J. DIFFERENTIAL GEOMETRY
15 (1980) 543–551

EXAMPLES OF CODIMENSION-ONE CLOSED MINIMAL SUBMANIFOLDS IN SOME SYMMETRIC SPACES. I

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1. Introduction

In the study of Riemannian geometry, the symmetric spaces constitute a natural family of nice testing spaces. They are characterized by a single neat property of being "symmetric with respect to any point" and, rather remarkably, can be classified via the structure-classification theory of semi-simple Lie groups [É. Cartan]. Therefore the study of geometry of submanifolds in symmetric spaces is a natural generalization to that of spaces of constant curvature which provides an ideal setting for in-depth investigation of the interaction between geometry and Lie group theory. However, such problems have been so far almost left unexplored. Hence let us begin with formulating some simple problems along the above lines.

In the case of compact symmetric spaces, the spheres S^n is one of the simplest and also the most well understood of Riemannian manifolds. Among all submanifolds of a given dimension $1 \leqslant r \leqslant n-1$ in S^n, the equator S^r is clearly the simplest and the "best" one. Therefore it is natural to pose the following problem

Problem 1. Let M^n be a given compact symmetric space. Among all r-dimensional submanifolds of M^n, $1 \leqslant r \leqslant n-1$, which one is the "simplest" and the "best" that one may consider it to be the "generalized r-dimensional equator" in M^n?

Of course, the above problem is as yet not precise because the "simplicity" and the "virtue" of submanifolds is, in fact, purely a matter of taste. Therefore one may adopt different "standards" to get possibly different generalized equators. For example, it is not too difficult to prove that the r-dimensional equator S^r is the unique closed r-dimensional minimal submanifold with the least total (r-dimensional) volume. Therefore the following precise problem is a natural variant of problem 1.

Received May 25, 1979. The work of the second author was partially supported by N.S.F. Grant MCS 77-23579.

Problem 2. Let M^n be a given compact symmetric space. For a given dimension r, $1 \leqslant r \leqslant n-1$, determine those closed r-dimensional minimal submanifolds with the least total volume.

In order to answer problems of the above type, the first step will be to find some simple examples of closed minimal submanifolds for each given dimension r, $1 \leqslant r \leqslant n-1$, in a given symmetric space M^n, they will then serve as "candidates" as well as "basis" for comparison or uniqueness type of theorem. Among examples of closed minimal submanifolds, those codimension-one examples are much harder to find and hence more interesting.

Problem 3. Let M^n be a given compact symmetric space. To find some simple examples of closed codimension-one minimal submanifolds.

The purpose of this paper is to construct some examples of closed codimension-one minimal submanifolds in some symmetric spaces of rank two following the formulation of [3]. One of the cases we study is the symmetric space E_6/F_4. Under the isometric action of F_4 on the above 26-dimensional symmetric space, the principal orbit type is $F_4/\mathrm{Spin}(8)$, and the orbit space is topologically a triangle. The general formulation of [3] asserts that one can equip the orbit space $F_4 \setminus E_6/F_4$ with a specific Riemannian metric by computing the distances between orbits and the 24-dimensional volume of orbits. Then the study of F_4-invariant codimension-one closed minimal submanifolds in E_6/F_4 can be reduced to the study of "closed" geodesics (may be closed by perpendicular to the boundary). By proving the existence of periodic solutions of certain specific ordinary differential equations on a triangular domain we are able to construct an F_4-invariant closed codimension-one minimal submanifold of the topological type of $S^1 \times (F_4/\mathrm{Spin}(8))$ in E_6/F_4. Moreover, the inverse image of the above submanifold in E_6 is a codimension-one closed minimal submanifold in E_6 of the topological type of $S^1 \times (F_4/\mathrm{Spin}(8)) \times F_4$. They are respectively the first examples of closed codimension-one minimal submanifolds in E_6/F_4 and E_6. In the case of sphere we also obtain some new examples of minimal imbedding of $S^1 \times S^{n-1} \times S^{n-1} \times S^{n-1}$ into S^{3n-1}, e.g., $T^4 \subset S^5$.

