N)$ for those g which send x to nearby points. Let U be a sufficiently small neighborhood of $\bar{x} \in N$ such that U is evenly covered by disjoint neighborhoods $\{U_i\}_{i=1}^q$ of $\{x_i\}_{i=1}^q = \Phi^{-1}(\bar{x})$, with $x_i \in U_i$ for all $1 \leqslant i \leqslant q$. We would like to show that g commutes with $\Gamma(N)$ if $g(x_1) \in U_1$. Let $\bar{y} = \Phi(g(x_1))$ and $\{y_i\}_{i=1}^q = \Phi^{-1}(\bar{y})$ such that $y_i \in U_i$. Clearly we need only to show that $g(x_i) = y_i$. By picking U sufficiently small and using the fact that g is an isometry, we have $g(x_i) \in U_i$. However g preserving fibers implies that $y_i = g(x_i)$ because $\{U_i\}$ are disjoint. This proves the theorem for those Φ which are the interior points of L. For the boundary points we can utilize a continuation argument. In fact, if we take a path through the interior of L to a boundary point Φ, then it is clear that by continuity the theorem also holds for the boundary points.

Remark. Any set of eigenfunctions from an eigenspace E_λ gives an isometric minimal immersion of M into $S^n(r)$ with $r^2 = m/\lambda$ iff they satisfy the algebraic criterion described in §1.

3. Applications

In case when M is also a standard sphere S^m of radius 1, Theorem 4 yields the following.

Theorem 5. *If $\Phi: S^m \to S^n(r)$ is an isometric minimal immersion, then $r^2 = m/\lambda$ for some $\lambda \in \operatorname{Spec}(S^m)$. Moreover $\Phi(S^m)$ is either an embedded sphere or an embedded projective space. In fact, if $\operatorname{Spec}(S^m) = \{0 = \lambda_0 < \lambda_1 < \lambda_2 \cdots\}$ (multiplicity not included), then Φ corresponds to embeddings of S^m if $r^2 = m/\lambda_{2i+1}$ for $0 \leqslant i < \infty$, and it corresponds to embedding of $\mathbf{RP}^m$ if $r^2 = m/\lambda_{2i}$ for $1 \leqslant i < \infty$.*

Proof. By Theorem 4, $\Phi(S^m)$ is an embedded homogeneous space covered by S^m with the induced homogeneous structure. This implies that the set of preimages of a point $z \in \Phi(S^m)$ is contained in the fixed point set of the isotropy subgroup of $x_0 \in \Phi^{-1}(z)$. Since the isotropy subgroup H_0 of $x_0 \in S^m$ has orbits homeomorphic to S^{m-1} with the exception of x_0 and its antipodal point, this means that $\Phi: S^m \to \Phi(S^m)$ is at most a 2-fold covering. Hence

$\Phi(S^m)$ is either S^m or $\mathbf{RP}^m$. However, it is known [1] that the eigenfunctions of S^m with eigenvalue λ_i are spanned by the harmonic homogeneous polynomials on $\mathbf{R}^{m+1}$ of degree i. Hence $-f(x) = f(-x)$ for $f \in E_{\lambda_{2i+1}}$ for $0 \leq i < \infty$, and $f(x) = f(-x)$ for $f \in E_{\lambda_{2i}}$ for $1 \leq i < \infty$. This proves the theorem.

Corollary 6. *Suppose N is a lens space which is isometrically k-fold covered by S^{2m-1}. Then N cannot be isometrically minimally immersed into any standard spheres if $k > 2$.*

Proof. Suppose on the contrary that $\Phi\colon \mathrm{N} \to S^n(r)$ is an isometric minimal immersion. Let $\pi\colon S^{2m-1} \to N$ be the covering map. Consider the composition $\Phi \circ \pi\colon S^{2m-1} \to S^n(r)$ which is clearly an isometric minimal immersion of S^{2m-1}. Moreover, the image $\Phi \circ \pi(S^{2m-1}) = \Phi(N)$ is at least k-fold covered by S^{2m-1}. But this contradicts Theorem 5 if $k > 2$.

Remark. In fact, the proof of Corollary 6 shows that if $\pi\colon M \to N$ is a covering map, then N can be isometrically immersed into some $S^n(r)$ iff N has the induced homogeneous structure of M.

