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SOME NEW RIEMANNIAN INVARIANTS

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Introduction

The purpose of this paper is to introduce some new riemannian invariants and to study their properties. In a future paper we will study riemannian manifolds whose invariants are large.

In the first section the invariants are defined and are related to the dimension of the group of isometries. In particular, we have

$$\dim I_p \leqslant \tfrac{1}{2} ATC_p(2n - ATC_p - 1),$$

where I_p is the isotropy group of isometries at a point p of an n-dimensional complete connected riemannian manifold M, and ATC_p is one of the invariants.

In the second section we show, using the invariants and the Rauch comparison theorem, that for manifolds whose diameter is small relative to their sectional curvature, the group I_p is finite for all p in M. We also study other properties of such "small diameter" manifolds.

In the third section we study how the invariants behave under products and coverings.

In the fourth section we compute the invariants on some riemannian manifolds.

In the fifth section we study in detail some of the properties the invariants possess. In particular we study the p-dependence.

In the sixth section we prove a result which relates the geometries of the submanifolds in question.

Throughout the paper a manifold will be a complete connected riemannian manifold unless otherwise stated. A submanifold will always be an embedded submanifold.

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1. Definitions and preliminary theorems

1.1. Definitions. A subset N of a complete riemannian manifold is said to be:

(a) *Totally Convex* (TC) if whenever $x, y \in N$ and γ is any geodesic from x to y, then $\gamma \subset N$.

(b) *Almost Totally Convex* (ATC) if whenever $x, y \in N$ and γ is any geodesic from x to y such that x is not conjugate to y along γ, then $\gamma \subset N$.

(c) *Completely Convex* (CC) if whenever $x, y \in N$ and γ is a unique minimizing geodesic from x to y such that x is not conjugate to y along γ, then $\gamma \subset N$.

It is clear from the definitions that

$$TC \Rightarrow ATC \Rightarrow CC.$$

1.2. Definitions. Let M be a complete connected riemannian manifold. For every linear $S \subset T_pM$, define N_S^{CC} to be the smallest topologically closed totally geodesic submanifold through p such that N_S^{CC} is completely convex and $S \subset T_p(N_S^{CC})$. Similarly, defined N_S^{ATC} and N_S^{TC}. Let N_S^{TG} be the smallest topologically closed totally geodesic submanifold such that $S \subset T_p(N_S^{TG})$.

The existence and uniqueness of these submanifolds follows from the fact that M satisfies all of the properties (except being the smallest) and the properties are closed under intersections.

The submanifolds are related by $N_S^{TG} \subseteq N_S^{CC} \subseteq N_S^{ATC} \subseteq N_S^{TC} \subseteq M$. The submanifolds N_S^{CC} and N_S^{ATC} are important in studying isometries as the following propositions show.

1.3. Proposition. *Let f: $M \to M$ be an isometry of a complete connected riemannian manifold, and S a linear subspace of T_pM. Then $f_*|_S$ determines $f|_{N_S^{CC}}$.*

Proof. Assume g: M $\to$ M is another isometry such that $g_*|_S = f_*|_S$, and let $h = g^{-1} \circ f$. Then $h_*|_S = \text{id}$. We need only show $h|_{N_S^{CC}} = \text{id}$. Let M^h be the fixed point set of h. We know that M^h is a topologically closed totally geodesic submanifold of M. We need only show M^h is completely convex. Let x and y be in M^h, and γ a unique minimizing geodesic from x to y. Since $h(x) = x$ and $h(y) = y$, $h(\gamma)$ is a geodesic from x to y. Since γ is minimizing, so is $h(\gamma)$. Since γ is the unique minimizing geodesic, $\gamma = h(\gamma)$. Since h preserves lengths, $\gamma(t) = h(\gamma(t))$ so $\gamma \subset M^h$ and M^h is completely convex. Further $S \subset T_pM^h$. Since N_S^{CC} is the smallest topologically closed totally geodesic completely convex submanifold, we have $N_S^{CC} \subset M^h$. So $h|_{N_S^{CC}} = \text{id}$.

For N_S^{ATC} we have a similar result.

1.4. Proposition. *Let $I(M)$ be the group of isometries of a complete connected riemannian manifold M, and let $I_S^f = \{ g \in I(M) s.t.\ g_*|_S = f_*|_S \}$ for S linear in T_pM and $f \in I(M)$. Then the set $\{h: N_S^{ATC} \to M\ s.t.\ h = g|_{N_S^{ATC}}$ for $g \in I_S^f\}$ is finite.*

Note. Proposition 1.3 says that the corresponding set for N_S^{CC} consists of one element.

Proof. $I_S^f = f \cdot I_S^{\mathrm{id}}$. I_S^{id} is a closed Lie subgroup of the isotropy subgroup at p and thus is compact. It is sufficient to show that the action of $g \in I_S^{\mathrm{id}}$ on N_S^{ATC} is determined by the component of I_S^{id} which g lies in, since there are only a finite number of components. Since I_S^{id} is a Lie group it is sufficient to show that if g is in the identity component of I_S^{id}, then $g|_{N_S^{ATC}} = \mathrm{id}$. So let g_t be a one-parameter subgroup of I_S^{id} such that $g = g_{t_0}$ for some t_0. Let M^{g_t} be the set of points fixed by all g_t. We know that M^{g_t} is a topologically closed totally geodesic submanifold of M. Further $g_{t*}|_S = \mathrm{id}$ for all t so $S \subset T_pM^{g_t}$. Thus in order to show $N_S^{ATC} \subset M^{g_t}$ we need only show M^{g_t} is almost totally convex. Let $x, y \in M^{g_t}$, and γ be a geodesic from x to y. If $\gamma \not\subset M^{g_t}$, then $g_t(\gamma)$ is a one-parameter group of geodesics from x to y. This implies that x is conjugate to y along γ. Therefore, if x is not conjugate to y along γ, then $\gamma \subset M^{g_t}$. Thus M^{g_t} is almost totally convex. q.e.d.

These propositions are most interesting, when S has small dimension, and N_S^{CC} or N_S^{ATC} is the whole manifold.

1.5. Definition. For M complete and connected, and $p \in M$, $CC_p(M) \equiv \min\{\dim S | S \subset T_pM$ and $N_S^{CC} = M\}$. This is clearly well defined since $N_{T_pM}^{CC} = M$.

Similarly, define $TC_p(M)$, $ATC_p(M)$, $TG_p(M)$.

We have the following relationship $0 \leqslant TC_p(M) \leqslant ATC_p(M) \leqslant CC_p(M) \leqslant TG_p(M) \leqslant n$, where n is the dimension of M. Further $1 \leqslant CC_p(M)$ as the point p is always a topologically closed totally geodesic completely convex submanifold.

These numbers do depend on the point p. See Section 4 for examples and Section 5 for discussion of the p-dependence.

The previous propositions lead us to the following relationships between ATC_p, CC_p and the dimension of the isotropy subgroup of isometries at p.

1.6. Theorem. *Let M be a complete connected riemannian manifold. For $p \in M$, let I_p be the isotropy subgroup of isometries at p. Then*

$$\dim(I_p) \leqslant \tfrac{1}{2} ATC_p(2n - ATC_p - 1) \leqslant \tfrac{1}{2} CC_p(2n - CC_p - 1).$$

Proof. Let $\rho: I_p \to O(n)$ be the isotropy representation. Let S be an ATC_p-dimensional linear subspace of T_pM such that $N_S^{ATC} = M$. Such a subspace exists by the definition of ATC_p. Now by Proposition 1.4 there are only a finite number of isometries whose differentials leave S fixed. Let $O(n\text{-}ATC_p)$ be the group of rotations which leave S fixed. Then $\rho(I_p) \cap O(n - ATC_p)$ is finite. Since ρ is injective, the result follows by checking the dimensions of the Lie algebras, i.e., $\dim I_p \leqslant \dim O(n) - \dim O(n - ATC_p) = \frac{1}{2}ATC_p(2n - ATC_p - 1)$. The other inequality follows from noticing that $ATC_p \leqslant CC_p$. q.e.d.

Note. The inequality $\dim(I_p) \leqslant \frac{1}{2}CC_p(2n - CC_p - 1)$ can be derived directly using a similar argument and Proposition 1.3. These inequalities can be improved by using representations of Lie groups.

The inequality $\dim I_p \leqslant \frac{1}{2}CC_p(2n - CC_{pp} - 1)$ can be made strict for most values of n and CC_p by noticing that $O(n)/O(n - CC_p)$ does not admit a Lie group structure so that the embedding f

$$\begin{tikzcd} I_p \arrow[r, "\rho"] \arrow[dr, "f"'] & O(n) \arrow[d] \\ & O(n)/O(n - CC_p) \end{tikzcd}$$

cannot be diffeomorphism.

The riemannian invariants ATC_p, CC_p, TC_p, TG_p give rise to differential invariants as follows.

1.7. Definition. If M is a smooth connected manifold and $p \in M$, define

$$\widetilde{CC} = \max\{CC_p(M, \rho) | \rho \text{ a complete metric}\}.$$

Likewise define $\widetilde{ATC}$, $\widetilde{TC}$, $\widetilde{TG}$.

Note. The differential invariants are independent of the point p. Let q be any other point of M. Then there is a diffeomorphism f of M such that $f(q) = p$. Thus

$$CC_p(M, \rho) = CC_q(M, f^*\rho).$$

The differential invariants are related to the Hsiang (or Compact) degree of symmetry by

1.8. Corollary. *If M^n is a smooth connected manifold, then*

$$h(M) \leqslant \tfrac{1}{2}\widetilde{ATC}(2n - \widetilde{ATC} - 1) + n,$$

where $h(M)$ is the Hsiang degree of symmetry.

Proof. Let G be a compact group of diffeomorphisms of dimension $h(M)$ acting effectively on M. Let ρ be any complete metric on M, and $\tilde{\rho}$ the G-averaged metric (i.e., $\tilde{\rho} = \int_G g^{-1}\rho \, dg$). Let G_p be the isotropy subgroup. With the averaged metric, M is a complete connected riemannian manifold on which G acts as a group of isometries. Since G is effective, $\dim G \leqslant \dim G_p + n$. By Theorem 1.6,

$$\dim G_p \leqslant \tfrac{1}{2} ATC_p(M)(2n - ATC_p(M) - 1) \leqslant \tfrac{1}{2}\widetilde{ATC}(2n - \widetilde{ATC} - 1).$$

Hence the corollary follows.

2. Manifolds with small diameter

The inequality of Theorem 1.6,

$$\dim(I_p) \leqslant \tfrac{1}{2} ATC_p(2n - ATC_p - 1),$$

tells us that $ATC_p = 0$ implies I_p is finite (since it is known to be compact Lie). In this chapter we will take advantage of this fact.

The following will be a useful corollary to the Rauch Comparison Theorem.

