J. DIFFERENTIAL GEOMETRY
16 (1981) 63–73

SOBOLEV SPACES OF DIFFERENTIAL FORMS AND DE RHAM-HODGE ISOMORPHISM

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0. Introduction

In this paper we examine conditions under which integration of forms over simplexes of a smooth triangulation of a complete oriented Riemannian manifold M induces an isomorphism between simplicial L^2-cohomology and the spaces of L^2 harmonic forms. A theorem of this type was proved in [5] for the case when M is an infinite Galois covering of a compact manifold. In this case various constructions and estimates could be done locally in the compact base and then pulled back to M. It was more or less clear that a similar proof could be given in a more general setting provided the manifold under consideration satisfies certain uniformity conditions. The conditions we shall use were formulated originally by Aubin [1] in his work on Sobolev spaces of functions on Riemannian manifolds.

Condition I. The manifold M has the injectivity radius $d > 0$.

Condition C_m. The curvature tensor R and its covariant derivatives $\nabla^l R$, $0 \leqslant l \leqslant m$, are uniformly bounded.

Clearly, if M is a covering of a compact manifold, Condition C_m holds for every $m \geqslant 0$ and so does Condition I. Existence of complete metrics satisfying condition C_m for an arbitrary $m \geqslant 0$ on every open manifold was established recently by R. Greene. In general if C_m holds for some $m \geqslant 0$ the two Sobolev norms involving derivatives of order up to $m + 2$ (one using the Laplacian, the other using covariant derivative) are equivalent. This is sufficient to obtain enough control over constants in Sobolev inequalities for the purpose of establishing the de Rham-Hodge isomorphism.

This paper consists of two sections. The first section treats various Sobolev spaces and inequalities. We obtain a generalization of certain results of Aubin [1]. The proofs are based on a theorem of Chernoff [3] which can be interpreted as stating that compactly supported C^∞ forms are dense in certain Sobolev spaces. We believe that the results in this section have

Received September 13, 1979. Partially supported by NSF Grant #MCS7802285 and Sonderforschungsbereich Theoretische Mathematik at University of Pennsylvania

significance beyond the application presented in the second section, which consists of the proof of the de Rham-Hodge isomorphism along the lines of [5].

During the work on this paper, I benefited a great deal from conversations with E. Calabi and J. Kazdan. In particular, Kazdan brought Aubin's work to my attention. Min-Oo helped me with an example showing that curvature assumptions are necessary in Theorem 1.3. I am grateful to all of them.

1. Sobolev spaces of differential forms

Throughout the paper M will denote a complete oriented Riemannian manifold of dimension N. The Riemannian metric induces inner products in fibers of various tensor bundles on M. We will denote by $\langle \cdot, \cdot \rangle$ these pointwise inner products, and by $|\cdot|$ the corresponding norms. The global (integrated) inner products and norms are given by

$$(\alpha, \beta)_0 = \int_M \langle \alpha, \beta \rangle dV,$$

$$\|\alpha\|_0^2 = \int_M |\alpha|^2 dV,$$

where dV is the volume element of the metric. If α, β are differential forms of the same degree, then

$$(\alpha, \beta)_0 = \int_M \alpha \wedge {*\beta},$$

where $*$ is the Hodge operator. A tensor α is square-integrable (in L^2) if it is measurable and $\|\alpha\|_0 < \infty$.

Definition 1.1. (a) The Sobolev space $A_1^{k,p}(M)$ is the space of differential forms on M of degree p, which are in L^2 together with their covariant derivatives of orders up to k. The Sobolev norm for $\omega \in A_1^{k,p}(M)$ is defined by

$$|||\omega|||_k = \sum_{l=0}^{k} \|\nabla^l \omega\|_0.$$

(b) The space $A^{k,p}(M)$ is defined for even integers k, $k = 2m$, and consists of p-forms ω on M for which both ω and $(I + \Delta)^m \omega$ are square integrable. The norm in $A^{k,p}$ is given by

$$\|\omega\|_k = \|(I + \Delta)^m \omega\|_0.$$

In this definition the derivatives are taken in the sense of distributions, and Δ denotes the Laplace operator.