2. Orbit structures and construction of invariant closed minimal submanifolds of codimension one

In this section we shall make some explicit computations of the geometry of orbit structure of the K-action on a given symmetric space G/K. For simplicity, we shall assume that G/K is simply connected, the general case can easily be reduced to the simply connected case by lifting to its universal covering. In order to study extremals of K-equivariant variation of the volume functional, the geometric structure which one needs consists of the following:

(i) A metric on the orbit space $K \setminus G/K$ which measures the distance between orbits.

(ii) A volume function defined on the orbit space which records the volume of principal orbits (its values on orbits of lower dimensions are defined to be zero).

Let us first recall the well-known case of compact connected semi-simple Lie group G considered as a symmetric spaced $G \times G/\Delta G$ where $\Delta G = \{(g, g^{-1}); g \in G\}$ acts on $G \simeq G \times G/\Delta G$ via conjugations. In this case, the maximal tori theorem of É. Cartan asserts that the principal orbit type is G/T. (In fact, historically, principal orbit type theorem of Montgomery-Samelson-Yang is a generalization of maximal tori theorem.) Let T be an arbitrary but fixed maximal torus of G, and $W = N(T)/T$ the Weyl group of G. Then the geometry of orbit structure of the adjoint G-action on itself, i.e., the geometry of conjugacy classes, can be concisely described as follows:

(i) The fixed point set of T is T itself, i.e., $F(T, G) = T$, and is a *flat totally geodesic* submanifold which intersects every orbit.

(ii) $T \subset G$ induces a bijection $T/W \cong G/Ad$ (the space of conjugacy classes), and W acts on T as a group generated by reflections. Therefore the orbit space $G/Ad \cong T/W$ can be identified with a chosen fundamental domain in T with respect to the W-action.

(iii) In the case that G is simply connected and in terms of Lie algebra terminology, one usually describes a chosen fundamental domain as those points in the Lie algebra of T (i.e., Cartan subalgebra of $\mathfrak{g}$) which satisfy the following inequalities, namely,

$$\alpha_j(x) \geqslant 0, j = 1, \cdots, r \quad \text{and} \quad \beta(x) \leqslant 1,$$

where $\{\alpha_j; j = 1, \cdots, r\}$ are the system of simple roots of $\mathfrak{g}$, and β is the highest root of $\mathfrak{g}$. Therefore the orbit space G/Ad with the metric of orbital distance is isomorphic to the above flat piece of polyhedron called Cartan polyhedron.

(iv) Let $\Sigma^+ = \{\alpha\}$ be the system of positive roots of $\mathfrak{g}$. Then the volume function $v(x) =$ volume of the orbit $G(x)$ is given as follows:

$$v(x) = c \cdot \prod_{\alpha \in \Sigma^+} \sin^2 \alpha(x),$$

where C is a fixed constant depending on the total volume of G.

Example. Suppose $G = SU(3)$ and $T = \{\mathrm{diag}(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}, e^{2\pi i\theta_3}); \theta_1 + \theta_2 + \theta_3 = 0\}$. Then the Cartan subalgebra is a 2-dimensional vector space parametrized by $(\theta_1, \theta_2, \theta_3)$ with condition $\theta_1 + \theta_2 + \theta_3 = 0$ and $\Sigma^+ = \{(\theta_1 - \theta_2), (\theta_2 - \theta_3), (\theta_1 - \theta_3)\}$. Therefore the orbit space $SU(3)/Ad$ can be geometrically identified with the following regular triangle:

