

SUBCARTESIAN SPACES

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Introduction

The notion of subcartesian spaces is a far reaching generalization of that of differentiable manifolds, and is designed explicitly to provide a framework for a study of manifolds with singularities. It includes as special cases piecewise manifolds, analytic and quasianalytic sets (in \mathbf{R}^n), and many others. Our motivation for studying subcartesian spaces has originated in the theory of differential problems on irregular domains and in the theory of Bessel potentials.

The basic concepts and results were introduced by the first author in [1] and independently, in a somewhat more restricted setting and with different motivation, in [9], [10]. Subcartesian spaces were also studied in [5], [6] where a theory of the de Rham cohomology was developed. Some elementary aspects of the theory were also summarized in [2] with the stress on subcartesian spaces of polyhedral type.

In the present paper we give a more detailed description of some basic aspects of the theory. In §1 we introduce the definitions of types, structures and subcartesian spaces as well as several examples. §2 deals to some extent with the question of uniqueness of structures determined on a topological space by an atlas and with metric aspects of subcartesian spaces. In §3 we study the concept of local dimension and some related topics. In §4 tangent spaces to a subcartesian space are introduced with a short discussion of tangent bundles of spaces with differentiable structures. A more detailed study of tangent bundles is left for another paper.

Due to the amount of time elapsed between the actual research and the preparation of this paper in its final form, some parts of the paper might have lost some of their novelty. Our attempts at establishing the priorities of some of the results and techniques in this paper might have not been as extensive as one would have desired.

1. Notations and basic definitions

Throughout this paper $(\mathfrak{R}, <)$ will denote a net of topological spaces with the order relation $<$ satisfying $K < M \Rightarrow K \subset M$ with homeomorphic inclusion.

The case of our main interest is when $\mathfrak{R} = \{\mathbf{R}^n\}_1^\infty = \mathbf{R}$ is the full sequence of cartesian spaces with the order relation defined by the natural inclusion; we will also consider the net $\{\mathbf{R}^k \times \mathbf{R}^l\}$ with partial ordering $\mathbf{R}^k \times \mathbf{R}^l < \mathbf{R}^m \times \mathbf{R}^n$ if $k < m, l < n$. It may also be of interest to consider nets of Banach or Hilbert spaces. We stress that the members of \mathfrak{R} are considered with fixed structures which are available on them, e.g., affine, metric, linear, etc., or as in the case of $\{\mathbf{R}^k \times \mathbf{R}^l\}$ the cartesian product structures.

Let X be a topological Hausdorff space. An \mathfrak{R} -atlas on X is a collection Φ of pairs (U_φ, φ) referred to as *charts* such that

(1.1) $\{U_\varphi\}_{\varphi \in \Phi}$ is an open cover of X ,

(1.2) φ is a homeomorphism of U_φ onto a subset $\varphi(U_\varphi)$ of a member $M = M_\varphi \in \mathfrak{R}$ determined by φ .

It should be stressed that $\varphi(U_\varphi)$ need not be open in M_φ . For simplicity we often write $\varphi = \{U_\varphi, \varphi\}$.

In the above context we refer to \mathfrak{R} as the *net of model spaces*, and to M_φ as the *coordinate space of the chart* φ .

A function in \mathfrak{R} is a continuous function f defined on an open set $\mathfrak{D}_f \subset M_f^1 \in \mathfrak{R}$ and the range in some $M_f^2 \in \mathfrak{R}$. A *homeomorphism in* \mathfrak{R} is a homeomorphism h between two open sets in an $M_h \in \mathfrak{R}$.

A *type* is a collection \mathfrak{H} of homeomorphisms in \mathfrak{R} satisfying the following conditions:

(1.3) $I_M \in \mathfrak{H}$ for every $M \in \mathfrak{R}$, I_M denoting the identity mapping on M .

(1.4) If $h_1, h_2 \in \mathfrak{H}$, then wherever defined, $h_1 \circ h_2 \in \mathfrak{H}$;

(1.5) (Local character of \mathfrak{H}), if $h \in \mathfrak{H}$, and $U \subset \mathfrak{D}_h$ is open, then $h|_U \in \mathfrak{H}$, $h|_U$ denoting the restriction of h to U . If h is a homeomorphism in \mathfrak{R} and for every $p \in \mathfrak{D}_h$ there is an open $U \subset \mathfrak{D}_h$ with $p \in U$ and $h|_U \in \mathfrak{H}$, then $h \in \mathfrak{H}$.

(1.6) If $h \in \mathfrak{H}$ then $h^{-1} \in \mathfrak{H}$.

The following definition is fundamental for our considerations.

An \mathfrak{R} atlas Φ on X defines on X a sub- \mathfrak{R} structure of type \mathfrak{H} (or \mathfrak{H} -structure) if for any two charts $\varphi, \psi \in \Phi$ and any $p \in U_\varphi \cap U_\psi$ there are a

neighborhood U of p in $U_\varphi \cap U_\psi$ and a homeomorphism $h \in \mathcal{K}$ such that $M_h \supset M_\varphi, M_\psi, \mathcal{D}_h \supset \varphi(U)$ and $h|_{\varphi(U)} = \psi \circ \varphi^{-1}|_{\varphi(U)}$.

The above property will be referred to as *local \mathcal{K} -extendability* (or *local extendability*) of connecting homeomorphisms of the atlas.

The space X endowed with an atlas Φ defining on X a structure of type \mathcal{K} is called a *sub- \mathcal{R} space* of type \mathcal{K} .

It is immediate to verify that if Φ defines on X a sub- \mathcal{R} structure of type \mathcal{K} , then so does any refinement of Φ . Two atlases defining on X structures of type \mathcal{K} are *compatible* (\mathcal{K} -*compatible*) if their union defines on X a structure of type \mathcal{K} . Similarly as in the case of manifolds, for any atlas Φ defining an \mathcal{K} -structure on X , there is a *maximal* atlas on X containing Φ and defining an \mathcal{K} -structure on X .

If $\mathcal{K}_1, \mathcal{K}_2$ are two types, then we say that \mathcal{K}_1 is *stronger* than \mathcal{K}_2 (or \mathcal{K}_2 is *weaker* than \mathcal{K}_1) if $\mathcal{K}_1 \subset \mathcal{K}_2$. Note that the intersection of any family of types is again a type—this is the weakest type stronger than all types of the family. If Φ defines on X a structure of type \mathcal{K} , then it also defines on X a structure of any weaker type. The strongest type is the *covering* type, here $\mathcal{K} = \mathcal{K}^c$ consists of all homeomorphisms of the form $I_{M|U}$ where $M \in \mathcal{R}$, and $U \subset M$ is open in M . The weakest type is the *topological* type, here \mathcal{K} consists of all homeomorphisms in \mathcal{R} , and the corresponding structures are called *topological* or C^0 -*structures*.

If X is a subset of some $M \in \mathcal{R}$, $X \subset M$, then X with its relative topology can be considered as a sub- \mathcal{R} space of covering type with the atlas consisting of a single chart $(X, I|_X)$ where $I|_X$ is the inclusion mapping. Note that different choices of M give rise to compatible atlases. As a consequence of the preceding remarks X can be viewed as a sub- \mathcal{R} space of any type.

We consider next some more specific classes of types and examples.

