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THE GAUSS MAP OF A THREE-DIMENSIONAL MINIMAL SURFACE

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1. Introduction

It is well known that the Gauss map of a connected two-dimensional minimal submanifold of $\mathbf{R}^3$ either is an open map or its image is just one point. This is based on the connection between two-dimensional minimal surfaces and analytic functions. It is natural to wonder to what extent the above result can be generalized to a connected three-dimensional minimal submanifold M of $\mathbf{R}^4$. Consideration of simple examples leads to the following conjecture: Either M is a portion of a cartesian product (of a two-dimensional minimal surface and a line) or a portion of a cone or the Gauss map of M is open. We will show this conjecture to be false.

The method of this paper is to derive, using an estimate from [6] and the assumed truth of the conjecture, certain conclusions about two-dimensional surfaces of least area. Specifically, we conclude that there is an oriented surface of least area T with boundary R, where R is as in §5(3), such that T is invariant under the transformation

$$(x, y, z) \to (-y, x, -z).$$

It is shown in §7 that no such T can exist. Thus the conjecture cannot be true.

We state the conjecture in a more convenient form. Let $\Omega \subset \mathbf{R}^n$ $(n \geqslant 2)$ be a connected open set. Suppose $f\colon \Omega \to \mathbf{R}$ is of class 2 and satisfies the minimal surface equation. Define the Gauss map $\zeta\colon \Omega \to \mathbf{S}^n$ by requiring, for each $x \in \Omega$,

(i) $\zeta(x) \cdot (\mathbf{e}_i + D_i f(x)\mathbf{e}_{n+1}) = 0, \quad i = 1, 2, 3, \cdots, n,$

(ii) $\zeta(x) \cdot \mathbf{e}_{n+1} > 0;$

throughout this paper, $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \cdots, \mathbf{e}_{n+1}$ will be the standard basis for $\mathbf{R}^{n+1}$.

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OM_n: Either we have

$$\zeta(\Omega) \subset \zeta(\Omega \sim K)$$

for each compact $K \subset \Omega$ or ζ is an open map.

Thus OM_2 is true and we will show OM_3 can fail to hold. Notice that whenever the graph of f is a portion of a cartesian product or a portion of a cone, then we have

$$\zeta(\Omega) \subset \zeta(\Omega \sim K)$$

for each compact $K \subset \Omega$.

2. Preliminaries

Except when otherwise stated, we will follow the notation and terminology of [1].

(1) Let n denote an integer ($n \geqslant 2$) and Ω a bounded open uniformly convex subset of $\mathbf{R}^n$. Set

$$\Gamma = \text{Bdry}\,\Omega, \quad \Gamma_0 = \partial(\mathbf{E}^n \llcorner \Omega).$$

(2) For each lipschitzian u: Clos $\Omega \to \mathbf{R}$ we write

$$\mathbf{G}[u] = \int_\Omega |Du| \, d\mathcal{L}^n,$$

$$\mathbf{A}[u] = \int_\Omega (1 + |Du|^2)^{1/2} d\mathcal{L}^n.$$

(3) For each lipschitzian $\phi: \Gamma \to \mathbf{R}$ we denote by $\mathfrak{B}(\phi)$ the set of lipschitzian u: Clos $\Omega \to \mathbf{R}$ such that $u|\Gamma = \phi$.

(4) For use in the next proposition, fix $\phi_0: \Gamma \to \mathbf{R}$ which satisfies the bounded slope condition (see [5, Definition 1.1]) and $u_0 \in \mathfrak{B}(\phi_0)$ with

$$\mathbf{G}[u_0] = \inf\{\mathbf{G}[u]: u \in \mathfrak{B}(\phi_0)\}$$

(u_0 exists by [6, 3(2)]).

(i) Set

$$T_r = \Gamma_0 \llcorner \{x: \phi_0(x) \geqslant r\} - \partial(\mathbf{E}^n \llcorner \{x: u_0(x) \geqslant r\}),$$

for $a = \inf\{u_0(x): x \in \Omega\} < r < b = \sup\{u_0(x): x \in \Omega\}$.

(ii) For each lipschitzian v: Clos $\Omega \to \mathbf{R}$ define

$$U_v: \text{Clos}\,\Omega \to \mathbf{R}, \quad N_v: \text{Clos}\,\Omega \to \Lambda^1(\mathbf{R}^n)$$

as in [6, 4(2)].

