J. DIFFERENTIAL GEOMETRY
15 (1980) 605–625

BOUNDARY BEHAVIOR OF HOLOMORPHIC FUNCTIONS ON PSEUDOCONVEX DOMAINS

DAVID CATLIN

1. INTRODUCTION

In the theory of several complex variables, one is led to the study of holomorphic functions on pseudoconvex domains. However, in this general context, many of the proofs of the basic results do not yield information about the behavioir of the functions at the boundary of the domain. In this paper under the additional assumption that the boundary of the domain is smooth, we show that some of the classical theorems can be stated in terms of the space of holomorphic functions which are smooth up to the boundary. Although our results hold on a class of pseudoconvex manifolds with smooth boundary, for the purposes of this introduction we shall simply state the results for pseudoconvex domains in $\mathbf{C}^n$.

Let Ω be a bounded pseudoconvex domain in $\mathbf{C}^n$ with smooth boundary. Denote by $A(\Omega)$ the set of holomorphic functions in Ω, and by $A^\infty(\Omega)$ the set $A(\Omega) \cap C^\infty(\overline{\Omega})$. The following theorem is the analogue of the Levi problem for $A^\infty(\Omega)$.

Theorem 3.3.1. *There exists $f \in A^\infty(\Omega)$ which does not extend analytically to a neighborhood of any boundary point.*

One can also study holomorphic convexity properties with respect to $A^\infty(\Omega)$. For K a compact subset of $\overline{\Omega}$, we define

$$\hat{K} = \left\{ z \in \overline{\Omega};\ |f(z)| \leq \sup_K |f|,\quad f \in A^\infty(\Omega) \right\}.$$

The set $\hat{K}$ is clearly a compact subset of $\overline{\Omega}$, and the holomorphic convexity properties at the boundary are exhibited by the set $\hat{K} \cap b\Omega$. This set has the property that it is determined solely by the set $K \cap b\Omega$, as is shown by the following theorem.

Theorem 3.1.7. *Let K_1 and K_2 be compact subsets of $\overline{\Omega}$. If $K_1 \cap b\Omega = K_2 \cap b\Omega$, then $\hat{K}_1 \cap b\Omega = \hat{K}_2 \cap b\Omega$.*

This has an immediate corollary.

Communicated by J. J. Kohn, December 1, 1979.

Corollary 3.1.8. *IF $K \subset\subset \Omega$, then $\hat{K} \subset\subset \Omega$.*

An analogue of the Oka-Weil approximation theorem is also given. However, instead of using the topology of uniform convergence, we use the topology given by L^2-Sobolev spaces. Thus for m a nonnegative integer and U an open subset of $\bar{\Omega}$, one defines the norm $\|u\|_{m,U}$ by

$$\|u\|^2_{m,U} = \sum_{|\alpha| < m} \int_U |D^\alpha u|^2 d\mu,$$

where $d\mu$ represents the standard volume element of Lebesgue measure on $\mathbf{C}^n$. Let $H_m(U)$ be the set of all functions u such that $\|u\|_{m,U}$ is finite. One can now state the approximation theorem.

Theorem 3.2.1. *Let K be a compact subset of Ω with $\hat{K} = K$. Suppose that U is an open subset of $\bar{\Omega}$ with $K \subset U$ and that $f \in A(U) \cap H_m(U)$. Then there is an open subset V with $K \subset V \subset U$, and functions $f_n \in A^\infty(\Omega)$ such that*

$$\lim_{n \to \infty} \|f_n - f\|_{m,V} = 0.$$

The methods of proof of the above theorems rely on the machinery of the $\bar{\partial}$-Neumann problem, in particular, on the global regularity theorem of Kohn [6], and on the techniques of Carleman estimates for the $\bar{\partial}$-operator introduced by Hormander [4], [5].

The author has recently received new proofs of Theorem 3.3.1 and Theorem 3.1.7. from Hakim and Sibony [3]. In fact, they even show that the spectrum of the Frechet algebra $A^\infty(\Omega)$ is exactly $\bar{\Omega}$. It is easy to show that this implies Theorem 3.3.1.

All of the above results hold if Ω is a complex manifold with smooth pseudoconvex boundary such that there is a function $\lambda \in C^\infty(\bar{\Omega})$ which is strongly plurisubharmonic in a neighborhood of the boundary of Ω. However, for simplicity, it will be assumed here that λ is strongly plurisubharmonic on all of $\bar{\Omega}$. For the general case, the necessary modifications may be found in [1].

It is a pleasure for me to thank my thesis advisor, J. J. Kohn, for his assistance in writing this paper. I would also like to thank Eric Bedford and John Erik Fornaess, who called my attention to some of the questions considered here and made several helpful suggestions.

2. THE $\bar{\partial}$-NEUMANN PROBLEM WITH WEIGHTS

2.1. Formulation of the $\bar{\partial}$-Neumann problem

Let Ω' be a complex hermitian manifold of dimension n, and let $\Omega \subset\subset \Omega'$ be an open submanifold of Ω' whose closure $\bar{\Omega}$ is compact. Denote by $b\Omega$ the boundary of Ω, and assume that there exist a neighborhood U of $b\Omega$ and a real-valued function $r \in C^\infty(U)$ such that $dr \neq 0$ on $b\Omega$ and $r(z) = 0$ if and only if $z \in b\Omega$. The sign of r is chosen so that $r < 0$ in Ω and $r > 0$ outside of $\bar{\Omega}$.

Denote by $C^\infty_{(0,q)}$ the space of forms of type $(0, q)$ on $\bar{\Omega}$ which are smooth up to and including the boundary. In terms of local coordinates $z_1, \cdots, z_n$ on a coordinate neighborhood V we can express $\phi \in C^\infty_{(0,q)}$ by

$$\phi = \sum \phi_J d\bar{z}^J,$$

where $\phi_J \in C^\infty(V \cap \bar{\Omega})$; $J = (j_1, \cdots, j_q)$ with $1 \leqslant j_1 < \cdots < j_q \leqslant n$; $d\bar{z}^J = dz_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$.

In a coordinate chart V, the hermitian metric has the form $\Sigma\, h_{jk} dz_j \otimes d\bar{z}_k$, where h_{jk} is a positive definite hermitian matrix with C^∞ coefficients. We keep the hermitian structure on Ω' fixed in all that follows. If f is a form of type $(1, 0)$ and $f = \Sigma f_j dz_j$ in a local coordinate system, we set $\langle f, f\rangle = \Sigma h^{jk} f_j \bar{f}_k$, where (h^{jk}) is the inverse of (h_{jk}). Every point in Ω' has a neighborhood U where there are n forms $\omega^1, \cdots, \omega^n$ of type $(1, 0)$ with C^∞ coefficients such that $\langle \omega^j, \omega^k\rangle = \delta_{jk}, j, k = 1, \cdots, n$. If we set $f = \Sigma f'_j \omega^j$, it follows that $\langle f, f\rangle = \Sigma |f'_j|^2$. More generally, a differential form of type $(0, q)$ can be written in a unique way in the form $f = \Sigma_{|J|} f_J \bar{\omega}^J$ and we can define $\langle f, f\rangle$ by $\langle f, f\rangle = |f|^2 = \Sigma\, |f_J|^2$. This definition is independent of the choice of the basis $\omega^1, \cdots, \omega^n$.

Let φ be a smooth real-valued function in $\bar{\Omega}$. We then define $L^2_{(0,q)}(\Omega, \varphi)$ as the space of all measurable forms f in Ω of type $(0, q)$ such that

$$\|f\|^2_\varphi = \int_\Omega |f|^2 e^{-\varphi} dV < \infty, \tag{2.1.1}$$

where dV represents the volume form associated with the hermitian metric.

The operator $\bar{\partial}$ defines, in the weak sense, closed densely defined operators

$$T\colon L^2_{(0,0)}(\Omega, \varphi) \to L^2_{(0,1)}(\Omega, \varphi),$$
$$S\colon L^2_{(0,1)}(\Omega, \varphi) \to L^2_{(0,2)}(\Omega, \varphi).$$

By T^* and S^* we shall mean the adjoints of T and S, respectively, with respect to the norm given by (2.1.1).