A type \mathcal{K} is *rigid* if for any h_1, h_2 with domains $\mathcal{D}_1, \mathcal{D}_2$ —open sets in the same $M \in \mathcal{R}$, and for any open $U \subset \mathcal{D}_1 \cap \mathcal{D}_2$ the condition $h_1|_U = h_2|_U$ implies $h_1|_{\mathcal{D}_1 \cap \mathcal{D}_2} = h_2|_{\mathcal{D}_1 \cap \mathcal{D}_2}$. Corresponding structures are referred to as *rigid structures*. An instance of rigid type occurs when \mathcal{R} is a net of topological vector spaces, and \mathcal{K} consists of linear or affine isomorphisms.

A type \mathcal{K} is *totally rigid* if for every $h \in \mathcal{K}$ with open domain D in M there is a unique $\tilde{h} \in \mathcal{K}$ with domain M such that $\tilde{h}|_{\mathcal{D}} = h$. Thus the type in the example above is actually not only rigid, but also totally rigid. Corresponding structures are called *totally rigid structures*. Rigidity conveys the concept of unique continuation of homeomorphisms in \mathcal{K} , total rigidity—that of unique continuation to the whole space containing the domain.

More examples will be given below.

If for every $M \in \mathfrak{R}$ the topology on M is given by a uniformity, and the inclusions are locally uniformly continuous, one can consider the type $\mathfrak{H}^{\text{unif.}} = \{h \in \mathfrak{H}^{\text{op}}; h, h^{-1} \text{ are locally uniformly continuous}\}$. A stronger Lipschitzian type can be introduced when the members of \mathfrak{R} are metric spaces and the inclusions are locally Lipschitzian mappings. In this case $\mathfrak{H} = \{h \in \mathfrak{H}^{\text{op}}; h, h^{-1} \text{ are locally Lipschitzian}\}$. Lipschitzian structures are also referred to as $C^{0,1}$ -structures.

We list next some examples of nets \mathfrak{R} which are or may be of interest in various applications.

(1) $\mathfrak{R} = \{\mathbf{R}^n\}_{n=0}^{\infty}$. Here, as already mentioned, we consider the spaces \mathbf{R}^n as subspaces of \mathbf{R}^{∞} of all sequences of real numbers, with natural inclusion $x = (x_1, \dots, x_n) \in \mathbf{R}^n \rightarrow (x_1, \dots, x_n, 0, 0 \dots) \in \mathbf{R}^{\infty}$. Functions and homeomorphisms in $\{\mathbf{R}^n\}$ are referred to as \mathbf{R} -functions, \mathbf{R} -homeomorphisms and sub- $\{\mathbf{R}^n\}$ structures are shortened to *sub- \mathbf{R} -structures* or *subcartesian structures*.

In the case when \mathfrak{R} consists of finite dimensional manifolds, a topological sub- \mathfrak{R} -structure on X gives rise to a sub- \mathbf{R} structure on X via coordinate charts of manifolds. Such structures are always uniform; in particular, the notion of uniform structure is superfluous in this instance.

$\mathfrak{R} = \{\mathbf{R}^k \times \mathbf{R}^l\}$ —in this case we choose not to identify $\mathbf{R}^k \times \mathbf{R}^l$ with \mathbf{R}^{k+l} and, as mentioned at the beginning, \mathfrak{R} is only partially ordered. The types of interest are *coupled types*; for instance, $\mathfrak{H}|_{\mathbf{R}^k \times \mathbf{R}^l}$ may consist of homeomorphisms of the form $(h, H) : (x, \xi) \in \mathfrak{D} \times \mathbf{R}^l \rightarrow (h(x), H(x)\xi)$ where \mathfrak{D} is an open set in \mathbf{R}^k , $h : \mathfrak{D} \rightarrow \mathbf{R}^k$ is a homeomorphism, and $H(x) \in GL(\mathbf{R}^l)$ for every $x \in \mathfrak{D}$. Additional conditions may be imposed on h and the mapping $x \rightarrow H(x)$.

We list next some examples of interesting sub- \mathbf{R} -structures.

C^k and $C^{k,1}$ structures. These arise from types consisting of homeomorphisms which together with their inverses are in C^k or, respectively, $C^{k,1}$ —the class of functions with (locally) Lipschitzian derivatives up to order k . A type stronger than $C^{0,1}$ -type is the *piecewise linear type*.

C^{∞} structures. \mathfrak{H} consists of all C^{∞} -homeomorphisms with C^{∞} inverses. Spaces with such structures were introduced in [3] under the name of differentiable spaces.

Real analytic type consists of all homeomorphisms analytic together with their inverses. This type is rigid.

A stronger (real entire) type is obtained if we consider homeomorphisms extendable to analytic homeomorphisms of the whole space (in which the original domain is open). This type is totally rigid.

There is a variety of types stronger than those listed above, of which we mention the *affine*, *linear*, *isometric* and *translation* types, each of them consisting of all the homeomorphisms described by the corresponding adjective and their restrictions to open subsets of their domains.

It is appropriate to mention at this state that an n -dimensional manifold can be regarded as a subcartesian space in more than one natural way. If Φ is an atlas defining on X a structure of an n -dimensional manifold, then we can consider the smallest type containing all the connecting homeomorphisms of Φ —this gives rise to the strongest structure defined by the atlas.

On the other hand, any embedding of X in \mathbf{R}^m , $m > n$, defines on X a covering structure and *a fortiori* any weaker structure.

(2) $\mathcal{R} = \{\mathbf{C}^n\}_{n=1}^\infty$. This sequence gives rise to *complex structures* or sub- \mathbf{C} structures. Again, the spaces \mathbf{C}^n are considered as naturally included in \mathbf{C}^∞ . Complex structures are of interest (i.e., are not superceded by sub- \mathbf{R} structures) only for sufficiently strong types, e.g., holomorphic type and various stronger types, e.g., algebraic type, entire type, complex affine type, complex linear type, etc. All these types are, of course, rigid; the last three are totally rigid.

(3) Additional examples of types are obtained if for \mathcal{R} we take the sequence of spheres S^n , $n = 1, 2, \dots$ or projective spaces \mathbf{P}^n , $n = 1, 2, \dots$. In the first case the type of interest consists of all the isometries, in the second the projectivities (with all restrictions to open sets). The corresponding structures are spherical and projective structures. Of course, for sufficiently weak types, the structures are superceded by sub- \mathbf{R} structures.

(4) To end this preliminary list of examples we mention the notions of *subhilbertian* and *subbanachian* structures arising when the net \mathcal{R} of model spaces consists of Banach or Hilbert spaces.

We discuss next types arising from classes of functions. If \mathcal{C} is a class of functions in \mathcal{R} , then we define

$$(1.7) \quad \mathcal{K}_{\mathcal{C}} = \{h; h \text{ is a homeomorphism in } \mathcal{R}, h, h^{-1} \in \mathcal{C}\}.$$

$\mathcal{K}_{\mathcal{C}}$ is a type provided \mathcal{C} satisfies the following conditions:

$$(1.8) \quad \text{For every } M \in \mathcal{R}, I_M \in \mathcal{C}.$$

$$(1.9) \quad \text{If } f_1, f_2 \in \mathcal{C}, \text{ then } f_1 \circ f_2 \text{ wherever defined, is also in } \mathcal{C}.$$

$$(1.10) \quad \text{If } f \in \mathcal{C}, \text{ then for every open } U \subset \mathcal{D}_f, f|_U \in \mathcal{C}.$$

Also if f is a function in \mathcal{R} , and for every $p \in \mathcal{D}_f$ there is an open $U \subset \mathcal{D}_f$ such that $f|_U \in \mathcal{C}$, then $f \in \mathcal{C}$.

