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RIEMANNIAN EXTENSIONS

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1. Introduction

An analytic Riemannian manifold E is an *extension* of an analytic Riemannian manifold M if there exists an isometry f: $M \to E$ imbedding M as an open submanifold of E. We shall always assume that the extension is proper, that is, $f(M) \neq E$. We shall usually suppress reference to the map f and identify M with $f(M)$.

A manifold is said to be *non-extendible* if it has no proper extension.

Suppose M is an analytic Riemannian manifold. If U is an arbitrary neighborhood on M, then the metric is known on M once it is known on U. If we can construct the class $\{E_\alpha\}$ of all extensions of U then M together with its extensions will be in $\{E_\alpha\}$. Hence there is no loss of generality in confining ourselves to the construction of extensions of simply connected manifolds which can be covered by a single chart.

Let M be a simply connected analytic Riemannian manifold such that

(i) M has an atlas consisting of a single chart,

(ii) the Killing equations on M have only trivial solutions.

The purpose of this paper is to consider the properties and methods of construction of inextendible extensions of M. The reason for insisting on condition (ii) will be clarified in §5 via an example.

Suppose E is a simply connected extension of M. If we require E to be Cauchy complete, then E is unique [3], but such an E need not exist. If we do not require E to be complete, then the following difficulty arises. Let E be an inextendible simply connected extension of a 2-dimensional manifold M. Select any point $p \in E$ with $p \notin M$ and construct the universal covering manifold of $E - \{p\}$. Then this covering manifold is also a simply connected inextendible extension of M so that non-extendibility is not a sufficient

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criterion for uniqueness. However, let us suppose that in addition to the above two conditions M also satisfies the further condition:

(iii) any neighborhood of an arbitrary point in M admits at most one reflection symmetry which leaves the point in question fixed.

Then we can show by construction that M possesses a unique simply connected extension which satisfies a certain completeness criterion which we discuss in §2.

2. Singularities and completeness

In order to arrive at a useful definition of completeness we need the concept of a quasiregular singularity. For a fuller discussion than we give here the reader is referred to the Review Article [2]. Let ∂M denote the Cauchy boundary of an analytic Riemannian manifold M. A point $q \in \partial M$ is a *nonsingular boundary point* if there exists an extension M' of M in which q is an interior point; otherwise q is said to be a *singular boundary point* of M. Roughly speaking, a *quasiregular singularity* of M is a singular boundary point $q \in \partial M$ for which the local geometry is perfectly well behaved as one approaches q; this occurs for example at the vertex of a cone. More precisely we have the following. Let q be a singular boundary point of M, and let $\gamma(t)$, $t \in [a, b]$, be a curve on $M \cup \partial M$ such that $\gamma(b) = q$ and for $t \in [a, b)$, $\gamma(t)$ is a semiopen geodesic arc on M with affine parameter t. Briefly, we say that γ is a geodesic arc on M terminating at q. Let $X_1, \cdots, X_N$ ($N = \dim M$) be a linearly independent set of vectors at $\gamma(a) \in M$ such that $X_1 = \dot{\gamma}(a)$. Let $X_1(t), \cdots, X_N(t)$, $a \leqslant t < b$, be the vectors obtained from $X_1, \cdots, X_N$ by parallel translation along γ from $\gamma(a)$ to $\gamma(t)$. Corresponding to γ and the vectors $X_1(t), \cdots, X_N(t)$ there exists a unique coordinate system $(x^1, \cdots, x^N)$ with the following properties:

(i) The equation of $\gamma(t)$, $t \in [a, b)$, is $x^1(t) = t$, $x^i(t) = 0$ for $i = 2, \cdots, N$.

(ii) The coordinates $x^2, \cdots, x^N$ are normal coordinates determined by the frame $X_2(t), \cdots, X_N(t)$ in each hypersurface $x^1 = c = \text{constant}$, $a \leqslant c < b$.

Let $g_{jk}(t)$ denote the component functions of the metric tensor in the coordinate system $(x^1, \cdots, x^N)$ at the point $\gamma(t)$. If $\lim_{t \to b} g_{jk}(t)$ exists and is analytic for all $j, k = 1, \cdots, N$, then q is called a *quasiregular singularity* of M.

If q is a quasiregular singularity of an analytic Riemannian manifold M, then ([2], [1]) M is locally extendible at q in the sense that any curve on M terminating at q is contained in an open set U which has an extension W in which q is an interior point. Note that $W \not\subset M$.

