J. DIFFERENTIAL GEOMETRY
15 (1980) 553–568

ASYMPTOTICS OF CURVATURE IN A SPACE OF POSITIVE CURVATURE

B. V. DEKSTER & I. KUPKA

We consider here a noncompact complete space M of positive curvature. As it is known, the minimum sectional curvature in a compact region vanishes as the region expands up to the whole space M. We estimate here (Remark 1.9 and Theorem 1.7) how fast it should vanish (and reestablish incidentally the fact of its vanishing; see (1.7.1)).

Other results are connected with the expanding family of Gromoll and Meyer's totally convex sets. We estimate the degree of their convexity and observe the decay of this degree as the family expands.

The proofs involve essentially an investigation of solutions of Jacobi equations. This investigation (§3) was conducted by Kupka, and the geometric part was written by Dekster.

1. The results

1.1. Let M be a noncompact complete Riemannian space of class C^∞ with positive sectional curvatures, and denote the distance between its subsets by $\rho(\cdot,\cdot)$. A curve will be said to be *normal* if it is parametrized with respect to the arc length. A normal geodesic c: $[0, \infty) \to M$ is called a *ray* if any segment of c is minimal between its end points.

1.2. Denote by $B_t(p)$ ($b_t(p)$) the closed (open) metric ball of radius t centered at $p \in M$. Recall the construction of Gromoll and Meyer's compact totally convex sets $C_t(p)$ as described in [1, Proposition 1.3]. Let c be a ray, $c(0) = p$. Put $b_c = \bigcup_{t>0} b_t(c(t))$. Denote by c_t: $[0, \infty) \to M$ the restricted ray from $c(t)$ to ∞ with $c_t(s) = c(t + s)$. Now put $C_t(p) = \bigcap_c (M \setminus b_{c_t})$ where the intersection is taken over all rays c emanating from p.

We will establish in §4.1 the following simple fact.

Remark. $C_t(p) \supset B_t(p)$ and

$$(1.2.1) \qquad \lim_{t\to\infty} R(t)/t = 1 \text{ where } R(t) = \max_{x \in C_t(p)} \rho(x, p).$$

Received February 24, 1977, and, in revised form, August 7, 1980. The work of the first author was supported by NRC of Canada Grants awarded to Professors W. H. Greub, P. Scherk, D. K. Sen and J. R. Vanstone.

1.3. An expanding family of compact sets $D_t \subset M$, $t \in [u, v]$, will be said to be *Lipschitzian* if there exists a constant c such that $\max_{x \in D_b} \rho(x, D_a)/(b - a) < c$ for any $a, b \in [u, v]$, $a < b$.

We prove in §§4.4–4.8 the following theorem.

Theorem. *The family* $C_t(p)$, $t \in [u, v]$, *is Lipschitzian if* $u > 0$.

1.4. We will say that a positive continuous function $k\colon R \to R$ has a *convexity function* $\varepsilon\colon (a, \infty) \to R$ if ε is the minimum solution of the equation

$$\phi' = -\phi^2 - k, \tag{1.4.1}$$

i.e., if any other solution $\phi\colon (b, \infty) \to R$ of (1.4.1) (finite in (b, ∞)) satisfies $b \geqslant a$ and $\phi(t) \geqslant \varepsilon(t)$ for $t > b$.

We will see in Remark 3.6 that if (1.4.1) has any solution $\phi\colon (b, \infty) \to R$, then ε exists and is positive. Obviously ε is unique and $\varepsilon' < 0$. We will see also that $\varepsilon(t) \to_{t\to\infty} 0$ (§3.7), and $\varepsilon(t) \to_{t \to a} \infty$ (Remark 3.6).

1.5. Denote by $k_b(t)$, $k_c(t)$ the minimum sectional curvatures in $B_t(p)$, $C_t(p)$, $t \geqslant 0$, respectively. Denote by A the class of functions $k\colon [0, \infty) \to R$ which are

(i) continuous, positive, nonincreasing and locally Lipschitzian in $(0, \infty)$,

(ii) such that the function

$$\tilde{k} \overset{\text{def}}{=} \begin{cases} k(t) \text{ if } t \geqslant 0, \\ k(0) \text{ if } t < 0 \end{cases} \tag{1.5.1}$$

has a convexity function $\varepsilon\colon (a, \infty) \to R$ with some $a \leqslant 0$.

We will call such ε a convexity function for both $\tilde{k}$ and k.

1.6. Theorem. $k_b \in A$, $k_c \in A$.

The proof is contained in §§4.9–4.11.

Remark. A statement close to the converse is also true: we will show in §5.1 that for any $k \in A$ there exists a surface M_0 of positive curvature having "polar" metric $ds^2 = dr^2 + G(r)d\theta^2$ with $G(r) \in C^2$ for $r > 0$, with no vertex at the pole and such that $k_b = k_c = k$ for the point p at the pole. (Moreover, $C_t(p) = B_t(p)$.)

So the study of k_b and k_c is reduced to a study of the class A.

1.7. Theorem. *Let* $k \in A$ *and* $\tilde{k}$ *be as in* (1.5.1). *Then*

$$K(t) \overset{\text{def}}{=} \int_t^\infty \tilde{k}(x)dx < \infty, t \in R, \left(\text{so}, k(t) \underset{t\to\infty}{\to} 0\right), \tag{1.7.1}$$

$$\int_t^\infty K^2(x)dx < \infty, t \in R, \tag{1.7.2}$$

$$\liminf_{t\to\infty} \frac{1}{K(t)} \int_t^\infty K^2(x)dx \leqslant \frac{1}{4}. \tag{1.7.3}$$

(1.7.1) and (1.7.2) are proved in §3.8, and (1.7.3) in §3.11.

1.8. Theorem. *Let k_0: $[0, \infty) \to R$ satisfy* §1.5(i), $k \in A$ *and* $k_0 \leqslant k$. *Then* $k_0 \in A$.

This is an obvious consequence of Lemma 3.9.

1.9. Remark. Theorems 1.7 and 1.8 allow us to establish some simple properties of a function $k \in A$. We show in §§2.1, 2.2 that there exist

$$\lim_{t\to\infty} k(t)\cdot t^{3/2} \geqslant 0, \quad \liminf_{t\to\infty} k(t)\cdot t^2 \leqslant \frac{1}{4}. \tag{1.9.1}$$

I.e., beginning with some t, the function $k(t)$ is less than $C/t^{3/2}$ with some constant $C > 0$. Moreover, "from time to time" as t grows, $k(t)$ becomes even less than, say, $0.251/t^2$.

1.10. The other results deal with convexity of the sets $C_t(p)$. A set $S \subset M$ will be said to be $[x, y]$-*convex*, $x > 0$, $y > 0$, if any C^∞-curve of the length $< y$ and of the curvature $< x$ having its ends in S belongs to S. A set $S \subset M$ will be said to be $[x]$-*convex* if for any $\delta \in (0, x)$ one can find $y > 0$ such that the set S is $[x - \delta, y]$-convex. Notice that if S is $[x, y]$-convex (respectively, $[x]$-convex), then it is $[\bar{x}, y]$-convex (respectively $[\bar{x}]$-convex) for any $\bar{x} \in (0, x)$.