2.1. Lemma. *Let M^n, M_0^n be complete riemannian manifolds such that $K_{M_0} \geqslant K_M$ (i.e., all sectional curvatures in M_0 are larger than those in M). Let $p \in M$ and $p_0 \in M_0$. Let I be an isometry from T_pM to $T_{p_0}M$. Assume further that there are no critical points of* Exp_p *or* Exp_{p_0} *in $B_r(0)$. If $\tau \subset B_r(0) \subset T_pM$ is a differentiable curve, then*

$$L[\operatorname{Exp}_p \tau] \geqslant L(\operatorname{Exp}_{p_0} I(\tau)),$$

where L represents length.

Proof. It is sufficient to show for every t that $\|\operatorname{Exp}_{p_*}\tau'(t)\| \geqslant \|\operatorname{Exp}_{p_{0*}} I(\tau'(t))\|$. Consider the variations

$$\alpha(s, t) = \operatorname{Exp}_p s \cdot \tau(t),$$

$$\alpha_0(s, t) = \operatorname{Exp}_{p_0} s \cdot I(\tau(t)).$$

Now for fixed t the variation vector fields V^t, V_0^t along the geodesics $\gamma(s) = \alpha(s, t)$ and $\gamma_0(s) = \alpha_0(s, t)$ are Jacobi fields with $V^t(0) = 0 = V_0^t(0)$, and further $I(V''(0)) = V''_0(0)$ and $I(\gamma'(0)) = \gamma_0'(0)$ so by the Rauch theorem (see [2, pp. 29, 30]), $\|V^t(s)\| \geqslant \|V_0(s)\|$. But $V^t(1) = \operatorname{Exp}_{p_*} \tau'(t)$ and $V_0^t(1) = \operatorname{Exp}_{p_{0*}} I(\tau'(t))$, so the lemma follows. q.e.d.

The following standard path lifting lemma will be useful in proving the main theorem of this chapter.

2.2. Lemma. *Let M be a complete connected riemannian manifold, and $\tau\colon [0, 1] \to M$ a piecewise differentiable curve. Let $p \in M$ and $v \in T_pM$ such that v is not in the conjugate locus in T_pM and that* $\operatorname{Exp}_p v = \tau(0)$. *Assume further that for $t \in [0, 1]$ there is an $\varepsilon > 0$ such that for all $s < t$ there is a unique lift $\tilde{\tau}_s\colon [0, s] \to T_pM$ starting at v (i.e., $\tilde{\tau}_s(0) = v$ and* $\operatorname{Exp}_p \tilde{\tau} = \tau|_{[0,s]}$*) such that the distance from $\tilde{\tau}_s(s)$ to the conjugate locus is $> \varepsilon$ (in the usual metric on T_pM). Then there is a unique lift $\tilde{\tau}\colon [0, t] \to T_pM$ of τ starting at v.*

Remark. The above lemma tells us that τ can be uniquely lifted to $\tilde{\tau}$ as long as $\tilde{\tau}$ does not approach the conjugate locus. This follows from the fact that if $\tilde{\tau}$: $[0, t] \to T_pM$ exists, and $\tilde{\tau}(t)$ is not in the conjugate locus, then near $\tilde{\tau}(t)$, Exp_p is a diffeomorphism, so for some $\varepsilon > 0$, $\tilde{\tau}$ can be uniquely extended to $[0, t + \varepsilon]$.

Proof of Lemma 2.2. All lengths of vectors in T_pM will be with respect to the usual metric while all distances of points in M will be with respect to the metric on M. Let L be the length of τ ($l(\tau)$). By the Gauss Lemma any partial lift $\tilde{\tau}$: $[0, s] \to T_pM$ must lie in $B_r(0)$ where $r = \|v\| + L$. Let $\bigcup_\varepsilon$ be the union of all $B_\varepsilon(w)$ for w in the conjugate locus in T_pM. Then $\bigcup_\varepsilon$ is open and for all $s < t$, $\tilde{\tau}(s) \in (T_pM - \bigcup_\varepsilon) \cap \overline{B_r(0)} \equiv C$. C is a compact subset of T_pM which contains no conjugate points. Consider Exp_{p_*} restricted to SC, the unit sphere bundle of $TC \subset TT_pM$. Since SC is compact and $\|\mathrm{Exp}_{p_*} \mathcal{V}\| \neq 0$ for $\mathcal{V} \in SC$, there is an $A > 0$ such that $\|\mathrm{Exp}_{p_*} \mathcal{V}\| \geq A$ for all $\mathcal{V} \in SC$. Thus for any piecewise differentiable curve $\gamma \subset C$ we have $l(\mathrm{Exp}_p\, C) \geq Al(C)$. Therefore $L/A \geq l(\tilde{\tau}|_{[0,t)})$. Since C is compact, $\{\tilde{\tau}(s) | s \in [0, t)\}$ has a limit point $\tilde{\tau}(t)$. This limit point is uniquie since $\tau|_{[0,t)}$ has finite length. Thus there is a unique lift $\tilde{\tau}$: $[0, t] \to T_pM$.

Remark. In the above lemma we ignored questions of differentiability of $\tilde{\tau}$. The necessary differentiability conditions follow from the fact that τ is piecewise differentiable, and Exp_p is a local diffeomorphism away from the conjugate locus.

2.3. Definition. A riemannian manifold M is said to have *small diameter* if M is compact connected with diameter d such that $d < \frac{1}{2}\pi/\sqrt{k}$ where k is some positive number with $k \geq K_M$.

2.4. Theorem. *If M is a manifold of small diameter, then I_p is finite for all $p \in M$.*

Remark. Let $\mathbf{RP}^n$ have the metric of constant curvature k. Then $d(\mathbf{RP}^n) = \frac{1}{2}\pi/\sqrt{k}$ and $I_p = O(n)$ for all $p \in \mathbf{RP}^n$ showing that the theorem is sharp.

The theorem contains the following well-known result.

2.5. Corollary. *If M is a connected compact manifold of nonpositive curvature, then I_p is finite for all $p \in M$.*

Proof. M is easily seen to have small diameter by letting $0 < k < (\frac{1}{2}\pi/d)^2$.

2.6. Corollary. *If a manifold M of small diameter also satisfies one of the following*:

(a) $\chi(M) \neq 0$, *where χ is the Euler characteristic*,

(b) *M is orientable and some Pontrjagin number is not* 0,

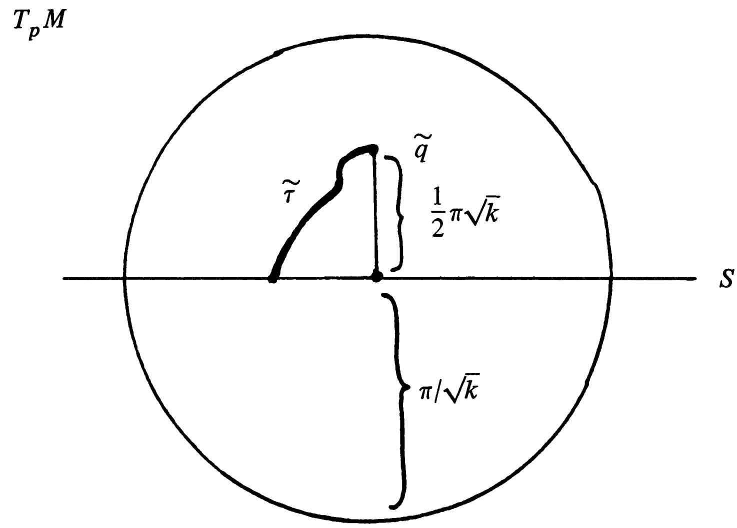
then $I(M)$ is finite.

Proof. These conditions imply that one-parameter groups of $I(M)$ have fixed points (see [3]).

Proof of Theorem 2.4. By the Rauch Comparison Theorem, the first conjugate point $\gamma(t_0)$ along any geodesic γ from p does not occur until a distance $\pi/\sqrt{k}$ along γ. Therefore the critical points of $\mathrm{Exp}_p\colon T_pM \to M$ lie outside the open ball $B_{\pi/\sqrt{k}}(0)$.

We claim that $\mathrm{Exp}_p^{-1}(p) \cap B_{\pi/\sqrt{k}}(0)$ does not lie in any $(n-1)$-dimensional linear subspace S of T_pM.

Assume such an S exists. Let $V \in T_pM$ be a unit vector normal to S. Let $\tilde{q} = \frac{1}{2}\pi/\sqrt{k} \cdot V \in T_pM$ and $q = \mathrm{Exp}_p\, \tilde{q}$. Let τ be a minimal geodesic from q to p. Since $\mathrm{diam}(M) < \frac{1}{2}\pi/\sqrt{k}$, $L(\tau) < \frac{1}{2}\pi/\sqrt{k}$. By the Gauss Lemma and Lemma 2.2 there is a unique lift $\tilde{\tau}$ of τ to T_pM such that $\tilde{\tau}(0) = \tilde{q}$. Further, the Gauss Lemma tells us $\tilde{\tau} \subset B_{\pi/\sqrt{k}}(0)$. Now $\mathrm{Exp}_p\, \tilde{\tau}(1) = p$, so $\tilde{\tau}(1) \in S$ by assumption.



Let $p_0 \in S^n$, the sphere of constant curvature k, and let $I\colon T_pM^n \to T_{p_0}S^n$ be an isometry. By Lemma 2.1, $L(\tau) = L(\mathrm{Exp}_p\, \tilde{\tau}) \geqslant L(\mathrm{Exp}_{p_0} I\tilde{\tau})$. If we let $\mathrm{Exp}_{p_0} I\tilde{q}$ be the north pole, then $\mathrm{Exp}_{p_0} I(S)$ is the equator, and since $\mathrm{Exp}_{p_0} I\tilde{\tau}$ is a curve from $\mathrm{Exp}_{p_0} I\tilde{q}$ to $\mathrm{Exp}_{p_0} I(S)$,

$$\tfrac{1}{2}\pi/\sqrt{k} > L(\tau) \geqslant L(\mathrm{Exp}_{p_0} I\tilde{\tau}) \geqslant \tfrac{1}{2}\pi/\sqrt{k}$$

giving the contradiction.

Now consider $N_0^{ATC} \cdot p \in N_0^{ATC}$. Let $V \in \mathrm{Exp}_p^{-1}(p) \cap B_{\pi/\sqrt{k}}$. Then $\mathrm{Exp}_p\, tV$ is a geodesic γ from p to p such that p is not conjugate to p along γ.

Thus $\gamma \subset N_0^{ATC}$ and $V \in T_pN_0^{ATC}$. By the claim the set of such V's span T_pM. Thus $T_pM = T_pN_0^{ATC}$ which implies that $N_0^{ATC} = M$ so that $ATC_p(M) = 0$. From Theorem 1.6 we get I_p is finite. q.e.d.

If M is a complete connected riemannian manifold, we will let $\tilde{M}$ denote its universal covering space with the induced metric from $\pi: \tilde{M} \to M$.