It is natural to refer to structures of type $\mathcal{K}_{\mathcal{C}}$ as \mathcal{C} -structures; this is compatible with the terminology used in some of the examples above. For

$M, N \in \mathfrak{R}$, \mathcal{C} a class of functions in \mathfrak{R} , we will use the notation $\mathcal{C}(M, N)$ for the functions in \mathcal{C} with (open) domains in M and values in N .

If X is a sub- \mathfrak{R} space with a \mathcal{C} -structure given by an atlas Φ and $M \in \mathfrak{R}$, then $\mathcal{C}(X, M)$ is the class of all functions $f: X \rightarrow M$ such that

(1.11) The domain \mathfrak{D}_f of f is open in X ,

For every $x \in \mathfrak{D}_f$ there are a neighborhood U of x in \mathfrak{D}_f ,

(1.12) a chart $\varphi \in \Phi$ and a function $\tilde{f} \in \mathcal{C}$, such that $U \subset U_\varphi$
and $f \circ \varphi^{-1}|_{\varphi(U)} = \tilde{f}|_{\varphi(U)}$.

(1.12) is a *local extendability* condition.

We now turn to some general remarks concerning sub- \mathfrak{R} spaces.

If a sub- \mathfrak{R} atlas on X satisfies the local extendability condition, then it defines on X a sub- \mathfrak{R} structure of certain type. Since the connecting homeomorphisms may be extended in many different ways, the same atlas may define on X structures which are not comparable. Note, however, that an atlas together with a fixed set of local extensions of connecting homeomorphisms defines a structure of a strongest possible type (compatible with the atlas and the set of extensions).

To illustrate the preceding remarks we consider the following example.

Let $X = \{(x, y) \in \mathbf{R}^2; y = |x|\}$; on X we consider the atlas consisting of the chart $\varphi: (x, |x|) \in X \rightarrow x \in \mathbf{R}^1$ and the inclusion mapping $\psi: X \rightarrow \mathbf{R}^2$. The connecting homeomorphism $\varphi \circ \psi^{-1}: (x, y) \in X \rightarrow x$ can be extended to $h(x, y) = (x, y - |x|)$ which is piecewise linear. Thus Φ together with this extension defines a piecewise linear structure. On the other hand $\varphi \circ \psi^{-1}$ also has an extension of the form $h_1(x, y) = (x, g(x, y))$ where $g \in C^\infty(\mathbf{R}|\mathbf{R}^2 \setminus (0, 0))$ and $\partial g / \partial y > 0$ for $(x, y) \neq (0, 0)$. With this extension the atlas Φ defines on X a structure of type \mathfrak{H} where \mathfrak{H} consists of homeomorphisms which together with their inverses are in C^∞ except possibly at 0. These two structures are not comparable; however, one can still consider the smallest type containing both h and h_1 and the corresponding structure on X .

Another possibility is the existence of several atlases on X defining on X structures of the same type \mathfrak{H} , which, however, are not \mathfrak{H} compatible. This, of course, may occur already in the case of manifolds.

Finally, an atlas on X may not define any sub- \mathfrak{R} structure on X . Thus, the following questions are of interest:

1. To describe intrinsically all possible structures defined on X by an atlas.
2. To describe all possible structures of a given type \mathfrak{H} on a space X .
3. To give conditions on an atlas in order that it define a structure of some type.

If the spaces $M \in \mathfrak{R}$ are metrizable, then any sub- \mathfrak{R} space is locally metrizable—the description asked for in question 1 should involve only topological and metric properties of X .

The questions 1, 2, 3 will be discussed to some extent in §2.

In some cases the local extendability of homeomorphisms is a consequence of local extendability of functions. The following proposition was proved in [4] under somewhat different assumptions.

The proof given below is essentially the same with some obvious modifications.

Proposition 1.1. *Let G_1, G_2 be topological groups, and h be a homeomorphism $h: A_1 \subset G_1 \rightarrow G_2$ such that h and h^{-1} can be extended to continuous functions on open sets $W_1 \supset A_1$ and $W_2 \supset h(A_1)$. Identify G_1 and G_2 with the subsets $G_1 \times \{e_2\}, \{e_1\} \times G_2$ of the product $G_1 \times G_2$, e_i denoting the unit in $G_i, i = 1, 2$. Then h can be extended to a homeomorphism between open subsets of $G_1 \times G_2$.*

Proof. Let $f_1: W_1 \rightarrow G_2, f_2: W_2 \rightarrow G_1$ be continuous extensions of h, h^{-1} ; define $h_1: W_1 \times G_2 \rightarrow W_1 \times G_2$ and $h_2: G_1 \times W_2 \rightarrow G_1 \times W_2$ by

$$(1.10) \quad h_1(g_1, g_2) = (g_1, g_2 f_1(g_1)), h_2(g_1, g_2) = (g_1^{-1} f_2(g_2), g_2).$$

By direct inspection we verify that h_1, h_2 are homeomorphisms of $W_1 \times G_2$ and $G_1 \times W_2$ onto themselves. It follows that $\tilde{h} = h_2 \circ h_1$ is a homeomorphism of $h_1^{-1}(W_1 \times W_2)$ onto $h_2(W_1 \times W_2)$. If $g_1 \in A_1$, then $h_1(g_1, e_2) = (g_1, f_1(g_1))$ and $\tilde{h}(g_1, e_2) = (g_1^{-1} f_2(f_1(g_1)), f_1(g_1)) = (e_1, f_1(g_1)) = (e_1, h(g_1))$, and \tilde{h} is an extension of h . q.e.d.

The following corollaries take advantage of the special form of the extension \tilde{h} constructed in the preceding proof.

Corollary 1.2. *If G_1 and G_2 are metric groups, and functions f_1, f_2 in Proposition 1.1 are Lipschitzian (or locally Lipschitzian), then so are the homeomorphism \tilde{h} and its inverse \tilde{h}^{-1} .*

Corollary 1.3. *If $G_1 = \mathbf{R}^{n_1}, G_2 = \mathbf{R}^{n_2}$ (with additive group structures), and f_1, f_2 are of class $C^k, C^{k,1}$ or C^∞ , then so are \tilde{h} and \tilde{h}^{-1} .*

2. Uniqueness of topological and Lipschitzian structures.

Metric aspects of sub- \mathfrak{R} spaces

We consider here certain types for which it is possible, at least to some extent, to answer questions raised at the end of the last section. Suppose that the directed family \mathfrak{R} consists of metric groups, which are absolute retracts and have the property that for any $M, N \in \mathfrak{R}$ there is a $K \in \mathfrak{R}$ such that K

is isometrically isomorphic to $M \times N$. We then have

Theorem 2.1. *If X is locally compact and Φ is an \mathcal{R} -atlas on X , then Φ defines on X a C^0 -structure. In particular, any two \mathcal{R} -atlases on X define compatible C^0 -structures.*