If $u \in C^\infty$ and the forms $\omega^1, \cdots, \omega^n$ are a local basis for forms of type $(1, 0)$ in an open coordinate patch U, we set

$$du = \sum \frac{\partial u}{\partial \omega^j} \omega^j + \sum \frac{\partial u}{\partial \bar{\omega}^j} \bar{\omega}^j$$

is a definition of the operators $\partial/\partial\omega^j$ and $\partial/\partial\bar{\omega}^j$ in U. Then we have

$$Tu = \bar{\partial}u = \sum \frac{\partial u}{\partial \bar{\omega}^j}\bar{\omega}^j,$$

and if $f = \Sigma f_k \bar{\omega}^k$, it follows that

$$Sf = \bar{\partial}f = \sum \frac{\partial f_j}{\partial \bar{\omega}^j}\bar{\omega}^j \wedge \omega^k + \cdots,$$

where the dots indicate terms in which no f_k is differentiated. They occur because $\bar{\partial}\omega^j$ and $\bar{\partial}\bar{\omega}^j$ need not be zero. After integration by parts, one can show that if $f \in D_{T^*}$, then

$$T^*f = -\sum \delta_j f_j + \cdots,$$

where $\delta_j w = e^{\varphi}\partial(\omega e^{-\varphi})/\partial\omega^j$, and the dots, as before, indicate terms where no f_j is differentiated. Moreover, f satisfies the boundary condition

$$\sum f_j \frac{\partial r}{\partial \omega^j} = 0 \quad \text{on } U \cap b\Omega. \tag{2.1.2}$$

We now formulate the $\bar{\partial}$-*Neumann problem*. In all that follows we shall be using a fixed weight $e^{-\varphi}$. We define the space $\mathfrak{D}^{0,1} \subset C^{\infty}_{(0,1)}$ by

$$\mathfrak{D}^{0,1} = C^{\infty}_{(0,1)} \cap D_{T^*}.$$

Thus $\mathfrak{D}^{0,1}$ can be characterized as the space of smooth forms of type (0, 1), which satisfy the boundary condition (2.1.2). Since $\mathfrak{D}^{0,1} \subset D_{T^*}$, we may define the hermitian form

$$Q\colon \mathfrak{D}^{0,1} \times \mathfrak{D}^{0,1} \to C$$

by

$$Q(\phi, \psi) = (T^*\phi, T^*\psi) + (S\phi, S\psi).$$

In the case which we shall consider, the following inequality holds:

$$Q(\phi, \phi) \geqslant \|\phi\|^2_{\varphi}, \quad \phi \in \mathfrak{D}^{0,1}. \tag{2.1.3}$$

Let $\tilde{\mathfrak{D}}^{0,1}$ be the Hilbert space obtained by completing $\mathfrak{D}^{0,1}$ under the norm $Q(\phi, \phi)^{1/2}$. By virtue of (2.1.3), there is a natural embedding of $\tilde{\mathfrak{D}}^{0,1}$ in $L^2_{(0,1)}(\Omega, \varphi)$. By well-known arguments in Hilbert space theory, there is a bounded self-adjoint N in $L^2_{(0,1)}(\Omega, \varphi)$ with the following properties:

(i) $R(N) \subset D_{T^*} \cap D_S$,

(ii) $T(T^*N) \subset D_T$ and $R(SN) \subset D_{S^*}$,

(iii) N is one-to-one,

(iv) if $\alpha \in L^2_{(0,1)}(\Omega, \varphi)$; then

$$Q(N\alpha, \psi) = (\alpha, \psi) \quad \text{for all } \psi \in \mathfrak{D}^{0,1}.$$

The last statement (iv) implies that $\square N\alpha = \alpha$, where $\square$ is the differential operator given by $TT^* + S^*S$. The operator N is usually called the Neumann operator, and is the inverse of $\square$. Our interest in the Neumann operator lies primarily in the following fact:

If $\alpha \in L^2_{(0,1)}(\Omega, \varphi)$ and $S\alpha = 0$, then $TT^*N\alpha = \alpha$. Thus $T^*N\alpha$ is a solution of $Tu = \alpha$, and it is uniquely characterized by the fact that $T^*N\alpha$ is orthogonal to the null space of T.

2.2. The basic estimate

In this section we present the basic estimate in Sobolev spaces. In the space $L^2_{(0,1)}(\Omega, \varphi)$ the estimate follows immediately from Hormander's estimates [4], and for higher order Sobolev spaces, the estimate and its proof are quite close to that of Kohn [6]. We shall therefore only briefly sketch the proof.

In what follows, we shall replace the weight function φ by $\chi(\varphi)$, where $\chi(\tau)$ is a smooth real-valued function. For $f \in C^\infty_{(0,1)}(U)$ with $\omega^1, \cdots, \omega^n$ as above, we shall use the expression $\|\bar{L}f\|^2_{\chi(\varphi)}$ to represent the sum $\Sigma\|\partial f_j/\partial\bar{\omega}^k\|^2_{\chi(\varphi)}$. This can be extended in an obvious way by means of a partition of unity to $f \in C^\infty_{(0,1)}(\bar{\Omega})$. Any two sums arising from different partitions will differ by at most $C\|f\|^2_{\chi(\varphi)}$, where C is a constant independent of the function $\chi(\varphi)$.

After using a partition of unity, Proposition 3.1.3. of [4] leads to the following proposition.

Proposition 2.2.1. *Suppose that the manifold Ω is pseudoconvex, and that there is a function $\varphi \in C^\infty(\bar{\Omega})$ that is strongly plurisubharmonic on all of $\bar{\Omega}$. Let χ be a smooth convex increasing function. Then there is a constant C independent of the weight function χ so that for all $f \in \mathfrak{D}^{0,1}$,*

$$(2.2.1)\quad \int \chi'(\varphi)|f|^2 e^{-\chi(\varphi)}dV + \|\bar{L}f\|^2_{\chi(\varphi)} \leqslant C\{\|T^*f\|^2_{\chi(\varphi)} + \|Sf\|^2_{\chi(\varphi)} + \|f\|^2\}.$$

We now choose a specific function $\chi_{s,t}$, depending on two parameters s and t, and lift the above L^2 estimates to higher order weighted Sobolev spaces. The proof, which follows that of Kohn [6], essentially involves only a determination of how the constants depend upon the above parameters. First we must give the precise definition of the weighted Sobolev spaces.

Let $\zeta_j, j = 1, 2, \cdots, N$, be smooth real-valued functions such that $\Sigma\,\zeta_j^2 = 1$, and the support of each function ζ_j is contained in a coordinate neighborhood of $\bar{\Omega}$. We define

$$\|f\|^2_{m,\chi(\varphi)} = \sum_{j=1}^{N} \sum_{|\alpha| < m} \|D^\alpha \zeta_j f\|^2_{\chi(\varphi)}.$$

Here D^α refers to $\partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_{2n}^{\alpha_{2n}}$ for the coordinate neighborhood of supp ζ_j. If ψ_k, $k = 1, 2, \cdots, N'$, is another set of functions with the above properties, then there exist constants C_1 and C_2, depending only on the functions ψ_k and ζ_j, $j = 1, \cdots, N$, $k = 1, \cdots, N'$, so that

$$\sum_{j=1}^{N} \sum_{|\alpha| \leqslant m} \|D^\alpha \zeta_j f\|^2_{\chi(\varphi)} \leqslant C_1 \sum_{k=1}^{N'} \sum_{|\alpha| \leqslant m} \|D^\alpha \psi_k f\|^2_{\chi(\varphi)}$$
$$\leqslant C_2 \sum_{j=1}^{N} \sum_{|\alpha| \leqslant m} \|D^\alpha \zeta_j f\|^2_{\chi(\varphi)}.$$

It is therefore enough to use any such partition of unity. Let $H_m(\Omega, \chi(\varphi))$ be the space of all $u \in L^2_{(0,1)}(\Omega, \chi(\varphi))$ such that $\|u\|_{m,\chi(\varphi)} < \infty$. In a similar way one can define norms for forms of type $(0, q)$. One simply requires that all components be contained in $H_m(\Omega, \chi(\varphi))$.

It will also be useful to define certain tangential operators defined in a coordinate neighborhood U of a point $z_0 \in b\Omega$. Choose real tangential coordinates $(t_1, t_2, \cdots, t_{2n-1}, r)$, where r is the boundary defining function. Let D_t^α denote the partial derivative $\partial^{|\alpha|}/\partial t_1^{\alpha_1} \cdots \partial t_{2n-1}^{\alpha_{2n-1}}$. We denote by A_t^k any differential operator which is of order k and supported in such a neighborhood U, and which is a sum of the operators D_t^α with C^∞ coefficients. We now specify a 2-parameter family of convex increasing functions $\chi_{s,t}$. For any μ, one can construct a function ψ with the following properties:

(2.2.2)
(a) $\psi \in C^\infty(\mathbf{R})$;
(b) ψ is convex and nondecreasing;
(c) $\psi(\tau) = 0$ for $\tau \leqslant \mu$, $\psi(\tau) > 0$ for $\tau > \mu$.