2.7. Proposition. *Let M be a manifold of small diameter. If for some $\tilde{p} \in \tilde{M}$ we have the cut locus to $\tilde{p}$ is equal to the first conjugate locus in $T_{\tilde{p}}\tilde{M}$, then $|\pi_1(M)| > \dim(M)$. Further $I_{\pi(\tilde{p})} \subset \mathrm{Aut}(\pi_1(M))$.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} T_{\tilde{p}}\tilde{M} & \xrightarrow{\mathrm{Exp}_{\tilde{p}}} & \tilde{M} \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi} \\ T_{\pi(\tilde{p})}M & \xrightarrow{\mathrm{Exp}_{\pi(\tilde{p})}} & M \end{array}$$

π_* will take the first conjugate locus in $T_{\tilde{p}}\tilde{M}$ to the first conjugate locus in $T_{\pi(\tilde{p})}M$. Since M is a manifold of small diameter, we know that the first conjugate locus lies outside $B_{\pi/\sqrt{k}}(0)$. Therefore the cut locus in $T_{\tilde{p}}\tilde{M}$ lies outside of $B_{\pi/\sqrt{k}}(0)$. Thus $\mathrm{Exp}_{\tilde{p}}|B_{\pi/\sqrt{k}}(0)$ is a diffeomorphism. Let $S = \mathrm{Exp}_{\pi(\tilde{p})}^{-1}\pi(\tilde{p}) \cap B_{\pi/\sqrt{k}}(0)$. By the claim in the proof of Theorem 2.6 we know that S lies in no $(n-1)$-dimensional linear subspace, so S contains at least $n+1$ points. Let $S' \subset \tilde{M}$ be $\mathrm{Exp}_{\tilde{p}} \circ \pi_*^{-1}(S)$. Since $\mathrm{Exp}_{\tilde{p}} \circ \pi_*^{-1}|B_{\pi/\sqrt{k}}(0)$ is a diffeomorphism, S' has at least $n+1$ points. Further $\pi(S') = \{\pi(\tilde{p})\}$, so $\pi_1(M) > n$. Now we know that each element of $I_{\pi(\tilde{p})}$ acts as an automorphism of $\pi_1(M)$, so we have a homomorphism $I_{\pi(\tilde{p})} \to \mathrm{Aut}(\pi_1(M))$. We need only show that this is injective. For each element $V \in S$, the loop $\mathrm{Exp}_p\, tV$ corresponds to an element of $\pi_1(M)$. The above argument shows that the function $S \to \pi_1(M)$ is one-to-one. Let $f \in I_{\pi(\tilde{p})}$ such that f corresponds to the identity in $\mathrm{Aut}(\pi_1(M))$. Now f_p acts as a permutation on S, and since f corresponds to $\mathrm{Id} \in \mathrm{Aut}(\pi_1(M))$, f_p leaves S fixed but since S spans T_pM, f_p leaves T_pM fixed. Therefore $f = \mathrm{Id}$ on M, so the map $I_{\pi(\tilde{p})} \to \mathrm{Aut}(\pi_1(M))$ is injective.

2.8. Corollary. *Let M be a manifold of small diameter. If for some $p \in M$ the first conjugate point along any geodesic eminating from p has multiplicity $\geqslant 2$, then $|\pi_1(M)| > \dim(M)$, and $I_p \subset \mathrm{Aut}(\pi_1(M))$.*

Proof. Let $\tilde{p} \in \tilde{M}$ be such that $\pi(\tilde{p}) = p$. Then the first conjugate point along any geodesic emanating from $\tilde{p}$ has multiplicity $\geqslant 2$. The corollary will follow from the theorem and the following lemma found in Warner [6].

2.9. Lemma. *Let M be a complete riemannian manifold. Let $p \in M$ such that the first conjugate point along any geodesic from p has multiplicity $\geqslant 2$. Then M is simply connected if and only if the first conjugate locus is equal to the cut locus in T_pM.*

Proof. Let q be any point in M such that q is not conjugate to p along any geodesic, and q is not in the cut locus to p. By Morse theory [5], $\Omega_{p,q}$ has the homotopy type of a C.W. complex with a cell of dimension λ for each geodesic from p to q to index λ. Since the first conjugate point along any geodesic has multiplicity $\geqslant 2$, we see there are no 1-cells in this C.W. complex. Thus M is simply connected if and only if for each such q there is a unique geodesic γ_q of index 0. γ_q must be the unique minimizing geodesic. If cut $=$ first conjugate, it is clear that the only geodesic from p to such a q of index 0 is the unique minimizing geodesic. Since the set of such q's is dense, we have that if the only geodesic from p to q of index 0 is the unique minimizing geodesic then the cut locus equals the first conjugate locus.

2.10. Corollary. *Let M^n be a complete simply connected riemannian manifold. Assume that there is a $k > 0$ such that $k \geqslant K_M$, and that for some $p \in M$ the first conjugate locus is equal to the cut locus in T_pM. Let G be a finite group acting freely on M through isometries. If $|G| \leqslant n$, then the orbit of $B_{\pi/2\sqrt{k}}(p)$ does not cover M.*

Proof. Assume the orbit did cover M. Studying the proofs of Theorems 2.4 and 2.7 we see that we can replace the condition on the diameter with a similar condition on the maximum distance from p to any point in M. In the current case the image of p in M/G will satisfy this condition. Hence $|\pi_1(M/G)| > n$, but $\pi_1(M/G) = G$ and $|G| \leqslant n$.

Remark. A similar statement can be made about free group actions where $|G| \leqslant mn$ only, then the disk will be smaller.

2.11. Corollary. *If M is a compact manifold of nonpositive curvature, then I_p is a subgroup of* $\mathrm{Aut}(\pi_1(M))$ *for all $p \in M$.*

Remarks. (1) If T^2 is the flat torus coming from the standard $Z \times Z$ action on $\mathbf{R}^2$, then for all $p \in T^2$, $I_p = \mathrm{Aut}(\pi_1(M))$.

(2) Applying Corollary 2.10 to S^n with constant curvature k we see that the orbit of the open upper hemisphere under a G action ($|G| \leqslant n$) does not cover. In fact, there is no way to cover S^n with n disks of radius $\pi/2\sqrt{k}$ even without a group action. Such a conclusion, however, is hard to make for other simply connected spaces with first conjugate locus equal to the cut locus. A simple volume argument will not suffice. The corollary may be saying more about the shape of such spaces than about free finite group actions.

3. Products and coverings

In this section we study how the invariants behave under coverings and products. In the next chapter we will give examples which show that the results of this section are the best possible.

We begin with some useful lemmas.

If M_1 and M_2 are complete connected riemannian manifolds, then so is $M_1 \times M_2$. Any geodesic γ from (p_1, p_2) is (γ_1, γ_2) where γ_1 and γ_2 are geodesics in M_1 and M_2 from the points p_1 and p_2.

3.1. Lemma. *Let $\gamma \subset M_1 \times M_2$ be a geodesic from (p_1, p_2) to (q_1, q_2). Then (p_1, p_2) is conjugate to (q_1, q_2) along γ if and only if p_i is conjugate to q_i along γ_i for some $i = 1$ or 2.*

Proof. If p_i is conjugate to q_i along γ_i, then there is a variation $\alpha(s, t) \to M_i$ through geodesics such that the variation vector field $\tilde{V}(t)$ along $\alpha(0, t) = \gamma_i(t)$ is not identically 0, but $V(0) = V(1) = 0$. Let $\tilde{\alpha}(s, t) \to M_1 \times M_2$ be defined by $\tilde{\alpha}(s, t) = (\alpha(s, t), \gamma_j(t))$, $i \neq j$. Then $\tilde{\alpha}(s, t)$ is a variation through geodesics, $\gamma(t) = \tilde{\alpha}(0, t)$ and the variation vector field $\tilde{V}(t)$ along $\gamma(t)$ is not identically 0, but $\tilde{V}(0) = \tilde{V}(1) = 0$. Thus (p_1, p_2) is conjugate to (q_1, q_2) along γ. If (p_1, p_2) is conjugate to (q_1, q_2) along γ, let $\alpha(s, t) \to M_1 \times M_2$ be an appropriate variation through geodesics. Now $\alpha(s, t)$ determines variations $\alpha_1(s, t) \to M_1$ and $\alpha_2(s, t) \to M_2$ through geodesics by projection. Now if $V(t)$ is the variation field along γ, then $V(t) = V_1(t) + V_2(t)$ where V_1 and V_2 correspond to the variations α_1 and α_2. Thus $V(t) \not\equiv 0$ implies $V_i \not\equiv 0$ for some i, and $V(0) = V(1) = 0$ implies $V_i(0) = V_i(1) = 0$. Therefore p_i is conjugate to q_i along $\gamma_i = \alpha_i(0, -)$.

3.2. Lemma. *If N_i is a topologically closed totally geodesic submanifold (resp. CC, ATC, TC) of M_i, $i = 1, 2$, then $N_1 \times N_2$ is a topologically closed totally geodesic submanifold (resp. CC, ATC, TC) of $M_1 \times M_2$.*

Proof. $N_1 \times N_2$ is clearly a closed submanifold. If $V \in T_{(p_1, p_2)} N_1 \times N_2$, then $V = V_1 + V_2$ where V_1 is tangent to N_1, and V_2 is tangent to N_2. Since N_i is totally geodesic, the geodesics γ_i such that $\gamma_i'(0) = V_i$ are in N_i. Thus the geodesic $\gamma = (\gamma_1, \gamma_2)$ is in $N_1 \times N_2$, and $\gamma'(0) = V$. Thus $N_1 \times N_2$ is totally geodesic. Now if γ is a unique minimizing geodesic from (p_1, p_2) to (q_1, q_2), then γ_i will be the unique minimizing geodesics from p_i to q_i. Thus if the N_i's are CC, $N_1 \times N_2$ is CC. If γ is any geodesic from (p_1, p_2) to (q_1, q_2), then γ_i will be a geodesic from p_i to q_i. Thus, if the N_i's are TC, then $N_1 \times N_2$ is TC. If γ is a geodesic from (p_1, p_2) to (q_1, q_2) such that (p_1, p_2) is not conjugate to (q_1, q_2) along γ, then Lemma 3.1 tells us that p_i is not conjugate to q_i along γ_i. Thus, if the N_i's are ATC, then $N_1 \times N_2$ is ATC. Hence the lemma follows.

3.3. Lemma. *Let $N \subset M_1 \times M_2$ be a topologically closed totally geodesic (resp. CC, ATC, TC) submanifold, and let $(p_1, p_2) \in N$. Then $N \cap M_i$ (M_i here is the copy of M_i going through the point (p_1, p_2)) is a topologically closed totally geodesic (resp. CC, ATC, TC) submanifold of M_i.*

Proof. Since M is a closed totally geodesic submanifold of $M_1 \times M_2$, so is $N \cap M_i$. Thus $N \cap M_i$ is a topologically closed totally geodesic submanifold of M_i. If $\gamma \subset M_i$ is a geodesic in M_i between two points in $N \cap M_i$, then γ is clearly a geodesic in $M_1 \times M_2$ between two points in N. Thus N being TC gives $N \cap M_i$ being TC. If $\gamma \subset M_i$ is a unique minimizing geodesic, then γ is a unique minimizing geodesic in $M_1 \times M_2$. Thus, if N is CC, then $N \cap M_i$ is CC. If $\gamma \subset M_i$ is a geodesic from p to $q \in M_i$ such that p is not conjugate to q along γ (thinking of conjugacy in M_i), then p is not conjugate to q along γ in $M_1 \times M_2$. This follows since $\gamma = (\gamma_1, \gamma_2)$ where γ_j, $j \neq i$, is the constant geodesic, so Lemma 3.1 tells us that γ_i not conjugate implies γ is not conjugate. Thus N being ATC gives $N \cap M_i$ being ATC.