Theorem 2.2. *If \mathcal{R} is a directed family of Hilbert spaces (e.g., $\mathcal{R} = \{\mathbf{R}^n\}$) and X is a metric space, then every Lipschitzian atlas defines on X a $C^{0,1}$ structure, and any two such atlases are $C^{0,1}$ compatible.*

Proof. If $p \in X$, $\varphi: U_\varphi \rightarrow M$, $\psi: U_\psi \rightarrow N$, $p \in U_\varphi \cap U_\psi$, then there is a compact neighborhood of p , $V \subset U_\varphi \cap U_\psi$. $\varphi(V) \subset M$, $\psi(V) \subset N$ are both compact and by Tietze extension theorem both $\varphi \circ \psi^{-1}: \psi(V) \rightarrow M$ and $\psi \circ \varphi^{-1}: \varphi(V) \rightarrow N$ can be extended to continuous functions defined on open subsets of M and N . Apply now Proposition 1.1 to obtain an extension of $\varphi \circ \psi^{-1}|_{\psi(V)}$ to a homeomorphism in $M \times N$ which can be identified with an element of \mathcal{R} .

The same proof remains valid in the case of Theorem 2.2 except for the following changes. V is chosen so that $\varphi, \psi|_V$ are Lipschitzian homeomorphisms; instead of Tietze's we use Kirszbruan's theorem (see [3], [8]).

We next consider some intrinsic conditions on the space X for existence on X of an \mathbf{R} -atlas and of a subcartesian structure. We begin with a definition: A *metric field* on a set X is a collection $(U_\varphi, d_\varphi)_{\varphi \in \Phi}$, where Φ is an indicial set, U_φ is open in X , $\cup_{\varphi \in \Phi} U_\varphi = X$, d_φ is a metric on U_φ , and

$$(2.1) \quad \begin{aligned} & \text{for any } \varphi, \psi \in \Phi \text{ with } U_\varphi \cap U_\psi \neq \emptyset \text{ the identity } I: U_\varphi \cap U_\psi \\ & \rightarrow U_\varphi \cap U_\psi \text{ is a homeomorphism from the metric space} \\ & (U_\varphi \cap U_\psi, d_\varphi) \text{ onto } (U_\varphi \cap U_\psi, d_\psi). \end{aligned}$$

For simplicity, we will often write $\{d_\varphi\}_{\varphi \in \Phi}$ or simply $\{d_\varphi\}$ instead of $\{U_\varphi, d_\varphi\}_{\varphi \in \Phi}$.

The case of interest is when X is a Hausdorff space, and the d_φ -topology on U_φ coincides with the X -topology. Then (2.1) is satisfied, X is locally metrizable and the metric field $\{d_\varphi\}$ is said to be *compatible* with the topology of X . Unless otherwise stated, we assume that the topology on X is defined by $\{d_\varphi\}$.

If \mathcal{R} is a directed set of metric spaces and Φ is an \mathcal{R} -atlas on X , then for $(\varphi, U_\varphi) \in \Phi$, $\varphi: U_\varphi \rightarrow M \in \mathcal{R}$, we set $d_\varphi(p, q) = d_M(\varphi(p), \varphi(q))$, d_M —the metric of M , $p, q \in U_\varphi$. In this way we get the metric field $\{d_\varphi\}$ induced by the atlas Φ .

We will show that certain structures given on X by an atlas Φ can be characterized by the properties of the induced metric field.

Let Φ be an indicial set, and $\{d_\varphi\}$ a metric field on X . We say that the field

is uniform if

(2.2) for $\varphi, \psi \in \Phi$ and every $p \in U_\varphi \cap U_\psi$ there is a neighborhood V of p in $(U_\varphi \cap U_\psi, d_\varphi)$ (or equivalently, $(U_\varphi \cap U_\psi, d_\psi)$), such that $V \subset U_\varphi \cap U_\psi$ and the identity mapping $V \subset (U_\varphi, d_\varphi) \rightarrow V \subset (U_\psi, d_\psi)$ is uniformly continuous. If “uniform” in the last condition is replaced by “Lipschitzian”, we say that the metric field $\{d_\varphi\}$ is Lipschitzian.

Proposition 2.1. *If \mathcal{R} is a directed family of metric spaces, and Φ an atlas on X defining on X a uniform topological structure, then the metric field $\{d_\varphi\}$ induced by Φ on X is uniform. If \mathcal{R} satisfies in addition the hypotheses in Theorem 2.1, members of \mathcal{R} are locally compact, and an \mathcal{R} -atlas Φ induces on X a uniform metric field, then Φ defines on X a topological \mathcal{R} -structure.*

Proof. If $U_\varphi \cap U_\psi \ni p$, then by the local extendability condition there is a both ways uniformly continuous homeomorphism h of a neighborhood V of $\varphi(p)$ in some $M \in \mathcal{R}$ onto a neighborhood V' of $\psi(p)$ in M such that $h|_{\varphi(U_\varphi) \cap V} = \psi \circ \varphi^{-1}|_{\varphi(U_\varphi) \cap V}$. Thus the identity from $(\varphi^{-1}(V), d_\varphi)$ onto $(\psi^{-1}(V'), d_\psi)$ can be written in the form $\psi^{-1} \circ h \circ \varphi$ which is uniformly continuous, φ, ψ being isometries.

On the other hand, if $p \in U_\varphi \cap U_\psi$, and $V_p \subset U_\varphi \cap U_\psi$ is a neighborhood of p such that $\text{id}(V_p, d_\varphi) \rightarrow (V_p, d_\psi)$ and $\text{id}(V_p, d_\psi) \rightarrow (V_p, d_\varphi)$ are uniformly continuous, then the connecting homeomorphisms $\varphi \circ \psi^{-1}: \psi(V_p) \rightarrow \varphi(V_p) \subset M_\varphi \in \mathcal{R}$, $\psi \circ \varphi^{-1}: \varphi(V_p) \rightarrow \psi(V_p) \subset M_\psi \in \mathcal{R}$ are uniformly continuous and can be extended to the closures $\overline{\psi(V_p)}, \overline{\varphi(V_p)}$ which can be assumed compact if M_φ, M_ψ are locally compact. Existence of an extension of $\varphi \circ \psi^{-1}$ to a homeomorphism of a neighborhood of $\psi(p)$ in some $M \in \mathcal{R}$, $M \supset M_\varphi, M_\psi$ follows now as in the proof of Theorem 2.1.

An analogous proposition can be stated concerning Lipschitzian metric fields. Remark next that if d and $\{d_\varphi\}$ are respectively a metric and a metric field on X , then it is meaningful to say that d and $\{d_\varphi\}$ are equivalent, uniformly equivalent or Lipschitz equivalent.

Our next objective is to show the following theorem.

Theorem 2.3. *Assume X is a paracompact (Hausdorff) space, and $\{d_\varphi\}$ a metric field on X . Then there is a metric d on X equivalent to $\{d_\varphi\}$. Moreover, if $\{d_\varphi\}$ is uniform or Lipschitzian, then d is uniformly or Lipschitz equivalent to $\{d_\varphi\}$.*

Proof. By paracompactness we can replace $\{d_\varphi\}$ by a refinement, denoted again by $\{d_\varphi\}$, with the following properties. (a) The cover $\{U_\varphi\}$ is locally finite, (b) for every $\varphi \in \Phi$ there is an open set V_φ such that $\overline{V_\varphi} \subset U_\varphi$ and

$\bigcup_{\varphi \in \Phi} V_\varphi = X$. Clearly this new field is equivalent (uniformly equivalent, Lipschitz equivalent) to the original one.