1.11. The following theorem proved in §§4.13–4.15 shows that the convexity of $C_t(p)$ is described by the properties of convexity functions.

Theorem. *Let* ε_b: $(a_b, \infty) \to R$ *and* ε_c: $(a_c, \infty) \to R$ *be the convexity functions for* k_b *and* k_c, *and let* $R(t)$ *be as in* §1.2. *Then* $C_t(p)$ *is* $[\varepsilon_b(R(t))]$-*convex and* $[\varepsilon_c(t)]$-*convex for* $t > 0$.

1.12. Remark. Theorem 1.11 is precise in the following sense. Let us return to the surface M_0 mentioned in Remark 1.6. Obviously $R(t) = t$ because $C_t(p) = B_t(p)$. It will be shown in §5.2 that the function $k \in A$ in §1.6 can be selected such that the geodesic curvature of the circumference $\operatorname{bd} B_t(p) = \operatorname{bd} C_t(p)$ is equal to its convexity function $\varepsilon(t)$.

1.13. Properties of convexity functions are collected in the following theorem proved in §4.12.

Theorem. (i) *Let* $k \in A$, ε: $(a, \infty) \to R$ *be its convexity function,* $K(t)$ *be as in* (1.7.1), *and* $\eta \in [0, 1/4]$ *be the left-hand part in* (1.7.3). *Then*

$$\varepsilon'(t) < 0 \text{ for } t \in (a, \infty);\ \varepsilon(t) \underset{t\to\infty}{\to} 0;\ \varepsilon(t) \underset{t\to a}{\to} \infty; \tag{1.13.1}$$

$$\varepsilon(t) > K(t) + \int_t^\infty K^2(x)\,dx,\ t > a; \tag{1.13.2}$$

$$1 \leqslant x_1 \leqslant \liminf_{t\to\infty} \frac{\varepsilon(t)}{K(t)} \leqslant x_2, \tag{1.13.3}$$

where x_1, x_2 *are the roots of the equation* $\eta x^2 - x + 1 = 0$ (*if* $\eta = 0$ *then* $x_1 = 1$, $x_2 = \infty$).

(ii) *Let* $\bar{\varepsilon}$: $(\bar{a}, \infty) \to R$ *be the convexity function for another function* $\bar{k} \in A$ *satisfying* $k \leqslant \bar{k}$. *Then*

$$a \leqslant \bar{a} \text{ and } \varepsilon(t) \leqslant \bar{\varepsilon}(t) \text{ for } t > \bar{a}. \tag{1.13.4}$$

2. A further discussion

2.1. Here and in §2.2, we prove (and specify) Remark 1.9. Denote by $k' \leqslant 0$ the derivative of k determined almost everywhere in $[0, \infty)$. As k is locally Lipschitzian in $(0, \infty)$, (see §1.5(i)), one can calculate $\int_u^v K^2(x)\,dx$, $0 < u < v$, by parts up to appearance of k'. By continuity of k at $t = 0$ and a simple limit reasoning, this integration can be also realized over any segment $[0, t]$, $t > 0$. Then

$$\int_0^t K^2\,dx = K^2(t) \cdot t + \int_t^\infty 2Kkx\,dx.$$

Passing to the limit as $t \to \infty$ and keeping in mind that $\int_0^t 2Kkx\,dx > 0$ increases by t and $K^2 t \geqslant 0$, we obtain by (1.7.2) the existence of

$$\lim_{t\to\infty} K^2(t) \cdot t < \infty, \quad \int_0^\infty 2Kkx\,dx < \infty. \tag{2.1.1}$$

As above, passing to the limit in the equality

$$\int_0^t 2Kkx\,dx = K(t) \cdot k(t) \cdot t^2 + \int_0^t k^2 x^2\,dx + \int_0^t K \cdot |k'| \cdot x^2\,dx,$$

we obtain by (2.1.1) the existence of

$$\lim_{t\to\infty} K(t) \cdot k(t) \cdot t^2 < \infty, \quad \int_0^\infty k^2 x^2\,dx < \infty, \quad \int_0^\infty K|k'|x^2\,dx < \infty. \tag{2.1.2}$$

In the same way, the equality

$$\int_0^t k^2 x^2\,dx = \frac{1}{3} k^2(t) \cdot t^3 + \frac{2}{3} \int_0^t k|k'|x^3\,dx$$

shows on the strength of (2.1.2) the existence of $\int_0^\infty k|k'|x^3\,dx$ and that

$$\lim_{t\to\infty} k(t) \cdot t^{3/2} = \left[3\int_0^\infty k^2 x^2\,dx - 2\int_0^\infty k|k'|x^3\,dx\right]^{1/2}. \tag{2.1.3}$$

Thus any constant greater than the right-hand part in (2.1.3) is good as C in §1.9.

2.2. Let us prove now that

$$\liminf_{t\to\infty} k(t) \cdot t^2 \leqslant \frac{1}{4}.$$

Suppose the contrary. Then for t larger than some $t_0 > 0$,

$$k(t) \cdot t^2 \geqslant c > \frac{1}{4}, \quad \text{i.e., } k(t) \geqslant \frac{c}{t^2}.$$

Put

$$k_0(t) = \begin{cases} c/t_0^2 & \text{for } t \in [0, t_0], \\ c/t^2 & \text{for } t > t_0. \end{cases}$$

Then $k_0 \leqslant k$, and $k_0 \in A$ by Theorem 1.8. Application of (1.7.3) to k_0 yields $c \leqslant 1/4$ which contradicts the choice of c.

2.3. A simple calculation based on (1.7.3) shows that if $k \in A$ and there exists $\lim_{t\to\infty} k(t) \cdot t^q > 0$, then $q \geqslant 2$ and, in the case $q = 2$, $\lim_{t\to\infty} k(t) \cdot t^q \leqslant 1/4$.

2.4. One can prove that if a function k: $[0, \infty) \to R$ satisfies §1.5(i) and $\limsup_{t\to\infty} \int_t^\infty K^2(x)dx/K(t) < 1/4$, then $k \in A$. We do not produce this proof, but notice instead that the function $1/(2t + 2)^2 \in A$ because $\phi(t) = 1/(2t + 2)$, $t > 0$, is a solution of (1.4.1) with $k = 1/(2t + 2)^2$. Then by Theorem 1.8, all functions in $[0, \infty)$ satisfying §1.5(i) and not exceeding $1/(2t + 2)^2$ belong to A.