Let us classify points of a manifold into two classes (one of which may be empty) according as they do or do not satisfy some specified property P. We require that P be chosen so that it is possible to decide whether or not a point satisfies property P once the metric on an arbitrary neighborhood of the point in question has been given. The previous paragraph shows that this classification can be extended to the quasiregular singularities of the manifold.

A manifold M is *P-complete* if

(i) all points of M satisfy property P,

(ii) no quasiregular singularity of M satisfies property P.

The significance of P-completeness lies in the following considerations:

(i) If a manifold possesses a P-complete extension, then there is a constructive method of obtaining this extension. If the construction fails, then no P-complete extension exists [5, §6].

(ii) If an arbitrary point is removed from a P-complete extension, then the covering manifold of the resulting manifold will not be P-complete since it will have a quasiregular singularity (corresponding to the removed point) which satisfies property P. Thus it is reasonable to conjecture that simply connected P-complete extensions are unique. An argument similar to [3, Theorem 3] shows that this is indeed the case.

Let Q be the property that an arbitrarily small neighborhood of a point possesses at most one reflection symmetry which leaves the point in question fixed. Our main result is the following.

Theorem. *Let M be a simply connected manifold satisfying conditions (i)–(iii) of* §1. *Then a unique simply connected Q-complete extension of M can be constructed.*

As an example let N be the 2-ellipsoid $a^2x^2 + b^2y^2 + c^2z^2 = 1$ (a, b, c distinct) with the six axis points removed. Let M be an open simply connected submanifold of N which can be covered by a single coordinate system. Then the universal covering manifold E of N is the unique simply connected Q-complete extension of M. By taking appropriate quotient manifolds of E other Q-complete extensions can be constructed, including N. We can extend N to a Cauchy complete extension of M by adjoining its six nonsingular boundary points.

Our reasons for considering this specific property Q are as follows:

(i) A solution always exists.

(ii) The essential ideas of the construction are exhibited.

(iii) The above example and geometric intuition suggest that adjoining all nonsingular boundary points to a Q-complete manifold results in an inextendible manifold. This is indeed the case, although we shall not consider the matter here [5].

3. Heuristic remarks on the construction

Let M be a simply connected analytic Riemannian manifold which has an atlas possessing a single chart. Assume that M possesses a simply connected extension E. If $x_0 \in M \subset E$ is a fixed point, then E can be identified with the set of homotopy classes of curves on E with base point x_0. Define a *chain* $C_0 f_0 \cdots C_k f_k$ to be a sequence $C_0, \cdots, C_k$ of curves in Euclidean space $\mathbf{R}^N$ ($N = \dim M$) together with a sequence $f_0, \cdots, f_k$ of analytic invertible $\mathbf{R}^N$-valued functions such that for $i = 0, \cdots, k-1, f_i$ maps the end point of C_i to the initial point of C_{i+1} and such that the domain of f_k includes the end points of C_k. In an obvious way a covering of a curve in E starting at x_0 by coordinate neighborhoods induces a chain in $\mathbf{R}^N$ with the f_j corresponding to coordinate transformations. Some chains (termed *admissible* chains) can be lifted to curves on E with base point x_0 in the sense that the given chain is induced by some such curve. We can construct E from the set of admissible chains if we have, firstly, a criterion to decide whether or not a given chain is admissible and, secondly, an equivalence relation among the admissible chains formulated so that homotopic curves on E with base point x_0 induce equivalent chains; E will then be the set of equivalence classes of admissible chains. Suppose C is a curve on E with base point x_0. Then the metric function at any point of C is the analytic continuation along C of the metric function at x_0. A chain induced by C will have a metric function defined at its initial point which can be analytically continued along the chain. This will be a necessary condition for admissibility. If E were complete (in the Cauchy sense), then it would also be sufficient. However, certain difficulties will become apparent which make it necessary to exclude some exceptional points from the extension.

4. Space elements and their extensions

Suppose we wish to extend an analytic Riemannian manifold M. It is more convenient to focus our attention on a neighborhood of some arbitrary point in M rather than M itself. Since M is analytic and satisfies condition (i) of §1, this is no loss of generality since the metric is known everywhere on M once it is known on a neighborhood of a single point of M.

Let $\mathfrak{F}_x(\mathbf{R}^N, \mathbf{R}^K)$ denote the set of $\mathbf{R}^K$-valued germs of analytic functions at $x \in \mathbf{R}^N$. An N-dimensional *elementary space element* is a set

$$S = \{x, \{g_{jk}\} | x \in \mathbf{R}^N; g_{jk} \in \mathfrak{F}_x(\mathbf{R}^N, \mathbf{R}); j, k = 1, \cdots, N\}$$

where the g_{jk} satisfy

(i) $g_{jk} = g_{kj}$, $j, k = 1, \cdots, N$,

(ii) $\sum_{j,k} g_{jk} u^j u^k > 0$ for all nonzero $(u^1, \cdots, u^N) \in \mathbf{R}^N$.