For each $s, t \geqslant 0$, we set $\chi_{s,t}(\tau) = t\tau + s\psi(\tau)$. The weight functions $\chi_{s,t}$ will play the role of the weight function χ in Proposition 2.2.1. One then has as a consequence that there is a constant $T_0 > 0$, such that for $t \geqslant T_0$, $s \geqslant 0$, and $f \in \mathfrak{D}^{0,1}$,

$$t\|f\|^2_{\chi_{s,t}(\varphi)} + \|\bar{L}f\|^2_{\chi_{s,t}(\varphi)} \leqslant CQ(f, f). \tag{2.2.3}$$

We now establish some convenient notation. If A and B are functions on a set of parameters S, we use the notation $A \lesssim B$ to mean that for some $C > 0$, $|A(\sigma)| \leqslant C|B(\sigma)|$ for all $\sigma \in S$. For the norm associated with the weight functions $\chi_{s,t}(\varphi)$, we shall write $\|f\|^2_{m,s,t}$ instead of $\|f\|^2_{m,\chi_{s,t}(\varphi)}$, and simply $\|f\|^2_{s,t}$ if $m = 0$. It will also be clear from the context that all of the operators arising in the $\bar{\partial}$-Neumann problem will be those associated with the weight function $\chi_{s,t}(\varphi)$. The estimates for the Neumann operator N are given in the following proposition.

Proposition 2.2.2. *Under the hypothesis of Proposition* 2.2.1 *for each integer* $m \geqslant 0$, *there exists a constant* T_m *such that for all* $t \geqslant T_m$, $s \geqslant 0$,

$$t\|u\|^2_{m,s,t} \lesssim \sum_{j=0}^{N} (1+s+t)^{2(m-j)} \cdot \|\Box u\|^2_{j,s,t}. \tag{2.2.4}$$

The proof of this proposition is based on (2.2.3) and is a straightforward but tedious verification that the constants obtained from the method of proof of Kohn's global regularity theorem [6] do indeed grow as in (2.2.4). In fact, for all applications we shall need only that there exist constants $C_m(t)$ and an integer N_m such that for $t \geqslant T_m$,

$$\|u\|^2_{m,s,t} \lesssim C_m(t)(1+s)^{N_m} \cdot \|\Box\|^2_{m,s,t};$$

that is, that the constant for fixed t has polynomial growth with respect to the parameter s.

By the method of elliptic regularization (see Kohn [6] or Folland-Kohn [2]), it follows under the above restrictions on t that if $\Box u \in H_m(\Omega)$, then $u = N\Box u \in H_m$ and that the above estimates hold. Moreover, if α is a form of type (0, 1) with $S\alpha = 0$, then $v = T^*N\alpha$ is the unique solution of $Tv = \alpha$ which is orthogonal to the null space of T. By the estimate (2.2.4), we know that if $\alpha \in H_m(\Omega)$, then $v \in H_{m-1}(\Omega)$. We now show that v is actually in $H_m(\Omega)$.

Proposition 2.2.3. *Under the hypotheses of Proposition* 2.2.1, *for each integer* $m \geqslant 0$, *there exists a constant* $T_m > 0$ *such that for all* $t \geqslant T_m$, $s \geqslant 0$, *the following estimate holds*:

$$\|T^*u\|^2_{m,s,t} + \|Su\|^2_{m,s,t} \lesssim \sum_{j=0}^{N} (1+s+t)^{2(m-j)} \|\Box u\|^2_{j,s,t}, \quad u \in D_\Box \cap C^\infty(\bar{\Omega}). \tag{2.2.5}$$

Proof. The proof is similar to that of the estimate for u. In the interior, one can estimate the derivatives of order m of T^*u and Su. As before, one then estimates the tangential derivatives. The remaining derivatives are controlled by showing that the boundary is non-characteristic with respect to a certain differential operator.

Let A be a differential operator of order m, supported in the interior of Ω. Applying A to T^*u for $u \in D$ gives

$$\|AT^*u\|^2_{s,t} = (ATT^*u, Au) + ([T, A]T^*u, Au) + (AT^*u, [A, T^*]u).$$

After performing the same operation for $\|ASu\|^2$ and adding both equations, we get

$$\begin{aligned}\|AT^*u\|_{s,t}^2 + \|ASu\|_{s,t}^2 = {} & (A\Box u, Au) + ([T, A]T^*u, Au) \\ & + (AT^*u, [A, T^*]u) + ([S^*, A]Su, Au) \\ & + (ASu, [A, S]u).\end{aligned}$$

Now in each of the last four terms on the right-hand side, on one side of the inner product there is an mth order differential operator applied to one of Su or T^*u. These mth order operators, which we denote by B^m, arise as commutators and satisfy the following estimate:

$$\|B^m v\|_{s,t}^2 \lesssim \sum_{j=0}^{m} (1 + s + t)^{2(m-j)} \|v\|_{j,s,t}^2.$$

Using the estimate $|(x, y)| \leqslant \varepsilon\|x\|^2 + C(\varepsilon)\|y\|^2$, with the ε in front of the terms $B^m T^* u$ and $B^m Su$, yields

$$\begin{aligned}\|AT^*u\|_{s,t}^2 + \|ASu\|_{s,t}^2 \lesssim {} & \|\Box u\|_{m,s,t}^2 + C(\varepsilon) \sum_{j=0}^{m} (1 + s + t)^{2(m-j)} \|u\|_{j,s,t} \\ & + \varepsilon \sum_{j=0}^{m} (1 + s + t)^{2(m-j)} \left[\|T^*u\|_{j,s,t}^2 + \|Su\|_{j,s,t}^2 \right].\end{aligned} \tag{2.2.6}$$

A similar inequality holds for A with support intersecting the boundary, provided we assume that the operators are tangential. Note that by (2.2.4), the term $C(\varepsilon)\sum_{j=0}^{m}(1 + s + t)^{2(m-j)}\|u\|_{j,s,t}^2$ appearing in (2.2.6) is dominated by the right-hand side of (2.2.5). Therefore, if we can only show how the nontangential derivatives can be estimated, then the proposition will follow by taking a partition of unity, adding up the inequalities and then taking ε sufficiently small to absorb the terms $\varepsilon\|T^*u\|_{m,s,t}^2 + \varepsilon\|Su\|_{m,s,t}^2$ into the left-hand side of (2.2.6). (The lower-order derivatives of T^*u and Su can be controlled by an elementary induction argument.)

Consider the operator L,

$$L: C_{(0,0)}^{\infty}(\bar{\Omega}) \oplus C_{(0,2)}^{\infty}(\bar{\Omega}) \to C_{(0,1)}^{\infty}(\bar{\Omega}) \oplus C_{(0,3)}^{\infty}(\bar{\Omega}),$$

given by $L(v, w) = (\bar{\partial}v + \vartheta w, \bar{\partial}w)$.

Since L is elliptic and of first order, one can write $(\partial/\partial r)(v, w)$ as a combination of $L(v, w)$ and tangential derivatives of (v, w). However,

$$L(T^*u, Su) = ((TT^* + S^*S)u, 0) = (\Box u, 0).$$

Therefore as in the proof of Proposition 2.2.3, one gets

$$\begin{aligned} &\|A_t^{m-k}\frac{\partial^k}{\partial r^k}T^*u\|_{s,t}^2 + \|A_t^{m-k}\frac{\partial^k}{\partial r^k}SU\|_{s,t}^2 \\ &\qquad \lesssim \|A_t^{m-k+1}\frac{\partial^{k-1}}{\partial r^{k-1}}T^*u\|_{s,t}^2 + \|A_t^{m-k+1}\frac{\partial^{k-1}}{\partial r^{k-1}}Su\|_{s,t}^2 \\ &\qquad\quad + \|A_t^{m-k}\frac{\partial^{k-1}}{\partial r^{k-1}}\Box u\|_{s,t}^2 \\ &\qquad\quad + (1+s+t)^2\big[\|T^*u\|_{m-1,s,t}^2 + \|Su\|_{m-1,s,t}^2\big]. \end{aligned} \tag{2.2.7}$$

After applying (2.2.7) m times, one obtains the estimate for the nontangential derivatives. By the earlier remark, this completes the proof.

Finally, by using the method of elliptic regularization, we can state the following theorem.

Theorem 2.2.4. *Let Ω be a pseudoconvex manifold with smooth boundary. Suppose that $\varphi \in C^\infty(\overline{\Omega})$ is strongly plurisubharmonic on all of $\overline{\Omega}$, and that $\chi_{s,t}(\varphi)$ is a 2-parameter family of weight functions constructed as in* (2.2.2).