2.5. When η in §1.13 increases from 0 to $1/4$, the root $x_1 = x_1(\eta)$ increases from 1 to 2, and $x_2 = x_2(\eta)$ decreases from ∞ to 2. Notice that the estimate $x_1 \leqslant \liminf_{t\to\infty} \varepsilon(t)/K(t) \leqslant x_2$ is not of geometric interest, since the number η, in the case $k = k_b$ or $k = k_c$, can depend on the point p and thus will not be a characteristic of the space M.

2.6. The estimates (1.13.2) and (1.13.3) are sharp in the following sense. There exists a function $k \in A$ such that

$$1 < \frac{\varepsilon(t)}{K(t) + \int_t^\infty K^2(x)\,dx} < \frac{\varepsilon(t)}{K(t)} \underset{t\to\infty}{\longrightarrow} 1. \tag{2.6.1}$$

One can check that a suitable

$$k = \begin{cases} t^{-4} & \text{for } t \geqslant 1, \\ 1 & \text{for } 0 \leqslant t < 1. \end{cases}$$

Then (1.4.1) is reduced to a special Riccati equation $\phi' = -\phi^2 - t^4$, whose general solution is

$$\phi(t, C) = \frac{1}{t} + \frac{1}{t^2}\tan\left(C + \frac{1}{t}\right).$$

A simple reasoning shows that

$$\varepsilon(t) = \phi\left(t, \frac{\pi}{2}\right) = \frac{1}{t} - \frac{1}{t^2}\cot\frac{1}{t} \quad \text{for } t \geqslant 1$$

and

$$\lim_{t\to\infty} \frac{\varepsilon(t)}{K(t)} = \lim_{t\to\infty} \varepsilon(t) \cdot 3t^3 = 1.$$

2.7. By Remark 1.2, $k_b(t) \geqslant k_c(t) \geqslant k_b(R(t))$, so that $1 \geqslant k_c(t)/k_b(t) \geqslant k_b(R(t))/k_b(t)$. Suppose there exists a $\lim_{t\to\infty} k_b(t) \cdot t^q \neq 0$ for some number q. Then passing to the limit as $t \to \infty$ in the equality

$$\frac{k_b(R(t))}{k_b(t)} = \frac{k_b(R(t)) \cdot R^q(t)}{k_b(t) \cdot t^q} \cdot \left(\frac{t}{R(t)}\right)^q,$$

we obtain

$$\lim_{t\to\infty} \frac{k_c(t)}{k_b(t)} = \lim_{t\to\infty} \frac{k_b(R(t))}{k_b(t)} = 1.$$

3. On solutions of Jacobi equations

3.1. Denote the class of positive continuous functions $k: R \to R$ by P. We consider here the differential equation

$$u'' = -k\,u, \tag{3.1.1}$$

where $k \in P$. By continuity of k, for any 3 numbers t_0, u_0 and $u_0' \in R$ there exists the unique solution $u(t)$, $t \in R$, of class C^2 with the initial data $u(t_0) = u_0$, $u'(t_0) = u_0'$.

Remark. By (3.1.1), a solution $u(t)$ of (3.1.1) is convex at any points t where $u(t) > 0$. Let $u(t_0) > 0$ and $u'(t_0) \geqslant 0$ ($u'(t_0) \leqslant 0$). Then obviously there exists a number $t_1 < t_0$ ($t_1 > t_0$) such that $u(t_1) = 0$, $u(t) > 0$ for $t \in (t_1, t_0]$ ($t \in [t_0, t_1)$). Moreover, if $u'(t_0) \neq 0$, then

$$t_1 \in \left(t_0 - \frac{u(t_0)}{u'(t_0)}, t_0\right) \quad \left(t_1 \in \left(t_0, t_0 - \frac{u(t_0)}{u'(t_0)}\right)\right).$$

3.2. Remark. Let $k_1, k_2 \in P$ and $k_1(t) \leqslant k_2(t)$, $t \in R$. Let $u_1(t)$, $u_2(t)$ be solutions of (3.1.1) with $k = k_1$, $k = k_2$ respectively, and let $t_1 < t_2$. Suppose further that

(1) $u_1(t_1) = u_2(t_1)$, $u_1(t_2) = u_2(t_2)$,

(2) $u_1(t) > 0$, $u_2(t) > 0$ for $t \in (t_1, t_2)$,

(3) either $u_1(t_1) \neq 0$ or $u_1(t_2) \neq 0$.

Then one can easily see that $u_2(t) \geqslant u_1(t)$ for $t \in (t_1, t_2)$. In fact, if $u_2 > u_1$ at a point in (t_1, t_2), then $u_2 > u_1$ in an interval $(x_1, x_2) \subset (t_1, t_2)$ such that $u_1(x_1) = u_2(x_1)$, $u_1(x_2) = u_2(x_2)$. Thus the formula $[u_1'u_2 - u_1u_2']_{x_1}^{x_2} = \int_{x_1}^{x_2}(k_2 - k_1)u_1u_2\,dt$ (see [3,10.31]) yields a contradiction.

Obviously if $k_1 \equiv k_2$, then $u_1 \equiv u_2$.

3.3. Denote by Ω_{t_0} the set of numbers u_0' such that the solution $u(t)$ of (3.1.1) with $u(t_0) = 1$, $u'(t_0) = u_0'$ is positive for $t > t_0$. By Remark 3.1,

$$u'(t) > 0 \text{ for } t \geqslant t_0. \tag{3.3.1}$$

It follows easily from Remark 3.2 that if $u_0' \in \Omega_{t_0}$ and $v_0' > u_0'$, then $v_0' \in \Omega_{t_0}$.

By (3.3.1), $i(t_0) \overset{\text{def}}{=} \inf \Omega_{t_0}$ exists if Ω_{t_0} is not empty. Let us show that $i(t_0) \in \Omega_{t_0}$. (Then, by (3.3.1), $i(t_0) > 0$.) Suppose the contrary. Then the solution $v(t)$ with $v(t_0) = 1$, $v'(t_0) = i(t_0)$ cuts t-axis at a point $t_1 > t_0$. Thus there exists $t_2 \in (t_0, t_1)$ such that $v'(t_2) < 0$. Therefore if $u_0' \in \Omega_{t_0}$ is sufficiently close to $i(t_0)$, then the corresponding solution u satisfies $u'(t_2) < 0$ and, by Remark 3.1, cuts the t-axis. This is impossible since $u_0' \in \Omega_{t_0}$. Thus we have proved

Proposition. *If Ω_{t_0} is not empty, then $\Omega_{t_0} = [i(t_0), \infty)$ for some $i(t_0) > 0$.*

3.4. Theorem. *Let Ω_{t_0} be nonempty for some $t_0 \in R$. Denote by $v(t)$ the solution of* (3.1.1) *with $v(t_0) = 1$ and $v'(t_0) = i(t_0)$, and by $a < t_0$ the number such that $v(a) = 0$, $v(t) > 0$ for $t > a$ (a exists by Remark* 3.1). *Then the following hold.*