Condition (ii) then implies

(iii) $\det(g_{jk}(x)) \neq 0$.

We define an equivalence relation "$\simeq$" on the set of N-dimensional elementary space elements (to be denoted by $\mathfrak{S}_N$) by letting $S = \{x, \{g_{jk}\}\} \simeq \{x', \{g'_{jk}\}\}$ if there exists $f \in \mathfrak{F}_x(\mathbf{R}^N, \mathbf{R}^N)$ with $f(x) = x'$ and with invertible Jacobian matrix J such that $G = J^t G' J$, where G (resp. G') denotes the matrix (g_{jk}) (resp. (g'_{jk})) and J^t is the transpose of J.

We write $f(S)$ for $\{x', \{g'_{jk}\}\}$, $\underset{\sim}{S}$ for the equivalence class of S in $\mathfrak{S}_N$ and $\underset{\sim}{\mathfrak{S}}_N$ for $\{\underset{\sim}{S} \mid S \in \mathfrak{S}_N\}$. Each $\underset{\sim}{S} \in \underset{\sim}{\mathfrak{S}}_N$ is called an *N-dimensional space element*.

Let γ: $[0, 1] \to \mathbf{R}^N$ parametrize a curve C in $\mathbf{R}^N$, and let $S = \{\gamma(0), \{g_{jk}\}\} \in \mathfrak{S}_N$. The curve C is *S-admissible*, if each g_{jk} can be analytically continued along C, and $\{\gamma(t), \{g_{jk}(t)\}\} \in \mathfrak{S}_N$ for all $t \in [0, 1]$, where $g_{jk}(t)$ denotes the continuation of g_{jk} along C to $\gamma(t)$.

Define a map π: $\mathfrak{S}_N \to \mathbf{R}^N$: $\{x, \{g_{jk}\}\} \mapsto x$. For some $S \in \mathfrak{S}_N$ let M be an open simply connected neighborhood of $\pi(S)$ such that any curve in M with initial point $\pi(S)$ is S-admissible. Such a neighborhood is a Riemannian manifold (denoted by $M(S)$) in an obvious way and is called a *local manifold* representing $\underset{\sim}{S}$. An *extension* of $\underset{\sim}{S} \in \underset{\sim}{\mathfrak{S}}_N$ is a Riemannian manifold E for which there is an isometry f: $M(S) \to E$ imbedding some local manifold $M(S)$ representing $\underset{\sim}{S}$ as an open submanifold of E.

5. Construction of extensions

In this section we make precise the heuristic discussion of the chain construction given in §3.

The initial (resp. terminal) point of a curve C on a manifold will be denoted by $\eta(C)$ (resp. $\tau(C)$). An invertible $f \in \mathfrak{F}_{\eta(C)}$, $(\mathbf{R}^N, \mathbf{R}^N)$ is *admissible* on C, if it can be analytically continued along C, and the continuation is invertible at each point of C. We write $C(f)$ for the continuation of f to $\tau(C)$, and if γ parametrizes C we write $f(C)$ for the curve with parametrization $t \mapsto f^t(\gamma(t))$, where f^t denotes the continuation of f to $\gamma(t)$.

An *elementary chain* $\mathcal{C} = C_0 f_0 \cdots C_k f_k$ in $\mathbf{R}^N$ is a finite sequence of curves $C_0, \cdots, C_k$ in $\mathbf{R}^N$ together with a sequence $f_0, \cdots, f_k$ of analytic $\mathbf{R}^N$-valued functions such that

(i) each f_i is an invertible function on a neighborhood of $\tau(C_i)$,

(ii) $f_{i-1}(\tau(C_{i-1})) = \eta(C_i)$, $i = 1, \cdots, k$.

We write $\eta(\mathcal{C})$, $\tau(\mathcal{C})$ for $\eta(C_0)$, $f_k(\tau(C_k))$ respectively.

Consider a curve C covered by open sets $U_1, \cdots, U_k$ on some manifold.

The curve C together with a choice of coordinate system on each U_i induces an elementary chain $C_0 f_0 \cdots C_k f_k$. A different choice of coordinates on each U_i induces a different elementary chain $C'_0 f'_0 \cdots C'_k f'_k$. These elementary chains are said to be *transforms* of each other. For the purposes of our construction it is necessary to be able to define a transform without reference to a given manifold.