3.4. Proposition. *Let M_1 and M_2 be connected complete riemannian manifolds. Then $CC_{p_1}(M_1) + CC_{p_2}(M_2) \geqslant CC_{(p_1,p_2)}(M_1 \times M_2) \geqslant \max\{CC_{p_1}(M_1), CC_{p_2}(M_2)\}$. Likewise for TC, ATC, and TG.*

Proof. (1) Assume $CC_{p_1}(M_1) \geqslant CC_{p_2}(M_2)$. Let $S \subset T_{(p_1,p_2)}(M_1 \times M_2)$ such that $\dim(S) < CC_{p_1}M_1$. Let S_1 be the projection of S onto $T_{(p_1,p_2)}M_1$. Then $\dim(S_1) \leqslant \dim S < CC_{p_1}M_1$. Let $N \subset M_1$ be $N_{S_1}^{CC}$. Since $\dim(S_1) < CC_{p_1}(M_1)$, $N \neq M_1$, thus $N \times M_2 \neq M_1 \times M_2$. Now $N \times M_2$ is a topologically closed complete convex totally geodesic submanifold of $M_1 \times M_2$ by Lemma 3.2. $S \subset T_{(p_1,p_2)}N \times M_2$, so $N_S^{CC} \subset N \times M_2 \neq M_1 \times M_2$. Since this was true for all S of dimension less than $CC_{p_1}M_1 \geqslant CC_{p_2}M_2$ we have $CC_{(p_1,p_2)}M_1 \times M_2 \geqslant \max\{CC_{p_1}(M_1), CC_{p_2}(M_2)\}$. The exact same argument works for TC, ATC and TG.

(2) Let $S_i \subset T_{p_i}M_i$ be such that $\dim(S_i) = CC_{p_i}(M_i)$ and $N_{S_i}^{CC} = M_i$, and $S \subset T_{(p_1,p_2)}M_1 \times M_2$ the direct sum $S_1 \oplus S_2$. Consider N_S^{CC}. By Lemma 3.3, $N_S^{CC} \cap M_i$ is a closed totally geodesic CC submanifold of M_i. Further, $S_i \subset T_{(p_1,p_2)}(N_S^{CC} \cap M_i)$, and since $N_{S_i}^{CC}(M_i) = M_i$ we have that $M_i \subset N_S^{CC}$. Thus $T_{(p_1,p_2)}M_1 \oplus T_{(p_1,p_2)}M_2 = T_{(p_1,p_2)}M_1 \times M_2$ is in $T_p(N_S^{CC})$. Therefore $N_S^{CC} = M_1 \times M_2$, so $CC_{(p_1,p_2)}(M_1 \times M_2) \leqslant \dim(S) = CC_{p_1}M_1 + CC_{p_2}M_2$, and the result follows. Again, the same proof works for TC, ATC, and TG.

In §5 we will see that if either M_1 and M_2 is compact, then $CC_{(p_1,p_2)}(M_1 \times M_2) = \max\{CC_{p_1}(M_1), CC_{p_2}(M_2)\}$. Likewise for TC and ATC.

If M is a connected complete riemannian manifold, we will denote by $\tilde{M}$ its universal covering space with the induced metric, and by $\pi\colon \tilde{M} \to M$ the covering projection.

3.5. Lemma. *If $N \subset M$ is a closed totally geodesic (resp. TC, ATC) submanifold, then so is $\pi^{-1}(N) \subset \tilde{M}$.*

Note. This lemma is false for CC. Let $p \in T^2$ (flat torus), then $\{p\}$ is CC, but $\pi^{-1}(p) \in E^2$ is not CC.

Proof of Lemma. Let $\tilde{N} = \pi^{-1}(N)$. $\tilde{N}$ is closed since N is closed. $\tilde{N}$ is a totally geodesic submanifold since N is, and this is a local property. Let $\tilde{\gamma}$ be geodesic in $\tilde{M}$ between two points in $\tilde{N}$. Then $\pi(\tilde{\gamma}) = \gamma$ is a geodesic between two points in N. Thus, if N is TC so is $\tilde{N}$. If $\tilde{\gamma}$ is a geodesic in $\tilde{M}$ between p and q such that p is not conjugate to q along $\tilde{\gamma}$, then $\gamma = \pi(\tilde{\gamma})$ is a geodesic from $\pi(p)$ to $\pi(q)$ such that $\pi(p)$ is not conjugate to $\pi(q)$ along γ. Thus if N is ATC, so is $\tilde{N}$.

3.6. Proposition. *$ATC_p(\tilde{M}) \geqslant ATC_{\pi(p)}$. Similarly for TC and TG.*

Note. The author cannot prove the corresponding result for CC, but has found no counterexample.

Proof. Let $S \subset T_p\tilde{M}$ such that $\dim(S) < ATC_{\pi(p)}(M)$, and let $S' = \pi_* S$. Then $N_{S'}^{ATC} \neq M$ since $\dim(S') < ATC_{\pi(p)}(M)$. Therefore $\pi^{-1}(N_{S'}^{ATC}) \neq \tilde{M}$, but Lemma 3.5 implies $N_S^{ATC} \subset \pi^{-1}(N_{S'}^{ATC}) \neq \tilde{M}$. Thus $ATC_p(\tilde{M}) \geqslant ATC_{\pi(p)}(M)$. Similarly for TC and TG.

Remark. The above obviously holds for any covering space.

4. Examples

In this section we examine some examples where the invariants can be computed. These examples serve to show the sharpness of the propositions in §3, and also illustrate some further properties of the invariants.

4.1. Example. If M is a manifold of small diameter, then

$$TC_p(M) = ATC_p(M) = 0$$

for all $p \in M$.

This follows from the results in §2 and the fact that $TC_p(M) \leqslant ATC_p(M)$.

4.2. Example. If $\mathbf{S}^n$, $\mathbf{RP}^n$, or $\mathbf{R}^n$ $(n > 1)$ have constant curvature, then, for all p,

$$\begin{gathered} TC_p(\mathbf{S}^n) = 0, ATC_p(\mathbf{S}^n) = CC_p(\mathbf{S}^n) = TG_p(\mathbf{S}^n) = n, \\ TC_p(\mathbf{RP}^n) = 0, ATC_p(\mathbf{RP}^n) = CC_p(\mathbf{RP}^n) = TG_p(\mathbf{RP}^n) = n, \\ TC_p(\mathbf{R}^n) = ATC_p(\mathbf{R}^n) = CC_p(\mathbf{R}^n) = TG_p(\mathbf{R}^n) = n. \end{gathered}$$

To show this let M be one of $\mathbf{S}^n$, $\mathbf{RP}^n$ or $\mathbf{R}^n$, and fix $p \in M$. If $S \subset T_pM$ is any linear subspace, then $\operatorname{Exp}_p S$ is a topologically closed almost totally convex totally geodesic submanifold of M. Thus $ATC_p(M) = n$. Since $ATC_p \leqslant CC_p \leqslant TG_p \leqslant n$, all of the above follow with the exception of the TC's. For $\mathbf{S}^n$ or $\mathbf{RP}^n$ the set of closed geodesics from p to p covers the space. Therefore $TC_p(\mathbf{S}^n) = TC_p(\mathbf{RP}^n) = 0$. In $\mathbf{R}^n$, $\operatorname{Exp}_p S$ is totally convex for any $S \subset T_pM$. Thus $TC_p(\mathbf{R}^n) = n$.

In a future paper we will show that these spaces are the only ones with $CC_p(M) = n$ for all $p \in M$.

The projection $\pi: \mathbf{S}^n \to \mathbf{RP}^n$ serves as an example where $ATC_p(\tilde{M}) = ATC_{\pi(p)}(M)$. While the projection $\pi: \mathbf{R}^n \to T^n$ where T^n is the flat torus is an example where $ATC_p(M) > ATC_{\pi(p)}(M)$. In fact these same examples work similarly for the other invariants.

Next we consider an example where the invariants depend on the point at which they are evaluated.

4.3. Example. Let M be a paraboloid of revolution with vertex v.

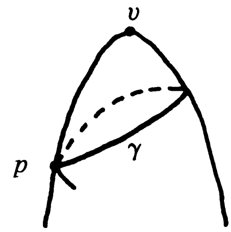


FIG. 4.1

$TC_v(M) = ATC_v(M) = 1$, $CC_v(M) = TG_v(M) = 2$, $TC_p(M) = ATC_p(M) = 0$, $CC_p(M) = TG_p(M) = 1$, for $p \neq v$.

First consider the vertex v. Since every geodesic from v does not return to v, we see that $\{v\}$ is totally convex, so that $N_0^{TC} \neq M$. If S is any one-dimensional subspace of T_vM then $\operatorname{Exp}_v S$ consists of the two geodesics running down on opposite sides of M. Since this set is completely convex, $CC_c(M) = 2$ and so $TG_v(M) = 2$. However there are geodesics running from one side of M to the other such that the endpoints are not conjugate along the geodesic. Thus $ATC_v(M) = 1$ and $TC_v(M) = 1$. Now if $p \neq v$, consider the closed geodesic γ represented in Fig. 4.1. Let S be the one-dimensional subspace at p generated by $\gamma'(0)$. It is clear that the only topologically closed totally geodesic submanifold N of M with $S \subset T_pM$ is M itself. Thus $TG_p(M) \leqslant 1$. But $1 \leqslant CC_p(M) \leqslant TG_p(M) \leqslant 1$. Further, since p is not conjugate to p along γ, we get $ATC_p(M) = 0$ and thus $TG_p(M) = 0$.

Now we will consider some products. For brevity we will consider only ATC_p.

4.4. Examples. (1) $ATC_p(S^r \times S^s) = \max\{r, s\}$.

(2) $ATC_p(\mathbf{R}^r \times \mathbf{R}^s) = r + s$; $\mathbf{R}^r$, $\mathbf{R}^s$ with flat metric.

(3) $ATC_p(S^r \times T^s) = r$ where T^s is the flat torus, and we assume that $r > 1$.

Example (2) is the same as Example 4.2, while Examples (1) and (3) will follow from Proposition 5.8.

In the above, Examples (1) and (2) serve to show the sharpness of Proposition 3.4. (3) shows that for every pair (n, r) of integers such that $0 \leqslant r \leqslant n$ there is an n-dimensional manifold M with $ATC_p(M) = r$.

In general it is not easy to compute these invariants. In the next example we will consider $\mathbf{CP}^n$.