The spaces $A_1^{k,p}$ are well suited to obtaining pointwise estimates of forms and their derivatives in terms of L^2 norms of high covariant derivatives (cf. [2] where the completion of the space of compactly supported C^∞ forms with respect to $||| \ |||_k$ was studied). The advantage of $A^{k,p}$ is that C^∞ forms with compact support are dense in these spaces. More precisely we have the following theorem due to Chernoff [3].

Theorem 1.2. *For every integer $m \geqslant 1$, the operator $(I + \Delta)^m$ is essentially self-adjoint as an operator on $A^{0,p}(M)$ (the Hilbert space of L^2 p-forms) with the domain consisting of all compactly supported C^∞ p-forms.*

Remark. Denote by $A_c^p(M)$ the space of C^∞ p-forms with compact support, and let $T = (I + \Delta)^m$. The theorem says that $\bar{T}$, the closure of T, satisfies $\bar{T} = \bar{T}^*$ and, clearly, $\bar{T}^* = T^*$. According to definitions, the domain of T^* is precisely $A^{2m,p}(M)$, and the domain of $\bar{T}$ is the closure in $A^{2m,p}(M)$, of $A_c^p(M)$, i.e., C^∞ forms with compact support are dense in $A^{2m,p}(M)$.

The following theorem is both less and more general than Theorem 2 of [1]. It is less general since we obtain only L^2 estimates rather than L^p ones for $\infty > p \geqslant 1$. However Aubin considers functions only, and the Sobolev spaces in his paper analogous to our $A_1^{k,p}(M)$ are completions of the spaces of C^∞ p-forms ω with $|||\omega|||_k < \infty$, whereas we consider all forms whose distributional derivatives are in L^2. Moreover, we make no assumptions about the injectivity radius.

Theorem 1.3. *If M is complete and satisfies Condition C_{2m}, then $A^{2m+2,p}(M) = A_1^{2m+2,p}(M)$ (i.e., they are equal as abstract vector spaces, and the two norms are mutually bounded). In particular $A_c^p(M)$ is dense $A_1^{2m+2,p}(M)$.*

Proof. It is very easy to see that for $\omega \in A_1^{2m,p}(M)$, $(I + \Delta)^m\omega$ is in L^2 and

$$(1.4) \qquad ||\omega||_{2m} \leqslant C|||\omega|||_{2m},$$

for every integer $m \geqslant 0$ with the constant C depending only on $N = \dim M$ and m. Thus the natural mapping of $A_1^{2m,p}(M)$, into $A^{2m,p}(M)$, is bounded. In view of Theorem 1.2 it will suffice to prove that there exists a constant $C_1 > 0$ depending only on the bounds of $|R|$, $|\nabla R|, \cdots, |\nabla^{2m}R|$, such that for every $\omega \in A_c^p(M)$

$$(1.5) \qquad |||\omega|||_{2m+2} \leqslant C_1||\omega||_{2m+2}$$

holds. It will be convenient to replace $|| \ ||_{2m+2}$ by an equivalent norm $|| \ ||'_{2m+2}$:

$$||\omega||'_{2m+2} = \sum_{l=0}^{m+1} ||\Delta^l\omega||_0,$$

and prove

$$(1.5'_m) \qquad |||\omega|||_{2m+2} \leqslant C'_1 \|\omega\|'_{2m+2}, \quad \text{if } \omega \in A^p_c(M),$$

where the constant C'_1 depends only on the bounds of $|\nabla^l R|$, $0 \leqslant l \leqslant m$. Consider first the case $m = 0$. The Weitzenbock identity $\Delta\omega = \nabla^*\nabla\omega + R_p\omega$, where ∇^* is the formal adjoint of ∇ and R_p is an algebraic operator involving the curvature tensor, implies

$$(1.6) \qquad (\nabla\omega, \nabla\omega)_0 = (\Delta\omega, \omega)_0 - (R_p\omega, \omega)_0$$

for $\omega \in A^p_c(M)$. Since Condition C_0 is assumed to hold, $\|\nabla\omega\|_0$ can be estimated in terms of $\|\omega\|'_2$. In order to estimate $\|\nabla^2\omega\|_0$ we introduce some notation. $\mathfrak{R}$ will denote an algebraic operator defined in terms of the curvature tensor R, and c will stand for a constant depending on the bound of $|R|$. The Bianchi identity yields