An elementary chain $\mathcal{C}' = C'_0 f'_0 \cdots C'_k f'_k$ is a *transform* of an elementary chain $\mathcal{C} = C_0 f_0 \cdots C_k f_k$ (both in $\mathbf{R}^N$) if there is a sequence $h_0, \cdots, h_{k+1}$ of analytic germs such that

(i) h_i is admissible on C_i, $i = 0, \cdots, k$,

(ii) $C'_i = h_i(C_i)$, $i = 0, \cdots, k$,

(iii) $f'_i = h_{i+1} \circ f_i \circ (C_i(h_i))^{-1}$, $i = 0, \cdots, k$,

(iv) h_{k+1} is analytic and invertible at $\tau(\mathcal{C})$.

We write $\underset{\sim}{h} = \{h_0, \cdots, h_{k+1}\}$, $\eta(\underset{\sim}{h}) = h_0$, $\tau(\underset{\sim}{h}) = h_{k+1}$, $\underset{\sim}{h}(\mathcal{C}) = \mathcal{C}$, and say that $\underset{\sim}{h}$ is *admissible* on $\mathcal{C}$.

We remark that the above definition may seem unnatural in that the two chains are based on the same number k of open sets. However, to give a definition of a transform which mirrors the covering of a curve in any two ways would be even more cumbersome than the above definition and, as we shall shortly show nothing essential would be gained by such a definition.

Roughly speaking, a chain $\mathcal{C}$ is S-admissible if $S \in \mathcal{S}_N$ can be analytically continued along $\mathcal{C}$. More precisely we have the following.

An elementary chain $\mathcal{C} = C_0 f_0 \cdots C_k f_k$ in $\mathbf{R}^N$ is *S-admissible* ($S \in \mathcal{S}_N$), if $\pi(S) = \eta(\mathcal{C})$, and if there exist $S_0, \cdots, S_{k+1} \in \mathcal{S}_N$ with $S_0 = S$ such that

(i) C_i is S_i admissible,

(ii) $S_{i+1} = f_i(C_i(S_i))$ where $C_i(S_i)$ denotes the analytic continuation of S_i along C_i to $\tau(C_i)$.

We denote S_{k+1} by $\mathcal{C}(S)$, and note that if $\underset{\sim}{h}$ is admissible on $\mathcal{C}$, then $\tau(\underset{\sim}{h})[\mathcal{C}(S)] = \underset{\sim}{h}(\mathcal{C})[\eta(\underset{\sim}{h})(S)]$, so that $\mathcal{C}(S) \simeq \underset{\sim}{h}(\mathcal{C})[\eta(\underset{\sim}{h})(S)]$.

An *N-dimensional chain* is a pair $(\mathcal{C}, S)$, where $S \in \mathcal{S}_N$, and $\mathcal{C}$ is an S-admissible elementary chain. A *transform* of a chain $(\mathcal{C}, S)$ is a chain $(\underset{\sim}{h}(\mathcal{C}), \eta(\underset{\sim}{h})(S))$ where $\underset{\sim}{h}$ is admissible on $\mathcal{C}$.

Suppose M is an open simply connected neighborhood of $\pi(S)$ ($S \in \mathcal{S}_N$) such that C is S-admissible for all curves C in M with $\eta(C) = \pi(S)$. Then any closed curve C in M with $\eta(C) = \pi(S) = \tau(C)$ is called a *null-loop* for S.

We are finally in a position to formulate an equivalence relation among chains in such a way that equivalent chains will correspond to homotopic curves on a simply connected extensions of a given space element.

For a fixed $\underset{\sim}{S} \in \underset{\sim}{\mathcal{S}}_N$ let $\chi(\underset{\sim}{S})$ denote the set of all chains $(\mathcal{C}, S)$ with $S \in \underset{\sim}{S}$. Arbitrary chains $(\mathcal{C}, S)$ and $(\mathcal{C}', S') \in \chi(\underset{\sim}{S})$ are *equivalent* (written

$(\mathcal{C}, S) \sim (\mathcal{C}', S')$) if $(\mathcal{C}, S)$ can be transformed into $(\mathcal{C}', S')$ by a finite sequence of the following operations:

E_1: Replacement of a chain $(C_0 f_0 \cdots C_i f_i \cdots C_k f_k, S)$ by a transform of the chain $(C_0 f_0 \cdots C_i' e C_i'' f_i \cdots C_k f_k, S)$ where e is the identity transformation and $C_i = C_i' C_i''$