In the general setting of an isometric covering $\pi\colon M \to N$, where M and N are only compact Riemannian manifolds, it is obvious that the eigenfunctions of N can be lifted to be eigenfunctions of M. If $\lambda \in \operatorname{Spec}(N)$, we denote the eigenspaces of N and M with eigenvalue λ by $\overline{E}_\lambda$ and E_λ respectively. Let $\pi^*(\overline{E}_\lambda)$ be the pulled back of $\overline{E}_\lambda$ to M, then $\pi^*(\overline{E}_\lambda) \subseteq E_\lambda$. It is natural to ask the following question: When does $\pi^*(\overline{E}_\lambda) = E_\lambda$? For the case where M is an irreducible homogeneous space, this question can be completely answered.

Theorem 7. *Let $\pi\colon M \to N$ be an isometric covering map. Then $\pi^*(\overline{E}_\lambda) = E_\lambda$ for all $\lambda \in \operatorname{Spec}(N)$ iff N inherits the homogeneous structure from M, i.e., $\Gamma(N) \subseteq Z(G)$.*

Proof. First we show that if there exists $\lambda \in \operatorname{Spec}(N)$ such that $\pi^*(\overline{E}_\lambda) = E_\lambda$, then $\Gamma(N) \subseteq Z(G)$. Let $\Phi\colon M \to S^n(r)$ be the standard immersion by an orthonormal basis of E_λ. However $E_\lambda = \pi^*(\overline{E}_\lambda)$ means that the eigenfunctions are invariant under $\Gamma(N)$. Theorem 4 then implies that there exists $\tilde{N}$ which is covered by M and $\Gamma(\tilde{N}) \subseteq Z(G)$. Moreover $\tilde{N}$ is the embedded image of Φ. On the other hand, since Φ is invariant under $\Gamma(N)$ we have the following diagram

$$M \xrightarrow{\pi} N \xrightarrow{\theta} \tilde{N}$$

with $\theta \circ \pi = \tilde{\pi}$ and $\Gamma(\tilde{N}) \supseteq \Gamma(N)$. However $\Gamma(\tilde{N}) \subseteq Z(G)$, hence $\Gamma(N) \subseteq Z(G)$.

Conversely, suppose $Z(G) \supseteq \Gamma(N)$. Then N is also an irreducible homogeneous manifold. Therefore for any $\lambda \in \operatorname{Spec}(N)$, E_λ gives an isometric minimal immersion $\Phi\colon N \to S^n(r)$ where $r^2 = m/\lambda$. This means that

$\Phi \circ \pi \colon M \to S^n(r)$ is an isometric minimal immersion of M into $S^n(r)$. By Theorem 4, we have

$$M \xrightarrow{\pi} N \xrightarrow{\Phi} \Phi(N) = \tilde{N}$$

where $\Gamma(\tilde{N}) \subseteq Z(G)$. However the proof of Theorem 4 implies that the image of the standard isometric minimal immersion of M into $S^k(r)$ by an orthonormal basis of E_λ is isometric to $\tilde{N}$. This implies that the eigenfunctions in E_λ are $\tilde{\Gamma}$-invariant, hence also Γ-invariant. This completes the proof of Theorem 7.

Remark. Theorem 7 actually shows that if $E_\lambda = \pi^*(\bar{E}_\lambda)$ for some $\lambda \in \operatorname{Spec}(N)$ then $E_\lambda = \pi^*(\bar{E}_\lambda)$ for all $\lambda \in \operatorname{Spec}(N)$.

When $M = S^{2m-1}$ and N a lens space k-fold covered by M. Then $E_\lambda \neq \pi^*(\bar{E}_\lambda)$ for all $\lambda \in \operatorname{Spec}(N)$ iff $k > 2$.

4. Embeddings

The above discussion gave us a rather clear picture of isometric minimal immersions of a compact irreducible homogeneous Riemannian manifold into a standard sphere. It is natural to ask if such a manifold M can always be isometrically minimally embedded into a standard sphere. By Theorem 4, this is equivalent to asking if there exists an eigenfunction on M which is not invariant under any subgroup of $Z(G)$. The next theorem gives conditions which guarantee the existence of infinitely such eigenfunctions

Theorem 8. *If $Z(G)$ is a cyclic group, then there exist infinitely many eigenfunctions which are not invariant under any subgroup of $Z(G)$.*

Since each eigenspace E_λ of M are of finite dimensions, we conclude

Corollary 9. *If $Z(G)$ is a cyclic group, then there exist infinitely many eigenspaces E_λ of M which give isometric minimal embeddings of M into $S^n(r)$.*

Before we prove Theorem 8, let us point out some elementary properties of $Z(G)$.