$$(1.7) \qquad \nabla^*\nabla = -\nabla\nabla^* = \mathfrak{R}.$$

To complete the proof of $(1.5'_0)$ we estimate the L^2 norm of $\nabla^2\omega$ using (1.6) and (1.7):

$$(1.8) \qquad \begin{aligned} \|\nabla^2\omega\|_0^2 &= (\nabla^*\nabla\nabla\omega, \nabla\omega)_0 = (\nabla\nabla^*\nabla\omega, \nabla\omega)_0 + (\mathfrak{R}\nabla\omega, \nabla\omega)_0 \\ &\leqslant \|\nabla^*\nabla\omega\|_0^2 + c\|\nabla\omega\|_0 \leqslant \|\nabla^*\nabla\omega\|_0^2 + c(\|\omega\|'_2)^2. \end{aligned}$$

On the other hand, by Weitzenbock identity we have

$$(1.9) \qquad \|\nabla^*\nabla\omega\|_0 \leqslant \|\Delta\omega\|_0 + \|\mathfrak{R}\omega\|_0 \leqslant \|\Delta\omega\|_0 + c\|\omega\|_0 \leqslant c\|\omega\|'_2.$$

$(1.5'_0)$ follows by combining (1.6), (1.8) and (1.9).

The proof of $(1.5'_m)$ for $m > 0$ is inductive and follows a similar pattern. In the estimates below c will denote a positive constant depending on bounds for $|R|, |\nabla R|, \cdots, |\nabla^{2m}R|$. To prove $(1.5'_m)$ if $(1.5'_{m-1})$ has been verified we have to estimate $\|\nabla^{2m+1}\omega\|_0$ and $\|\nabla^{2m+2}\omega\|_0$ in terms of $\|\omega\|'_{2m+2}$ for $\omega \in A^p_c(M)$.Thus

$$(1.10) \qquad \begin{aligned} \|\nabla^{2m+1}\omega\|_0^2 &= (\nabla^*\nabla^{2m+1}\omega, \nabla^{2m}\omega)_0 \leqslant c\|\nabla^{2m+2}\omega\|_0 \cdot \|\nabla^{2m}\omega\|_0 \\ &\leqslant \frac{c}{2}\left(\varepsilon^2\|\nabla^{2m+2}\omega\|_0^2 + \frac{1}{\varepsilon^2}\|\nabla^{2m}\omega\|_0^2\right). \end{aligned}$$

By the inductive assumption, $\|\nabla^{2m}\omega\|_0 \leqslant c\|\omega\|'_{2m} \leqslant c\|\omega\|'_{2m+2}$ and therefore

$$(1.11) \qquad \|\nabla^{2m+1}\omega\|_0^2 \leqslant c\left(\varepsilon^2\|\nabla^{2m+2}\omega\|_0^2 + \frac{1}{\varepsilon^2}\|\omega\|'_{2m+2}\right).$$

In this inequality $\varepsilon > 0$ is arbitrary and will be specified shortly. Because of this last inequality it suffices now to estimate $\|\nabla^{2m+2}\omega\|_0$. Now

$$\|\nabla^{2m+2}\omega\|_0^2 = (\nabla^{2m+2}\omega, \nabla^{2m+2}\omega)_0 = ((\nabla^*)^{2m+1}\nabla^{2m+2}\omega, \nabla\omega)_0.$$

Using the Bianchi identity and integration by parts repeatedly as in (1.8) we obtain

$$\|\nabla^{2m+2}\omega\|_0^2 = \|(\nabla^*\nabla)^{2m+1}\omega\|_0^2 + E, \tag{1.12}$$

where E is quadratic in covariant derivatives of ω of order not exceeding $2m + 1$, and linear in covariant derivatives of R of order less or equal $2m$ (cf. (1.8)). The terms in E involving derivatives of ω of order $2m + 1$ can be estimated using (1.11). It is here that one has to take $\varepsilon > 0$ sufficiently small in order to absorb $\|\nabla^{2m+2}\omega\|_0^2$ appearing on the right-hand side inequality on its left-hand side. The inductive assumption $(1.5'_{m-1})$ and Conditions C_m enable us to estimate the remaining terms in E yielding

$$\|\nabla^{2m+2}\omega\|_0^2 \leqslant c\big(\|(\nabla^*\nabla)^{m+1}\omega\|_0 + \|\omega\|'_{2m+2}\big). \tag{1.13}$$