4.5. Example. For $\mathbf{CP}^n$ with the symmetric space metric we have

$$TC_p(\mathbf{CP}^n) = 0, ATC_p(\mathbf{CP}^n) = CC_p(\mathbf{CP}^n k) = TG_p(\mathbf{CP}^n) = n.$$

(Note: $n = \frac{1}{2}$ real dimension.) Every geodesic emanating from p returns to p, thus $TC_p(\mathbf{CP}^n) = 0$.

The strategy in computing the remaining numbers is as follows. First we will show that if $S \subset T_p\mathbf{CP}^n$ is a complex subspace, then $\operatorname{Exp}_p S$ is a topologically closed, almost totally convex, totally geodesic submanifold. Then since every subspace of dimension less than n is contained in a complex subspace $S \neq T_p\mathbf{CP}^n$, we have $ATC_p(\mathbf{CP}^n) \geqslant n$. Next, by examining Lie triple systems we construct an n-dimensional subspace $\tilde{S} \subset T_p(\mathbf{CP}^n)$ such that the only closed totally geodesic submanifold n with $\tilde{S} \subset T_pN$ is $\mathbf{CP}^n$ itself. Therefore $TG_p(\mathbf{CP}^n) \leqslant n$. Furthermore, $n \leqslant ATC_p(\mathbf{CP}^n) \leqslant CC_p(\mathbf{CP}^n) \leqslant TG_p(\mathbf{CP}^n) \leqslant n$, and the result will follow.

For the first part consider the fibration:

$$\begin{array}{c} S^1 \to S^{2n+1} \\ \downarrow\pi \\ \mathbf{CP}^n \end{array}$$

π is a riemannian submersion where S^{2n+1} has the usual metric induced from $\mathbf{C}^{n+1}$. If $\tilde{p} \in S^{2n+1}$, let $\tilde{T}_{\tilde{p}} \subset T_{\tilde{p}}S^{2n+1}$ be the subspace perpendicular to $i\tilde{p}$ (ip is the vector at p obtained by parallel translation in $\mathbf{C}^{n+1}$ of $ip \in T_0\mathbf{C}^{n+1}$). Then $T_p(\mathbf{CP}^n)$ can be identified with $\tilde{T}_{\tilde{p}}$ ($\pi(\tilde{p}) = p$) through $\pi_{*\tilde{p}}$. If γ is a geodesic from p in $\mathbf{CP}^n$, then the geodesic $\tilde{\gamma}$ from $\tilde{p}$ in S^{2n+1} with corresponding initial tangent vector has the property that $\pi(\tilde{\gamma}) = \gamma$.

Kobayashi and Nomizu (see [3, pp. 273–278]) show that for each complex m-dimensional subspace $S \subset T_p\mathbf{CP}^n$ there is a complex totally geodesic submanifold $N \subset \mathbf{CP}^n$ such that $S = T_pN$ (in fact $N = \mathbf{CP}^m$). We need to show that N is ATC. Let γ be a geodesic between two points $q_1, q_2 \in N$ such that $\gamma \not\subset N$. It is sufficient to show that q_1 is conjugate to q_2 along γ.

Let $\tilde{q}_1 \in S^{2n+1}$ such that $\pi(\tilde{q}_1) = q_1$, and let $S^1 = T_{q_1}N$ and $S^1 \subset \tilde{T}_{\tilde{q}_1}$ be the corresponding subspace. We claim that $\pi^{-1}(N) \subset S^{2n+1}$ is equal to $S^{2n+1} \cap \mathbf{Q}^{\mathbf{C}}$ where $\mathbf{Q}^{\mathbf{C}}$ is the complex linear subspace of $\mathbf{C}^{n+1}$ spanned by $\tilde{q}_1$ and S^1, S^1 being translated to $0 \in \mathbf{C}^{n+1}$.

Proof of claim. Since N is totally geodesic, we know that all geodesics from $\tilde{q}_1$ with initial tangent vectors in S^1 must lie in $\pi^{-1}(N)$. This tells us that $S^{2n+1} \cap \mathbf{Q}^{\mathbf{R}} \subset \pi^{-1}(N)$ where $\mathbf{Q}^{\mathbf{R}}$ is the real span of $\tilde{q}_1$ and S^1. By the definition of $\mathbf{C}^{n+1} \to \mathbf{CP}^n$ we see that $S^{2n+1} \cap \mathbf{Q}^{\mathbf{C}} \subset \pi^{-1}(N)$. For dimension reasons and the fact that $\pi^{-1}(N)$ is a connected closed $2m + 1$ submanifold, we see $S^{2n+1} \cap \mathbf{Q}^{\mathbf{C}} = \pi^{-1}(N)$.

Now let $\tilde{\gamma} \subset S^{2n+1}$ be the geodesic from $\tilde{q}_1$ corresponding to γ. Since $\gamma \not\subset N$, $\tilde{\gamma} \not\subset \pi^{-1}(N)$. Since $S^{2n+1} \cap \mathbf{Q}^{\mathbf{C}} = \pi^{-1}(N)$ is ATC, we see that $\tilde{q}_1$ is conjugate to $\tilde{q}_2$ along $\tilde{\gamma}$ ($\tilde{q}_2 = \tilde{\gamma}(1)$ so $\pi(\tilde{q}_2) = q_2$). In fact, we can find a variation $\tilde{\gamma}_s$ of geodesics such that $\tilde{\gamma}_0 = \gamma$, $\tilde{\gamma}_s(0) = \tilde{q}_1$, $\tilde{\gamma}_s(1) = \tilde{q}_2$ and $\langle \tilde{\gamma}_s'(0), i\tilde{q}_1 \rangle = 0$. In particular $\tilde{\gamma}_s'(0) \in \tilde{T}_{\tilde{q}_1}$. Therefore $\gamma_s = \pi(\tilde{\gamma}_s)$ is a variation through geodesics in $\mathbf{CP}^n$ with $\gamma_0 = \gamma$, $\gamma_s(0) = q_1$, and $\gamma_s(1) = q_2$, and q_1 is conjugate to q_2 along γ.

For the second part we first consider a subspace $T \subset T_pCP^n$ such that $T = T_pN$ where N is a totally geodesic submanifold of $\mathbf{CP}^n$. We know that if $\xi_1, \xi_2, \xi_3 \in T$, then $R(\xi_1, \xi_2)\xi_3 \in T$.

We claim that if $\xi_1, \xi_2 \in T$ such that $\langle \xi_2, J\xi_1 \rangle \neq 0$, then $J\xi_2 \in T$. We know that $R(\xi_1, \xi_2)\xi_2 \in T$. Using the formula for curvature given by Kobayashi and Nomizu (see [4, p. 277]),

$$\begin{aligned} R(\xi_1, \xi_2)\xi_2 &= h(\xi_2, \xi_2)\xi_1 - h(\xi_2, \xi_1)\xi_2 + h(\xi_1, \xi_2)\xi_2 - h(\xi_2, \xi_1)\xi_2 \\ &= h(\xi_2, \xi_2)\xi_1 - g(\xi_2, \xi_1)\xi_2 - [2g(\xi_2, J\xi_1) - g(\xi_1, J\xi_2)]J\xi_2 \\ &= h(\xi_2, \xi_2)\xi_1 - g(\xi_2, \xi_1)\xi_2 - 3g(\xi_2, J\xi_1)J\xi_2, \end{aligned}$$

where h is the hermitian inner product. The first two terms are clearly in T, and since $g(\xi_2, J\xi) \neq 0$ we get $J\xi_2 \in T$.

We now construct an n-dimensional subspace S by describing a basis $\{\xi_1, \xi_2, \cdots, \xi_n\}$. Choose ξ_1 arbitrarily, and ξ_2 outside the subspace spanned by ξ_1 and $J\xi_1$ but close enough to $J\xi_1$ such that $g(\xi_2, J\xi_1) \neq 0$. In general choose ξ_i outside the subspace spanned by $\xi_1, J\xi_1, \xi_2, J\xi_2$k, $\cdots, \xi_{i-1}, J\xi_{i-1}$ but close enough to $J\xi_{i-1}$ such that $g(\xi_i, J\xi_{i-1}) \neq 0$. We can certainly choose n vectors this way. Let $T = T_pN_S^{TG}$, so that $S \in T$. For $i \geqslant 1$, $g(\xi_{i+1}, J\xi_i) \neq 0$ so $J\xi_{i+1} \in T$. Further $g(\xi_1, J\xi_2) = -g(\xi_2, J\xi_1) \neq 0$, so $J\xi_1 \in T$. Thus $T = T_p\mathbf{CP}^n$ so that $N_S^{TG} = CP^n$, and hence $TG_p(\mathbf{CP}^n) \leqslant n$.

It should be possible to compute the invariants for the other symmetric spaces of rank 1 in a similar way.

One would expect $\mathbf{CP}^n$ to have large invariants where in fact we get only half the real dimension. In a future paper we will see that for normal homogeneous spaces M if $ATC_p(M) \geqslant \frac{1}{2}(n+3)$, then $\tilde{M}$ is isometric to $M_1^r \times M_2^s$ where M^r is a constant curvature space and $r \geqslant \frac{1}{2}(n+3)$. Thus irreducible normal homogeneous spaces other than S^n, $\mathbf{RP}^n$, or $\mathbf{R}^n$ have $ATC_p < \frac{1}{2}(n+3)$.

We have been able to show, with the assistance of Allen Back, that for simple lens spaces $ATC_p(L_q^n) = \frac{1}{2}(n-1)$. We have also computed the invariants for generalized Lens spaces and compact Lie groups with bi-invariant metrics.

5. Continuity properties

For a smooth manifold M let $G_r(M)$ be the Grassman bundle of r-planes. Define the bundle $G(M) \xrightarrow{\pi} M$ by $G(M) = G_0(M) + G_1(M) + \cdots + G_n(M)$, where $+$ is disjoint union. For M connected complete riemannian consider the following functions:

$$\begin{tikzcd} G(M) \arrow[r, "f_{CC}"] \arrow[d, "\pi"] & G(M) \arrow[d, "\pi"] \\ M \arrow[r, "\mathrm{id}"] & M \end{tikzcd} \qquad \begin{tikzcd} G(M) \arrow[r, "d"] & \mathbf{Z} \\ M \arrow[r, "CC"] & \mathbf{Z} \end{tikzcd}$$

where $d(S)$ = dimension of S, $CC(p)$ is CC_p, and $f_{CC}(S) = T_{\pi(S)}(N_S^{CC})$. These functions are related as follows: $CC(p) = \min\{d(S) | S \in \pi^{-1}(p) \cap (d \circ f_{CC})^{-1}(n)\}$. We can likewise define ATC, f_{ATC} (resp. TC, TG).

In this section we wish to consider the continuity properties of these functions. We have seen that CC need not be constant, thus CC is not necessarily continuous. We will show it is upper semi-continuous. Also f_{CC} need not be continuous, for there could exist S_1, S_2 such that $d(S_1) = d(S_2)$ but $d(f_{CC}S_1) \neq d(f_{CC}S_2)$. We will show that this is the only way in which f_{CC} is not continuous. Using the results we will then show $CC_{(p_1,p_2)}(M_1 \times M_2) = \max\{CC_{p_1}(M_1), CC_{p_2}(M_2)\}$ (resp. TC, ATC) whenever M_1 or M_2 is compact.