Lemma 10. *$Z(G)$ is a finite group, and $Z(G) \cap H = \{e\}$.*

Proof. Let x_0 be any point in M, and denote the orbit of x_0 under $Z(G)$ by $Z(x_0)$. Clearly $Z(x_0)$ is contained in the fixed point set of H_0. Indeed, if $h \in H_0$ and $z \in Z(G)$, then

$$hz(x_0) = zh(x_0) = z(x_0). \tag{4.1}$$

Hence if $Z(G)$ is not finite, by compactness there exist $z, z' \in Z(G)$ which are sufficiently close to each other. Let $x_0 \in M$ be the point which represents the coset zH. Then $z'(x_0)$ will be sufficiently close to x_0. If γ is the unique minimizing geodesic joining x_0 and $z'(x_0)$, then γ is invariant under H_0, since

x_0 and $z'(x_0)$ are invariant and γ is unique. However this implies the vector tangent to γ at x_0 is invariant under H_0, which contradicts the irreducibility assumption of H.

To prove that $Z(G) \cap H = \{e\}$, it suffices to show that if $z \in Z(G)$ where $z \neq e$, then z has no fixed point. Assume $x \in M$ is a fixed point of z. By the effectiveness of G, there exist points y_1 and y_2 in M such that $z(y_1) = y_2$. Let $g \in G$ be the isometry which sends y_2 to x. Now consider

$$z^{-1}gz(y_1) = z^{-1}g(y_2) = z^{-1}(x) = x. \tag{4.2}$$

On the other hand, since $z \in Z(G)$,

$$z^{-1}gz(y_1) = g(y_1), \tag{4.3}$$

which implies $g(y_1) = x$. However $g(y_2) = x$ and $y_1 \neq y_2$, which is a contradiction. Thus the proof is complete.

In general, let Z be a finite abelian group, and $S = \{K_\alpha\}_{\alpha=1}^q$ be the set of proper subgroups of Z. We denote $K_{\alpha_1 \cdots \alpha_p}$ to be the subgroup generated by $\bigcup_{i=1}^p K_{\alpha_i}$.

Proposition 11. *The equation*

$$|Z| = \sum_\alpha \frac{|Z|}{|K\alpha|} - \sum_{\alpha_1<\alpha_2} \frac{|Z|}{|K_{\alpha_1\alpha_2}|} + \sum_{\alpha_1<\alpha_2<\alpha_3} \frac{|Z|}{|K_{\alpha_1\alpha_2\alpha_3}|} - \cdots \pm \frac{|Z|}{|K_{12\cdots q}|}$$

is equivalent to the statement

$$|Z| = \textit{order of} \bigcup_\alpha K_\alpha.$$

Proof. Let Z^* be the dual group of Z, i.e., $Z^* = \mathrm{End}_{\mathbf{Z}}(Z, \mathbf{C}^*)$. It is well-known that $Z^* \approx Z$. Consider $K_\alpha \in S$, and define $K_\alpha^\perp = \{\varphi \in Z^* | \varphi(K_\alpha) = 1\}$. Clearly $(Z/K_\alpha)^* \approx K_\alpha^\perp$. Then

$$\frac{|Z|}{|K_\alpha|} = |K_\alpha^\perp|. \tag{4.4}$$

If η: $Z^* \to Z$ is an isomorphism, then for $K_\alpha \in S$ let $\hat{K}_\alpha = \eta(K_\alpha^\perp \mathrm{k})$. Hence

$$\frac{|Z|}{|K_\alpha|} = |\hat{K}_\alpha|. \tag{4.5}$$

Also for $K_{\alpha_1}, K_{\alpha_2} \in S$ we have

$$\bar{K}_{\alpha_1\alpha_2} = \bar{K}_{\alpha_1} \cap \bar{K}_{\alpha_2}, \tag{4.6}$$

since

$$\hat{K}_{\alpha_1\alpha_2} = \eta(K_{\alpha_1\alpha_2}^\perp) = \eta(K_{\alpha_1}^\perp \cap K_{\alpha_2}^\perp) = \eta(K_{\alpha_1}^\perp) \cap \eta(K_{\alpha_2}^\perp).$$

Hence the sum is

$$(4.7)\qquad \sum_{\alpha}|\hat{K}_{\alpha}| - \sum_{\alpha_1<\alpha_2}|\hat{K}_{\alpha_1}\cap\hat{K}_{\alpha_2}| + \cdots = |\bigcup_{\alpha} K_{\alpha}|$$

as claimed.

