J. DIFFERENTIAL GEOMETRY
16 (1981) 101–104

PARALLEL VECTOR FIELDS ON MANIFOLDS WITH BOUNDARY

PETER PERCELL

1. Introduction

For closed manifolds it is apparently a nontrivial problem to find necessary and sufficient topological conditions for the existence of a vector field which is parallel with respect to some Riemannian metric. It is known that if a closed Riemannian n-manifold admits a parallel vector field, then its Euler-Poincare characteristic is zero, its Betti numbers satisfy

$$b_1 \geqslant 1 \text{ and } b_{k+1} \geqslant b_k - b_{k-1} \text{ for } 1 \leqslant k \leqslant n-1$$

(Chern [2] and Karp [3]) and all of its Pontryagin numbers are zero (a consequence of a result of Bott [1]).

In this note we shall show that every compact manifold with boundary admits a vector field which is parallel with respect to some Riemannian metric by showing that every transient vector field is parallel with respect to some Riemannian metric. In the language of differentiable dynamical systems, a vector field is transient if its non-wandering set is empty. It was shown in [5] that a vector field on a compact manifold with boundary is transient if and only if it is a nowhere-zero gradient vector field with respect to some Riemannian metric. The proof we shall give of the fact that a transient vector field is parallel is just a slight modification of the proof in [5] that a transient vector field is a nonvanishing gradient field. The question of whether or not a transient vector field must be parallel was raised by Rene Thom [6].

2. Definitions and results

Throughout this note, M is a compact connected smooth $(n+1)$-manifold with nonempty boundary, ∂M, N is the double of M, and $\mathfrak{X}(P)$ is the collection of smooth vector fields on a smooth manifold P. We shall work with the following definition of transient.

Received, October 27, 1979, and, in revised form, September 15, 1980. Research supported by a grant from the University of Houston.

Definition. $X \in \mathfrak{X}(M)$ is said to be *transient* if each integral curve for X leaves M in finite positive and negative time.

Thus each trajectory of a transient vector field is a compact arc whose endpoints are on ∂M. Note that a transient vector field can never vanish, but it is still possible for a trajectory to be a single point–which must be on ∂M. Such a point is called a point of exterior tangency because if the vector field is extended to N, then locally the trajectory through the point is a curve which is tangent to ∂M from outside of M.

Definition. $X \in \mathfrak{X}(P)$ is said to be *strongly parallel* if there exist a positive integer K and an embedding $\alpha: P \to \mathbf{R}^K$ such that $\alpha_* X = \partial/\partial z | \alpha(P)$, where α_* is the differential map of α, and z is a nontrivial linear functional on $\mathbf{R}^K$, e.g., the last coordinate function. (Note that such an X is parallel in the usual sense with respect to the Riemannian metric on P which is the pull-back by α of the standard Riemannian metric on $\mathbf{R}^K$.)

Theorem. *If $X \in \mathfrak{X}(M)$ is transient, then X is strongly parallel.*

The proof of the theorem will be given in the next section.

Corollary. *The set of strongly parallel vector fields on M is nonempty.*

Proof. The set of transient vector fields on M is nonempty [5], since the gradient with respect to any Riemannian metric of a smooth function on M with no critical points is easily seen to be transient. Such a function can be produced by the following device. Let f be a Morse function on N, and let $x_1, \cdots, x_k$ be the critical points of f in M. Choose nonintersecting simple arcs $\gamma_1, \cdots, \gamma_k$ such that γ_j starts at x_j and meets ∂M just once where it ends. Use the arcs γ_j as guides for "fingers" which push ∂M into M in such a way that the arcs γ_j, and hence the critical points x_j, then lie outside of ∂M. In effect, this pushes all the critical points of f into $N - M$.

3. Proof of the Theorem

The idea of the proof is simply to combine the flow tube technique in [5] with the well-known proof that a compact manifold embeds in some Euclidean space–see, for example, [4].

Let $\tilde{X} \in \mathfrak{X}(N)$ be an extension of X to N, and let $\phi: N \times \mathbf{R} \to N$ be the flow associated to $\tilde{X}$. Since X is transient, for every $x \in M$ there exists an open interval $I = (a, b) \subset \mathbf{R}$ such that $0 \in I$, $\phi | \{x\} \times \bar{I}$ is 1–1 and

$$\phi(x, a), \phi(x, b) \in N - M.$$

It follows that there exists an embedded open n-ball $\Sigma \subset N$ such that $x \in \Sigma$, Σ is transverse to $\tilde{X}$, $\phi | \Sigma \times I$ is an embedding, and

$$\phi(\Sigma \times \{a, b\}) \subset N - M.$$

In [5], the set $\phi(\Sigma \times I)$ was called a flow tube for $\tilde{X}$ with ends outside of M. Since M is compact, there exists a finite set of points $x_1, \cdots, x_k \in M$ such that the associated flow tubes $\phi(\Sigma_i \times I_i)$, $i = 1, \cdots, k$, cover M.