Finally $(\nabla^*\nabla)^{m+1}$ is the leading term of Δ^{m+1}, and the Weitzenbock identity, Condition C_m and $(1.5'_{m-1})$ imply that

$$\|(\nabla^*\nabla)^{m+1}\omega\|_0^2 \leqslant C \cdot \|\omega\|'_{2m+2}. \tag{1.14}$$

Now (1.14), (1.13) and (1.11) prove $(1.5'_m)$, which completes the proof of Theorem 1.3.

We now give an example of a surface M whose curvature is unbounded and for which $A_1^{2,p} \neq A^{2,p}$. Let M be $\mathbf{R}^2$ equipped with the metric

$$ds^2 = dr^2 + f(r)^2 d\theta^2.$$

We shall specify the function f later. The Hodge $*$ operator satisfies $*dr = fd\theta$, $*d\theta = -dr/f$. It follows that if ω is a C^∞ form of degree one such that

$$\omega = \frac{1}{f(r)}dr + d\theta \quad \text{for } r \geqslant 1,$$

then $\Delta\omega = 0$ for $r \geqslant 1$. Therefore if $\omega \in A^{0,p}$, then $\omega \in A^{2,p}$. On the other hand, $|\omega|^2 = 2/f^2$ on $r \geqslant 1$ and $dV = f\, dr\, d\theta$. Thus ω is in L^2 if and only if

$$\int_1^\infty \frac{dr}{f} < \infty.$$

By Weitzenbock identity $\Delta\omega = \nabla^*\nabla\omega - (f''/f)\omega$. Thus if $\Delta\omega$ is in L^2 and $\nabla^2\omega$ is square-integrable, then $\nabla^*\nabla\omega$ and $(f''/f)\omega$ are in L^2. But $(f''/f)\omega$ is in L^2 if and only if

$$\int_1^\infty \frac{(f'')}{f^3}dr < \infty.$$

Hence to construct an example in which ω, $\Delta\omega$ are in L^2 but $\nabla^2\omega$ is not we have to exhibit a function for which $\int_1^\infty dr/f(r) < \infty$ and $\int_1^\infty (f'')^2/f^3\, dr = \infty$. An example of such a function was provided by Min-Oo. We can take for f any C^∞ function such that

$$f(r) = \begin{cases} r & \text{if } r \leqslant \frac{1}{2}, \\ r^2 + \dfrac{1}{r}\sin(r^4) & \text{if } r \geqslant 1. \end{cases}$$

The above example shows that some curvature assumptions are necessary for the conclusion of Theorem 1.3 to hold. We remark that the method of §4 of [3] implies that $\nabla^*\nabla = \Delta - \mathfrak{R}$ is essentially self-adjoint on $A_c^p(M)$ together with all its powers provided $\Delta - \mathfrak{R}$ is bounded from below (i.e., if $(\Delta\omega, \omega)_0 - (\mathfrak{R}\omega, \omega)_0 \geqslant c(\omega, \omega)_0$ for every $\omega \in A_c^p$). This semiboundedness would be implied by the boundedness from above of the highest eigenvalue of $\mathfrak{R}$ and in particular by our Condition C_0. This however is not sufficient for our purpose.

The following result of Cantor [2] will be needed in the second section.

Proposition 1.15. *Suppose M satisfies Conditions C_0 and I. Let $d > 0$ be the lower bound of the injectivity radius of M, and let $0 < r < d$, $l \geqslant 0$, $k \geqslant 0$, $l + N/2 < k$. Then there exists a constant $c > 0$ such that for every $\omega \in A_1^{k,p}(M)$ and every $x \in M$*

$$|\nabla^l\omega(x)| \leqslant c \sum_{j=0}^{k} \left(\int_{B_r(x)} |\nabla^j\omega|^2 dV \right)^{1/2},$$

where $B_r(x)$ is the ball of radius r around x.