Proof of Theorem 8. Assume the contrary that all but finitely many eigenfunctions are invariant under some nontrivial subgroup of $Z(G)$. Let $S = \{K_{\alpha}\}_{\alpha=1}^{q}$ be the set of proper subgroups of $Z(G)$. This is a finite set because of Lemma 10. To each $\lambda_i \in \mathrm{Spec}(M)$, we associate an eigenfunction φ_i with eigenvalue λ_i such that the set $\{\varphi_i\}_{i=1}^{\infty}$ form an orthonormal basis for $L^2(M)$, where the λ_i are ordered as follows $0 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots$ (including multiplicities). We denote n^{λ} to be the number of eigenfunctions in $\{\varphi_i\}$ with eigenvalues less than or equal to λ, and n_{α}^{λ} (respectively, n_0^{λ}) be the number of such eigenfunctions which are (respectively, are not invariant under the group K_{α}. A simple counting argument shows)

$$(4.8)\quad n^{\lambda} - n_0^{\lambda} = \sum_{\alpha} n_{\alpha}^{\lambda} - \sum_{\alpha_1<\alpha_2} n_{\alpha_1\alpha_2}^{\lambda} + \sum_{\alpha_1<\alpha_2<\alpha_3} n_{\alpha_1\alpha_2\alpha_3}^{\lambda} - \cdots \pm n_{12\cdots q}^{\lambda},$$

where $n_{\alpha_1\cdots\alpha_p}^{\lambda}$ = number of eigenfunctions in $\{\varphi_i\}$ with eigenvalues less than or equal to λ and are invariant under the subgroup $K_{\alpha_1\cdots\alpha_p}$ of $Z(G)$ generated by $\bigcup_{i=1}^{p} K_{\alpha_i}$. Let $M_{\alpha_1\cdots\alpha_p} = M/K_{\alpha_1\cdots\alpha_p}$ be the manifold which is covered by M with $K_{\alpha_1\cdots\alpha_p}$ as its group of deck transformations. The eigenfunctions on $M_{\alpha_1\cdots\alpha_p}$ are the $K_{\alpha_1\cdots\alpha_p}$-invariant ones on M. Dividing (4.8) by $\lambda^{m/2}$ yields

$$(4.9)\qquad \frac{n^{\lambda}}{\lambda^{m/2}} - \frac{n_0^{\lambda}}{\lambda^{m/2}} = \sum_{\alpha}\frac{n_{\alpha}^{\lambda}}{\lambda^{m/2}} - \sum_{\alpha_1<\alpha_2}\frac{n_{\alpha_1\alpha_2}^{\lambda}}{\lambda^{m/2}} + \cdots \pm \frac{n_{12\cdots q}^{\lambda}}{\lambda^{m/2}}.$$

Taking the limit as $\lambda \to \infty$, the Weyl formula gives

$$(4.10)\qquad C_m V = \sum_{\alpha} C_m V_{\alpha} - \sum_{\alpha_1<\alpha_2} C_m V_{\alpha_1\alpha_2} + \cdots \pm C_m V_{12\cdots q},$$

where V = volume of M, $V_{\alpha_1\cdots\alpha_p}$ = volume of $M_{\alpha_1\cdots\alpha_p}$, and C_m = constant depending only on m. Here we have used the fact that $\lim_{\lambda\to\infty} n_0^{\lambda}$ is finite. Since $M \to M_{\alpha_1\cdots\alpha_p}$ is a covering map with the number of sheets equal to $|K_{\alpha_1\cdots\alpha_p}|$,

$$(4.11)\qquad V_{\alpha_1\cdots\alpha_p} = \frac{V}{|K_{\alpha_1\cdots\alpha_p}|}.$$

Therefore (4.10) becomes

$$(4.12)\qquad 1 = \sum_{\alpha}\frac{1}{|K_{\alpha}|} - \sum_{\alpha_1<\alpha_2}\frac{1}{|K_{\alpha_1\alpha_2}|} + \cdots \pm \frac{1}{|K_{12\cdots q}|}.$$

Multiplying both sides by $|Z| = |Z(G)|$, we have

$$|Z| = \sum_{\alpha} \frac{|Z|}{|K_\alpha|} - \sum_{\alpha_1 < \alpha_2} \frac{|Z|}{|K_{\alpha_1\alpha_2}|} + \cdots \pm \frac{|Z|}{|K_{12\cdots q}|}. \tag{4.13}$$

By Proposition 11, this is equivalent to the fact that the order of $Z(G)$ is equal to the order of the union of all its proper subgroups. But this is true iff $Z(G)$ is not cyclic. Hence this contradicts the assumption.

Remark. In fact, we have shown that

$$n_0^\lambda \sim C_m \Lambda^{m/2}\left[1 - \frac{\text{order of } \bigcup_\alpha K_\alpha}{|Z|}\right].$$

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