(1) *Ω_t is empty for $t \leqslant a$, and nonemtpy for $t > a$. (Therefore a does not depend on t_0.)*

(2) *For any nontrivial solution $w(t)$ of* (3.1.1) *with $w(a) = 0$,*

$$i(t) = \frac{w'(t)}{w(t)}, \quad t > a. \tag{3.4.1}$$

(3) *k has a convexity function (see* §1.4) *$\varepsilon(t) = i(t)$, $t > a$.*

3.5. *Proof.* (1) Let $t_1 \leqslant a$. We need to prove that Ω_{t_1} is empty. Suppose the contrary, i.e., a solution $u(t)$ with $u(t_1) = 1$ is positive for $t \geqslant t_1$. Let us consider the solution $u(t)/u(t_0)$, so that $[u(t)/u(t_0)]|_{t=t_0} = 1$. By definition of $i(t_0)$, $[u(t)/u(t_0)]'|_{t=t_0} \geqslant i(t_0)$. Then $u(\bar{t})/u(t_0) = v(\bar{t})$ for some $\bar{t} \in (a, t_0)$ and, by Remark 3.2, $u(t)/u(t_0) = v(t)$ which is impossible as $u(a) > 0$. Obviously, Ω_{t_2} is not empty for $t_2 > a$ because the solution $v(t)/v(t_2)$ with $[v(t)/v(t_2)]|_{t=t_2} = 1$ is positive for $t \geqslant t_2$, and therefore $v'(t_2)/v(t_2) \in \Omega_{t_2}$.

(2) The mentioned solution $w(t)$ can be represented as $cv(t)$ where the constant $c \neq 0$. So we need to prove that $i(t_2) = v'(t_2)/v(t_2)$ for $t_2 > a$. It was noted above that $v'(t_2)/v(t_2) \in \Omega_{t_2}$; therefore $i(t_2) \leqslant v'(t_2)/v(t_2)$. Suppose $i(t_2) < v'(t_2)/v(t_2)$. Let $u(t)$ be the solution with $u(t_2) = 1$, $u'(t_2) = i(t_2)$. By Remark 3.1 there exists $d < t_2$ satisfying $u(d) = 0$, $u(t) > 0$ for $t > d$. It follows easily from Remark 3.2 that $d < a$. Then the solution $u(t)/u(a)$ is positive for $t > a$ so that $u'(a)/u(a) \in \Omega_a$ which contradicts (1).

(3) Direct calculation based on (3.4.1) and (3.1.1) shows that $i(t)$ is a solution of (1.4.1). Let now ϕ: $(b, \infty) \to R$ be another solution, and $t_0 \in (b, \infty)$. Then the function $u(t) = \exp \int_{t_0}^{t} \phi(t)\, dx$ defined for $t \in (b, \infty)$ is the solution of (3.1.1) with the initial data $u(t_0) = 1$, $u'(t_0) = \phi(t_0)$ satisfying $u(t) > 0$ for $t \geqslant t_0$. That is why Ω_{t_0} is nonempty for any $t_0 > b$. Then $b \geqslant a$ by (1), and $\phi(t_0) \geqslant i(t_0)$ by definition of $i(t_0)$.

3.6. Remark. Let $k \in P$, and let (1.4.1) have a solution $\phi: (b, \infty) \to R$ (perhaps the convexity function). Take some $t_0 > b$ and consider again the function $u(t) = \exp \int_{t_0}^{t} \phi(x)\, dx$, $t > b$. As above, $u'(t_0) \in \Omega_{t_0}$ and Theorem 3.4 can be applied. Then

(i) By (3), k has a convexity function $\varepsilon: (a, \infty) \to R$ with some $a < t_0$. Arbitrariness of $t_0 > b$ implies $a \leqslant b$.

(ii) By (1), Ω_t is nonempty if and only if $t > a$.

(iii) By (3) and Proposition 3.3, $\Omega_t = [\varepsilon(t), \infty)$ and $\varepsilon(t) > 0$.

(iv) By (3) and (3.4.1), $\varepsilon(t) = w'(t)/w(t)$, $t > a$, where w is any nontrivial solution of (3.1.1) with $w(a) = 0$. Therefore $\varepsilon(t) \to_{t \to a} \infty$.

3.7. By (1.4.1), $\varepsilon' < 0$. Therefore there exists $l \overset{\text{def}}{=} \lim_{t\to\infty} \varepsilon(t) \geqslant 0$. If $l > 0$, then $\varepsilon'(t) \leqslant -l^2$ for $t \in (a, \infty)$, and ε vanishes at a point which is impossible. So $l = 0$.

3.8. Putting $\varepsilon(t)$ in (1.4.1) and integrating over an interval $[t, t']$ we have

$$\varepsilon(t) = \varepsilon(t') + \int_t^{t'} \varepsilon^2\, dx + \int_t^{t'} k\, dx. \tag{3.8.1}$$

Passage to the limit as $t' \to \infty$ yields

$$\varepsilon(t) = \int_t^{\infty} \varepsilon^2\, dx + \int_t^{\infty} k\, dx, \tag{3.8.2}$$

where both integrals exist. Therefore

$$\varepsilon(t) > \int_t^{\infty} k(x)\, dx \overset{\text{def}}{=} K(t), \quad t > a. \tag{3.8.3}$$

It follows from (3.8.3) and the existence of $\int_t^\infty \varepsilon^2\, dx$ that $\int_t^\infty K^2(x)\, dx < \infty$. Then by (3.8.2), (3.8.3),

$$\varepsilon(t) > \int_t^{\infty} K^2(x)\, dx + K(t), \quad t > a. \tag{3.8.4}$$

3.9. Lemma. *Let $k_1, k_2 \in P$, $k_1 \leqslant k_2$, and let k_2 have a convexity function $\varepsilon_2: (a_2, \infty) \to R$. Then k_1 has a convexity function $\varepsilon_1: (a_1, \infty) \to R$. Moreover, $a_1 \leqslant a_2$ and $\varepsilon_1(t) \leqslant \varepsilon_2(t)$ for $t > a_2$.*

Proof. Take $t_0 > a_2$. Since ε_2 is a solution of (1.4.1) with $k = k_2$ and, by Remark 3.6(iii), Ω_{t_0} (constructed for k_2) is the segment $[\varepsilon_2(t_0), \infty)$. Let $u(t)$ be the solution of (3.1.1) with $k = k_2$ and initial data $u(t_0) = 1$, $u'(t_0) = \varepsilon_2(t_0)$, so that $u_2(t) > 0$ for $t \geqslant t_0$. Denote by $v(t)$ the solution of (3.1.1) with $k = k_1$ and the initial data $v(t_0) = 1$, $v'(t_0) = 2\varepsilon_2(t_0)$. It follows easily from Remark 3.2 that $v(t) > 0$ for $t \geqslant t_0$. So Ω_{t_0} (for k_1) is not empty for any $t_0 > a_2$. Now by Theorem 3.4(3), k_1 has a convexity function $\varepsilon_1: (a_1, \infty) \to R$ with $a_1 < t_0$. Arbitrariness of $t_0 > a_2$ implies $a_1 \leqslant a_2$.