For $p \in X$ we introduce the following sets of indices φ :

$$(2.3) \quad \Phi_p = \{ \varphi \in \Phi; p \in \bar{V}_\varphi \}, \Phi'_p = \{ \varphi \in \Phi, p \in V_\varphi \}.$$

Clearly Φ_p is finite and $\Phi'_p \subset \Phi_p$. Let further

$$(2.4) \quad U_p = \left(\bigcap_{\varphi \in \Phi_p} U_\varphi \right) \cap \left(\bigcap_{\varphi \in \Phi'_p} V_\varphi \right) \setminus \bigcup_{\varphi \notin \Phi_p} \bar{V}_\varphi.$$

U_p is an open neighborhood of p . On U_p define two metrics

$$(2.5) \quad d_p(a, b) = \left(\sum_{\varphi \in \Phi_p} d_\varphi(a, b)^2 \right)^{1/2},$$

$$d'_p(a, b) = \left(\sum_{\varphi \in \Phi'_p} d_\varphi(a, b)^2 \right)^{1/2}, \quad a, b \in U_p.$$

We have $d'_p(a, b) \leq d_p(a, b)$, also the metric fields $\{U_p, d_p\}_{p \in X}$, $\{U_p, d'_p\}_{p \in X}$ are both equivalent (uniformly equivalent, Lipschitz equivalent) to the original field. For $a, b \in X$ let

$$(2.6) \quad d(a, b) = \inf \left\{ \sum_{k=0}^n d_{p_k}(a_k, a_{k+1}); \right. \\ \left. a = a_0, a_{n+1} = b, a_i, a_{i+1} \in U_{p_i}, i = 1, \dots, n \right\},$$

with the usual convention that $d(a, b) = \infty$ if there are no finite sequences $\{a_j\}, \{p_i\}$ with indicated properties.

We will show that d is a metric on X with the desired properties. The symmetry and the triangle inequality are obvious. Also if $(a, b) \in U_p$ for some $p \in X$ then

$$(2.6') \quad d(a, b) \leq d_p(a, b),$$

showing that d -topology on X is not stronger than $\{d_\varphi\}$ -topology.

We next show that for every $a \in X$ there is an $\varepsilon_a > 0$ such that

$$(2.7) \quad d(a, b) < \varepsilon_a \Rightarrow b \in U_a, d'_a(a, b) \leq d(a, b),$$

$$(2.8) \quad d(a, b) < \varepsilon_a, d(b, c) < \delta \leq \varepsilon_a \Rightarrow b, c \in U_a, d'_a(b, c) < \delta.$$

To this effect we note the following properties of the neighborhoods U_p :

$$(2.9) \quad q \in U_p \Rightarrow \Phi'_p \subset \Phi'_q, \Phi_q \subset \Phi_p.$$

In fact, $q \in U_p \Rightarrow q \in V_\varphi$ for all $\varphi \in \Phi'_p$, i.e., $\Phi'_p \subset \Phi'_q$. Also if $q \in U_p$ then $q \notin \bar{V}_\varphi$ for all $\varphi \notin \Phi_p$, i.e., $\Phi_p \subset \Phi_q$. As a consequence we note

$$(2.10) \quad s \in U_p \cap U_q \Rightarrow \Phi_s \subset \Phi_p \cap \Phi_q.$$

We choose now $\varepsilon_a > 0$ so that

$$(2.11) \quad \bigcap_{\varphi \in \Phi'_a} \{q \in U_\varphi; d_\varphi(a, q) < 2\varepsilon_a\} \subset U_a.$$

To prove (2.7) let $0 < \delta \leq \varepsilon_a$ and $d(a, b) < \delta$. By (2.5) we can find $a_0 = a, a_1, \dots, a_{n+1} = b, p_0, \dots, p_n$ such that $\sum_0^n d_{p_k}(a_k, a_{k+1}) < \delta$.

Since $a \in U_{p_0}$, we get by (2.9) $\Phi_a \subset \Phi_{p_0}$ and $d_\varphi(a, a_1) \leq d_a(a, a_1) \leq d_{p_0}(a, a_1) < \delta$ for all $\varphi \in \Phi_a$ (and consequently for all $\varphi \in \Phi'_a$); consequently $a_1 \in U_a$ (by (2.11)) and $d'_a(a, a_1) \leq d_a(a, a_1) < \delta$. Suppose that we have already shown that $a_1, \dots, a_k \in U_a, d_a(a, a_i) < \delta, 1 \leq i \leq k < n$. Using (2.9), (2.10), $a_i \in U_{p_i} \cap U_a$ implies $\Phi'_a \subset \Phi'_{a_i} \subset \Phi_{a_i} \subset \Phi_a \cap \Phi_{p_i}, i = 0, \dots, k$, in particular $\Phi'_a \subset \Phi_{p_0} \cap \dots \cap \Phi_{p_k}$. Hence for $\varphi \in \Phi'_a$ we have $\varphi \in \Phi_{p_k}, d_\varphi(a_k, a_{k+1}) \leq d_{p_k}(a_k, a_{k+1}) < \delta$ and, by the induction hypothesis and the triangle inequality,

$$d_\varphi(a, a_{k+1}) \leq d_\varphi(a, a_k) + d_\varphi(a_k, a_{k+1}) < d'_a(a, a_k) + \delta < 2\delta \leq 2\varepsilon_a$$

showing that $a_{k+1} \in U_a$. Also

$$\begin{aligned} d'_a(a, a_{k+1}) &= \left(\sum_{\varphi \in \Phi'_a} (d_\varphi(a, a_{k+1}))^2 \right)^{1/2} \\ &\leq \left(\sum_{\varphi \in \Phi'_a} \left(\sum_{i=0}^k d_\varphi(a_i, a_{i+1}) \right)^2 \right)^{1/2} \\ &\leq \sum_{i=0}^k \left(\sum_{\varphi \in \Phi'_a} d_\varphi(a_i, a_{i+1})^2 \right)^{1/2} \leq \sum_{i=0}^k d_{p_i}(a_i, a_{i+1}) < \delta \end{aligned}$$

since $\Phi'_a \subset \Phi_{p_i}, i = 0, \dots, k$. This proves (2.7).