If N is a connected topologically closed totally geodesic submanifold of M and $p \in N$, then $\mathrm{Exp}_p(T_pN) = N$. With this the following lemma comes immediately from the definition of f_{CC}.

5.1. Lemma. (a) $S \subset f_{CC}S$.

(b) *Image of* $f_{CC} = \{S | \mathrm{Exp}_{\pi(S)}(S) = N_S^{CC}\}$.

(c) $f_{CC} \circ f_{CC} = f_{CC}$.

Similarly for TC, ATC, and TG.

5.2. Proposition. *The image of f_{CC} is closed in $G(M)$.*

Proof. Let $S_i \to S$ be a convergent sequence in $G(M)$ such that $S_i \in \mathrm{Im}(f_{CC})$. We can assume that $d(S_i) = d(S) = r$ for all i. Let $p_i = \pi(S_i)$ and $p = \pi(S)$. We know $p_i \to p$. We need only show that $\mathrm{Exp}_p\, S = N_S^{CC}$, i.e., that $\mathrm{Exp}_p\, S$ is a topologically closed, completely convex, totally geodesic submanifold. Let $N_i = \mathrm{Exp}_p S_i$ and $N = \mathrm{Exp}_p\, S$. For every $q \in N$ we will show:

(1) $\exists S^q \subset T_qM$ such that $d(S^q) = r$ and $\mathrm{Exp}_q(S^q) \subset N$.

(2) $\forall q' \in N$ if γ is a unique minimizing geodesic from q to q' such that q is not conjugate to q' along γ, then $\gamma \subset N$.

(1) Let $q \in N$. Then there is a $V \in S$ such that $q = \mathrm{Exp}_p\, V$. Let $V_i \in S_i$ such that $V_i \to V$ and $|V_i| = |V|$. Let $q_i = \mathrm{Exp}_{p_i}\, V_i$, and $S^{q_i} = T_{q_i}(N_i)$. Let $\gamma_i(t) = \mathrm{Exp}_{p_i}(tV_i)$. Then since each N_i is totally geodesic, S^{q_i} is the parallel

translate of S_i along γ_i to q_i. If $\gamma(t) = \operatorname{Exp}_p(tV)$, then $\gamma_i \to \gamma$, and thus $S^{q_i} \to S^q$ where S^q is the parallel translate of S along γ. Now let $W \in S^q$, and choose $W_i \in S^{q_i}$ such that $W_i \to W$ and $|W_i| = |W|$. We need to show that $\operatorname{Exp}_q W \in N$. Let $z = \operatorname{Exp}_q W$ and let $z_i = \operatorname{Exp}_{q_i} W_i$. Thus $z_i \to z$. Since N_i is totally geodesic and $W_i \in T_{q_i}N_i$, we have $z_i \in N_i$. Thus there is a $\overline{W}_i \in S_i$ such that $z_i = \operatorname{Exp}_{p_i} \overline{W}_i$. We can choose $\overline{W}_i$ such that $|\overline{W}_i| \leqslant |V_i| + |W_i| = |V| + |W|$. Therefore some subsequence of the $\overline{W}_i$ converge to $\overline{W} \in S$. Thus we get $\operatorname{Exp}_p \overline{W} = \lim_{i\to\infty} \operatorname{Exp}_{p_i} \overline{W}_i = \lim_{i\to\infty} z_i = z$. So property (1) is shown.

(2) Let $q' \in \operatorname{Exp}_p S$ and let γ be a unique minimizing geodesic from q to q' such that q is not conjugate to q' along γ. That is, the cut point along γ (if it occurs at all) happens after q'. Similar to part (1) choose $q_i \to q$, $q_i' \to q'$, and S^{q_i}, S^q. Let γ_i be a minimizing geodesic from q_i to q_i'. Since the distance to the cut locus is a continuous function on the unit sphere bundle (see [2, p. 94]), for i large enough there is a unique minimizing geodesic τ_i from q_i to q_i' such that q_i is not conjugate to q_i' along τ_i. Since N_i is completely convex $\tau_i \subset N_i$, so $\tau_i'(0) \in T_{q_i}N_i = S^{q_i}$. Some subsequence of the $\tau_i'(0)$ converge to a $V \in S^q$. Let $\tau = \operatorname{Exp}_q tV$. By part (1), $\tau \subset N$. Since τ is a limit of minimizing geodesics, τ is minimizing from q to q'. Since γ is the unique minimizing geodesic, $\tau = \gamma$. Therefore $\gamma \subset N$ and (2) is shown.

To complete the proof of the proposition it is sufficient to show that for every $q \in N$ there is an $\varepsilon > 0$ such that $N \cap B_\varepsilon(q) = \operatorname{Exp}_q(S^q \cap B_\varepsilon(0))$. Choose ε so small that Exp_q is a diffeomorphism on $B_\varepsilon(0)$ and that for every $q', q'' \in B_\varepsilon(q) = \operatorname{Exp}_q(B_\varepsilon(0))$ there is a unique geodesic $\gamma_{q''}^{q'}$ from q' to q'' in $B_\varepsilon(q)$, further $\gamma_{q''}^{q'}$ will be minimizing and q' will not be conjugate to q'' along $\gamma_{q''}^{q'}$. We know from (1) that $\operatorname{Exp}_q(S^q \cap B_\varepsilon(0)) \subset N \cap B_\varepsilon(q)$. Assume there was a $q' \in N \cap B_\varepsilon(q)$ such that $q' \notin \operatorname{Exp}_q(S^q \cap B_\varepsilon(0))$. By (2) all the geodesics $\gamma_{q''}^{q'}$ will be in N for $q'' \in \operatorname{Exp}_q(S^q \cap B_\varepsilon(0))$. But this means that N contains an open subset of dimension $r + 1$. But N is the image of S by the exponential map, so by Sard's theorem this cannot happen. Thus the proposition follows.

Next we consider the image of f_{ATC}. We will use the following lemma. The author would like to thank Allen Back for the proof.

5.3. Lemma. *Let $V \in TM$ such that V is not a critical point of* $\operatorname{Exp}_{\pi(V)}$. *Then there are open sets U, U', U'', with $V \in U \subset TM$, $\pi(V) \in U' \subset M$, and* $\operatorname{Exp}(V) \in U'' \subset M$, *such that $f\colon U \to U' \times U''$ is a diffeomorphism where $f\colon TM \to M \times M$ by $f(W) = (\pi(W), \operatorname{Exp}(W))$.*

Proof. The function f is clearly differentiable. Consider $f_{*V}\colon T_V TM \to T_{\pi(V)}M \oplus T_{\operatorname{Exp}(V)}M$. Since TM is a bundle over M, the image of f_{*V} contains $T_{\pi(V)}M$, and since V is not a critical point of $\operatorname{Exp}_{\pi(V)}$ we see that $T_{\operatorname{Exp}(V)}(M)$ is in the image of f_{*V}. For dimension reasons f_{*V} is an isomorphism. By the inverse function theorem there is an open set $O \subset TM$ such that $f|_0$ is a diffeomorphism onto its image. Choose U', U'' such that $U' \times U'' \subset f(0)$, and let $U = f^{-1}(U' \times U'')$.

5.4. Proposition. *The image of f_{ATC} is closed.*

Proof. We first note that $\operatorname{Im} f_{ATC} \subset \operatorname{Im} f_{CC}$. This follows because if $\operatorname{Exp}_p S = N_S^{ATC}$ then $\operatorname{Exp}_p S = N_S^{CC}$. Let $S_i \to S$ in $G(M)$ such that $S_i \in \operatorname{Im} f_{ATC}$. By Proposition 5.2, $S \in \operatorname{Im} f_{CC}$. Therefore we need only show that $\operatorname{Exp}_p S$ is almost totally convex ($p = \pi(S)$). Let $q, q' \in N \equiv \operatorname{Exp}_p S$. Let γ be a geodesic from q to q' such that q is not conjugate to q' along γ. Let $V \in T_qM$ be such that V is tangent to γ and $\operatorname{Exp} V = q'$. We need to show $V \in T_qV$. Since q is not conjugate to q' along γ, V is not a critical point of Exp_q. Choose subsets $V \in U \subset TM$, $q \in U' \subset M$, and $q' \in U'' \subset M$ as in Lemma 5.3. As in the proof of Proposition 5.2 choose sequences $q_i \to q$, $q_i' \to q'$ such that $q_i, q_i' \in N_i \equiv \operatorname{Exp}_{p_i} S_i$. For i sufficiently large $q_i \in U'$ and $q_i' \in U''$. Thus by the lemma there is a unique $V_i \in U$ such that $\pi(V_i) = q_i$ and $\operatorname{Exp}(V_i) = q_i'$. Further $V_i \to V$. Since the function f of the lemma is nonsingular in U, V_i is not a critical point of Exp_{q_i}. Now the geodesic $\gamma_i(t) = \operatorname{Exp}_{q_i} tV_i$ is a geodesic from q_i to q_i' such that q_i is not conjugate to q_i' along γ. Therefore since N_i is almost totally convex, $\gamma_i \subset N_i$ so $V_i \in T_{q_i}N_i$. Since $T_{q_i}N_i \to T_qN$ (see proof of Proposition 5.2) and $V_i \to V$, we have $V \in T_qN$. Therefore N is almost totally convex and the proposition follows.

Remark. The author suspects that $\operatorname{Im}(f_{TC})$ is always closed while $\operatorname{Im}(f_{TG})$ is not always closed.

Next we consider the functions $d \circ f_{CC}$ and $d \circ f_{ATC}$.

5.5. Proposition. *The functions $d \circ f_{CC}$ and $d \circ f_{ATC}$ are lower semi-continuous.*

Proof. We need to show that for every $q \in \{0, 1, \cdots, n\}$ the set $Q = (d \circ f_{CC})^{-1}\{0, 1, \cdots, q\}$ is closed. Let $S_i \to S$ in $G(M)$ where $S_i \in Q$. We have $d \circ f_{CC}(S_i) \in \{0, 1, \cdots, q\}$ and therefore $d \circ f_{CC}(S_i) = r$ for an infinite number of i's and some $r \leqslant q$. Thus $f_{CC}(S_i) \in G_rM$. By the compactness of the fibres in G_rM some subsequence $f_{CC}(S_j)$ converges to an $\tilde{S} \in G_r(M)$. Since the image of f_{CC} is closed, $\tilde{S}$ is in the image of f_{CC}. By Lemma 5.1 we have $\operatorname{Exp}_p \tilde{S} = N_{\tilde{S}}^{CC}$ and $S_j \subset f_{CC}(S_j)$. Therefore $S \subset \tilde{S}$. So we get $N_S^{CC} \subset N_{\tilde{S}}^{CC}$ and $d \circ f_{CC}(S) \leqslant r \leqslant q$. Thus $S \in Q$ implying that Q is closed. The same argument works for $d \circ f_{ATC}$.

We are now in a position to study the function CC.