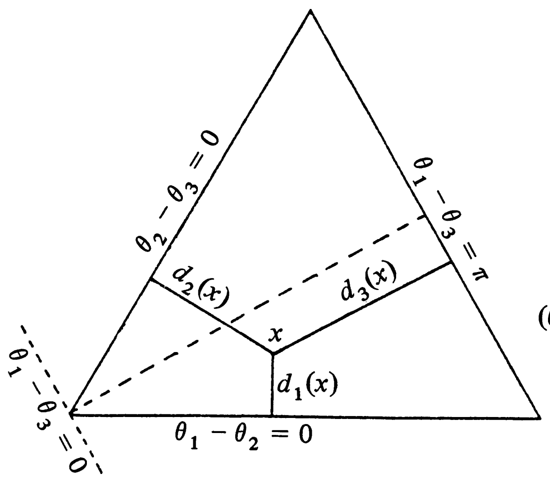

FIG. 1

Let $d_j(x), j = 1, 2, 3$, be respectively the distance of x to its three sides. Then the volume function $v(x) = c \cdot \prod \sin^2 d_j(x)$. ($d_3(x) = \pi - (\theta_1 - \theta_3)(x)$ and hence $\sin^2 d_3(x) = \sin^2(\theta_1 - \theta_3)(x)$.)

The above geometric structure of conjugacy classes is of basic importance in the classical Lie group theory. In the case of simply connected compact symmetric spaces G/K, one has the following generalization on the geometry of K-orbits which is essentially a reformulation of some results known to É. Cartan.

Let G/K be a given simply connected compact symmetric space. Then the geometry of orbit structure of the K-action on G/K can be concisely described as follows:

(i) Let $K(x_0)$ be a principal orbit of the above K-action on G/K. Then the exponential of the space of all *normal* vectors of $K(x_0)$ at x_0 forms a *flat totally geodesic* subtorus which *intersects every K-orbit*. Such tori are *maximal* among all flat totally geodesic subtori of G/K and hence called maximal tori of G/K. All maximal tori of G/K are conjugate under the action of G, and their rank is called the rank of the symmetric space G/K.

(ii) For each fixed maximal torus T of G/K, there is an action of group generated by reflections with respect to restricted roots of G/K such that $T/W' \cong K \setminus G/K$. Therefore the orbit space $K \setminus G/K$ with the metric of orbital distance can be identified with a flat piece bounded by suitable hyperplanes.

(iii) Again, the volume function $v(x)$ = volume of the principal orbit $K(x)$ is given by

$$v(x) = c \cdot \prod |\sin \alpha(x)|,$$

where α runs through the system of restricted roots of the pair (G, K).

We refer to [1], [2], [4] for more detail discussion of the above basic important fact of the geometry of symmetric spaces. For the purpose of this paper, we shall only need the following explicit descriptions for the special cases of $SU(3)/SO(3)$, $SU(6)/Sp(3)$ and E_6/F_4.

(i) The metrics of orbital distance of the above three symmetric space are the same as that of $SU(3)$ which can be identified with a flat regular triangle of height π.

(ii) The volume function $v(x)$ for the following four symmetric spaces: $SU(3)/SO(3)$, $SU(3)$, $SU(6)/Sp(3)$ and E_6/F_4 are respectively

$$v(x) = c \cdot \left[\sin d_1(x) \cdot \sin d_2(x) \cdot \sin d_3(x)\right]^k, \quad k = 1, 2, 4, 8.$$

Let Δ be the regular triangle in (x, y)-plane given as follows:

$$\Delta = \left\{(x, y) : y \geqslant 0,\ \sqrt{3x} + y \leqslant \pi \text{ and } -\sqrt{3x} + y \leqslant \pi\right\}.$$