3. Proposition. *Suppose* $\|Du_0(x)\| > 0$ *holds for* $\mathfrak{L}^n$ *almost every* $x \in$ Clos Ω.

(1) *For* $v \in \mathfrak{B}(\phi_0)$, $\mathbf{G}[v] = \mathbf{G}[u_0]$ *implies* $v = u_0$.

(2) *Let* $\omega \in \mathbf{O}(n)$ *be such that*

(i) $\omega(\Omega) = \Omega$,

(ii) $\omega_{\#}\mathbf{E}^n = -\mathbf{E}^n$,

(iii) $-\phi_0 \circ \omega(x) = \phi_0(x)$ *for* $x \in \Gamma$.

If $\mathcal{H}^{n-1}[\Gamma \cap \phi_0^{-1}(0)] = 0$ *holds, then we have* $\omega_{\#}T_0 = T_0$.

Proof. (1) Suppose $v \in \mathfrak{B}(\phi_0)$ satisfies $\mathbf{G}[v] = \mathbf{G}[u_0]$. By [6, 10(1)] we have

$$\int_a^b \int_{u_0^{-1}(r)} |N_v(x)| \, d\mathcal{H}_x^{n-1} d\mathfrak{L}_r^1 = 0.$$

Applying [6, 5, 8(2)] we obtain

$$\int_a^b \int \|\vec{T}_r(x) \llcorner D(v - u_0)(x)\| \, d\|T_r\|_x \, d\mathfrak{L}_r^1 = 0.$$

By [6, 7(1)] we see that [2, 2] is applicable for $\mathfrak{L}^1$ almost every r, so we have

$$\int_a^b \int |v - u_0| \, d\|T_r\|_x \, d\mathfrak{L}_r^1 = 0.$$

Conclusion (1) now follows by applying [6, 5] and [1, 3.2.12].

(2) Using (1) we obtain

$$-u_0 \circ \omega(x) = u_0(x) \quad \text{for } x \in \text{Clos}\, \Omega.$$

Noting also $\omega_{\#}\Gamma_0 = -\Gamma_0$, we compute

$$\omega_{\#}T_0 = \partial(\mathbf{E}^n \llcorner \{x: u_0(x) \leqslant 0\}) - \Gamma_0 \llcorner \{x: \phi_0(x) \leqslant 0\},$$

and hence

$$T_0 - \omega_{\#}T_0 = \Gamma_0 \llcorner \{x: \phi_0(x) = 0\} - \partial(\mathbf{E}^n \llcorner \{x: u_0(x) = 0\}).$$

Conclusion (2) now follows from [1, 2.9.11].

4. Lemma. *Let* $f_1, f_2, f_3, \cdots$ *be a sequence of class* 2 *functions on* $U \subset \mathbf{R}^n$ (U *open*) *which converge uniformly on compact subsets of* U *to the lipschitzian function* f. *If there is* $d > 0$ *such that* $|Df_k(x)| \geqslant d$ *holds for each* $x \in U$ *and each* $k = 1, 2, 3, \cdots$, *then* $|Df(x)| \geqslant d$ *holds for* $\mathfrak{L}^n$ *almost every* $x \in U$.

Proof. Fix $x \in U$ and $\varepsilon > 0$ so that $Df(x)$ exists and $\mathbf{B}(x, \varepsilon) \subset U$. By solving the initial value problem

$$\langle \mathbf{e}_1, Du(t) \rangle = \operatorname{grad} f_k(u(t)), \; u(0) = x,$$

we see easily that there exists $y_k \in \mathbf{B}(x, \varepsilon)$ with $f_k(y_k) - f_k(x) \geqslant d\varepsilon$, for $k = 1, 2, 3, \cdots$. It follows that there exists $z \in \mathbf{B}(x, \varepsilon)$ with $f(z) - f(x) \geqslant d\varepsilon$. Since $\varepsilon > 0$ can be chosen arbitrarily small, we have $|Df(x)| \geqslant d$.