Suppose that α is a $\bar\partial$-closed form of type (0, 1) *and that $\alpha \in H_m(\Omega)$. Let v be the solution of $Tv = \alpha$, which is provided by the $\bar\partial$-Neumann problem with weight $\chi_{s,t}(\varphi)$. Then there exist positive constants T_m and C_m such that for all $t \geqslant T_m$ and all $s \geqslant 0$, $v \in H_m(\Omega)$ and satisfies*

$$\|v\|_{m,s,t}^2 \leqslant C_m(1+s+t)^{2m}\|\alpha\|_{m,s,t}^2. \tag{2.2.8}$$

3. APPLICATIONS TO BOUNDARY BEHAVIOR OF HOLOMORPHIC FUNCTIONS

3.1. Holomorphic convexity at the boundary

In this section we shall study the holomorphic convexity of pseudoconvex manifolds, with special emphasis given to the convexity properties at the boundary. We shall define the holomorphic hull with respect to the holomorphic functions which are smooth up to the boundary, denoted by $A^\infty(\Omega)$. Thus $A^\infty(\Omega)$ is defined by

$$A^\infty(\Omega) = A(\Omega) \cap C^\infty(\overline{\Omega}).$$

Since functions in $A^\infty(\Omega)$ have smooth boundary values, we can allow both K and $\hat K$ to have boundary points as well.

Definition 3.1.1. If K is a compact subset of $\overline{\Omega}$, we define the hull $\hat K$ by

$$\hat K = \left\{ z \in \overline{\Omega};\ |f(z)| \leqslant \sup_K |f|, f \in A^\infty(\Omega)\right\}.$$

For the class of manifolds we shall consider, it will be shown that if K is a compact subset of the interior, then the above holomorphic hull and the one taken with respect to $A(\Omega)$ coincide. It will also be useful to consider the hull $\hat{K}_P$, taken with respect to $P^\infty(\Omega)$, the set of functions $\varphi \in C^\infty(\bar{\Omega})$ which are plurisubharmonic in Ω.

Definition 3.1.2. For K a compact subset of $\bar{\Omega}$, the hull $\hat{K}_P$ is given by

$$\hat{K}_P = \left\{ z \in \bar{\Omega};\ \varphi(z) \leqslant \sup_K \varphi,\ \varphi \in P^\infty(\Omega) \right\}.$$

Assumption 3.1.3. It will be assumed in all which follows that Ω is a pseudoconvex manifold with smooth boundary, and that there is a function $\varphi \in C^\infty(\bar{\Omega})$ such that φ is strongly plurisubharmonic on $\bar{\Omega}$.

We shall make frequent use of the following density theorem.

Theorem 3.1.4. *Let f be a function in $A(\Omega) \cap H_m(\Omega)$, where m is a nonnegative integer. Then for any $\varepsilon > 0$, there exists $g \in A^\infty(\Omega)$ with $\| g - f\|_m < \varepsilon$.*

Proof. Using the function φ given by Assumption 3.1.3, we apply the estimate given by Theorem 2.2.3 for $s = 0$, and t any number larger than T_m. First let $U_j, j = 0, 1, \cdots, k$, be a finite collection of open sets with the following properties.

(1) $\bar{\Omega} = \bigcup_{j=1}^k U_j$,

(2) $U_0 \subset\subset \Omega$,

(3) On each $U_j, j = 1, 2, \cdots, k$, there are holomorphic coordinates $z_1^j, \cdots, z_n^j$ with $\partial r/\partial x_n^j > 0$, where $z_n^j = x_n^j + iy_n^j$.

Let $\zeta_j, j = 0, 1, \cdots, k$, be a partition of unity subordinate to the covering $\{U_j\}$. For sufficiently small $\delta > 0$, let f_δ be given by

$$f_\delta(z) = \zeta_0(z)f(z) + \sum_{j=1}^k (\zeta_j f)(z_1^j, \cdots, z_n^j + \delta).$$

Observe that $f_\delta \in C^\infty(\bar{\Omega})$,

$$\lim_{\delta\to 0} \|f_\delta - f\|_m = 0, \quad \lim_{\delta\to 0} \|\bar{\partial} f_\delta\|_m = 0.$$

Let $m_l, l = 1, 2, \cdots,$ be an increasing sequence of positive integers with $m_1 > m$. Choose $t_1 > \max(T_m, T_{m_1})$. Let v_δ be the solution of $\bar{\partial} v_\delta = \bar{\partial} f_\delta$ which is obtained from Theorem 2.2.4 with $t = t_1$ and $s = 0$. Since $t_1 > T_{m_1}$, it is clear that $v_\delta \in H_{m_1}(\Omega)$ and since $t_1 > T_m$, v_δ satisfies

$$\|v_\delta\|^2_{m,0,t_1} \leqslant C_m(1 + t_1)^{2m} \cdot \|\bar{\partial} f_\delta\|^2_{m,0,t_1}.$$

Hence $\lim_{\delta\to 0}\|v_\delta\|_m^2 = 0$. Setting $u_\delta = f_\delta - v_\delta$, it follows that $u_\delta \in H_{m_1}$ and $\lim_{\delta\to 0}\|u_\delta - f\|_m = 0$ Therefore there exists $g_1 = u_{\delta_1} \in H_{m_1}(\Omega)$ such that

$\| g_1 - f \|_m < \varepsilon/2$. By the same reasoning, for every l there exists a holomorphic function g_l such that $\| g_l - g_{l-1} \| < 2^{-l}\varepsilon$. Set $g = \lim g_l$. Clearly, $g \in A^\infty(\Omega)$ and $\| g - f \|_m < \varepsilon$. The proof is complete.

We now apply the above theorem to show that the two hulls defined above are identical. The idea of the proof has been taken from Theorem 5.2.10 of [5].

Theorem 3.1.5. *If K is a compact subset of $\bar{\Omega}$, then $\hat{K} = \hat{K}_P$.*

Remark. Hakim and Sibony [3] have strengthened this theorem by showing that the above result holds also if one takes the hull with respect to plurisubharmonic functions which are only continuous on $\bar{\Omega}$.

Proof of Theorem 3.1.5. Since it is obvious that $\hat{K}_P \subset \hat{K}$, it is sufficient to prove the opposite inclusion. So assume K is a compact subset of $\bar{\Omega}$ and $z_0 \notin \hat{K}_P$. It follows that there is a function $\varphi \in P^\infty(\Omega)$ with $\varphi(z_0) > \sup_K \varphi$. We can assume that φ is strongly plurisubharmonic on $\bar{\Omega}$, for if not consider $\varphi' = \varphi + \varepsilon\tilde{\varphi}$ for sufficiently small $\varepsilon > 0$, where $\tilde{\varphi}$ is any smooth strongly plurisubharmonic function. We shall construct a family of functions $g_s \in A^\infty(\Omega)$ such that $\lim_{s\to\infty} g_s(z_0) = 1$, and $\lim_{s\to\infty} g_s(z) = 0$ for all z with $\varphi(z) \leqslant \varphi(z_0)$, $z \neq z_0$. Thus for large s, $|g_s(z_0)| > \sup_K |g_s|$, and hence $z_0 \notin \hat{K}$.

Since φ is strongly plurisubharmonic at z_0, there exist a neighborhood V, $z_0 \in V$, and a function $u_0 \in A^\infty(V)$ with $u_0(z_0) = 0$ and such that $\operatorname{Re} u_0(z) < 0$ whenever z satisfies $z \in V$, $\varphi(z) \leqslant \mu_0$, $z \neq z_0$. Choose a function $\zeta \in C_0^\infty(V)$ such that $\zeta(z) = 1$ for z in a neighborhood of z_0. It follows that there exists numbers a and μ' with $a > 0$ and $\mu' > \mu_0$ such that

$$\operatorname{Re} u_0(z) < -a \quad \text{if } z \in \operatorname{supp} \bar{\partial}\zeta \text{ and } \varphi(z) < \mu'. \tag{3.1.1}$$

Choose μ and $\mu_0 < \mu < \mu'$, and let $\chi_{s,t}(\varphi)$ be the 2-parameter family of weight functions constructed as in (2.2.2). Set $m = n + 2$ and fix $t_0 > T_{n+2}$. Then the hypotheses of Theorem 2.2.4 are satisfied and it follows that, for all $s \geqslant 0$, one can solve the $\bar{\partial}$-equation so that the estimate (2.2.8) holds.