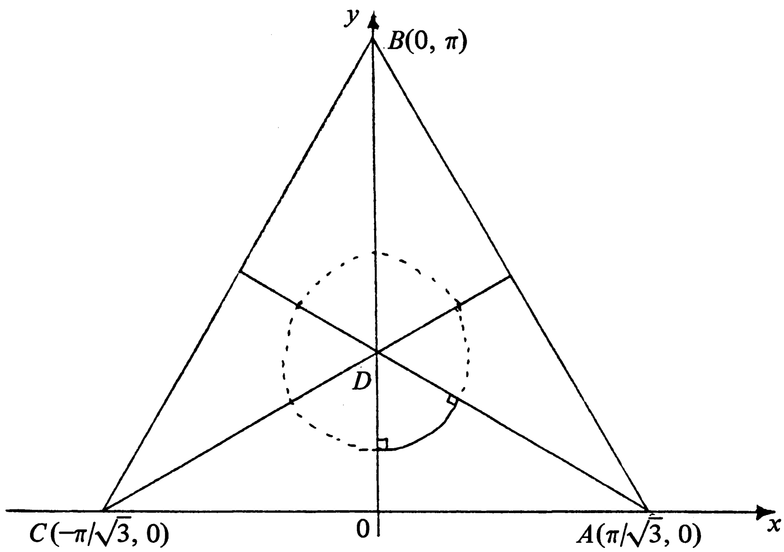

FIG. 2

Following the formulation of [4], one may define a new metric on Δ with $ds^2 = v(x, y)^2 \cdot (dx + dy^2)$ where $v(x, y)$ is the volume function of $G/K \xrightarrow{p} \Delta$, namely,

$$v(x, y) = c \cdot \left[\sin d_1 \cdot \sin d_2 \cdot \sin d_3\right]^k, \quad k = 1, 2, 4, 8$$

for $G/K = SU(3)/SO(3)$, $SU(3)$, $SU(6)/Sp(3)$ and E_6/F_4 respectively. Then it is not difficult to see that the length of a curve in Δ always equals the volume of its inverse image in G/K, geodesics in Δ correspond to K-invariant codimension-one minimal submanifolds in G/K; and "closed" geodesics correspond to closed K-invariant codimension-one minimal submanifolds in G/K. In each of the case $k = 1, 2, 4, 8$ we shall prove in §3 that there exists at least one regularly embedded closed geodesic with respect to the above modified metric on Δ. Therefore one has the following nice examples of K-invariant closed codimension-one minimal submanifolds in the above four symmetric spaces.

Theorem 1. *In the symmetric spaces*

$$G/K = SU(3)/SO(3), \quad SU(3), SU(6)/Sp(3) \quad \text{or } E_6/F_4$$

there exists an K-invariant closed codimension-one minimal submanifold of the type

$$S^1 \times \left(\frac{SO(3)}{Z_2^2}\right), S^1 \times \left(\frac{SU(3)}{T^2}\right), S^1 \times \left(\frac{Sp(3)}{Sp(1)^3}\right) \text{ or } S^1 \times \left(\frac{F_4}{\mathrm{Spin}(8)}\right)$$

respectively.

Corollary. *In the symmetric spaces $SU(3)$, $SU(6)$ or E_6 there exists a closed codimension-one minimal submanifold of the type*

$$S^1 \times \left(\frac{SO(3) \times SO(3)}{Z_2^2}\right), S^1 \times \left(\frac{Sp(3) \times Sp(3)}{Sp(1)^3}\right) \text{ or } S^1 \times \left(\frac{F_4 \times F_4}{\mathrm{Spin}(8)}\right)$$

respectively.

Proof. They are respectively the inverse images of those closed codimension-one submanifolds with respect to the following fibrations with isometric fibres:

$$SU(3) \to SU(3)/SO(3), \quad SU(6) \to SU(6)/Sp(3), \quad E_6 \to E_6/F_4.$$

Remark. The above example of minimal submanifold of the type $S^1 \times (F_4 \times F_4/\mathrm{Spin}(8))$ in E_6 is so far the *only* known example of *codimension-one* closed minimal submanifolds in E_6.