To verify (2.8) choose $p_0, \dots, p_n, b = a_0, a_1, \dots, a_n = c$, so that $d(b, c) \leq \sum_{i=1}^n d_{p_i}(a_i, a_{i+1}) < \delta. d(a, b) < \varepsilon_a$ implies by (2.7) that $b \in U_a$ and $d'_a(a, b) < \varepsilon_a$; in particular (by (2.9)), $\Phi'_a \subset \Phi'_b$. Since $b, a_1 \in U_{p_0}$, we have $\Phi_b \subset \Phi_{p_0}$; in particular, $\Phi'_a \subset \Phi_{p_0}$ which implies $d'_a(b, a_1) \leq d_{p_0}(b, a_1) < \delta$. Thus for all $\varphi \in \Phi'_a, d_\varphi(a, a_1) \leq d_\varphi(a, b) + d_\varphi(b, a_1) < \varepsilon_a + \delta \leq 2\varepsilon_a$ implying that $a_1 \in U_a$. Suppose we have shown that $a_1, \dots, a_k \in U_a$ and $d'_a(b, a_k) < \delta$. Then $\Phi'_a \subset \Phi'_{a_1} \cap \dots \cap \Phi'_{a_k}$ and since $a_i \in U_{p_i}, \Phi_{a_i} \subset \Phi_{p_i}$ and $\Phi'_a \subset \Phi_{p_0} \cap \dots \cap \Phi_{p_k}$. Now, since $a_{k+1} \in U_{p_k}$ and $\Phi'_a \subset \Phi_{p_k}$, we can write for every $\varphi \in \Phi'_a$,

$$\begin{aligned} d_\varphi(a, a_{k+1}) &\leq d_\varphi(a, b) + d_\varphi(b, a_{k+1}) < \varepsilon_a + \sum_0^k d_\varphi(a_j, a_{j+1}) \\ &\leq \varepsilon_a + \sum_0^k d_{p_j}(a_j, a_{j+1}) < \varepsilon_a + \delta \leq 2\varepsilon_a \end{aligned}$$

implying $a_{k+1} \in U_a$ and

$$d'_a(b, a_{k+1}) = \left(\sum_{\varphi \in \Phi'_a} d_\varphi(b, a_{k+1})^2 \right)^{1/2} \\ \leq \left[\sum_{\varphi \in \Phi'_a} \left(\sum_{j=0}^k d_\varphi(a_j, a_{j+1})^2 \right) \right]^{1/2} < \sum_{j=0}^k d_{p_j}(a_j, a_{j+1}) < \delta$$

which proves (2.8).

(2.7) shows that d is a metric. (2.7), (2.6') imply that d and $\{d_\varphi\}$ define equivalent topologies, i.e., d and $\{d_\varphi\}$ are equivalent. (2.6') and (2.8) imply that for any $a \in X$ the identity on $W_a = \{p; d(a, p) < \epsilon_a\}$ is Lipschitz continuous from (W_a, d) to (W_a, d'_a) and from (W_a, d'_a) to (W_a, d) . Since, as already remarked, d_a, d'_a are uniformly or Lipschitz equivalent, if the original field $\{d_\varphi\}$ is uniform or Lipschitzian, d is uniform or Lipschitz equivalent to $\{d_\varphi\}$. q.e.d.

An obvious corollary to Theorem 2.3 is that a paracompact locally metrizable space is metrizable. This is, of course, well known and the emphasis of the theorem lies in its second part.

If Φ is an atlas defining on X a structure of a differentiable manifold, then the metric d of Theorem 2.1 constructed from the corresponding metric field is Lipschitz equivalent to the Riemannian metric on X induced by Φ .

We next consider the problem of completing spaces with sub- \mathfrak{R} structures.

Suppose that \mathfrak{R} is a directed family of complete metric spaces, and X is a paracompact space with a uniform topological sub- \mathfrak{R} structure defined by a locally finite atlas Φ , and denote by $\{d_\varphi\}$ the metric field induced on X by Φ . Let d be a metric on X uniformly equivalent to $\{d_\varphi\}$, as given by Theorem 2.3. For every $a \in X$ there is a neighborhood G_a of a in X such that

(2.12 i) $G_a \subset U_\varphi$ for all $\varphi \in \tilde{\Phi}_a = \{\varphi; a \in U_\varphi\}$,

(2.12 ii) $\varphi, \psi \in \tilde{\Phi}_a \Rightarrow \varphi \circ \psi^{-1}|_{\psi(G_a)}$ can be extended to a uniformly continuous homeomorphism h of an open neighborhood of $\overline{\psi(G_a)}$ in some $M \in \mathfrak{R}$.

(2.12 iii) For every $\varphi \in \tilde{\Phi}_a$ the identity $(G_a, d) \rightarrow (G_a, d_\varphi)$ is uniformly bicontinuous.

Let \tilde{X} be the abstract completion of X with metric d . It is then clear that for every $\varphi \in \tilde{\Phi}_a, \varphi: U_\varphi \rightarrow M_\varphi \in \mathfrak{R}$, the homeomorphism $\varphi|_{G_a}$ can be extended to a homeomorphism $\tilde{\varphi}$ of $\overline{G_a}$ (closure in \tilde{X}) onto $\overline{\varphi(G_a)}$. Also, for $\varphi, \psi \in \tilde{\Phi}_a, h$ in (2.12 ii) is an extension to a neighborhood of $\overline{\psi(G_a)}$ of

$\tilde{\varphi} \circ \tilde{\psi}^{-1}|_{\overline{\psi(G_a)}}$. Let $X_\Phi = \bigcup_{a \in X} \overline{G_a}^{int}$. Then X_Φ is open and dense in \tilde{X} , and the homeomorphisms $\tilde{\varphi}, \varphi \in \tilde{\Phi}$, form an \mathcal{R} -atlas on X_Φ defining a uniform topological structure. We refer to X_Φ as the *local completion* of X determined by the atlas $\tilde{\Phi}$.

Since, in the case when the members of \mathcal{R} are locally compact, it can be assumed that, in addition to (2.12), $\overline{\varphi(G_a)}$ is compact for all $\varphi \in \tilde{\Phi}_a$, it follows that X_Φ is locally compact. We refer to X_Φ as *local compactification* of X induced by the atlas $\tilde{\Phi}$.

Uniformly equivalent atlases (or in locally compact case, equivalent atlases) do not lead to the same local completion or local compactification. However, it is easy to see that they lead to *equivalent* local completions or local compactifications in the following sense.

Two local completions $(X_\Phi, d), (X_{\Phi'}, d')$ of X are *equivalent* if there are open dense subsets $Y \subset X_\Phi, Y' \subset X_{\Phi'}$ such that $X \subset Y, X \subset Y'$, and the identity on X can be extended to a locally uniform homeomorphism of (Y, d) onto (Y', d') . This together with Theorem 2.1 leads to the following proposition:

Proposition 2.2. *If \mathcal{R} satisfies the hypotheses of Theorem 2.1 and consists of locally compact spaces, then two \mathcal{R} -atlases on X define the same topological structure on X if and only if they induce equivalent local compactifications of X .*

In the case when $\mathcal{R} = \{\mathbf{R}^n\}$ (or \mathcal{R} is a family of separable Hilbert spaces) it is possible to characterize at least some structures on a space X by means of metric fields.

A metric d on a set E is said to be Euclidean if for some n there is an isometry of (E, d) onto a subset of \mathbf{R}^n . We recall the following result of K. Menger [7].

Let d be a metric on a set E , and define, for $p_0, \dots, p_k \in E$, the $(k + 2) \times (k + 2)$ determinant:

$$(2.13) \quad D(p_0, \dots, p_k) = \begin{vmatrix} 0 & 1 & & \dots & 1 \\ 1 & & & & \\ & \{d(p_i, p_j)^2\}_{i,j=0}^k & & & \\ \vdots & & & & \\ \vdots & & & & \\ 1 & & & & \end{vmatrix}.$$

Proposition 2.3. *d is a Euclidean metric on E if and only if $\text{sign } D(p_0, \dots, p_k) = (-1)^k$ or 0. If $n = \max\{k; D(p_0, \dots, p_k) \neq 0 \text{ for some } p_0, \dots, p_k \in E\}$, then (E, d) is isometric to a subset of \mathbf{R}^n .*

For $p_0, \dots, p_n \in \mathbf{R}^m$ the determinant $D(p_0, \dots, p_k)$ is equal to $(-1)^k k! 2^k \times (\text{volume of the simplex } (p_0, \dots, p_k))^2$.