Let u be an analytic function in V and set $\alpha_s = \bar{\partial}(\zeta u e^{\tau s u_0})$, where τ is a constant whose value is yet to be determined. Denote by V_s the solution of $\bar{\partial} V_s = \alpha_s$, given by the $\bar{\partial}$-Neumann problem with weight $\chi_{s,t_0}(\varphi)$. By Theorem 2.2.4, v_s satisfies

$$\| v_s \|^2_{n+2,s,t_0} \leqslant C_{n+2}(1 + t_0 + s)^{2(n+2)} \| \alpha_s \|^2_{n+2,s,t_0}. \tag{3.1.2}$$

Since $\operatorname{supp} \alpha_s \subset \operatorname{supp} \bar{\partial}\zeta$, it follows that for some constant C independent of s,

$$\| \alpha_s \|^2_{n+2,s,t_0} \leqslant \int_{\operatorname{supp} \bar{\partial}\zeta} C(1 + s + \tau s)^{2(n+2)} \exp(2\tau s u_0 - \chi_{s,t_0}(\varphi))\, dV. \tag{3.1.3}$$

We wish to show that there exist positive numbers δ and τ such that for sufficiently large s,

$$\operatorname{Re}(2\tau s u_0 - \chi_{s,t_0}(\delta))(z) < -\delta s, \quad z \in \operatorname{supp} \bar{\partial}\zeta.$$

Toward this end, set

$$M_1 = \sup_{z \in \operatorname{supp} \bar{\partial}\zeta} \{\operatorname{Re} u_0(z)\}, \quad M_2 = \sup_{s > 0, z \in \operatorname{supp} \bar{\partial}\zeta} \{-\chi_s(\varphi)(z)\}.$$

Recall from (2.2.2) that $\chi_{s,t_0}(\varphi) = t_0\varphi + s\psi(\varphi(z))$, where ψ is a convex increasing function with $\psi(\tau) > 0$ for $\tau > \mu$. Suppose that $\varphi(z) \geqslant \mu'$. Then we have $\chi_{s,t_0}(\varphi) \geqslant t_0\mu' + s\psi(\mu')$, and consequently

$$\operatorname{Re}(2\tau s u_0 - \chi_{s,t_0}(\varphi))(z) < s(2\tau M_1 - \psi(\mu')) - t_0\mu'.$$

Fix $\tau > 0$ so that $2\tau M_1 - \psi(\mu') < 0$. Then for large s, $\operatorname{Re}(2\tau s u_0 - \chi_{s,t_0}(\varphi))(z)$ approaches $-\infty$ if $\varphi(z) \geqslant \mu'$. If on the other hand, $\varphi(z) < \mu'$, then by (3.1.1), $\operatorname{Re} u_0(z) < -a$, and hence

$$\operatorname{Re}(2\tau s u_0 - \chi_{s,t_0}(\varphi)) \leqslant -2a\tau s + M_2.$$

Thus we see that $\delta > 0$ can be chosen so that, for large s,

$$\operatorname{Re}(2\tau s u_0 - \chi_{s,t_0}(\varphi))(z) \leqslant -\delta s, \quad z \in \operatorname{supp} \bar{\partial}\zeta.$$

Hence by (3.1.3), for a new constant C,

$$\|\alpha_s\|^2_{n+2,s,t_0} \leqslant C(1+s)^{2(n+2)} e^{-\delta s}.$$

Since this also decreases exponentially to zero, it follows by (3.1.2) that

$$\lim_{s \to \infty} \|V_s\|^2_{n+2,s,t_0} = 0.$$

By the Sobolev Lemma and the fact that the weight functions $\chi_{s,t_0}(\varphi)$ are independent of φ for $\varphi \leqslant \mu$,

$$\sup_{z \in \Omega, \varphi(z) < \mu_0, |\alpha| \leqslant 1} \{|D^\alpha V_s(z)|\} \leqslant C \|V_s\|^2_{n+2,s,t_0}.$$

If we now set $g_s = \zeta u e^{\tau s u_0} - V_s$, then it is clear that $g_s \in A(\Omega) \cap H_{n+2}$ and

$$\lim_{s \to \infty} g_s(z_0) = u(z_0), \quad \lim_{s \to \infty} g_s(z) = 0$$

for z with $\varphi(z) \leqslant \mu_0$, $z \neq z_0$. If we set $u \equiv 1$ in V, then the existence of such functions g_s immediately implies that $\hat{K} \subset \hat{K}_P$, except for the fact that the functions g_s are in $H_{n+2}(\Omega)$ and not necessarily in $C^\infty(\bar{\Omega})$. But then after approximating g_s sufficiently closely (Theorem 3.1.4), the result follows.

Remark. If we choose local holomorphic coordinates $z_1, \cdots, z_n$ in V, which vanish at z_0, and in the above proof set $u = z_k$, $k = 1, 2, \cdots, n$, then the corresponding function g_s^k has the property that

$$\lim_{s\to\infty} dg_s^k(z_0) = dz_k(z_0).$$

Hence for large s their differentials are linearly independent. As before the functions g_s^k are in $H_{n+2}(\Omega)$, and it remains only to approximate them by functions $f_k \in A^\infty(\Omega)$. This means that for every point $z_0 \in \bar{\Omega}$, there exist n functions $f_1, \cdots, f_n \in A^\infty(\bar{\Omega})$ such that their differentials at z_0 are linearly independent. This fact will be needed later.

We now give a construction with plurisubharmonic functions which is useful for studying convexity properties at the boundary.

Proposition 3.1.6. *Suppose that $\varphi \in C^\infty(\bar{\Omega})$ is strongly plurisubharmonic in a neighborhood of $b\Omega$. Then for all sufficiently small $a > 0$, there exist smooth strongly plurisubharmonic functions φ_a on $\bar{\Omega}$ which satisfy the following properties*:

(a) $\varphi_{a|_{b\Omega}} = \varphi_{|b\Omega}$,

(b) $\varphi_a \leqslant \varphi$,

(c) *for all $z \in \Omega$, $\lim_{a\to 0^+} \varphi_a(z) = -\infty$ uniformly on compact subsets of Ω.*

Proof. The proof is just a slight twist of Theorem 3.7 of Kohn [6]. This theorem states that for sufficiently large C, the function $\mu = -\log|r| + C\varphi$ is strongly plurisubharmonic in Ω near $b\Omega$. (r is the boundary-defining function of $\bar{\Omega}$). Hence if we set $s(z) = -e^{-\mu}$, then $s(z)$ is a new boundary-defining function such that the level sets of s are pseudoconvex near $b\Omega$. With both a and $b > 0$, set $\lambda_{a,b} = -\log(a - s) + b\varphi$. By imitating the proof of Theorem 3.7 in [6], it follows that if b is chosen sufficiently large, say $b = b_0$, then for all $a > 0$, the functions λ_{a,b_0} are strongly plurisubharmonic in a neighborhood U of the boundary, where U is independent of a. Observe that there is a constant γ independent of a, such that if $\lambda_{a,b_0}(z) \geqslant \gamma$, then $z \in U$. Let χ be a smooth convex function such that $\chi(t) = \gamma + 1$ for $t \leqslant \gamma$ and $\chi(t) = t$ for $t \geqslant \gamma + 2$. Set

$$\zeta_a(z) = \frac{1}{b_0}\left[\chi(\lambda_{a,b_0}(z)) + \log a\right].$$

Since χ is convex and nondecreasing, the functions ζ_a are plurisubharmonic on $\bar{\Omega}$. They satisfy the conclusions of the proposition except that they may fail to be strongly plurisubharmonic in the interior of Ω. An examination of the proof of Theorem 3.7 and the above formula for ζ_a shows that there is a neighborhood V (in the relative topology of $\bar{\Omega}$) of $b\Omega$ such that the eigenvalues of the complex Hessian of ζ_a are bounded below in V by a fixed positive constant independent of a. Let $\psi \in C_0^\infty(\Omega)$ be chosen so that $\psi(z) = 1$ for all $z \notin V$. Set

$$\varphi_{a,\delta}(z) = \zeta_a(z) + \delta\psi(z)\cdot\tilde{\varphi}(z),$$

where φ is any smooth strongly plurisubharmonic function on $\bar{\Omega}$. For sufficiently small δ, say $\delta = \delta_0$, the functions $\varphi_a = \varphi_{a,\delta_0}$ are strongly plurisubharmonic on $\bar{\Omega}$ and satisfy the conclusions of the proposition.

Proposition 3.1.6 is the main step in the following theorem.

Theorem 3.1.7. *Let K_1, K_2 be compact subsets of $\bar{\Omega}$ such that $K_1 \cap b\Omega = K_2 \cap b\Omega$. Then $\hat{K}_1 \cap b\Omega = \hat{K}_2 \cap b\Omega$.*

Remark. Hakim and Sibony [3] have recently found a new proof of Theorem 3.1.7.

Proof. By Theorem 3.1.5, it suffices to show that $\hat{K}_{1P} \cap b\Omega = \hat{K}_{2P} \cap b\Omega$. By symmetry, it suffices to show that $\hat{K}_{1P} \cap b\Omega \subset \hat{K}_{2P} \cap b\Omega.^{\infty}(\Omega)$

Suppose then that $z_0 \notin \hat{K}_{1P}$, $z_0 \in b\Omega$. Then, as in the proof of Theorem 3.1.5, there exists a smooth strongly plurisubharmonic function g with $g(z_0) > 1$, $\sup_{K_1} g < 1$. Since $K_1 \cap b\Omega = K_2 \cap b\Omega$, there exists a set $K \subset\subset \Omega$ with

$$\hat{K}_2 \subset K \cup \{z \in \bar{\Omega}; g(z) \leqslant 1\}.$$

By Proposition 3.1.6, one can choose a sufficiently small such that $g_a(z) \leqslant 1$ for $z \in K$. But since $g_a \leqslant g$, this implies that $g_a \leqslant 1$ on $\hat{K}_2$. On the other hand, $g_a(z_0) = g(z_0) > 1$. Thus $z_0 \notin \hat{K}_{2P}$. This completes the proof.