3. The proof of existence of periodic solutions for certain specific ordinary differential equations

In this section we shall prove the existence of a nice periodic solution of the geodesic equation on Δ with the above conformally modified metric $ds^2 = v^2 \cdot (dx^2 + dy^2)$. In view of the symmetries of (Δ, ds^2) with respect to the three bisectors, one simple-minded way of constructing a periodic solution will be to seek a geodesic arc which is perpendicular to two bisectors (see Fig. 2, as indicated by the solid arc), and then it is easy to obtain a closed geodesic by reflections with respect to the three bisectors. Therefore we shall carefully investigate the behavior of those geodesics in the domain ΔODA perpendicular to $\overline{OD}$. From the point of view of differential equations we shall study solutions of the following second order equation with given initial conditions, namely,

$$\begin{cases} y'' = \dfrac{k(1+y'^2)}{\sin y(\cos y + \cos\sqrt{3}x)} \\ \qquad \cdot \left(\cos 2y + \cos y \cos\sqrt{3}x + \sqrt{3}\, y' \sin\sqrt{3}x \sin y\right), \\ y(0) = \lambda, \quad 0 < \lambda < \dfrac{\pi}{3}, \quad y'(0) = 0. \end{cases}$$

As usual, one may consider the above equation as a system of first order equations: $dy/dx = p$, $dp/dx = f(x, y, p)$ with $y(0) = \lambda$, $p(0) = 0$. Let Ω_ε be the following domain in (x, y, p)-space:

$$\Omega_\varepsilon = \left\{(x, y, p): x \geqslant 0, \varepsilon \leqslant y \leqslant \frac{\pi}{3} - \frac{x}{\sqrt{3}}, 0 \leqslant p \leqslant 3\right\}.$$

Let $y = \phi(x, \lambda)$, $p = \phi'(x, \lambda)$ be the unique solution satisfying the above initial condition which is parametrized by λ. We shall study those solution curves γ_λ in Ω_ε whose projections on (x, y)-plane, namely $\{\bar{\gamma}_\lambda: y = \phi(x, \lambda)\}$, intersects the boundary line $y = \pi/3 - x/\sqrt{3}$. Observe that $f(x, y, p)$ is an everywhere positive function on Ω_ε, namely, for each given $\varepsilon \leqslant \lambda < \pi/3$, $\phi''(x, \lambda) > 0$ and $\phi'(x, \lambda) > 0$. Therefore there are only the following two possibilities for the boundary behavior of $\{\gamma_\lambda\}$: either the slope $p = \phi'(x, \lambda)$ reaches the bound 3 before the curve $\bar{\gamma}_\lambda$ interests $\overline{AD}$, or $p = \phi'(x, \lambda) \leqslant 3$ and $\bar{\gamma}_\lambda$ intersects $\overline{AD}$. In the later case, we shall denote the coordinates of the point of intersection by x_λ, y_λ, and its slope by p_λ.

Theorem 2. *For each of the four cases $k = 1, 2, 4, 8$, there exists a suitable initial value λ, say $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ respectively, such that $\bar{\gamma}_\lambda$ intersects $\overline{AD}$ and $p_\lambda = \sqrt{3}$, i.e., $\bar{\gamma}_\lambda$ intersects $\overline{AD}$ perpendicularly.*

Proof. (i) In view of the fact that $f(x, y, p)$ has a factor of $\sin y$ in the denominator, the value of $f(x, y, p)$ becomes rather large when y gets small. Therefore for small initial value λ, the curve $\bar{\gamma}_\lambda$ rapidly turns upward and hence the slope $p = \phi(x, \lambda)$ will reach the bound 3 before $\bar{\gamma}_\lambda$ intersects $\overline{AD}$. We shall choose and then fix such a sufficiently small $\varepsilon > 0$. On the other hand, for an initial value λ very close to $\pi/3$, $\bar{\gamma}_\lambda$ is simply a very short segment slightly bending upward. Hence $\phi'(x, \lambda)$ is small, and $\bar{\gamma}_\lambda$ intersects $\overline{AD}$ with an angle slightly larger than $\pi/6$.