Suppose now $\varepsilon_1(t_0) > \varepsilon_2(t_0)$. Let $\bar{v}(t)$ be the solution of (3.1.1) with $k = k_1$ and initial data $\bar{v}(t_0) = 1$, $\bar{v}'(t_0) = \frac{1}{2}(\varepsilon_1(t_0) + \varepsilon_2(t_0))$. The solution $\bar{v}$ should cut the t-axis, and therefore it cuts $u(t)$ at a point $t > t_0$. This contradicts Remark 3.2. Thus $\varepsilon_1(t_0) \leqslant \varepsilon_2(t_0)$, $t_0 > a_2$.

3.10. Remark. Let $k \in P$ have a convexity function ε: $(a, \infty) \to R$, and let $u(t)$ be a nontrivial solution of (3.1.1) with $u(t^*) = 0$, $t^* \geqslant a$. Then $u(t) \neq 0$ for $t > t^*$. In fact, suppose the contrary. Then there exists $t_1 > t^*$ such that $u(t_1) = 0$, $u(t) \neq 0$ for $t \in (t^*, t_1)$. Let $t_0 \in (t^*, t_1)$, and let the constant c satisfy $cu(t_0) = 1$. Then $cu'(t_0) < \min \Omega_{t_0} = \varepsilon(t_0)$. Let now $v(t)$ be as in Theorem 3.4. Obviously there exists $t_2 \in [a, t_0)$ such that $cu(t_2) = v(t_2) \geqslant 0$ and $cu(t) > v(t)$ for $t \in (t_2, t_0)$. This contradicts Remark 3.2.

3.11. Lemma. *Let a function $k \in P$ have a convexity function ε: $(a, \infty) \to R$, and $K(t)$ be defined by* (3.8.3). *Then*

$$\liminf_{t\to\infty} \frac{1}{K(t)} \int_t^\infty K^2(x)\,dx \leqslant \frac{1}{4}. \tag{3.11.1}$$

Proof. Suppose the contrary. Then there exists $l > 1/4$ and $t_1 \in R$ such that $[1/K(t)]\int_t^\infty K^2(x)\,dx \geqslant l$ when $t \geqslant t_1$. Let $t_1 > a$ and $q = \inf_{t\geqslant t_1} [\varepsilon(t)/K(t)]$. By (3.8.3), $q \geqslant 1$. By (3.8.2),

$$\frac{\varepsilon(t)}{K(t)} = \frac{1}{K(t)} \int_t^\infty \varepsilon^2(x)\,dx + 1 \geqslant \frac{q^2}{K(t)} \int_t^\infty K^2(x)\,dx + 1 \geqslant q^2 l + 1$$

for $t \geqslant t_1$. Hence

$$q = \inf_{t\geqslant t_1} \frac{\varepsilon(t)}{K(t)} \geqslant q^2 l + 1, \tag{3.11.2}$$

i.e., $q^2 l - q + 1 \leqslant 0$ which is impossible with $l > 1/4$.

4. Proofs of the results

4.1. Proof of Remark 1.2. Suppose $C_t(p) \not\supset B_t(p)$, i.e., there exists $q \in M$ such that $\rho(p, q) \leqslant t$ and $q \notin C_t(p)$. Then there exists a ray c emanating from p such that $q \in b_{c_t}$ (see §1.2). So $\rho(q, c(t + s)) < s$ for some $s > 0$. Now

$$\rho(p, c(t + s)) \leqslant \rho(p, q) + \rho(q, c(t + s)) < t + s, \tag{4.1.1}$$

which is not possible since c is a ray, and $\rho(p, c(t + s)) = t + s$. So $C_t(p) \supset B_t(p)$.

4.2. To prove (1.2.1) it is enough to show that for any sequence $t_j \to_{j\to\infty} \infty$ there exists a subsequence t_i such that $\lim_{i\to\infty}[R(t_i)/t_i] = 1$. Let a subsequence t_i be such that the directions of the shortest paths pq_i, $q_i \in C_{t_i}(p)$, of the length $R(t_i)$ converge to some direction at the point p (if pq_i is not unique, by pq_i we mean one of them). It is easy to see (and is known; see Proof of Proposition 1.3 in [1]) that the geodesic c: $[0, \infty) \to M$ emanating from p in the limit direction is a ray.

Let us consider the triangle pq_ir_i, where $r_i = c(t_i)$ and the side q_ir_i is a shortest path with the ends q_i, r_i. By the construction, the angle

(4.2.1) $$\alpha_i = \measuredangle\, r_ipq_i \to 0 \quad \text{as } i \to \infty.$$

Let us show that

(4.2.2) $$\beta_i = \measuredangle\, pr_iq_i \leqslant \frac{\pi}{2} \quad (\text{if } q_i \neq r_i).$$

Suppose the contrary. Then obviously the points in r_iq_i sufficiently close to r_i belong to the open ball of radius s centered at $c(t + s)$, $s > 0$, and therefore not to $C_t(p)$. But this is impossible by the total convexity of $C_t(p)$.

Let $p'q_i'r_i'$ be a triangle in Euclidean plane with $p'q_i' = pq_i$, $q_i'r_i' = q_ir_i$, $r_i'p' = r_ip$. By Toponogov comparison theorem,

(4.2.3) $$\alpha_i' = \measuredangle\, r_i'p'q_i' \leqslant \alpha_i; \quad \beta_i' = \measuredangle\, p'r_i'q_i' \leqslant \beta_i.$$

By the law of sines and the inclusion $C_t(p) \supset B_t(p)$,

$$\frac{\sin(\alpha_i' + \beta_i')}{\sin \beta_i'} = \frac{p'r_i'}{p'q_i'} = \frac{t_i}{R(t_i)} \leqslant 1;$$

$$\cos \alpha_i' + \sin \alpha_i' \cot \beta_i' \leqslant 1; \quad \cot \beta_i' \leqslant \tan \frac{\alpha_i'}{2}.$$

By (4.2.2) and (4.2.3),

$$0 \leqslant \cot \beta_i \leqslant \cot \beta_i' \leqslant \tan \frac{\alpha_i'}{2} \leqslant \tan \frac{\alpha_i}{2}.$$

Then, by (4.2.1), $\cot \beta_i' \to 0$, and $\alpha_i' \to 0$ as $i \to \infty$. So

$$\lim_{i\to\infty} \frac{t_i}{R(t_i)} = \lim_{i\to\infty} \left\{ \begin{matrix} \cos \alpha_i' + \sin \alpha_i' \cot \beta_i', & \text{if } q_i \neq r_i \\ 1, & \text{if } q_i = r_i \end{matrix} \right\} = 1.$$

4.3. Lemma. *Let $D_t \subset M$, $t \in [u, v]$, be a Lipschitzian expanding family of compact sets, and $k(t)$ be the minimum sectional curvature in D_t. Then the mapping $k\colon [u, v] \to R$ is Lipschitzian.*

The proof seems to be obvious.