5.6. Theorem. *The image of the function CC (resp. TC, ATC, TG) consists of at most two consecutive integers r and $r + 1$. Further CC (resp. ATC) is upper semi-continuous (i.e., $CC^{-1}(r)$ is open).*

Proof. To show the first part we will show that for every p and q in M, $CC(q) \leqslant CC(p) + 1$. Let $S \subset T_pM$ be a linear subspace of dimension $CC(p)$, such that $N_S^{CC} = M$. Let γ be any geodesic from p to q. Let $S' \subset T_qM$ be $S' = \gamma_q^p(S) + \gamma'(q)$, where γ_q^p represents parallel translation. Dimension of $S' \leqslant$ dimension of $S + 1 = CC(p) + 1$. We need only show $N_{S'}^{CC} = M$. Since $\gamma'(q) \in S'$, we know that γ is in $N_{S'}^{CC}$, so that $p \in N_{S'}^{CC}$. Since $\gamma_q^p(S) \subset S' \subset T_qN_{S'}^{CC}$ and $N_{S'}^{CC}$ is totally geodesic, $S \subset T_pN_{S'}^{CC}$, but M is the only topologically closed, completely convex, totally geodesic submanifold through p with S in its tangent space at p. Therefore $N_{S'}^{CC} = M$. The same argument works for TC, ATC, and TG.

For the second part, assume the image of CC consists of the points r and $r + 1$. Then $CC^{-1}(r) = \pi(G_r(M) \cap (d \circ f_{CC})^{-1}(n))$. By Proposition 5.5, $(d \circ f_{CC})^{-1}(n)$ is open in $G(M)$. Thus $G_r(M) \cap (d \circ f_{CC})^{-1}(n)$ is open in $G_r(M)$. Since $\pi\colon G_r(M) \to M$ is an open map, $\pi(G_r(M) \cap (d \circ f_{CC})^{-1}(n))$ is open. The same argument works for ATC.

Now we consider the functions f_{CC} and f_{ATC} in greater detail. We have noted earlier that these functions need not be continuous since subspaces of the same dimension can have images of different dimensions. We will put a new topoplogy on $G(M)$ to take care of this, and the resulting functions will be continuous.

Let $G^{CC}{}^r_s(M) = G_r(M) \cap f_{CC}^{-1}(G_s(M))$, and define $G^{CC}(M)$ to be the disjoint union of the $G^{CC}{}^r_s$. We have the following commutative diagrams of functions:

$$\begin{tikzcd} G^{CC} \arrow[r, "f_{CC}"] \arrow[d, "\pi"] & G \arrow[d, "\pi"] \\ M \arrow[r, "\mathrm{id}"] & M \end{tikzcd}$$
1

$$\begin{tikzcd} G^{CC} \arrow[r, "f_{CC}"] \arrow[d, "\pi"] & G^{CC} \arrow[d, "\pi"] \\ M \arrow[r, "\mathrm{id}"] & M \end{tikzcd}$$
2

$$\begin{tikzcd} G^{CC} \arrow[r, "i"] \arrow[d, "\pi"] & G \arrow[d, "\pi"] \\ M \arrow[r, "\mathrm{id}"] & M \end{tikzcd}$$
3

where i is the identification (ignoring topologies). Similar spaces and diagrams can be constructed for ATC.

5.7. Theorem. *The diagrams* (1), (2), *and* (3) *are commutative diagrams of continuous functions.*

Proof. We need to show:

(a) $G^{CC} \xrightarrow{i} G$ is continuous.

(b) $G^{CC} \xrightarrow{\pi} M$ is continuous.

(c) $G^{CC} \xrightarrow{f_{CC}} G$ is continuous.

(d) $G^{CC} \xrightarrow{f_{CC}} G^{CC}$ is continuous.

(a) is continuous since G^{CC} has a finer topology than G. (b) is continuous from the commutative diagram:

$$\begin{tikzcd} G^{CC} \arrow[rr, "\pi"] \arrow[dr, "i"'] & & M \\ & G \arrow[ur, "\pi"'] & \end{tikzcd}$$

In order to show that a function from G^{CC} is continuous it is sufficient to show that its restriction to each $G_s^{CC^r}$ is continuous. f_{CC}: $G_s^{CC^r} \to G_s(M)$. Let C be a closed set in $G_s(M)$. We need to show that $D = f_{CC}^{-1}(C)$ is closed in $G_s^{CC^r}$. Let $S_i \to S$ be a convergent sequence in $G_s^{CC^r}$ such that $S_i \in D$. Since $S_i \to S$ in $G_s^{CC^r}$, $S_i \to S$ in G. By a previous argument some subsequence $f_{CC}(S_j)$ converges to a subspace $\tilde{S} \in G$. Since $f_{CC}(S_j) \in G_s(M)$, $\tilde{S} \in G_s(M)$. Since $f_{CC}(S_j) \in C$ a closed set, $\tilde{S} \in C$. By the same argument as before $f_{CC}(S) \subset \tilde{S}$, but for dimension reasons (i.e., $S \in G^{CC^r_s}$ so $f_{CC}(S) \in G_s(M)$) $f_{CC}(S) = \tilde{S} \in C$ so $S \in D = f_{CC}^{-1}(C)$. Therefore (c) follows. In order to show (d) we need only note that since $f_{CC} \circ f_{CC} = f_{CC}$ we have $f_{CC}(G_s^{CC^r}) \subset G_s^{CC^s}$, thus (c) implies (d). All of the arguments above work for ATC.

Remarks. In Theorem 5.6 we see that the points with highest CC form a closed set F. The author suspects that they form a closed submanifold of codimension at least 2.

We are now in a position to prove

5.8. Proposition. *Let M_1 be a compact riemannian manifold, and M_2 a complete riemannian manifold. Then for $p_1 \in M_1$, $p_2 \in M_2$, $CC_{(p_1,p_2)}(M_1 \times M_2) = \max\{CC_{p_1}(M_1), CC_{p_2}(M_2)\}$ (resp. TC, ATC).*

Remark. This does not hold for TG as $TG_p(\mathbf{R}^1) = 1$, $TG_q(S^1) = 1$, while $TG_{(p,q)}(\mathbf{R}^1 \times S^1) = 2$.

Proof. Let $r_i = CC_{p_i}(M_i)$. Let $A^i = \{S \in G^{r_i}(T_{p_i}M_i) | N_S^{CC} = M_i\}$. By proposition 5.5, A^1 and A^2 are open in $G^{r_1}(T_{p_1}M_1)$ and $G^{r_2}(T_{p_2}M_2)$ respectively. Since the set of geodesics γ from p such that $\gamma(t)$ is a cut point of p for all $t \geqslant t_0$ is nowhere dense (that is to say the set of $\gamma'(0)$'s is nowhere dense in the unit sphere) and since A^1 is open, we can choose $S^1 \in A^1$ and a basis

$\{X_1, \cdots, X_{r_1}\}$ of S^1 such that the geodesics $\gamma_i(t) = \operatorname{Exp}_p tX_i$ are not in the above set. Choose $S^2 \in A^2$ and a basis $\{Y_1, \cdots, Y_{r_2}\}$. We will assume $|X_i| = |Y_i| = 1$. Since A^1 is open, there is an $\varepsilon > 0$ such that for any set $\{Z_1, Z_2, \cdots, Z_{r_1}\}$ of unit vectors in $T_{p_1}M_1$ the subspace spanned by $\{X_1 + a_1Z_1, \cdots, X_{r_1} + a_{r_1}Z_{r_1}\}$ is in A^1 whenever $|a_i| < \varepsilon$ for all i.

Let d be the diameter of M_1. Choose $b_i > 0$ such that $d < \varepsilon b_i$ and $\gamma_i(b_i)$ is not a cut point of p_1. Let $c > 0$ be some number less than the distance from p_2 to its cut locus. Let $S \subset T_{(p_1,p_2)}M_1 \times M_2$ be the span of $\{b_1X_1 + cY_1, \cdots, b_{r_1}X_{r_1} + cY_{r_1}, cY_{r_1+1}, \cdots, cY_{r_2}\}$ (or the other way around if $r_1 > r_2$). We need only show by 3.4 that $N_S^{CC} = M_1 \times M_2$. Let γ^i be the geodesic $(\gamma_1^i, \gamma_2^i) = \operatorname{Exp}_{(p_1,p_2)}t(b_iX_i + cY_i)$. $\gamma^i(t)$ lies in N_S^{ATC} for all t. By the choice of b_i and c, $\gamma^i(1) = (\gamma_1^i(1), \gamma_2^i(1))$ is not on the cut locus to (p_1, p_2) in $M_1 \times M_2$. Thus the unique minimizing geodesic $\sigma^i(t)$ from (p_1, p_2) to $\gamma^i(1)$ must lie in N_S^{CC}. If we parameterize $\sigma^i(t)$ so that $\sigma^i(1) = \gamma^i(1)$, then $\sigma^{i\prime}(0) = (e_iZ_i, cY_i) \in T_{(p_1,p_2)}N_S^{CC}$ where $e_i < d$ and $|Z_i| = 1$. Thus

$$\gamma^{i\prime}(0) - \sigma^{i\prime}(0) = b_iX_i + cY_i - e_iZ_i - cY_i = b_iX_i - e_iZ_i \in T_{(p_1,p_2)}N_S^{CC},$$

which implies that $X_i - (e_i/b_i)Z_i$ is in $T_{(p_1,p_2)}M_1 \times M_2$, so that the subspace S' spanned by $\{X_i - (e_1/b_1)Z_1, \cdots, X_{r_1} - (e_{r_1}/b_{r_1})Z_{r_1}\}$ is contained in $T_{(p_1,p_2)}N_S^{CC}$. Since $|-e_i/b_i| < \varepsilon$, $S' \in A^1$ where S' is considered as a subspace of $T_{p_1}M_1$. By Lemma 3.3, $N_S^{CC} \cap M_1$ must contain $N_{S'}^{CC} = M_1$, so that $X_i \in T_{(p_1,p_2)}N_S^{CC}$. Thus $Y_i \in T_{(p_1,p_2)}N_S^{CC}$ and $S^2 \subset T_{(p_1,p_2)}N_S^{CC}$, and hence $M_2 \subset N_S^{CC}$. The result now follows. The exact same argument works with CC replaced with ATC or TC.

6. A geometric relationship

The purpose of this section is to prove the following result.

6.1. Theorem. *Let τ: $[0, 1] \to G^r(T_pM)$ be a piecewise C^∞ path such that $\tau(t)$ is in the image of f_{CC} (resp. TC, ATC). Then $\operatorname{Exp}_p(\tau(0))$ is isometric to $\operatorname{Exp}_p(\tau(t))$.*

We will first need a series of lemmas.

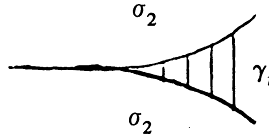
6.2. Lemma. *Let M be a complete riemannian manifold. Let $\sigma_1(t)$, $\sigma_2(t)$ be C^∞ curves in M such that $\sigma_1(0) = \sigma_2(0)$. Assume further that for each $t \in [0, 1]$ there is a topologically closed totally geodesic CC submanifold N_t such that*

(i) $\sigma_i(t) \in N_t$,

(ii) $\sigma_i'(t)$ *is perpendicular to* N_t.