A simple but important special case of Theorem 3.1.7 is when $K_1 \subset\subset \Omega$ and $K_2 = \varnothing$. Then we have $\hat{K}_1 \cap b\Omega = \hat{K}_2 \cap b\Omega = \varnothing$. This gives

Corollary 3.1.8. *If $K \subset\subset \Omega$, then $\hat{K} \subset\subset \Omega$.*

Remark. The hypothesis that the boundary of Ω is smooth is essential. Sibony [7] has constructed a bounded pseudoconvex domain in C^2 with nonsmooth boundary such that $\hat{K}_{\infty}$, defined as the holomorphic hull with respect to the bounded holomorphic of the domain, does not satisfy the conclusions of Corollary 3.1.8.

Remark. Using an approximation theorem, Theorem 3.2.1, and Corollary 3.1.8, one can easily show that the definition of $\hat{K}$ given here and the usual definition coincide if $K \subset\subset \Omega$.

3.2. An approximation theorem

In this section we shall prove, for the Sobolev spaces H_m, an approximation theorem of the Oka-Weil type. the main feature of this theorem is that the region on which the approximation takes place is allowed to intersect the boundary of the domain. In order to measure the H_m-norm of a function f restricted to a given open set G, we define the norm $\| \ \|_{m,G}$ by

$$\|f\|_{m,G}^2 = \sum_j \sum_{|\alpha| < m} \int_{V_j \cap G} |D^{\alpha} f|^2 \, dV.$$

where $V_j, j = 1, 2, \cdots, N$, is any covering of $\bar{G}$ by coordinate neighborhoods, and D^α refers to the derivative D^α in the coordinate chart V_j. The topology defined by this norm is independent of the covering. In [5], Hörmander gave a proof of the Oka-Weil theorem, based on estimates for the solution of the $\bar{\partial}$-equation in an infinite family of weighted spaces. This idea is also used in the following approximation theorem.

Theorem 3.2.1. *Let Ω be a complex manifold satisfying Assumption 3.1.3. Suppose that K is a compact subset of $\bar{\Omega}$ and that $\hat{K} = K$. Let G be a neighborhood of K in the relative topology of Ω. Suppose that $f \in A(\Omega \cap G)$ and that $\|f\|_{m,G} < \infty$. Then there exist a sequence of functions $f_n \in A^\infty(\Omega)$ and an open subset G' with $K \subset\subset G' \subset\subset G$ such that $\lim_{n\to\infty}\|f_n - f\|_{m,G'} = 0$.*

Proof. Since $\hat{K} = \hat{K}_P = K$, there exist a function φ, which is strongly plurisubharmonic and smooth on $\bar{\Omega}$, and a number μ such that $\varphi(z) \leqslant \mu$ for $z \in K$, and $\{z \in \Omega;\ \varphi(z) \leqslant \mu\} \subset\subset G$. Let $\chi_{s,t}(\varphi)$ be a 2-parameter family of weighted functions constructed as in (2.2.2). Fix $t_0 > T_m$, where T_m is the constant given by Theorem 2.2.4. We may choose $\psi \in C_0^\infty(G)$ so that $\psi = 1$ in a neighborhood of $\{z \in \Omega;\ \varphi(z) \leqslant \mu\}$. Set $\alpha = \bar{\partial}(\psi f)$. Since $\varphi(z) > \mu$ on supp α, there exist positive constants C and b, independent of s, so that

$$\|\alpha\|^2_{m,s,t_0} \leqslant Ce^{-bs}\|\psi_1 f\|, \tag{3.2.1}$$

where $\psi_1 \in C_0^\infty(G)$ is chosen so that $\psi_1 \equiv 1$ on supp ψ.

Let P_s denote the projection of $L^2(\Omega)$ onto $L^2(\Omega) \cap A(\Omega)$ with respect to the norm $\|\ \|_{s,t_0}$. Then $P_s(\psi f) = \psi f - V_s$, where V_s is the solution of $\bar{\partial} V_s = \alpha$ given by Theorem 2.2.4. By (3.2.1) and (2.2.8),

$$\begin{aligned}\|P_s(\psi f) - \psi f\|^2_{m,s,t_0} &= \|V_s\|^2_{m,s,t_0}\\ &\leqslant C_m(1 + s + t_0)^{2m}\cdot\|\alpha\|^2_{m,s,t_0}\\ &\leqslant C_m(1 + s + t_0)^{2m}\cdot Ce^{-bs}\|\psi_1 f\|^2_m.\end{aligned}$$

Set $G' = \{z \in \bar{\Omega};\ \varphi(z) < \mu\}$. The weight functions $\chi_{s,t_0}(\varphi)$ are independent of s in G'. By (3.2.2) and the fact that $\psi \equiv 1$ in G', this gives

$$\|P_s(\psi f) - f\|^2_{m,G'} \leqslant C_m(1 + s + t_0)^{2m}\cdot Ce^{-bs}\|\psi_1 f\|^2_m. \tag{3.2.3}$$

Since the right-hand side of (3.2.3) approaches 0 as s approaches ∞, the functions $P_s(\psi f)$ clearly satisfy the conclusions of the theorem, except that $P_s(\psi f)$ are in $H_m(\Omega)$ and not necessarily $A^\infty(\Omega)$. However, by Theorem 3.1.4 we may approximate $P_s(\psi f)$ arbitrarily closely by $f_s \in A^\infty(\Omega)$. The proof is then complete.

3.3. Domains of existence for $A^\infty(\Omega)$

We now apply the method of the above approximation theorem to show that if Ω is a complex manifold satisfying Assumption 3.1.3, then there is a function $u \in A^\infty(\Omega)$ which cannot be extended analytically to a neighborhood of any boundary point. A new proof of this fact has recently been found by Hakim and Sibony [3]. I would like to take this opportunity to thank Professor E. Bedford, who originally suggested this problem to me, and who pointed out the possible approach of studying the points on the boundary around which the Levi form has constant rank. As in Corollary 3.1.8, the assumption of certain minimal smoothness properties of the boundary would appear to be essential, for Sibony has constructed in [7] a bounded pseudoconvex domain $\Omega \subset \mathbf{C}^2$ such that if $f \in H^\infty(\Omega)$, the set of bounded holomorphic functions in Ω, then f extends analytically to a larger domain.

The proof below of Theorem 3.3.1 has roughly two parts. In the first part, holomorphic functions are constructed in a neighborhood of a boundary point where the rank of the Levi form is locally constant. These functions satisfy certain inequalities which essentially show that the theorem is true locally. In the second part, one then uses the argument of Theorem 3.2.1 to approximate the locally defined functions by globally defined functions for which similar inequalities still hold. From these inequalities the result follows in a straightforward manner.

Theorem 3.3.1. *Let Ω be a complex manifold satisfying Assumption* 3.1.3. *Then there exists a function $u \in A^\infty(\Omega)$ which cannot be extended analytically to a neighborhood of any boundary point.*

Before giving the proof of Theorem 3.3.1, we shall prove a lemma which is useful in the local part of the proof.

Lemma 3.3.2. *Let z_0 be a boundary point of an n-dimensional complex manifold with smooth pseudoconvex boundary. Suppose that there is an l-dimensional complex manifold $\gamma \subset b\Omega$, with $z_0 \in \gamma$, and that the rank of the Levi form at z_0 is $n - l - 1$. Then there exists a coordinate neighborhood V of z_0, with holomorphic coordinates $z_1, \cdots, z_n$ satisfying the following properties*:

(1) $V \subset \gamma = \{z \in V;\ z_{l+k} = 0,\ k = 1, 2, \cdots, n - l\}$.