4.4–4.8. Proof of Theorem 1.3.

4.4. Let $\bar{k} > 0$ be the maximum sectional curvature in the set $N = \{q \in M | \rho(q, C_v(p)) \leqslant 1\}$. Denote a two-dimensional sphere with the curvature $\bar{k}$ by S; the length of its meridian is $\pi/\sqrt{\bar{k}}$. Put $\delta = \min\{\pi/2\sqrt{\bar{k}}, 1, u\}$. It is known (and can be easily proved on the basis of Rauch comparison theorem and Toponogov theorem on comparison of triangles) that any geodesic intersecting $C_v(p)$ and not longer than δ is a unique shortest path, and its ends are not conjugate points in it.

4.5. Let us show that there exists a number $\beta > 0$ such that for any $t \in [u, v]$, any $q \in C_t(p)$, any shortest path qp, and any $s \in (0, \delta/2)$ the closed ball

$$B_{s\beta}(m) \subset C_t(p), \tag{4.5.1}$$

where $m \in qp$ and $qm = s$. (The point m exists since $qp \geqslant u \geqslant \delta > s$ by Remark 1.2.) It means that $\partial C_t(p)$ does not contain "too sharp edges". Suppose the contrary. Then there exist sequences (1) $1 > \beta_i \underset{i\to\infty}{\to} 0$, $i = 1, 2, \cdots$, (2) $t_i \in [u, v]$, (3) $q_i \in C_{t_i}(p)$, (4) shortest paths q_ip and (5) numbers $s_i \in (0, \delta/2)$ such that $B_{s_i\beta_i}(m_i) \not\subset C_{t_i}(p)$, where $m_i \in q_ip$, $q_im_i = s_i$. Selecting a subsequence, if necessary, one may assume that

$$t_i \to t_0 \in [u, v],\ q_i \to q_0 \in C_{t_0}(p),\ q_ip \to q_0p \text{ as } i \to \infty, \tag{4.5.2}$$

where q_0p is a shortest path between q_0 and p. Let $r_i \in B_{s_i\beta_i}(m_i) \setminus C_{t_i}(p)$. Then (4.5.3)

$$\rho(r_i, q_i) \leqslant \rho(q_i, m_i) + \rho(m_i, r_i) \leqslant s_i + s_i\beta_i < \frac{\delta}{2}(1 + \beta_i) < \delta.$$

So the shortest path $r_iq_i \subset N$, and is unique.

4.6. Let us show that the angle $\gamma_i \overset{\text{def}}{=} \measuredangle r_iq_im_i \to_{i\to\infty} 0$. To this end, we construct on the sphere S a triangle $r_i'q_i'm_i'$ whose sides are respectively equal to the sides of the triangle $r_iq_im_i$. Then the angle $\gamma_i' \overset{\text{def}}{=} \measuredangle r_i'q_i'm_i' \to_{i\to\infty} 0$ because

$$\frac{r_i'm_i'}{m_i'q_i'} = \frac{r_im_i}{m_iq_i} \leqslant \frac{s_i\beta_i}{s_i} = \beta_i \underset{i\to\infty}{\to} 0.$$

Now by Toponogov comparison theorem,

$$\gamma_i \leqslant \gamma_i' \underset{i\to\infty}{\to} 0. \tag{4.6.1}$$

4.7. Let μ_0 be the direction (unit vector) of the shortest path q_0p at the end q_0, μ_i be the direction of q_ip at q_i, and ν_i be the direction of q_ir_i at q_i, so that $\langle \mu_i, \nu_i \rangle = \cos \gamma_i$. (Notice that $q_ir_i \geqslant q_im_i - m_ir_i = s_i - s_i\beta_i > 0$.) It follows easily from (4.5.2) and (4.6.1) that

$$(q_i, \nu_i) \underset{i\to\infty}{\to} (q_0, \mu_0) \quad \text{in } TM, \tag{4.7.1}$$

so that

$$p_i \overset{\text{def}}{=} \exp(q_i, \nu_i \cdot q_ip) \underset{i\to\infty}{\to} \exp(q_0, \mu_0 \cdot q_0p) = p.$$

By Remark 1.2 and Proposition 1.3 (1) in [1], for large i

$$p_i \in B_u(p) \subset C_u(p) \subset C_{t_i}(p). \tag{4.7.2}$$

Now by the total convexity of $C_{t_i}(p)$ the normal geodesic g_i: $[0, q_0p] \to M$ with $g_i(x) = \exp(q_i, v_i \cdot x)$ lies in $C_{t_i}(p)$. By (4.5.3), $q_ir_i < \delta \leqslant u \leqslant q_0p$. So $r_i = g_i(q_ir_i) \in C_{t_i}(p)$ which contradicts the choice of r_i.

4.8. Suppose now Theorem 1.4 is false. Then there exist sequences $a_i < b_i$, $a_i, b_i \in [u, v]$, such that

$$\frac{d(a_i, b_i)}{b_i - a_i} \underset{i\to\infty}{\to} \infty, \quad \text{where } d(a, b) = \max_{x \in C_b(p)} \rho(x, C_a(p)). \tag{4.8.1}$$

Thus $b_i - a_i \underset{i\to\infty}{\to} 0$ because $d(a, b)$, $a, b \in [u, v]$, does not exceed the diameter of $C_v(p)$. So for large i,

$$\Delta \overset{\text{def}}{=} b_i - a_i < \frac{\delta}{2}\beta. \tag{4.8.2}$$

Let $q \in C_{b_i}(p)$ satisfy $\rho(q, C_{a_i}(p)) = d(a_i, b_i)$, and let qp be a shortest path with the ends q, p. Let $m \in qp$ be such that $qm = \Delta/\beta < \delta/2$ by (4.8.2). By (4.5.1), $B_\Delta(m) \subset C_{b_i}(p)$. Then by Proposition 1.3(1) in [1], $m \in C_{a_i}(p)$. Now

$$\frac{d(a_i, b_i)}{b_i - a_i} \leqslant \frac{\rho(q, m)}{\Delta} = \frac{\Delta/\beta}{\Delta} = \frac{1}{\beta}$$

which contradicts (4.8.1).

4.9–4.11. Proof of Theorem 1.6.

4.9. By Theorem 1.3 and Lemma 4.3, k_c is Lipschitzian in any segment $[u, v]$, $u > 0$. It follows easily from Proposition 1.3(1) in [1] that, for sufficiently small $t > 0$, the set $C_t(p)$ lies in an arbitrary small neighborhood of $C_0(p)$. Therefore k_c is continuous at $t = 0$, and k_c satisfies §1.5(i). Obviously k_b satisfies §1.5(i).