Then $\sigma_1(t) = \sigma_2(t)$ for all t.

Proof. Let $A = \{t \in [0, 1] | \sigma_1(t) = \sigma_2(t)\}$. Clearly $A \neq \varnothing$ and A is closed. For t close to A, $\sigma_1(t)$ is not a cut point of $\sigma_2(t)$, so there exists a unique minimizing geodesic $\gamma_t(s)$ from $\sigma_1(t)$ to $\sigma_2(t)$:



Since $\sigma_i(t) \in N_t$ and N_t is completely convex, $\gamma_t(s) \in N_t$. Therefore by (ii), $\langle \gamma_t'(0), \sigma_1'(t) \rangle = \langle \gamma_t'(1), \sigma_2'(t) \rangle = 0$. The first variation formula allows us to conclude that A is open, and the lemma follows.

6.3. Lemma. *Let $N \subset M$ be a topologically closed totally geodesic submanifold of M. For $p \in N$, $A \in T_pN$, and $X \in T_AT_pM$ such that X is perpendicular to T_pN, we have* $\mathrm{Exp}_{p*} X$ *is perpendicular to N.*

Proof. If $\mathrm{Exp}_{p*} X = 0$, the result holds trivially. Otherwise, let $J(t)$ be the Jacobi field along $\mathrm{Exp}_p\, tA$ such that $J(0) = 0$, and $J'(0)$ is the translation of X to 0 in T_pM. Then $J(1) = \mathrm{Exp}_{p_*}(X)$, but since N is totally geodesic and both $J(0)$ and $J'(0)$ are perpendicular to N, we have $J(t)$ perpendicular to N, and the result follows.

6.4. Lemma. *Let F: $N^n \times I \to M^m$ $(n < m)$ be a C^∞ function where N is a smooth manifold and M is a complete riemannian manifold. Assume*:

(i) *F_t: $N \to M$ is a smooth embedding,*

(ii) *$F_t(N) \equiv N_t$ is totally geodesic* (*not necessarily closed*),

(iii) *for every $(p, t) \in N \times I$; $F_{*(p,t)}\partial/\partial t$ is perpendicular to N_t.*

Then N_0 is isometric to N_t in the induced metric.

Proof. For $p \in N$ and $X, Y \in T_pN$, define $X(t)$, $Y(t) \in T_{(p,t)}N \times I$ in the obvious way. We need only show $\partial/\partial t(F_t^*g)(X(t), Y(t)) = 0$ where F_t^*g is the pulled back metric.

Let $A = \{(p, t) \in N \times I | F_{*(p,t)}\partial/\partial t \neq 0\}$. By continuity it is sufficient to show that the above holds on A and on the complement of the closure of A.

Let (p, t) be in the complement of the closure of A. There are open sets $p \in U \subset N$ and $t \in V \subset I$ such that for all $(q, s) \in U \times V$, $F_*(q, s)\partial/\partial t = 0$. Therefore $F_{t_1}|_U = F_{t_2}|_U$ for $t_1, t_2 \in V$. Thus for $X(t)$, $Y(t)$ at (p, t) we have $d/dt(F_t^*g)(X(t), Y(t)) = 0$.

Let $(p, t) \in A$. There is an open set $U \times V \subset N \times I$ such that $F|_{U\times V}$ is a C^∞ embedding. Extend $X(t)$, $Y(t)$ on $U \times I$ such that $[X, \partial/\partial t] = [Y, \partial/\partial t] = 0$. Let X, Y and T be the vector fields on $F(U \times V)$ induced by F_*. Then we have $\partial/\partial t(F_t^*g)(X(t), Y(t)) = Tg(X, Y) = \nabla_T g(X, Y) = g(\nabla_T X, Y) + g(x, \nabla_T Y) = g(\nabla_X T, Y) + g(X, \nabla_Y T)$. On the other hand, N_t is totally geodesic, and X and Y are tangent to N_t, while T is perpendicular to N_t. Thus $\nabla_X T$ and $\nabla_Y T$ are perpendicular to N_t. Hence $\partial/\partial t(F_t^*g)(X(t), Y(t)) = 0$, and the lemma follows.

Proof of Theorem 6.1. Clearly we can assume that τ is C^∞. Let $ST^r(T_pM)$ be the stiefel manifold of orthonormal r-frames in T_pM with the normal homogeneous metric. Let $ST^r(T_pM) \xrightarrow{\pi} G^r(T_pM)$ be the principal bundle. Define a connection on π by taking as horizontal subspaces the subspace perpendicular to the fibre. Let $\tilde{\tau}(t)$ be any horizontal lift of $\tau(t)$. Let L: $\mathbf{R}^r \times I \to T_pM$ be $L(X, t) = \tilde{\tau}(t)X$ where $\tilde{\tau}(t)$: $\mathbf{R}^r \to T_pM$ is the orthogonal transformation induced by $\tilde{\tau}(t)$ as an element of $ST^r(T_pM)$. Since $\tilde{\tau}$ is horizontal, $L_{*(x,t)}\partial/\partial t$ is perpendicular to the linear subspace $\tau(t) = \tilde{\tau}(t)(\mathbf{R}^r) \subset T_pM$. Let $N_t = \operatorname{Exp}_p(\tau(t))$. By assumption, N_t is a topologically closed totally geodesic CC submanifold. By Lemma 6.3, $\operatorname{Exp}_{p*} L_{*(X,t)}\partial/\partial t$ is perpendicular to N_t for all $(x, t) \in \mathbf{R}^r \times I$.

We now define F: $N_0 \times I \to M$. For $q \in N_0 \subset M$ let $\tilde{q} \in T_pN_0$ be such that $\operatorname{Exp}_p \tilde{q} = q$. Let $F(q, t) = \operatorname{Exp}_p(L(\tilde{\tau}0)^{-1}(\tilde{q}), t)$. To show F is well defined let $\tilde{\tilde{q}}$ be another point in T_pN_0 such that $\operatorname{Exp}_p \tilde{\tilde{q}} = q$. Let $\sigma_1(t) = \operatorname{Exp}_p[L(\tilde{\tau}_{(0)}^{-1}(\tilde{q}), t)]$ and let $\sigma_2(t) = \operatorname{Exp}_p[L(\tilde{\tau}_{(0)}^{-1}(\tilde{\tilde{q}}), t)]$. Now $\sigma_1(0) = \sigma_2(0) = q$ and $\sigma_1(t), \sigma_2(t) \in N_t$. Further we have $\sigma_1'(t) = \operatorname{Exp}_{p*} L_{*(\tilde{\tau}_{(0)}^{-1}\tilde{q},t)}\partial/\partial t$ which is perpendicular to N_t. Similarly $\sigma_2'(t)$ is perpendicular to N_t. Lemma 6.2 now tells us that $\sigma_1(t) = \sigma_2(t)$, so F is well defined. A similar argument shows that F_t is $1-1$. F_t is clearly onto N_t as the image of F_t is $\operatorname{Exp}_p \tau(t) = N_t$.

If $q \in N_0$ is such that there is more than one minimizing geodesic in N_0 from p to q, then the image $F_t(q)$ will have more than one minimizing geodesic in N_t from p (this follows from the definition of F_t which takes geodesics from p to geodesics from p). Since the continuous function $L(-, t) \circ \tilde{\tau}_{(0)}^{-1} = F_{t*}$ takes ordinary tangent cut points in T_pN_0 to ordinary tangent cut points in T_pN_t and since the ordinary tangent cut points are dense in the tangent cut locus in T_pN_0 (see [1, p. 133]), F_{t*} takes tangent cut points in N_0 to cut points in N_t. Now by the definition we see that F_t is a diffeomorphism when restricted to the complement of the cut locus. Thus Lemma 6.4 tells us that F_t is an isometry when restricted to the complement of the cut locus to p.

In fact the theorem will follow from Lemma 6.3 if we show F_t is a diffeomorphism. This will follow if we show that F_t is a diffeomorphism when restricted to the complement of the cut locus to q for all q in a small neighborhood of p.

Let c be the distance from p to its cut locus in N_0 (and hence N_t). Let $q \in N_0$ be any point in the ball $B_{\frac{1}{3}c}(p)$. Let $q_t = F_t(q)$. Let B_t be the orthonormal frame at q_t obtained by the parallel translation of $\tilde{\tau}(t)$ along the unique minimizing geodesic from p to q_t. Note $B_t = F_{t*}B_0$ by the fact that F_t is a local isometry. Let L^q: $\mathbf{R}^r \times I \to TM$ be the transformation induced from the B_t's. For $x \in \mathbf{R}^r$, $|x| < \frac{1}{3}c$, we see by the local isometry of F_t that there is a $y \in \mathbf{R}^r$ such that for all $t \in I$, $\mathrm{Exp}_{q_t} L^q(x, t) = \mathrm{Exp}_p L(Y, t)$. Thus we have $\mathrm{Exp}_{q_t *} L^q_{*(x,t)} \partial/\partial t$ is perpendicular to N_t for all $x \in \mathbf{R}^r$ such that $|x| < \frac{1}{3}C$. Along the geodesic $\gamma_t(s) = \mathrm{Exp}_{q_t} L^q(sx, t) \subset N_t$ consider the field $\mathrm{Exp}_{q_t *} L^q_{*(x,t)} \partial/\partial t \equiv J(s)$. $J(s)$ is the variation field of the variation $\alpha(s, t) = \gamma_t(s)$ and thus is Jacobi. By the above for small s, $J(s)$ is perpendicular to N_t and is always so since N_t is totally geodesic. Thus $\mathrm{Exp}_{q_t *} L^q_{*(x,t)} \partial/\partial t$ is perpendicular to N_t for all $(x, t) \in \mathbf{R}^r \times I$.

Now define F_t^q from L^q as we defined F_t from L. All the facts about F_t now hold for F_t^q. In particular, F_t^q is a diffeomorphism when restricted to the complement of the cut locus to q in N_0. We need only show $F_t^q = F_t$. From the definition, $F_0^q = F_0$ (they correspond to the identity map on N_0). Let σ be in N_0. Let $\sigma_1(t) = F_t^q(\sigma)$ and $\sigma_2(t) = F_t(\sigma)$. From the above, $\sigma_i'(t)$ is perpendicular to N_t and $\sigma_i(t) \in N_t$, so the result follows from Lemma 6.2. The result follows for ATC and TC since the image of f_{ATC} or f_{TC} is contained in the image of f_{CC}.

Remark. Theorem 6.1 is false for TG. Let $M = S^1 \times \mathbf{R}$. Then all geodesics through a given point are TG (their images are closed). All but one are isometric to $\mathbf{R}$ while one is isometric to S^1.

Added in proof. The author has recently noticed that arguments similar to those in the second section of this paper show that if M^n is a manifold of small diameter such that $1 \geqslant K_M > 1/4$, then $|\pi_1(M)| > n$ and I_p injects into $\mathrm{Aut}(\pi_1(M))$ for all $p \in M$.

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