5. Notation

(1) Set

$$x(r, \phi, \theta) = (r \sin \phi \cos \theta, r \sin \phi \sin \theta, r \cos \phi),$$
$$v(\phi, \theta) = (-\cos \phi \cos \theta, -\cos \phi \sin \theta, \sin \phi).$$

(2) For use in the next proposition, fix $0 < d$, $0 < \phi_0 < \pi/2$, $F: \mathbf{S}^2 \to \mathbf{R}$ of class 3, and affine functions $A_+, A_-: \mathbf{R}^3 \to \mathbf{R}$. Suppose

(i) If $\phi_0 \leqslant \phi \leqslant \pi - \phi_0$ holds, then we have, for each $0 \leqslant \theta \leqslant 2\pi$,

$$\langle v(\phi, \theta), DF[x(1, \phi, \theta)]\rangle \geqslant d,$$

(ii) $DA_+ = DA_-$, $\langle \mathbf{e}_3, DA_+\rangle \geqslant d$, and

$$\langle \mathbf{e}_1, DA_+\rangle = \langle \mathbf{e}_2, DA_+\rangle = 0,$$

(iii) $A_-(x) \leqslant F(x) \leqslant A_+(x)$, for each $x \in \mathbf{S}^2$,

(iv) $F|U_+ = A_+|U_+$ and $F|U_- = A_-|U_-$,

where

$$U_+ = \{x(1, \phi, \theta): 0 \leqslant \phi \leqslant \phi_0\},$$
$$U_- = \{x(1, \phi, \theta): \pi - \phi_0 \leqslant \phi \leqslant \pi\}.$$

(3) For each $\theta \in \mathbf{R}$, define $f_\theta: \mathbf{R} \to \mathbf{R}^3$ by setting

$$f_\theta(\phi) = x(1, \phi, \theta).$$

Put

$$R = R_0 - R_{\pi/2} + R_\pi - R_{3\pi/2},$$

where

$$R_\theta = f_{\theta\#}[0, \pi].$$

(4) Define $\tau, \mu, \sigma \in \mathbf{O}(3)$ by setting

$$\tau(x, y, z) = (x, y, -z),$$
$$\mu(x, y, z) = (-y, x, z),$$
$$\sigma(x, y, z) = (y, x, z),$$

for each $(x, y, z) \in \mathbf{R}^3$. Note that $(\mu \circ \tau)_\# \mathbf{E}^3 = -\mathbf{E}^3$.

6. Proposition. *Suppose OM_3 holds.*

(1) *Let $f \in \mathfrak{B}(F)$ satisfy*

$$A[f] = \inf\{\mathbf{A}[u]: u \in \mathfrak{B}(F)\}.$$

Then $|Df(x)| \geqslant d$ holds for each $x \in \mathbf{U}(0, 1)$.

(2) *Let $g \in \mathfrak{B}(F)$ satisfy*

$$\mathbf{G}[g] = \inf\{\mathbf{G}[u]: u \in \mathfrak{B}(F)\}.$$

Then $|Dg(x)| \geqslant d$ holds for $\mathfrak{L}^3$ almost every $x \in \mathbf{U}(0, 1)$.

Proof. (1) The Gauss map ζ: $\mathbf{U}(0, 1) \to \mathbf{S}^3$ defined in §1 extends continuously to $\mathbf{B}(0, 1)$ (see [4, Lemma 4]). We write $\mathbf{R}^4 = \mathbf{R}^3 \times \mathbf{R}$ and set $\xi = \rho \circ \zeta$, where ρ is projection on the first factor. Using the planes defined by A_+ and A_- as barriers (§5(2iii), §5(2iv), and [5, Lemma 2.2]), we see that

$$D_3 f[x(1, \phi, \theta)] \geqslant d$$

holds for $0 \leqslant \phi \leqslant \phi_0$ and $\pi - \phi_0 \leqslant \phi \leqslant \pi$. Combining this with §5(2i), we easily see that

$$(*) \qquad |\xi(x)| \geqslant d/(1 + d^2)^{1/2}$$

holds for $x \in \mathbf{S}^2$. This implies by OM_3 that $(*)$ holds for $x \in \mathbf{B}(0, 1)$, because, as is easily checked, for $x \in \mathbf{S}^2$, if $\mathbf{e}_1 \cdot \xi(x) = \mathbf{e}_2 \cdot \xi(x) = 0$ holds, then $\mathbf{e}_3 \cdot \xi(x) < 0$ holds. The condition $(*)$ clearly implies $|Df(x)| \geqslant d$.