If Φ is a sub- \mathbf{R} atlas on X , then $\{d_\varphi\}$ —the metric field induced by Φ —is Euclidean, and the metric on X defined by (2.6) is equivalent to a Euclidean metric field. On the other hand, if X is a metric space with a metric d equivalent to a Euclidean metric field, then there is on X an \mathbf{R} -atlas. This leads to the following proposition.

Proposition 2.4. *In order that a metric space X admit an \mathbf{R} -atlas, it is necessary and sufficient that the metric be equivalent to a Euclidean metric field.*

Given a Euclidean metric field on X , $\{d_\varphi\}$ say, it is possible to construct an \mathbf{R} -atlas on X either by use of Proposition 2.3 or by explicit formulas as follows.

We may assume, choosing a refinement of $\{d_\varphi\}$ if necessary, that $\{d_\varphi\}$ has the following property: for every φ , U_φ is either infinite or consists of a single point. If $U_\varphi = \{p\}$, we set $\varphi(p) = 0 \in \mathbf{R}^0$. If U_φ is infinite, then by hypothesis about d_φ being Euclidean there is an isometry $i_\varphi: U_\varphi \rightarrow \mathbf{R}^n$. Also, we can find n , $p_0, \dots, p_n \in U_\varphi$, such that $D(p_0, \dots, p_n) \neq 0$ and $D(q_0, \dots, q_k) = 0$ for $k > n$ and $q_0, \dots, q_k \in U_\varphi$. We can assume that $n = n_\varphi$. We now define $\varphi(p_0) = 0 \in \mathbf{R}^n$, and for every $p \in U_\varphi$

$$(2.14) \quad (\tilde{\varphi}(p))_k = \frac{1}{2}(d_\varphi(p, p_0)^2 + d_\varphi(p_k, p_0)^2 - d_\varphi(p, p_k)^2), \quad k = 1, \dots, n.$$

If we let $i_\varphi(p_k) - i_\varphi(p_0) = v_k$, then the vectors $v_k \in \mathbf{R}^n$ are linearly independent and $(\varphi(p))|_k = \langle i_\varphi(p) - i_\varphi(p_0), v_k \rangle_n$, showing that $\varphi: U_\varphi \rightarrow \mathbf{R}^n$ is a homeomorphism differing from i_φ by an affine isomorphism $\varphi(p) = Ai_\varphi(p) - Ai_\varphi(p_0)$, where A is determined by the condition $A'e_k = v_k$, $\{e_k\}$ being the standard basis in \mathbf{R}^n .

A sub- \mathbf{R} atlas on X may or may not define on X a sub- \mathbf{R} structure. We have the following.

Proposition 2.5. *In order that a sub- \mathbf{R} atlas Φ define on X a sub- \mathbf{R} structure it is necessary and sufficient that the metric field induced by the atlas be uniform.*

Proof. The necessity of the condition was already noted. The sufficiency follows from the observation that the uniformity of the metric field implies that for any $\varphi, \psi \in \Phi$ and $p \in U_\varphi \cap U_\psi$ there is a neighborhood U of p such that $\varphi \circ \psi^{-1}$ can be extended to a homeomorphism of $\overline{\psi(U)} (\subset \mathbf{R}^n)$ onto $\overline{\varphi(U)} (\subset \mathbf{R}^n)$, and without loss of generality we can assume that $\overline{\psi(U)}, \overline{\varphi(U)}$ are compact, and then we can apply Proposition 1.1.

Remark 1. The statement of Proposition 2.5 remains valid for Lipschitzian structures if the condition of uniformity of the metric field is replaced by its Lipschitz character.

Remark 2. A metric space admits a subcartesian structure if and only if the metric is uniformly equivalent to a Euclidean metric field. The structure is Lipschitzian if and only if the metric is Lipschitz equivalent to a Euclidean metric field.

Remark 3. (2.14) may define on X a structure stronger than the topological one. Since connecting homeomorphisms can be considered as mapping between closed subsets of \mathbf{R}^n (for some n), the results of the kind of Whitney's theorem together with Proposition 1.1 can be used to check if the structure is e.g. differentiable.

3. Tangential charts, local dimension, homogeneous and regular parts of a subcartesian space

Let X be a subcartesian space with structure of type \mathcal{C} given by a maximal atlas Φ . The *local dimension* of X at $p \in X$ is defined by

$$(3.1) \quad \dim_p X = \min\{n_\varphi; \varphi \in \Phi, p \in U_\varphi\}.$$

Note that $\dim_p X$ depends on the type \mathcal{C} . If $X = \{(x_1, x_2); x_2 = |x_1|\} \subset \mathbf{R}^2$ with the covering structure defined by inclusion, then $\dim_p X = 2$ for every $p \in X$. In any differentiable structure with \mathcal{C} containing linear mappings $\dim_p X = 1$ for $p \in X, p \neq 0, \dim_0 X = 2$, but in Lipschitzian structure $\dim_p X = 1$ for every $p \in X$.

Any chart φ for which the minimum in (3.1) is attained is called a *tangential chart* at p . It is obvious that such chart always exists, the atlas Φ being maximal.

Remark. The notion of a tangential chart can be defined in a more general setting: a chart $\varphi: U_\varphi \rightarrow M, p \in U_\varphi$, in a maximal \mathcal{R} -atlas Φ is tangential at p if for any other chart $\psi \in \Phi$ at $p, \psi: U_\psi \rightarrow N$, we have $M < N$. In such settings, however, there is no reason for tangential charts to exist in general.

Proposition 3.1. *The function $p \rightarrow \dim_p X$ is upper semicontinuous.*

Proof. If φ is a tangential chart at $p: U_\varphi \rightarrow \mathbf{R}^n$, then $\dim_q X < n = \dim_p X$ for every $q \in U_\varphi$. q.e.d.

A point $p \in X$ is a *point of homogeneity* of X if there is a neighborhood U of p in X such that $\dim_q X = \dim_p X$ for all $q \in U$. The set of all points of homogeneity of X is the *homogeneous part* of X ; its complement is the *nonhomogeneous part* of X .

Proposition 3.2. *The homogeneous part of X is an open dense subset of X .*

Proof. It is obvious that the homogeneous part is open. The second assertion follows easily from Proposition 3.1. q.e.d.

The homogeneous part of X is the union of disjoint open subsets of X on which the local dimension is constant; these are referred to as *homogeneous components* of X .

The above concepts depend, of course, on the choice of the particular structure on X . For the sake of illustration we consider some examples.

Example 1. Let $X \subset \mathbf{R}^3$ consist of the boundary of the cube with vertices $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, $\varepsilon_i = 0, 1$, and of its four diagonals. The inclusion $X \subset \mathbf{R}^3$ defines on X a covering structure and a fortiori any subcartesian structure.