2. DeRham–Hodge isomorphism

The main result of this section is a generalization of Theorem 1 of [5]. The proofs are very similar to those in [5] and will be only sketched.

We shall need a cohomological description of the spaces $\mathcal{H}^p(M)$ of L^2 harmonic p-forms, $0 \leqslant p \leqslant N$ (cf. [5, (2.8)] but note the difference in notation):

$$\begin{aligned} Z^{k,p} &= \{\omega \in A^{2k,p} \mid d\omega = 0\}, \\ B^{k,p} &= dA^{2(k+1),p-1}, \\ H^{k,p} &= Z^{k,p} / \overline{B^{k,p}}, \end{aligned} \tag{2.1}$$

where $k \geqslant 0$, $0 \leqslant p \leqslant N$, and the closure is taken in the topology of $A^{2k,p}$. The following proposition was proved in [5]. In view of Theorem 1.2 the proof is valid in a more general setting and yields

Proposition 2.2. *The spaces $H^{k,p}$ are independent of k. More precisely, for every $k \geqslant 1$, $0 \leqslant p \leqslant N$, $Z^{k,p}$ admits an orthogonal direct sum decomposition*

$$Z^{k,p} = \mathcal{H}^p \oplus \overline{B^{k,p}}\,.$$

Furthermore, when $k = 0$,

$$Z^{0,p} = \mathcal{H}^p \oplus \overline{B^p}\,,$$

where $B^p = \{\omega \in A^{0,p} | \omega = d\eta$ for some $\eta \in A^{0,p-1}\}$.

Let M be a complete, oriented Riemannian manifold satisfying Conditions I and C_{2k} with $2k > N/2 - 1$. We will prove below that for such M the spaces $\mathcal{H}^p(M)$ are isomorphic to simplicially defined L^2-cohomology spaces. For such an isomorphism to hold, the triangulation of M must be sufficiently regular. The appropriate class of triangulations $h\colon K \to M$ are the triangulations satisfying the following conditions:

(2.3)

(i) There exists $\theta_0 > 0$ such that for every simplex σ of K of maximal dimension the fullness $\theta(\sigma)$ of σ satisfies $\theta(\sigma) \geqslant \theta_0$,

(ii) there exist $c_1 > c_2 > 0$ such that for every simplex σ of dimension N
$$c_2 \leqslant \operatorname{vol}(\sigma) \leqslant c_1,$$

(iii) there exists a constant $c > 0$ such that for every vertex v of K the barycentric coordinate function $\varphi_v\colon M \to \mathbf{R}$ satisfies $|\nabla \varphi_v| \leqslant c$.

Remarks. (a) To simplify notation we identify K with M via h.

(b) A discussion of fullness can be found in [8, p. 125] or [6, §2].

(c) In view of (i), (ii) is equivalent to having both upper and lower bound for diameters of simplexes of K. (i), (ii) together are equivalent to simplexes of K having a positive lower bound of the volume and an upper bound of the diameter.

(d) (i) and (ii) imply that the number of simplexes of dimension N meeting at a vertex of K is bounded independently of the vertex.

Conditions (2.3) are analogous to Conditions C_0 and I. As a matter of fact, it follows from an unpublished work of Calabi that a complete Riemannian manifold satisfying Conditions C_0 and I admits triangulations satisfying (2.3) of arbitrarily small mesh.

The simplicial L^2-cohomology is defined as follows (cf. [5]). An oriented real-valued cochain $f \in C^p(K, \mathbf{R})$ is said to be in L^2 if and only if

$$\|f\|^2 = \sum_{\sigma^p \in K} |f(\sigma)|^2 < \infty. \tag{2.4}$$

The space $C_2^p(K)$ of L^2 cochains is a Hilbert space with the inner product

$$(f, g) = \sum_{\sigma} f(\sigma) \cdot g(\sigma). \tag{2.5}$$

In view of Remark (d) above the restriction of simplicial coboundary operator d_c to $C_2^*(K)$ is a bounded operator from $C_2^*(K)$ to $C_2^{*+1}(K)$. One can define the combinatorial Laplacian $\Delta_c = d_c d_c^* + d_c^* d_c$ and the L^2-cohomology as its kernel. The following is an equivalent definition which is more convenient in the present context:

$$\begin{aligned} Z^p(K) &= \{f \in C_2^p(K) | d_c f = 0\}, \\ B^p(K) &= d_c C_2^{p-1}(K), \\ H^p(K) &= Z^p(K)/\overline{B^p(K)}. \end{aligned} \tag{2.6}$$

The theorem alluded to above takes the following form.