E_2: Replacement of a chain $(C_0 f_0 \cdots C_i' e C_i'' f_i \cdots C_k f_k, S)$ by a transform of the chain $(C_0 f_0 \cdots C_i f_i \cdots C_k f_k, S)$ (Same notation as E_1)

E_3: Replacement of a chain $(C_0 f_0 \cdots C_i' f_i' D f_i'' \cdots C_k f_k, S)$ by a transform of the chain $(C_0 f_0 \cdots C_i f_i \cdots C_k f_k, S)$ where D is a null loop for $(C_0 f_0 \cdots C_i f_i' \mathrm{i})(S)$ and $f_i = f_i'' \circ f_i'$

E_4: Replacement of a chain $(C_0 f_0 \cdots C_i f_i \cdots C_k f_k, S)$ by a transform of the chain $(C_0 f_0 \cdots C_i f_i' D f_i'' \cdots C_k f_k, S)$ (Same notation as E_3)

The equivalence class of $(\mathcal{C}, S) \in \chi(\underset{\sim}{S})$ is written $\langle \mathcal{C}, S \rangle$.

We remark that operations E_1 and E_2 are essentially equivalent to the alternative definition of a transform discussed previously.

Let $E(\underset{\sim}{S}) = \{\langle \mathcal{C}, S \rangle | (\mathcal{C}, S) \in \chi(\underset{\sim}{S})\}$. From a naive point of view one would expect that $E(\underset{\sim}{S})$ is an inextendible simply connected extension of $\underset{\sim}{S}$. In fact, choose any $(\mathcal{C}, S) \in \langle \mathcal{C}, S \rangle \in E(\underset{\sim}{S})$, and let U be an open simply connected neighborhood of $\pi(\mathcal{C}(S))$ such that every curve on U with initial point $\pi(\mathcal{C}(S))$ is $\mathcal{C}(S)$-admissible. Put $V = \{\langle \mathcal{C}Ce, S \rangle | C \subset U\}$, where e is the identity transformation. If the map ϕ: $V \to U$: $\langle \mathcal{C}Ce, S \rangle \mapsto \tau(C)$ is an injection for all $(\mathcal{C}, S)$ and suitable choice of U, then $E(\underset{\sim}{S})$ is an analytic manifold (cf. Theorem below). However, if there exists $\langle \mathcal{C}, S \rangle \in E(\underset{\sim}{S})$ such that every local manifold representing $\mathcal{C}(S)$ contains at least one pair of isometric points (points are said to be *isometric* if they have isometric neighborhoods), then the chain construction may break down. As an example consider the metric

$$g = \frac{1}{3}\left(2 + \frac{z}{\sqrt{z^2 + \rho^2}}\right)\rho^2 d\phi \otimes d\phi + d\rho \otimes d\rho + dz \otimes dz. \tag{1}$$

If for all integers k we identity the points (ρ, ϕ, z) and $(\rho, \phi + 2k\pi, z)$ for all ϕ, z and all $\rho > 0$ by the coordinate transformation $x = \rho \cos \phi$, $y = \rho \sin \phi$, $z = z$, then we have

$$\begin{aligned} g = {} & \frac{x^2 + \lambda y^2}{x^2 + y^2} dx \otimes dx + \frac{\lambda x^2 + y^2}{x^2 + y^2} dy \otimes dy \\ & + \frac{xy(1 - \lambda)}{x^2 + y^2} (dx \otimes dy + dy \otimes dx) + dz \otimes dz, \end{aligned}$$

where

$$\lambda = \frac{1}{3}\left(2 + \frac{z}{\sqrt{z^2 + \rho^2}}\right).$$

As $\rho \to 0$ we have $\lambda \to 1$ for $z > 0$, so that the boundary points $z > 0$, $\rho = 0$ can be included in an extension of (1) provided we perform the above identification.

If we perform the transformation

$$x = \rho \cos \frac{\phi}{\sqrt{3}}, \quad y = \rho \sin \frac{\phi}{\sqrt{3}}, \quad z = z,$$

then

$$g = \frac{x^2 + \alpha y^2}{x^2 + y^2} dx \otimes dx + \frac{\alpha x^2 + y^2}{x^2 + y^2} dy \otimes dy$$
$$+ \frac{2xy(1 - \alpha)}{x^2 + y^2}(dx \otimes dy + dy \otimes dx) + dz \otimes dz,$$

where

$$\alpha = 2 + \frac{z}{\sqrt{z^2 + \rho^2}}.$$

As $\rho \to 0$ we have $\alpha \to 1$ for $z < 0$ (for now $\sqrt{z^2} = -z$), so that the boundary points $z < 0$, $\rho = 0$ can be included in an extension of (1) provided we identify the points $(\rho, \phi + 2\sqrt{3}\, k\pi, z)$ and (ρ, ϕ, z) for all integers k and all ϕ, z, and $\rho > 0$.