4.10. Let us show that any nontrivial solution of (3.1.1) with $k \in \{k_b, k_c\}$ has not more than one zero in $[0, \infty)$. Suppose the contrary. Then one can find a solution $u(t) \not\equiv 0$ such that $u(t_1) = u(t_2) = 0$, $0 \leqslant t_1 < t_2$, $u(t) > 0$ for $t \in (t_1, t_2)$ and $u'(t_1) = 1$.

Let $u_m(t)$ be the solution of the equation

$$u'' = m \cdot k\, u \tag{4.10.1}$$

with $u_m(t_1) = 0$, $u_m'(t_1) = 1$ where the constant $m \in (0, 1)$. As $u'(t_2) < 0$ and by continuity reasoning, u_m has a zero t_3 close to t_2 if m is sufficiently close to 1.

Let now c: $[0, \infty) \to M$ be a ray, $c(0) = p$. Let $X(t) \perp \dot c(t)$ be a parallel unit vector field along $c|_{[t_1, t_3]}$, and let $Y(t) = u_m(t)X(t)$, $t \in [t_1, t_3]$. The second variation L'' of the arc length for the variation $V(t, \delta) = \exp(\delta \cdot Y(t))$ satisfies

$$L'' = \int_{t_1}^{t_3} (\langle Y', Y'\rangle - \langle R(Y, \dot c)\dot c, Y\rangle)dt = \int_{t_1}^{t_3} (u_m'^2 - K(t)u_m^2)dt,$$

where $K(t)$ is the section curvature in the plane of the vectors $\dot{c}(t)$ and $X(t)$. By Remark 1.2, $c(t) \in B_t(p) \subset C_t(p)$, $t \geqslant 0$. Then $K(t) \geqslant k(t) > mk(t)$, $k \in \{k_b, k_c\}$. By (4.10.1) we therefore obtain

$$L'' < \int_{t_1}^{t_3} (u_m'^2 - m\,k\,u_m^2)dt = \int_{t_1}^{t_3} (u_m'^2 + u_m'' u_m)\,dt = u_m' u_m \Big|_{t_1}^{t_3} = 0,$$

which is impossible since c is a ray.

4.11. Let $k \in \{\tilde{k}_b, \tilde{k}_c\}$ where $\tilde{}$ means the extension as in (1.5.1). Thus $k \in P$; see §3.1. Take $t_0 > 0$ and let $u(t)$ be the solution of (3.1.1) with $u(t_0) = 1$, $u'(t_0) = 1/t_0$. Then the equation $u(t) = 0$ has a root $t_1 \in (0, t_0)$ by Remark 3.1, and has no roots in $[t_0, \infty)$ by §4.10. Therefore $1/t_0 \in \Omega_{t_0}$; see 3.3. By Theorem 3.4(3), $k(t)$ has a convexity function ε: $(a, \infty) \to R$, $a < t_0$. Arbitrariness of $t_0 > 0$ implies $a \leqslant 0$. Therefore $k_b, k_c \in A$.

4.12. Proof of Theorem 1.13. (1.13.1) has been proved in §§3.7, 3.6. (1.13.2) coincides with (3.8.4).

Put $x = \liminf_{t\to\infty} [\varepsilon(t)/K(t)]$ (possibly, $x = \infty$) and $q(\tau) = \inf_{t\geqslant\tau} [\varepsilon(t)/K(t)]$. Then $x = \lim_{\tau\to\infty} q(\tau)$. By (3.8.2), for $t \geqslant \tau$,

$$\frac{\varepsilon(t)}{K(t)} = \frac{1}{K(t)} \int_t^\infty \varepsilon^2(x)\,dx + 1 \geqslant \frac{q^2(\tau)}{K(t)} \int_t^\infty K^2(x)\,dx + 1.$$

Applying $\liminf_{t\to\infty}$ to this inequality, one has

$$x \geqslant q^2(\tau)\eta + 1.$$

Passing to the limit as $\tau \to \infty$, we have

$$x \geqslant x^2\eta + 1, \tag{4.12.1}$$

so that $x_1 \leqslant x \leqslant x_2$. A simple consideration shows that $x_1 \geqslant 1$ when $\eta \in [0, 1/4]$ according to (1.7.3). Thus (1.13.3) has been proved.

(1.13.4) is a part of Lemma 3.9.

4.13–4.15. Proof of Theorem 1.11.

4.13. Let, for short, $k \in \{k_b, k_c\}$ and $\varepsilon \in \{\varepsilon_b, \varepsilon_c\}$, $a \in \{a_b, a_c\}$, $\sigma \in \{R(t), t\}$. (So $\sigma > 0$.)

Let $\delta \in (0, \varepsilon(\sigma))$ and $y = y(\delta, \sigma)$ satisfy

$$\varepsilon(\sigma + y) > \varepsilon(\sigma) - \delta. \tag{4.13.1}$$

It is sufficient to prove that the set $C_t(p)$ is $[\varepsilon(\sigma) - \delta, y]$-convex.

Suppose the contrary. Then there exists a normal curve d: $[0, l_0] \to M$, $0 < l_0 < y$ with the maximum curvature $\bar{\varepsilon} < \varepsilon(\sigma) - \delta$ satisfying $d(0) \in C_t(p)$, $d(l_0) \in C_t(p)$, $d(l_1) \notin C_t(p)$ for some $l_1 \in (0, l_0)$. Let $\varepsilon^* \in (\bar{\varepsilon}, \varepsilon(\sigma) - \delta) \subset (\bar{\varepsilon}, \varepsilon(\sigma + y))$. By Remark 3.6(iii), $\Omega_{\sigma+y} = [\varepsilon(\sigma + y), \infty)$. Then the solution $v(\tau)$ of (3.1.1) with $v(\sigma + y) = 1$, $v'(\sigma + y) = \varepsilon^*$ has roots larger than $\sigma + y$. Let τ^* be the least of them, so that $v(\tau) > 0$ when $\tau \in (\sigma + y, \tau^*)$.

4.14. Obviously there exists a ray c emanating from p such that $d(l_1) \in b_{c_t}$ (see §1.2) meanwhile $d(0)$ and $d(l_0) \in M \setminus b_{c_t}$. Then, if a point q in the ray c is sufficiently far from p, the point $d(l_1)$ belongs to the ball $b_\rho(q)$ of radius $\rho = \rho(q, c(t))$ centered at q. Let q be also such that $L \overset{\text{def}}{=} \rho(q, d[0, l_0]) > \tau^* - \sigma - y$.

Let l_2 satisfy $\rho(q, d(l_2)) = L$. Then $d(l_2) \in b_\rho(q) \subset b_{c_t}$, so that $l_2 \neq 0$, $l_2 \neq l_0$.