Theorem 2.7. *Let M be a complete oriented manifold satisfying Conditions I and C_k for an integer $k > N/2 - 1$. Let (K, h) be a triangulation of M satisfying* (2.3). *The integration of forms over simplexes of K induces an isomorphism*

$$\int : \mathcal{H}^* \to H^*.$$

The proof will be broken down into a sequence of steps as in [5]. First, the invariance of $H^*(K)$ under subdivisions has to be checked, but as is usual in this context, it will only hold for subdivisions which are uniform in a rather technical sense. For our purpose it will suffice to consider the standard subdivisions of K (cf. [8, p. 358] or [4]).

Lemma 2.8. *Let K_1 be the standard subdivision of K. The natural cochain mapping s: $C^*(K_1) \to C^*(K)$ restricts to a bounded operator s: $C_2^*(K_1) \to C_2^*(K)$ and induces an isomorphism on L^2-cohomology.*

The proof usually given for finite complexes (cf. [7, Chapter 5], [5, Proposition 1.2]) works in this case because all operations employed in it are defined locally, and therefore induce bounded maps on L^2 cochains. Lemma 2.8 is, of course, valid also for barycentric subdivisions.

Integration defines a mapping of continuous differential forms into cochains. We will denote the cochain associated to a differential form ω by $\int \omega$ (or $\int_K \omega$ if it is necessary to indicate the triangulation being used). Until further notice the only triangulation considered will be the one appearing in the statement of Theorem 2.7.

Lemma 2.9. *Suppose $\omega \in A^{2(k+1),p}$. Then $\int \omega$ is an L^2 cochain. Moreover*

$$\int : A^{2(k+1),p} \to C_2^p(K)$$

is bounded, and

$$\int \overline{B^{k,p}} \subset \overline{d_c C_2^{p-1}}\,.$$

This lemma is proved exactly as Lemma 3.2 of [5]. The main point is to use Proposition 1.15 and Theorem 1.3 to show that $\int$ is a bounded operator.

As in [5], we obtain

Corollary 2.10. *Integration induces a mapping*

$$\int : \mathcal{H}^p(M) \to H^p(K).$$

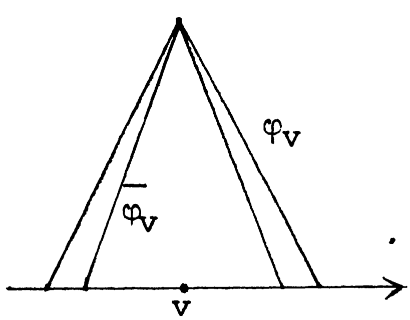
To show that the above mapping is an isomorphism we use the Whitney mappings (cf. [8, p. 138], [5], [6]) constructed with the aid of various partitions of unity. We shall use the barycentric coordinate functions and other functions obtained by smoothing the barycentric coordinates. The following lemma asserts the existence of well-behaved partitions of unity.

Lemma 2.11. *There exists a partition of unity $\{\psi_v\}_{v\in K}$ indexed by vertices of K such that for every vertex $v \in K$*

$$\begin{array}{ll} \text{(i)} & \psi_v \in C^\infty(M), \\ \text{(ii)} & \operatorname{supp} \psi_v \subset \operatorname{Star}(v), \\ \text{(iii)} & |\nabla^l \psi_v| \leqslant C \qquad 0 \leqslant l \leqslant 2k+2, \end{array}$$

where the constant C is independent of the vertex v.