The chain construction would force both identifications, leading to a contradiction. One way to circumvent this difficulty is to insist on the condition that every point p of the manifold M to be extended possesses a neighborhood in which no two points are isometric. This implies [4] that M possesses no Killing vectors which is our reason for assuming condition (ii) in the introduction. In fact, as we show below, the above condition can be relaxed slightly (cf. condition (iii)) in the introduction, but not to the extent of allowing nontrivial Killing vectors.

A space element $\underset{\sim}{S}$ is *regular* if there exists a local manifold $M(S)$ representing $\underset{\sim}{S}$ in which no two distinct points are isometric. A chain $(\mathcal{C}, S)$ is *regular* if S and its continuation to any point of $\mathcal{C}$ are regular. Suppose U is any open simply connected neighborhood of $\pi(S)$ such that $C(S)$ is regular for all curves C in U with $\eta(S) = \pi(S)$. Then any closed curve C in U with $\eta(C) = \pi(S) = \tau(C)$ is a *regular null loop* for S. If in operations $E_1, \cdots, E_4$ above we restrict ourselves to regular chains and regular null loops, then

(Theorem below), $E(\underset{\sim}{S})$ is an extension of $\underset{\sim}{S}$ for any regular space element $\underset{\sim}{S}$. However, this restriction is too drastic. As an indication of how to proceed consider the ellipsoid $a^2x^2 + b^2y^2 + c^2z^2 = 1$ with a, b, c distinct. At any point p on the ellipsoid with $x(p), y(p), z(p)$ all nonzero we obtain a regular space element $\underset{\sim}{S}_p$ from the usual metric. The space $E(\underset{\sim}{S}_p)$ (constructed by restriction to regular chains) is isometric to the octant $x > 0, y > 0, z > 0$ of the ellipsoid. To this octant we attach the Cauchy boundary, but omit the three axis points. We paste together copies of this octant-with-boundary along corresponding boundary components to obtain the universal covering manifold of the ellipsoid without axis points. From this covering manifold one can by appropriate identifications obtain the ellipsoid without axis points. Since these are nonsingular boundary points the original ellipsoid can be recovered.

A space element $\underset{\sim}{S}$ is *semiregular* if there exists a local manifold $M(S)$ representing $\underset{\sim}{S}$ such that

(i) there is a nontrivial isometry $f\colon M(S) \to M(S)$

(ii) each point of $M(S)$ is isometric to at most one other point of $M(S)$,

(iii) the matrix of $(df)_{\pi(S)}$ in a suitable coordinate system is

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & -1 \end{bmatrix}.$$

Theorem. *Every regular or semiregular space element possesses a unique simply connected Q-complete extension.*

Proof. Let $\underset{\sim}{S}$ be a regular space element, and let $E = \{\langle \mathcal{C}, S\rangle | (\mathcal{C}, S) \text{ is regular}, S \in \underset{\sim}{S}\}$. For any $\langle \mathcal{C}, S\rangle \in E$ choose an arbitrary $(\mathcal{C}, S) \in \langle \mathcal{C}, S\rangle$, and let U be an open simply connected neighborhood for $\pi(\mathcal{C}(S))$ such that the local manifold $U(\mathcal{C}(S))$ contains no pair of isometric points. (Such a U is called a *regular* neighborhood for $\mathcal{C}(S)$). Let $V = \{\langle \mathcal{C}Ce, S\rangle | C \subset U\}$, where e is the identity transformation. The map

$$\phi\colon V \to U\colon \langle \mathcal{C}Ce, S\rangle \mapsto \tau(C)$$

is an injection as a consequence of $\mathcal{C}(S)$ being regular and the easily proved fact that if $\mathcal{C}_1(S_1) \neq \mathcal{C}_2(S_2)$ then necessarily $(\mathcal{C}_1, S_1) \not\sim (\mathcal{C}_2, S_2)$. By this method an analytic atlas can be constructed for E. A Riemannian metric can be defined on each coordinate neighborhood, so that E will be a Riemannian manifold if it is Hausdorff.