(2) Writing z'-$(z_1, \cdots, z_l)$ and $z'' = (z_{l+1}, \cdots, z_n)$, the Taylor expansion of the boundary-defining function $r(z)$ in the variables z'' has the form

$$r(z) = r(z', z'') = \frac{\partial r}{\partial z_n}(z', 0)z_n + \frac{\partial r}{\partial \bar{z}_n}(z', 0)z_n + O(|z''|)^2.$$

(3) For $z \in V \cap \bar{\Omega}$

$$2\,\mathrm{Re}\, z_n - |Im z_n| + \frac{1}{2}\sum_{k=l+1}^{n-1} |z_k|^2 \leqslant r(z) \leqslant \frac{1}{2}\,\mathrm{Re}\, z_n + |Im z_n| + 2\sum_{k=l+1}^{n-1} |z_k|^2.$$

Proof. Choose complex coordinates $w_1, \cdots, w_n$ in a neighborhood W of z_0, such that

$$W \cap \gamma = \{z \in W;\ w_{l+k} = 0, k = 1, 2, \cdots, n - l\}.$$

Writing $w = (w', w'')$ as above, we take the first-order Taylor expansion of the boundary-defining function $r(w)$ in the variables w'' around the point $(w', 0)$:

$$r(w) = \sum_{k=l+1}^{n} \frac{\partial r}{\partial w_k}(w', 0)w_k + \sum_{k=l+1}^{n} \frac{\partial r}{\partial \overline{w}_k}(w', 0)\overline{w}_k + O(|w''|^2).$$

Let $M = \{w \in W; \Sigma_{k=l+1}^{n} \partial r/\partial w_k(w', 0)w_k = 0\}$. We claim that M is a complex manifold of dimension $n - 1$. To see this we must show that for $j = 1, \cdots, n$,

$$\frac{\partial}{\partial \overline{w}_j} \sum_{k=l+1}^{n} \frac{\partial r}{\partial w_k}(w', 0)w_k = 0$$

at all points of M. For $j = l + 1, \cdots, n$ this is obvious. For $j = 1, \cdots, l$ and $w_l \in M$, we note that $\Sigma_{k=l+1}^{n} w_k\partial/\partial w_k$ is a tangential vector of type $(1, 0)$ at the point $(w', 0)$ and that $\Sigma_{k=l+1}^{n}(\partial^2 r/\partial \overline{w}_j\partial w_k)(w', 0)w_k$ is simply

$$\mathfrak{L}_r\left((w', 0);\ \sum_{k=l+1}^{n} w_k\frac{\partial}{\partial w_k}, \frac{\partial}{\partial w_j}\right),$$

i.e., the Levi form of r at the point $(w', 0)$ evaluated at the vectors $\Sigma_{k=l+1}^{n} w_k(\partial/\partial w_k)$ and $\partial/\partial w_j$. Since $\partial/\partial w_j$ is in the null space of the Levi form at $(w', 0)$, and the Levi form is positive semi-definite, we conclude that

$$\sum_{k=l+1}^{n} \frac{\partial^2 r}{\partial \overline{w}_j\partial w_k}(w', 0)w_k = 0.$$

Therefore M is an $(n - 1)$-dimensional complex manifold which is tangent to $b\Omega$ at each point of $\gamma \cap W$. If we take coordinates $z_1, \cdots, z_n$ in a neighborhood V of z_0 such that M is given by $\{z \in V; z_n = 0\}$ and γ is given by $\{z \in V; z_{l+k} = 0, k = 1, \cdots, n - l\}$, then the Taylor series of $r(z)$ with respect to the variables z'' has the form (2).

Finally, by making coordinate changes only in the variables z'', and assuming, as we may, that the point z_0 is mapped to the origin, we can write the second-order Taylor series of $r(z)$ in the variables z'' around z_0 in the form

$$r(0, z'') = z_n + \bar{z}_n + \sum_{j=l+1}^{n-1} |z_j|^2 + O(|z''|^3).$$

Under this coordinate change, properties (1) and (2) still hold. It is then trivial to show that after possibly shrinking V somewhat, property (3) holds.

Proof of Theorem 3.3.1. Let z_0 be a point in $b\Omega$ such that for some neighborhood V_0 of z_0, the Levi form has constant rank $n - l - 1$ on $V_0 \cap b\Omega$. The heart of the proof of Theorem 3.3.1 is contained in the following claim.

Claim. For any neighborhood U of z_0, and any integer $m \geqslant 0$, there exists an infinite family of functions $f_\delta \in A^\infty(\Omega)$, $\delta > 0$, such that

$$\lim_{\delta \to 0} \|f_\delta\|_m = 0, \quad \lim_{\delta \to 0} \|f_\delta\|_{m+1,U} = +\infty.$$

Proof of Claim. We wish to construct a weight function φ which behaves like $|z - z_0|^2$ near z_0. Let $g_1, \cdots, g_N \in A^\infty(\Omega)$ be constructed as in the remark following Theorem 3.1.5 so that $g_k(z_0) = 0$, $k = 1, \cdots, N$, and n of their differentials are linearly independent at every point of $\bar{\Omega}$. Let $\zeta \in C^\infty(\bar{\Omega})$ be constructed so that $\zeta \equiv 0$ near z_0, and $\zeta(z) > 0$ at all points $z, z \neq z_0$, where $g_k(z) = 0$ for all k, $k = 1, 2, \cdots, N$. Then for sufficiently small $\varepsilon > 0$, the function

$$\varphi(z) = \sum_{k=1}^{N} |g_k(z)|^2 + \varepsilon\zeta(z)$$

vanishes only at z_0 and is strongly plurisubharmonic on all of $\bar{\Omega}$. Suppose that $\mu > 0$ and that $\chi_{s,t}(\varphi)$ is the 2-parameter family of weight functions constructed as in (2.2.2). Then the hypotheses of Theorem 2.2.4 are satisfied.

By a well-known result, since the Levi form has constant rank $n - l - 1$ near z_0, there exists a neighborhood V_0' of z_0 such that $b\Omega \cap V_0'$ foliates into complex manifolds. In particular, there is an l-dimensional complex manifold passing through z_0, and the hypotheses of Lemma 3.3.2 are satisfied. Hence there is a coordinate neighborhood V of z_0 with coordinate functions $z_1, \cdots, z_n$ such that properties (1), (2), and (3) hold in V.

Let $\psi \in C_0^\infty(V)$ with $\psi \equiv 1$ in a neighborhood of z_0. It follows that there exists $\mu > 0$ so that $\psi(z) = 1$ for all z in a neighborhood of $\{z \in \bar{\Omega}; \varphi(z) \leqslant \mu\}$. This μ will now be fixed in all that follows. For $\delta > 0$, we define a function u_δ in V by

$$u_\delta = \left(\frac{1}{z_n - \delta}\right)^n.$$

The function ψu_δ may be considered as a smooth function on all of $\bar{\Omega}$ if we set $(\psi u_\delta)(z) = 0$ for $z \notin V$. Let U be any neighborhood of z_0 with $U \subset V$ such that $\varphi(z) < \mu$ for $z \in U$. Then by using property (3) of the lemma, one easily verifies that for any integer $m \geqslant 0$, there exists a constant C_m, independent of δ, such that

$$\|\psi u_\delta\|_m \leqslant C_m \|u_\delta\|_{m,U}, \tag{3.3.1}$$

$$\|\bar{\partial}(\psi u_\delta)\|_m \leqslant C_m \|u_\delta\|_{m,U}, \tag{3.3.2}$$

$$\lim \frac{\|u_\delta\|_{m+1,U}}{\|u_\delta\|_{m,U}} = +\infty. \tag{3.3.3}$$

Proceeding exactly as in Theorem 3.2.1, we show that the functions u_δ can be approximated by globally defined functions g_δ such that

$$\lim_{\delta\to 0} \frac{\|g_\delta\|_{m+1,U}}{\|g_\delta\|_m} = +\infty. \tag{3.3.4}$$

Fix $t_0 > \operatorname{Max}(T_m, T_{m+1})$, where T_m and T_{m+1} are the constants given by Theorem 2.2.4. Denoting by P_s the projection operator of $L^2(\Omega)$ into $L^2(\Omega)$ $A(\Omega)$ with respect to the norm $\|\ \|_{s,t_0}$, we observe that

$$P_s(\psi u_\delta) = \psi u_\delta - v_{\delta,s},$$

where $v_{\delta,s}$ is the solution of $\bar{\partial} v_{\delta,s} = \bar{\partial}(\psi u_\delta)$ given by Theorem 2.2.4. Since $\psi(z) = 1$ for all z in a neighborhood of $\{z \in \Omega;\ \varphi(z) \leqslant \mu\}$, it follows that $\varphi(z) > \mu$ when $z \in \operatorname{supp} \bar{\partial}(\psi u_\delta)$. Hence there exist constants $b > 0$, $C' > 0$ such that

$$\|\bar{\partial}(\psi u_\delta)\|^2_{m+1,s,t_0} \leqslant C' e^{-bs} \|\bar{\partial}(\psi u_\delta)\|^2_{m+1}. \tag{3.3.5}$$

Combining the estimates in (2.2.8), (3.3.2), and (3.3.5), one obtains that

$$\begin{aligned}
\|P_s(\psi u_\delta) - \psi u_\delta\|^2_{m+1,s,t_0} &= \|v_{\delta,s}\|^2_{m+1,s,t_0} \\
&\leqslant C_{m+1}(1 + s + t_0)^{2m+2} \|\bar{\partial}(\psi u_\delta)\|^2_{m+1,s,t_0} \\
&\leqslant C' C_{m+1}(1 + s + t_0)^{2m+2} e^{-bs} \|\bar{\partial}(\psi u_\delta)\|^2_{m+1} \\
&\leqslant C_{m+1} C' C_{m+1}(1 + s + t_0)^{2m+2} e^{-bs} \|u_\delta\|^2_{m+1,U}.
\end{aligned}$$