(2) For $k = 1, 2, 3, \cdots$ define F_k: $\mathbf{S}^2 \to \mathbf{R}$ by setting

$$F_k(x) = kF(x).$$

The conditions of §5(2) hold for F_k with d, A_+, A_- replaced by kd, kA_+, kA_-, respectively. Let $f_k \in \mathfrak{B}(F_k)$ satisfy

$$\mathbf{A}[f_k] = \inf\{\mathbf{A}[u]: u \in \mathfrak{B}(F_k)\},$$

and set $g_k = k^{-1}f_k$. By [5, Propositions 3.1 and 6.2] we have $\mathrm{Lip}(g_k) \leqslant M$ (M independent of k), and by (1) we have $|Dg_k(x)| \geqslant d$ for each $x \in \mathbf{U}(0, 1)$. By the Ascoli Theorem, the proof of [6, 3(2)], and Lemma 4, we obtain $g \in \mathfrak{B}(F)$ such that

$$\mathbf{G}[g] = \inf\{\mathbf{G}[u]: u \in \mathfrak{B}(F)\},$$

and $|Dg(x)| \geqslant d$ holds for $\mathfrak{L}^3$ almost every $x \in \mathbf{U}(0, 1)$. Conclusion (2) now follows from Proposition 3(1).

7. Proposition. *There exists no absolutely area minimizing $T \in \mathfrak{R}_2(\mathbf{R}^3)$ with $\partial T = R$ and*

$$T = (\mu \circ \tau)_\# T.$$

Proof. Suppose such a T exists. Set

$$\begin{aligned} T_k &= T \llcorner \{x(r, \phi, \theta): 0 < r \leqslant 1, 0 < \phi < \pi, \\ &\quad 2^{-1}(k-1)\pi \leqslant \theta < 2^{-1}k\pi\}, \quad k = 1, 2, 3, 4, \\ W_1 &= \partial T_1 \llcorner \{(x, y, z): y = 0\}, \\ W_2 &= \partial T_1 \llcorner \{(x, y, z): x = 0, y > 0\}, \\ W_3 &= = \partial T_1 \llcorner \{(x, y, z): x = y = 0\}, \\ W_4 &= \partial T_1 - W_3. \end{aligned}$$

Using [1, 4.1.15], we obtain

$$(\mu \circ \tau)_\#(W_1 + W_2) + (\mu \circ \tau \circ \mu^2)_\#(W_1 + W_2)$$
$$= (\sigma \circ \tau)_\#(W_1 + W_2) + (\sigma \circ \tau \circ \mu^2)_\#(W_1 + W_2).$$

From this we see that

$$T' = T_1 + (\sigma \circ \tau)_\# T_1 + T_3 + (\sigma \circ \tau)_\# T_3$$

is absolutely area minimizing and satisfies $\partial T' = R$. We compute

$$\partial T' = 2W_3 + 2(\sigma \circ \tau)_\# W_3 + W_4 + (\sigma \circ \tau)_\# W_4 + \mu^2{}_\# W_4 + (\sigma \circ \tau \circ \mu^2)_\# W_4.$$

Consequently, we have

$$W_3 + (\sigma \circ \tau)_\# W_3 = 0,$$

and thus $\partial T'' = R_0 - R_{\pi/2}$ where

$$T'' = T' \llcorner \{(x, y, z): y > -x\} = T_1 + (\sigma \circ \tau)_\# T_1.$$

Note that T'' is absolutely area minimizing.

There is but one absolutely area minimizing $Q \in \mathcal{R}_2(\mathbf{R}^3)$ with $\partial Q = R_0 - R_{\pi/2}$ (see [3, p. 1063]); further, $\Theta^2(\|Q\|, x) = 1$ holds for $\|Q\|$ almost all $x \in \mathbf{R}^3$, spt $Q \sim$ spt ∂Q is diffeomorphic to a connected open subset of $\mathbf{R}^2$, and we have

$$\operatorname{spt} Q \cap \{(x, y, z): xy = 0\} \subset \operatorname{spt}(R_0 - R_{\pi/2}).$$

We conclude that $\operatorname{spt} T_1 \subset \operatorname{spt} Q$, since $\mathbf{M}[T''] = \mathbf{M}[T_1] + \mathbf{M}[(\sigma \circ \tau)_\# T_1]$ holds; hence we have

$$\operatorname{spt} \partial T_1 \subset \operatorname{spt}(R_0 - R_{\pi/2}),$$

and, by the constancy theorem (see [1, 4.1.7]), $T_1 = lQ$ for some integer l. This contradicts $\partial T'' = R_0 - R_{\pi/2}$.