Let $x_i = \langle \mathcal{C}_i, S_i\rangle$, $i = 1, 2$, be points of E which cannot be separated by disjoint open neighborhoods. Suppose $U_i \subset \mathbf{R}^N$ ($N = \dim \underset{\sim}{S}$) is a regular neighborhood for $\mathcal{C}_i(S_i)$, and let

$$V_i = \{\langle \mathcal{C}_i Ce, S_i \rangle | C \subset U_i\},$$
$$\phi_i: V_i \to U_i: \langle \mathcal{C}_i Ce, S_i \rangle \mapsto \tau(C).$$

Further let

$$(\mathcal{C}, S) \in V_1 \cap V_2,$$

and suppose

$$\phi: V_1 \cap V_2 \to \mathbf{R}^N$$

is a chart. Since $\mathcal{C}_1(S_1) \simeq \mathcal{C}_2(S_2)$ we may assume

$$\phi_1(V_1) = U = \phi_2(V_2) \quad \text{for some } U \subset \mathbf{R}^N$$

and

$$\phi_1|(V_1 \cap V_2) = \phi = \phi_2|(V_1 \cap V_2).$$

Let C be a curve in U joining $\pi(\mathcal{C}(S))$ to $\pi(\mathcal{C}_1(S_1)) = \pi(\mathcal{C}_2(S_2))$. Since each ϕ_i is an injection, we have $\langle \mathcal{C}_i, S_i \rangle = \langle \mathcal{C}Ce, S \rangle$, $i = 1, 2$. Hence $x_1 = x_2$, and E is Hausdorff.

We next show that E is simply connected. Let $x_0 = \langle C_* e, S \rangle$, where C_* is the constant curve at $\pi(S)$, ($S \in \underset{\sim}{S}$). For $(\mathcal{C}, S) = (C_0 f_0 \cdots C_k f_k, S)$ let C_q have parametrization $t \mapsto \alpha_q(t)$, $q = 0, \cdots, k$; $0 \leqslant t \leqslant 1$. For any fixed $\tau \in [0, 1]$ let C_q^τ be the curve parametrized by $t \mapsto \alpha_q(\tau t)$, $0 \leqslant t \leqslant 1$, and put $\mathcal{C}_q^\tau = C_0 f_0 \cdots f_{q-1} C_q^\tau$. Then $\gamma_q: [q, q+1] \to E: t \mapsto \langle \mathcal{C}_q^{t-q}, S \rangle$ parametrizes a curve $\underset{\sim}{\mathcal{C}}_q$ in E, and the curve $\underset{\sim}{\mathcal{C}} = \underset{\sim}{\mathcal{C}}_0 \underset{\sim}{\mathcal{C}}_1 \cdots \underset{\sim}{\mathcal{C}}_k$ is a curve in E with $\eta(\underset{\sim}{\mathcal{C}}) = x_0$. Every regular chain $(\mathcal{C}, S)$ induces such a curve on E, and every curve on E with initial point x_0 is induced by a chain. We note that $\langle \mathcal{C}, S \rangle = \tau(\underset{\sim}{\mathcal{C}})$. Since the operations $E_1, \cdots, E_4$ do not change the homotopy class of the induced curve, we see that curves $\underset{\sim}{\mathcal{C}}_1$ and $\underset{\sim}{\mathcal{C}}_2$ in E with $\eta(\underset{\sim}{\mathcal{C}}_1) = x_0 = \eta(\underset{\sim}{\mathcal{C}}_2)$, $\tau(\underset{\sim}{\mathcal{C}}_1) = \tau(\underset{\sim}{\mathcal{C}}_2)$ are homotopic. (Because $\langle \mathcal{C}_1, S_1 \rangle = \tau(\underset{\sim}{\mathcal{C}}_1)$, $\langle \mathcal{C}_2, S_2 \rangle = \tau(\underset{\sim}{\mathcal{C}}_2)$ and hence $(\mathcal{C}_1, S_1) \sim (\mathcal{C}_2, S_2)$). Hence E is simply connected.

The next step is to attach a suitable boundary to E and to paste together copies of this manifold with boundary to yield a Q-complete extension.