Denote by g: $[0, L] \to M$ a normal shortest path with $g(0) = d(l_2)$, $g(L) = q$. $\dot{d}(l_2) \perp \dot{g}(0)$ because $l_2 \in (0, l_0)$. Let $X(s)$, $s \in [0, L]$, be the parallel unit vector field along g with $X(0) = \dot{d}(l_2)$. Denote by $K(s)$ the sectional curvature in the plane of the vectors $X(s)$ and $\dot{g}(s)$. By triangle inequality, $\rho(g(s), p) < R(t) + y + s$, so that

$$K(s) \geqslant k_b(\rho(g(s), p)) \geqslant k_b(R(t) + y + s), \quad s \in [0, L]. \tag{4.14.1}$$

Obviously, $\rho(g(s), C_t(p)) \leqslant s + l_0 < s + y$. It follows easily from Proposition 1.3 in [1] that $g(s) \in C_{t+s+y}(p)$. Hence

$$K(s) \geqslant k_c(t + y + s), \quad s \in [0, L]. \tag{4.14.2}$$

Now (4.14.1) and (4.14.2) can be rewritten as

$$K(s) \geqslant k(\sigma + y + s), \quad s \in [0, L]. \tag{4.14.3}$$

4.15. The following calculation was influenced by Lemma 1 in [2]. Let us consider a variation of the shortest path g corresponding to the vector field

$$Y(s) = \begin{cases} X(s) \cdot v(\sigma + y + s) & \text{for } s \in [0, \tau^* - \sigma - y], \\ 0 & \text{for } s \in [\tau^* - \sigma - y, L], \end{cases} \tag{4.15.1}$$

and such that its end $g(0)$ slides along the curve d. The second variation L'' of its length L satisfies

$$L'' = l_d(Y(0), Y(0)) + \int_0^L (\langle Y'Y' \rangle - \langle R(Y, \dot{g})\dot{g}, Y \rangle) ds,$$

where l_d is the second quadratic form of the curve d with respect to its normal $\dot{g}(0)$. Since $|Y(0)| = |X(0)| \cdot v(\sigma + y) = 1$, we have $l_d(Y(0), Y(0)) \leqslant \bar{\varepsilon}$. Then by (4.15.1),

$$\begin{aligned} L'' &\leqslant \bar{\varepsilon} + \int_0^{\tau^* - \sigma - y} \big[v'^2(\sigma + y + s) \\ &\qquad - v^2(\sigma + y + s) \langle R(X(s), \dot{g}(s))\dot{g}(s), X(s) \rangle \big] ds \\ &= \bar{\varepsilon} + \int_0^{\tau^* - \sigma - y} \big[v'^2(\sigma + y + s) - v^2(\sigma + y + s) \cdot K(s) \big] ds \\ &\leqslant \bar{\varepsilon} + \int_0^{\tau^* - \sigma - y} \big[v'^2(\sigma + y + s) - k(\sigma + y + s) \cdot v^2(\sigma + y + s) \big] ds \\ &\qquad \text{(by (4.14.3))} \end{aligned}$$

$$= \bar{\varepsilon} + \int_0^{\tau^* - \sigma - y} \Big[(v \cdot v')' \Big|_{\tau = \sigma + y + s} - v(\sigma + y + s) \cdot v''(\sigma + y + s)$$

$$- k(\sigma + y + s) \cdot v^2(\sigma + y + s) \Big] ds$$

$$= \bar{\varepsilon} + v \cdot v' \Big|_{\sigma + y}^{\tau^*} - \int_0^{\tau^* - \sigma - y} v(\sigma + y + s) \cdot \big[v''(\sigma + y + s)$$

$$+ k(\sigma + y + s) \cdot v(\sigma + y + s) \big] ds.$$

By (3.1.1), the last integral is zero. Hence $L'' \leqslant \bar{\varepsilon} - \varepsilon^* < 0$ which is impossible since $d(l_2) = g(0)$ is the point in the curve d closest to the point q.

5. Construction of a space with given k_b, k_c

5.1. We construct here the surface M_0 mentioned in Remark 1.6. Let $k \in A$, $\tilde{k}$ be as in (1.5.1), and ε: $(a, \infty) \to R$ be their convexity function. Denote by $u(t)$ the solution of the equation

(5.1.1) $$u'' = -\tilde{k} u$$

with the initial data $u(0) = 0$, $u'(0) = 1$. By Remark 3.10, $u(t) > 0$ for $t > 0$. We mean by M_0 the "polar" metric $ds^2 = dr^2 + G(r) d\theta^2$ where $G(r) = u^2(r)$, $r \geqslant 0$. Let us show that this metric is the one described in §1.6.

As a solution of (5.1.1), $u \in C^2$ and therefore $G(r) \in C^2$. The length $l(r)$ and the geodesic curvature $g(r)$ of the circumference $r = \text{const}$ satisfy

(5.1.2) $$l(r) = \sqrt{G(r)} \cdot 2\pi = 2\pi u(r), \quad g(r) = \frac{G'(r)}{2G(r)} = \frac{u'(r)}{u(r)}.$$

Then its total curvature $l \cdot g = 2\pi u'(r)$ goes to 2π as $r \to 0$. Thus there is no curvature at the pole.

One can easily see that the geodesics $\theta = \text{const}$, $r \geqslant 0$, are rays. Then $C_t(p) = B_t(p)$ for the point p at the pole. Due to (5.1.1), the curvature at a point (r, θ) is $-[\sqrt{G(r)}\,]''/\sqrt{G(r)} = -u''(r)/u(r) = \tilde{k}(r) = k(r)$. Thus $k_b(t) = k_c(t) = \min_{r \in [0, t]} k(r) = k(t)$.

5.2. Let us prove now Remark 1.12. Let $k_0 \in A$, and ε_0: $(a_0, \infty) \to R$, $a_0 \leqslant 0$, be its convexity function. Put $k(t) = \tilde{k}_0(t + a_0)|_{[0, \infty)}$; see (1.5.1). Obviously $\tilde{k}(t) = \tilde{k}_0(t + a_0)$. Therefore the solutions of the equation $\phi' = -\phi^2 - \tilde{k}$ are obtained from the solutions of $\phi' = -\phi^2 - \tilde{k}_0$ by means of their "parallel shift by the distance $|a_0|$ in the postive direction of the t-axis". It implies that k has the convexity function $\varepsilon(t) = \varepsilon_0(t + a_0)$, $t \in (0, \infty)$. (I.e., $a = 0$ for k.) By Remark 3.6(iv), the last quotient in (5.1.2) is $\varepsilon(r)$. Therefore the geodesic curvature $g(t)$ of bd $B_t(p) =$ bd $C_t(p)$ is equal to $\varepsilon(t)$.

References

[1] J. Cheeger & Gromoll, *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math. **96** (1972) 413–443.

[2] D. Gromoll & W. Meyer, *On complete open manifolds of positive curvature*, Ann. of Math. **90** (1969) 75–90.

[3] E. L. Ince, *Ordinary differential equations*, Dover Publications, New York, 1956.

UNIVERSITY OF TORONTO