Proof. Let $v \in K$, and let φ_v be the corresponding barycentric coordinate function. Define



$$\bar{\varphi}_v(x) = \begin{cases} 0 & \text{if } \varphi_v(x) \leqslant \frac{1}{3}, \\ \frac{1}{2}(3\varphi_v(x) - 1) & \text{if } \varphi_v(x) \geqslant \frac{1}{3}. \end{cases}$$

The function $\bar{\varphi}_v$ has a pyramid-like graph similar to the graph of φ_v, but its support is well inside Star(v). Let

$$\delta = \sup_{v, x \in \operatorname{supp} \bar{\varphi}_v,\, y \in \partial \operatorname{Star}(v)} r(x, y),$$

where $r(x, y)$ is the geodesic distance between x and y. It follows from (2.3) that $\delta > 0$. Let $\eta(r)$ be a C^∞ function of $r \in \mathbf{R}$ such that $0 \leqslant \eta \leqslant 1$ and

$$\eta(r) = \begin{cases} 1 & \text{if } |r| < \frac{\delta}{2}, \\ 0 & \text{if } |r| > \frac{3}{4}\delta. \end{cases}$$

Define $\tilde{\varphi}_v$ by the formula

$$\tilde{\varphi}_v(x) = \int_M \eta(r(x, y))\bar{\varphi}_v(y)dy.$$

Since M satisfies Condition C_{2m}, the function $r^2(\cdot, y)$ has bounded covariant derivatives of order up to $2m + 2$ (cf. [1]). Moreover, it follows from the definition of $\bar{\varphi}_v$ that

$$\sum_{w \in K} \bar{\varphi}_w(x) \geqslant \tfrac{1}{2}$$

for every $x \in M$. It is now easy to check that the functions ψ_v defined by

$$\psi_v(x) = \Big(\sum_{w \in K} \tilde{\varphi}_w(x)\Big)^{-1} \tilde{\varphi}_v(x)$$

have the required properties.

Using the smoothed partition of unity $\{\psi_v\}$ we can construct a Whitney mapping W which associates a C^∞ form Wc to every cochain $c \in C^*(K)$. Because of (2.11)(iii) W gives a bounded operator from $C_2^p(K)$ to $A_1^{2k+2}(M) = A^{2k+2}(M)$. As a corollary we obtain (cf. [5, Lemma 3.8]).

Lemma 2.12. *W induces a mapping of $H^p(K)$ into $\mathcal{H}^p(M)$ and*

$$\int \circ\, W = \mathrm{Id}.$$

Therefore $\int : \mathcal{H}^p(M) \to H^p(K)$ is surjective.

Finally, to prove the injectivity of $\int$ on L^2-cohomology one uses an approximation technique as in [5]. Let $K = \{K_n\}_{n=0}^\infty$ be the sequence of standard subdivisions of $K = K_0$. We remark that the sequence of barycentric subdivisions cannot be used here. The standard subdivisions have the advantage that Conditions (2.3) will still hold for all subdivided complexes (cf. [8, p. 348]).

We shall denote by W_n the Whitney mapping constructed using the barycentric coordinate functions of the complex K_n.

Lemma 2.13. *For every* $\omega \in \mathcal{H}^*(M)$ *and every* $\varepsilon > 0$ *there exists* $n \geqslant 0$ *such that*

$$\|\omega - W_n \circ \int_{K_n} \omega\|_0 \leqslant \varepsilon.$$

Proof. Let σ be a closed simplex of dimension N, $x \in \sigma$. In view of [6, Proposition 2.4], $|\omega(x) - (W_n \circ \int_{K_n} \omega)(x)|$ can be estimated in terms of $\sup_{y \in \sigma} |\nabla \omega(y)|$ which in turn, by Proposition 1.15, can be bounded in terms of L^2 norms of covariant derivatives of ω of order up to $2k + 2$ on $\mathrm{Int}(\bigcup_{\sigma \cap \tau \neq \varnothing} \tau)$. This, together with the observation that $\omega \in A_1^{2k+2,p}$ by Theorem 1.3, allows one to complete the proof (cf. [5, Lemma 3.9]).

Injectivity of $\int: \mathcal{H}^*(M) \to H^*(K)$ is an easy consequence of Lemma 2.10 (the proof of Lemma 3.10 in [5] carries over verbatim). The proof of de Rham-Hodge isomorphism is thus completed.

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