Since $\varphi(z) < \mu$ for $z \in U$, the weight functions $\chi_{s,t_0}(\varphi)$ are independent of s for $z \in U$. Since $\psi(z) = 1$ on U, we obtain, for new constants C and $b' > 0$,

$$\|P_s(\psi u_\delta) - u_\delta\|^2_{m+1,U} \leqslant C e^{-b's} \|u_\delta\|^2_{m+1,U}.$$

Now choose $s = s_0$ so that $Ce^{-b's_0} \leqslant 1/4$. By the triangle inequality, we have

$$\|P_{s_0}(\psi u_\delta)\|_{m+1,U} \geqslant \frac{1}{2} \|u_\delta\|_{m+1,U}.$$

Both s_0 and t_0 are now fixed. Since the norm $\| \ \|_{m,s_0,t_0}$ is equivalent to $\| \ \|_m$, we have

$$\begin{aligned}\|P_{s_0}(\psi u_\delta)\|_m &\leqslant \|\psi u_\delta\|_m + \|v_{\delta,s_0}\|_m \\ &\leqslant \|\psi u_\delta\|_m + C(s_0, t_0)\|\bar{\partial}(\psi u_\delta)\| \\ &\leqslant C'(s_0, t_0)\|u_\delta\|_{m,U},\end{aligned}$$

where the last inequality follows by (3.3.1) and (3.3.2).

Thus we have constructed an infinite family of holomorphic functions $g_\delta = P_{s_0}(\psi u_\delta)$ which satisfy

$$\lim_{\delta\to 0} \frac{\|g_\delta\|_{m+1,U}}{\|g_\delta\|_m} \geqslant \frac{1}{2C'(s_0, t_0)} \lim_{\delta\to 0} \frac{\|u_\delta\|_{m+1,U}}{\|u_\delta\|_{m,U}} = +\infty. \tag{3.3.6}$$

We use Theorem 3.1.4 to approximate the functions g_δ by $f_\delta \in A^\infty(\Omega)$ so that (3.3.6) still holds for the functions f_δ. After normalizing the functions appropriately we have

$$\lim_{\delta\to 0} \|f_\delta\|_{m+1,U} = +\infty, \quad \lim_{\delta\to 0} \|f_\delta\|_m = 0.$$

This proves the claim.

We now proceed to construct the functions $u \in A^\infty(\Omega)$. By elementary continuity considerations, one can choose a sequence of points $z_k \in b\Omega$ such that $\{z_k, k = 1, 2, \cdots\}$ is dense in $b\Omega$, and such that the Levi form has constant rank near each point z_k. Let U_k^j be a sequence of neighborhoods of z_k such that $d(z, z_k) < 1/j$, where $z \in U_k^j$. Choose sequences $j(m)$ and $k(m)$, $m = 1, 2, \cdots$ such that each neighborhood is represented exactly once in the sequence $U_{k(m)}^{j(m)}$. Suppose that f is a function in $A^\infty(\Omega)$ and that there exists a neighborhood V (in the ambient space Ω') of a point $w \in b\Omega$ such that f extends analytically to $\Omega \cup V$. Since $\{z_k, k = 1, 2, \cdots\}$ is dense in $b\Omega$, there exist an integer k_0, a constant $C > 0$, and an open set $V' \subset\subset V$ such that $z_{k_0} \in V'$ and $\|f\|_{m,V'} < m!C^m$ for all $m = 1, 2, \cdots$. Therefore if we construct a function $u \in A^\infty(\Omega)$ such that for each pair of postive integers (j, k) there exists an integer m such that $\|u\|_{m,U_k^j\cap\Omega} \geqslant m!j^m$, then the theorem will be proved. Since there is a one-to-one correspondence between the pairs (j, k) and the positive integers m, given by $j = j(m)$, $k = k(m)$, it will suffice to construct a function $u \in A^\infty(\Omega)$ satisfying

$$\|u\|_{m,U_{k(m)}^{j(m)}\cap\Omega} \geqslant m!\,(j(m))^m. \tag{3.3.7}$$

By the claim proven above, for each integer $m > 0$ and $\delta > 0$ there exists a function $f_{\delta,m}$ such that

$$\lim_{\delta\to 0} \|f_{\delta,m}\|_{m,U_{k(m)}^{j(m)}\cap\Omega} = +\infty,$$
$$\lim_{\delta\to 0} \|f_{\delta,m}\|_{m-1} = 0.$$

Choose $\delta(1)$ sufficiently small so that for $u_1 = f_{\delta(1),1}$, we have

$$\|u_1\|_{1,U^{j(1)}_{k(1)}\cap\Omega} > j(1).$$

Assume inductively that we have chosen $\delta(l)$ for $l \leqslant m-1$ so that the following two conditions are satisfied.

(1) $\|u_l - u_{l-1}\|_{l-1} < 2^{-l}$, where $u_l = \sum_{\nu=1}^{l} f_{\delta(\nu),\nu}$,

(2) $\|u_l\|_{\nu,U^{j(\nu)}_{k(\nu)}\cap\Omega} > \nu!(j(\nu))$ for $\nu = 1, \cdots, l$.

We seek u_m in the form $u_m = u_{m-1} + f_{\delta(m),m}$, where $\delta(m)$ is yet to be determined. There are three conditions which must be satisfied.

(a) $\|u_m - u_{m-1}\|_{m-1} < 2^{-m}$.

Since $u_m - u_{m-1} = f_{\delta(m),m}$ and $\lim_{\delta\to 0}\|f_{\delta,m}\|_{m-1} = 0$, this is accomplished by taking $\delta(m)$ sufficiently small.

(b) $\|u_m\|_{\nu,U^{j(\nu)}_{k(\nu)}\cap\Omega} > \nu!(j(\nu))^{\nu}$ for $\nu = 1, \cdots, m-1$.

Since $u_m = u_{m-1} = f_{\delta(m),m}$ and we have $\|u_{m-1}\|_{\nu,U^{j(\nu)}_{k(\nu)}\cap b\Omega} > \nu!(j(\nu))^{\nu}$ and $\lim_{\delta\to 0}\|f_{\delta,m}\|_{m-1} = 0$, and since the norm $\|\ \|_{m-1}$ dominates $\|\ \|_{\nu}$ for $\nu = 1, \cdots, m-1$, (b) is also satisfied by taking $\delta(m)$ sufficiently small.

(c) $\|u_{m-1} + f_{\delta(m),m}\|_{m,U^{j(m)}_{k(m)}\cap\Omega} > m!(j(m))^{m}$.

This follows immediately since u_{m-1} is already fixed and $\lim_{\delta\to 0}\|f_{\delta,m}\|_{m,U^{j(m)}_{k(m)}\cap\Omega} = +\infty$. Thus if we choose $\delta = \delta(m)$ sufficiently small, then the inductive step is complete.

By (1), $u = \lim_{l\to\infty} u_l$ exists and is in $A^{\infty}(\Omega)$. By (2), the function u satisfies

$$\|u\|_{\nu,U^{j(\nu)}_{k(\nu)}\cap\Omega} \geqslant \nu!\,(j(\nu))^{\nu}, \quad \nu = 1, 2, \cdots.$$

By the remarks preceding (3.4.6), the above inequalities imply that u does not extend analytically to a neighborhood of any boundary point. This completes the proof.

References

[1] D. Catlin, *Boundary behavior of holomorphic functions on pseudoconvex domains*, Dissertation, Princeton University, 1978.

[2] G. B. Folland & J. J. Kohn, *The Neumann problem for the Cauchy-Riemann complex*, Annals of Math. Studies, No. 75, Princeton University Press, Princeton, 1972.

[3] M. Hakim & N. Sibony, *Spectre de $A(\overline{\Omega})$ pour des domaines bornés faiblement pseudoconvexes réguliers*, J. Functional Analysis **37** (1980) 127–135.

[4] L. Hörmander, *L^2 estimates and existence theorems for the $\bar\partial$ operator*, Acta Math. **113** (1965) 89–152.

[5] ______, *An introduction to complex analysis in several variables*, Van Nostrand, Princeton, 1966.

[6] J. J. Kohn, *Global regularity for $\bar\partial$ on weakly pseudoconvex manifolds*, Trans. Amer. Math. Soc. **181** (1973) 273–292.

[7] N. Sibony, *Prolongement analytique des fonctions holomorphes bornées*, C. R. Acad. Sci. Paris, Ser. A **275** (1972) 973–976.

University of Chicago