Let E^* be the manifold E together with its Cauchy boundary ∂E. A chain $(\mathcal{C}, S)$ is a *boundary chain* if S is regular, $\mathcal{C}(S)$ is semiregular, and the continuation of S to any point of $\mathcal{C}$ other than $\tau(\mathcal{C})$ is regular. In a similar way to that described above each boundary chain can be identified with a curve $\underset{\sim}{\mathcal{C}}$ on E^* with $\eta(\underset{\sim}{\mathcal{C}}) = x_0$, $\tau(\underset{\sim}{\mathcal{C}}) \in \partial E$. Delete from ∂E those points not accessible by boundary chains. The resulting manifold with boundary is denoted by R. Since E has a countable basis for its topology (being a

Riemannian manifold) the number of boundary components of R is countable. Let $\{B^\alpha\}_{\alpha\in A}$ be the set of boundary components of R, A being a countable index set. Define Φ to be the free group with generators $\{\alpha|\alpha \in A\}$ and generating relations $\alpha^2 =$ identity, for all $\alpha \in A$. For each $\phi \in \Phi$ associate a copy R_ϕ of R, and put $R = R_e$. Let λ_ϕ: $R \to R_\phi$ be the natural map. In the disjoint union $\bigcup_{\phi\in\Phi} R_\phi$ we identify B^α_ϕ and $B^\alpha_{\phi\alpha}$ by the map $\lambda_{\phi\alpha} \circ (\lambda_\phi)^{-1}$, where $\{B^\alpha_\phi\}_{\alpha\in A}$ is the set of boundary components of R_ϕ. The resulting quotient space is the unique simply connected Q-complete extension of $\underset{\sim}{S}$. This completes the proof of the theorem when $\underset{\sim}{S}$ is regular. Since any semiregular space element can be continued to a regular space element, the theorem is also true when $\underset{\sim}{S}$ is semiregular.

We denote the Q-complete simply connected extension of $\underset{\sim}{S}$ by $K(\underset{\sim}{S})$. With the above notation we have the following.

Corollary. *Let G denote the group of isometries of R, and let $H = \{(\phi, e) \in \Phi \times G|\phi^2 = e\}$.*

(i) *The group of isometries of $K(\underset{\sim}{S})$ is a semidirect product of G and Φ.*

(ii) (a) *If F is a subgroup of $\Phi \times G$ with $F \cap H = e$, then $K(\underset{\sim}{S})/F$ is a Q-complete extension of $\underset{\sim}{S}$. Here $\Phi \times G$ denotes the semidirect product in* (i).

(b) *Every Q-complete extension of $\underset{\sim}{S}$ has the form $K(\underset{\sim}{S})/F$ where F is a subgroup of $\Phi \times G$ with $H \cap F = e$.*

Proof. For $q \in G$, $\sigma(g)$ denotes the permutation on A given by $g(B^\alpha) = B^{\sigma(g)(\alpha)}$. The group G acts on Φ by $g(\alpha\beta \cdots \gamma) = [\sigma(g)(\alpha)][\sigma(g)(\beta)] \cdots [\sigma(g)(\gamma)]$. Suppose $x \in K(\underset{\sim}{S})$. Then $x \in R_\psi$ for some $\psi \in \Phi$. For any $\phi \in \Phi$, $g \in G$ we define isometries ϕ, g and (ϕ, g) of $K(\underset{\sim}{S})$ by

$$\phi\colon x \mapsto (\lambda_{\phi\psi} \circ \lambda_\psi^{-1})(x),$$
$$g\colon x \mapsto (\lambda_{g(\psi)} \circ g \circ \lambda_\psi^{-1})(x),$$
$$(\phi, g)\colon \alpha \mapsto \phi(g(x)).$$

A straightforward calculation shows that

$$(\phi_1, g_1)[(\phi_2, g_2)](x) = (g_1(\phi_2) \circ \phi_1, g_1 \circ g_2)(x),$$

so that the group of isometries of $K(\underset{\sim}{S})$ appears as a semidirect product of Φ and G.

(ii) (a) follows from the fact that $F \cap H = e$ iff F acts properly discontinuously on $K(\underset{\sim}{S})$.

(b) follows from the uniqueness of $K(\underset{\sim}{S})$.

References

[1] C. J. S. Clark, *Local extensions in singular space–times*, Comm. Math. Phys. **32** (1973) 205–214.

[1] G. F. R. Ellis & B. G. Schmidt, *Singular space times*, General Relativity and Gravitation **8** (1977) 915–953.

[3] B. S. Myers, *Riemannian manifolds in the large*, Duke Math. J. **1** (1935) 39–49.

[4] A.Nijenhuis, *A theorem on sequences of local affine collineations and isometries*, Nieuw Arch. Wisk. **2** (1954) 200–212.

[5] G. H. Smith, *Analytic extension of Riemannian manifolds*, Ph.D. Thesis, University of N.S.W., Australia, 1977.

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