Let

$$\psi_i\colon \phi(\Sigma_i \times I_i) \to \Sigma_i \times I_i$$

be the inverse of $\phi|\Sigma_i \times I_i$, let

$$\pi_i\colon \phi(\Sigma_i \times I_i) \to \Sigma_i$$

be ψ_i followed by projection of $\Sigma_i \times I_i$ onto Σ_i, and let

$$\zeta_i\colon \phi(\Sigma_i \times I_i) \to I_i$$

be ψ_i followed by projection of $\Sigma_i \times I_i$ onto I_i. When f_i is a map from $\phi(\Sigma_i \times I_i)$ to a Euclidean space, let f_i^* be the map from M to the same Euclidean space defined by

$$f_i^* = \begin{cases} f_i & \text{on } \phi(\Sigma_i \times I_i) \cap M, \\ 0 & \text{on } M - \phi(\Sigma_i \times I_i). \end{cases}$$

Let $\xi_i\colon \Sigma_i \to \mathbf{R}$, $i = 1, \cdots, k$, be a set of smooth functions with the following properties:

(a) $0 \leqslant \xi_i \leqslant 1$,

(b) ξ_i has compact support in Σ_i, and

(c) if Λ_i is the interior of $\{x \in \Sigma_i | \xi_i(x) = 1\}$ (with respect to the topology on Σ_i), then the flow tubes $\phi(\Lambda_i \times I_i)$, $i = 1, \cdots, k$, still cover M.

Let $\rho_i = \xi_i \circ \pi_i$ and

$$\zeta = \sum_{i=1}^{k} \rho_i^* \zeta_i^* \Big/ \sum_{i=1}^{k} \rho_i^*.$$

Then ζ is a smooth function on M and $X\zeta \equiv 1$, since $\tilde{X}\zeta_i \equiv 1$ on $\phi(\Sigma_i \times I_i)$ and ρ_i^* is constant on trajectories of X in M. Thus ζ is a globally defined time coordinate along trajectories of X.

Define $\alpha\colon M \to \mathbf{R}^k \times (\mathbf{R}^n)^k \times \mathbf{R}$ by

$$\alpha(x) = (\rho_1^*(x), \cdots, \rho_k^*(x), \rho_1^*(x)\pi_1^*(x), \cdots, \rho_k^*(x)\pi_k^*(x), \zeta(x)),$$

where π_i is now considered to map into $\mathbf{R}^n$ by identifying its target space Σ_i with the open unit ball in $\mathbf{R}^n$. We shall show that α is an embedding and that $\alpha_* X = \partial/\partial z | \alpha(M)$, where z is the coordinate function for the product space $\mathbf{R}^k \times (\mathbf{R}^n)^k \times \mathbf{R}$ corresponding to the last factor $\mathbf{R}$.

To prove that α is 1–1, suppose $\alpha(x) = \alpha(y)$. Let J be the (necessarily nonempty) set of indices i for which $\rho_i^*(x) \neq 0$. Then x and y lie on the same trajectory segment, call it S, in

$$\bigcap_{j \in J} \phi(\Sigma_j \times I_j)$$

because $\rho_i^*(x) = \rho_i^*(y)$, $i = 1, \cdots, k$, and $\pi_j^*(x) = \pi_j^*(y)$, $j \in J$. Let ζ_S: $S \to \mathbf{R}$ be defined by

$$\zeta_S = \sum_{j \in J} \rho_j \zeta_j \Big/ \sum_{j \in J} \rho_j.$$

Then $\zeta_S(x) = \zeta(x) = \zeta(y) = \zeta_S(y)$. But, for $j \in J$, ρ_j is constant on S and ζ_j is a time coordinate along S, so ζ_S is 1–1 on S. Thus $x = y$, so α is 1–1.

Since α is smooth and M is compact, it follows that α is a homeomorphism onto its image. The proof that α has rank $n + 1$ at each point is straightforward. Thus we can conclude that α is an embedding.

Finally, $\alpha_* X = \partial / \partial z | \alpha(M)$ because α takes trajectories of X to line segments parallel to the z-axis, since ρ_i^* and π_i^*, $i = 1, \cdots, k$, are constant on trajectories of X, and because

$$(\alpha_* X)(z) = X(z \circ \alpha) = X\zeta \equiv 1.$$

References

[1] R. Bott, *Vector fields and characteristic numbers*, Michigan Math. J. **14** (1967) 231–244.

[2] S. S. Chern, *The geometry of G-structures*, Bull. Amer. Math. Soc. **72** (1966) 167–219.

[3] L. Karp, *Parallel vector fields and the topology of manifolds*, Bull. Amer. Math. Soc. **83** (1976) 1051–1053.

[4] J. Munkres, *Elementary differential topology*, Annals of Math. Studies, No. 54, Princeton University Press, Princeton, 1963.

[5] P. Percell, *Structural stability on manifolds with boundary*, Topology **12** (1973) 123–144.

[6] R. Thom, Private communication.

UNIVERSITY OF HOUSTON