8. Lemma. *Fix* $0 < d_1 < 2^{-1}$ *and* $0 < \phi_0 < 2^{-1}\pi$. *There exists a function* $f(a, \phi)$, *defined and of class* ∞ *for* $\phi_0 < a < \pi - \phi_0$ *and* $-\infty < \phi < \infty$, *satisfying*

(i) $$-f(\pi - a, \pi - \phi) = f(a, \phi),$$

(ii) $0 \leqslant \phi \leqslant \phi_0$ *implies* $f(a, \phi) = 1 - d_1 + d_1 \cos \phi$,

(iii) $\pi - \phi_0 \leqslant \phi \leqslant \pi$ *implies* $f(a, \phi) = -1 + d_1 + d_1 \cos \phi$,

(iv) $$-1 + d_1 + d_1 \cos \phi \leqslant f(a, \phi) \leqslant 1 - d_1 + d_1 \cos \phi, \quad \text{for } 0 \leqslant \phi \leqslant \pi,$$

(v) $$f(a, a) = 0,$$

(vi) $$D_2 f(a, \phi) \leqslant -d_1 \sin \phi, \quad \text{for } \phi_0 \leqslant \phi \leqslant \pi - \phi_0.$$

Proof. Construction of such a function is routine.

9. Theorem. *OM_3 can fail.*

Proof. We suppose OM_3 is valid. Let $\varepsilon > 0$ be arbitrary. Choose ν: $\mathbf{R} \to \mathbf{R}$ of class ∞ so that

$$0 < \inf\{\nu(\theta): \theta \in \mathbf{R}\}, \nu(\theta + \pi/2) = \pi - \nu(\theta),$$
$$\mathcal{F}_{\mathbf{B}(0, 1)}(R - \nu^*_\#[0, 2\pi]) < \varepsilon,$$

where $\nu^*(\theta) = x(1, \nu(\theta), \theta)$. Applying Lemma 8, with $\phi_0 < \inf\{\nu(\theta): \theta \in \mathbf{R}\}$, we define F: $\mathbf{S}^2 \to \mathbf{R}$ by setting

$$F[x(1, \phi, \theta)] = f[\nu(\theta), \phi]$$

for $0 \leqslant \phi \leqslant \pi$ and $0 \leqslant \theta \leqslant 2\pi$. One checks easily that F satisfies the conditions of §5(2) (with $d = d_1 \sin \phi_0$). Let $u \in \mathfrak{B}(F)$ satisfy

$$\mathbf{G}[u] = \inf\{\mathbf{G}[v]: v \in \mathfrak{B}(F)\}.$$

By Proposition 6(2), Propostion 3 is applicable to u, so, replacing ω in Proposition 3(2) by $\mu \circ \tau$ and ϕ_0 by F, we have $(\mu \circ \tau)_\# T = T$, where

$$T = [\partial(\mathbf{E}^3 \llcorner \mathbf{U}(0, 1))] \llcorner \{x: F(x) \geqslant 0\} - \partial(\mathbf{E}^3 \llcorner \{x: u(x) \geqslant 0\}).$$

By [6, 7(1)], $T \in \mathfrak{R}_2(\mathbf{R}^3)$ is absolutely area minimizing. Clearly,

$$\partial T = \nu^*_\#[0, 2\pi] \quad \text{and} \quad \mathbf{M}[T] \leqslant 4\pi$$

hold. Since $\varepsilon > 0$ was arbitrary, the compactness theorem (see [1, 4.2.17]) leads to a contradiction of Proposition 7.

References

[1] H. Federer, *Geometric measure theory*, Springer, New York, 1969.

[2] ______, *A minimizing property of extremal submanifolds*, Arch. Rational Mech. Anal. **59** (1975) 207–217.

[3] W. Fleming, *An example in the problem of least area*, Proc. Amer. Math. Soc. **7** (1956) 1063–1074.

[4] H. Jenkins & J. Serrin, *The Dirichlet problem for the minimal surface equation in higher dimensions*, J. Reine Angew. Math. **229** (1968) 170–187.

[5] M. Miranda, *Un teorema di esistenza e unicità per il problema dell'area minima in n variabili*, Ann. Scuola Norm. Sup. Pisa, Ser. III, **19** (1965) 233–249.

[6] H. Parks, *Explicit determination of area minimizing hypersurfaces*, Duke Math. J. **44** (1977) 519–534.

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