(ii) Let Λ be the set of those λ_0, $\varepsilon \leqslant \lambda_0 \leqslant \pi/3$, such that $\bar{\gamma}_\lambda$ intersects $\overline{AD}$, and $0 \leqslant \phi'(x, \lambda) \leqslant 5/2$ for all $\lambda_0 \leqslant \lambda \leqslant \pi/3$ and $\phi(x, \lambda) \leqslant \pi/3 - x/\sqrt{3}$. It follows from the continuous dependence of solutions with respect to the parameter λ that Λ is a *closed interval* $[a, \pi/3]$ properly contained in $[\varepsilon, \pi/3]$. Suppose $p_a \geqslant \sqrt{3}$. Then it follows again from the continuity of p_λ with respect to $\lambda \in \Lambda$ that there exists a $\lambda \in \Lambda$ with $p_\lambda = \sqrt{3}$ and hence a perpendicular geodesic arc between $\overline{OD}$ and $\overline{AD}$.

(iii) Suppose that $p_a < \sqrt{3}$. Then it follows from the usual estimate that

$$|\phi(x, \lambda) - \phi(x, a)| < |\lambda - a| \cdot e^{2Mx},$$
$$|\phi'(x, \lambda) - \phi'(x, a)| < |\lambda - a| \cdot e^{2Mx},$$

where M is the Lipschitz constant for $f(x, y, p)$ in Ω_ε and $a, \lambda \in [\varepsilon, \pi/3]$, $y = \phi(x, \lambda) < \pi/3 - x\sqrt{3}$. Therefore there exists a $\delta > 0$ such that all λ in the open interval $(a - \delta, a + \delta)$ belong to Λ which is a contradiction to the fact that a is a boundary point of Λ. Hence p_a must be $\geqslant \sqrt{3}$, and this completes the proof of Theorem 2.

Concluding remarks. (i) Following the basic idea in the proof of existence in Theorem 2, it is not difficult to use computor to carry out numerical estimate for the approximate value of initial values $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ for the four cases corresponding to $k = 1, 2, 3, 4$. Their approximate values are as follows:

$$\lambda_1 = 0.399, \quad \lambda_2 = 0.566, \quad \lambda_3 = 0.699, \quad \lambda_4 = 0.799.$$

(ii) Let S^{3n-1} be the unit sphere of $\mathbf{R}^{3n} = \mathbf{R}^n \oplus \mathbf{R}^n \oplus \mathbf{R}^n$, and $G = O(n) \times O(n) \times O(n)$ acting orthogonally on $\mathbf{R}^{3n}$ via outer direct sum of standard $O(n)$-actions on the above three copies of $\mathbf{R}^n$ separately. Then the principal orbit type is $S^{n-1} \times S^{n-1} \times S^{n-1}$, the orbit space S^{3n-1}/G is metrically an octant of the unit 2-sphere, i.e., $\{(|x|, |y|, |z|) \text{ with } |x|^2 + |y|^2 + |z|^2 = 1\}$ and the volume function $v = c \cdot |x|^{n-1} \cdot |y|^{n-1} \cdot |z|^{n-1}$. The above geometric data are clearly symmetric with respect to its three bisectors. Therefore a similar proof will show that existence of a closed minimal codimension-one submanifold in S^{3n-1} of the type $S^1 \times S^{n-1} \times S^{n-1} \times S^{n-1}$. In the special case of $n = 2$, one obtains a minimal imbedding of $T^4 = S^1 \times S^1 \times S^1 \times S^1$ into S^5. This example can also be considered as periodic minimal immersion of $\mathbf{R}^4$ into S^5. One does not know whether there are minimal imbeddings of codimension-one torus into S^m for $m \neq 3, 5$.

(iii) In a succeeding paper, we shall discuss the existence of closed geodesics for the modified orbital metrics which are perpendicular to the boundary of triangle. Due to the fact that the modified metric becomes degenerate at the boundary, such discussion is technically more involved. However, the reward is also much more because one can then obtain more interesting examples of codimension-one closed minimal submanifolds in a wider variety of symmetric spaces. For example, one can obtain a minimal embedding of S^{25} in E_6/F_4.

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