With the covering structure $\dim_p X = 2$ for every p in the interior of the face lying in the plane $x_3 = 0$. The homogeneous part of X is the union of the above set and the set of all points with the third coordinate $x_3 > 0$. The nonhomogeneous part is the boundary of the face lying in $x_3 = 0$.

With the topological structure $\dim_p X = 1$ if p is in the interior of any of the segments joining $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ to a vertex, $\dim_p X = 3$ if p is a vertex and $\dim_p X = 2$ otherwise. The nonhomogeneous part consists of the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and the vertices.

The same is true in $C^{0,1}$ -structure.

With a C^1 -structure the nonhomogeneous part consists of $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, the vertices and edges all of local dimension 3, the local dimension at points of homogeneity is as in the topological structure.

Example 2. We consider an example of a simple arc in $\mathbf{R}^\infty = \bigcup \mathbf{R}^n$ with unbounded local dimension. Let $f(t)$ be a periodic piecewise linear function of the real variable of period 1 with $f(0) = f(1) = 0$, $f(1/3) = f(2/3) = 1/3$. For $n \geq 1$ and $k = sn + r$, $r = 1, \dots, n$, $s = 0, 1, \dots$, we define $A_k \subset \mathbf{R}^{n+1}$ as $A_k = \{t, x(t); t \in [2^{-k-1}, 2^{-k}]; x = (x_1^k(t), \dots, x_n^k(t))\}$ where $x_l^k(t) = 2^{-(k+1)}f(2^{k+1}t)$ if $l = r$, $x_l^k(t) = 0$, $l \neq r$. Let $A = \bigcup A_k \cup \{0\}$. Then A is a Lipschitzian simple arc in \mathbf{R}^{n+1} . With the covering structure $\dim_0 A = n + 1$, we claim that the same is true for any differentiable structure. To see this we use the following lemma, which will be proved later (Proposition 4.3).

Lemma. Let $A \subset \mathbf{R}^n$ have C^k -structure induced by inclusion and $p \in A$. Then $\dim_p A = n$ if and only if for every C^k -function defined in a neighborhood U of p we have $f|_{U \cap A} = 0 \Rightarrow Df(p) = 0$.

The condition of the lemma is immediately verified in the present case.

Patching in $\bigcup \mathbf{R}^n$ the arcs A as constructed above for $n \rightarrow \infty$ one easily gets a simple Lipschitzian graph of unbounded local dimension. The $C^{0,1}$ -dimension of the arc is 1 at every point.

Example 3. This example is meant to illustrate the point that in general we cannot expect the connecting homeomorphism between two charts φ, ψ of

an atlas giving a subcartesian structure on X to be extendable to a homeomorphism with open domain in \mathbf{R}^n with $n = \max(n_\varphi, n_\psi)$.

Let $A \subset \mathbf{R}^3$ be the set consisting of $(0, 0, 0)$ and circles A_n with centers at points $(2^{-n}, 0, 0)$ and radii 2^{-n-1} ; for n even A_n is taken in the plane $x_3 = 0$ and for n odd in $x_2 = 0$. Thus A_n 's are pairwise linked. On A we consider the atlas consisting of the inclusion $A \subset \mathbf{R}^3$ and the chart φ which is defined by $\varphi|_{A_n} = \text{identity}$ for even n and, for odd n , maps isometrically A_n onto the circle about $(-2^{-n}, 0, 0)$ radius 2^{-n-1} in the plane $x_3 = 0$ (e.g., reflection in the x_3 -axis and rotation by $\pi/2$ about x_1 -axis). Note that φ is Lipschitzian with a constant < 9 , and φ^{-1} is Lipschitzian with constant < 1 . By Proposition 1.1, φ can be extended to a Lipschitzian homeomorphism of an open neighborhood of A in \mathbf{R}^5 ; i.e., the atlas defines a $C^{0,1}$ -structure on A . However, φ cannot be extended to a homeomorphism of any neighborhood of 0 in \mathbf{R}^3 (this would unlink the circles in \mathbf{R}^3). This shows, by the way, that the above atlas does not define on A any differentiable structure (see Proposition 4.6).

Example 4. The same as Example 3 except that instead of circles we consider sufficiently thin tori. Then the charts defined in Example 3 are actually tangential. The remainder of the discussion can be repeated verbatim.

Let X be subcartesian space with a differentiable structure given by an atlas Φ . A point $p \in X$ is said to be *regular* if there is a neighborhood $U \subset X$ of p such that $\Phi|_U = \{\varphi|_U; \varphi \in \Phi\}$ defines on U a structure of a differentiable manifold. The set of regular points of X is the *regular part* of X . Points of X which are not regular are called singular points.

In Example 1, with differentiable structure, the regular part of X coincides with the homogeneous part.

If $X = \{(x_1, x_2) \in \mathbf{R}^2; |x_2| \leq |x_1|\}$, then for any structure the regular part of X is the set $\{(x_1, x_2) \in \mathbf{R}^2; |x_2| < |x_1|\}$ but the homogeneous part of X is X . It is clear from the definition that the regular part of X is open and contained in the homogeneous part of X . It is also easily seen that the regular part may be empty.

The concept of regular part is of special interest in the case of subcartesian spaces of polyhedral type introduced and discussed in [2]. If X is a space of polyhedral type, then its regular part X_1 is dense in X . At least locally, X can be written as $X = X_1 \cup X_2 \cup \cdots \cup X_N$ where X_i is the regular part of $X_i \cup X_{i+1} \cup \cdots \cup X_N$ for $i = 1, \dots, N$.

In the case of coupled structures introduced in §1, it is meaningful to introduce the notion of coupled dimension, $\dim_p X$; this is a pair of nonnegative integers (m, n) such that there is a chart φ at p with $\varphi(U_\varphi) \subset \mathbf{R}^m \times \mathbf{R}^n$

and for any chart ψ at p , $\psi(U_\psi) \subset \mathbf{R}^k \times \mathbf{R}^l$ we have $k \geq m$, $l \geq n$. This notion does not seem to be very useful.

4. Differentiable structures and tangent spaces

In this section we consider structures defined by a class \mathcal{C} of functions in \mathbf{R} (satisfying (1.8), (1.9), (1.10)) contained in C^1 . To simplify the presentation we will actually restrict the statements of the main result to the cases when $\mathcal{C} = C^k$, and indicate more general cases in remarks.

If $f \in C^1(\mathbf{R}^m, \mathbf{R}^k)$; then by $Df(x; \xi)$ we denote the differential of f at x with increment ξ . Similar notation will be used later with reference to higher order differentials and partial differentials. Thus we write $D^2f(x; \xi; \eta)$, etc.

For a class \mathcal{C} of functions and a set $A \subset \mathbf{R}^m$ we will denote by \mathcal{N}_A the set of all $f \in \mathcal{C}$ such that f is defined in an open neighborhood of A and $f|_A = 0$. There should be no confusion caused by suppressing the symbol \mathcal{C} in the notation \mathcal{N}_A .

We will use the following version of Proposition 1.1.

Proposition 4.1. *If $h: A \subset \mathbf{R}^k \rightarrow B \subset \mathbf{R}^l$ is a homeomorphism of A onto B such that both h and h^{-1} can be extended to C^k -functions defined on neighborhoods respectively of A and B , then h can be extended to a C^k -homeomorphism \tilde{h} of a neighborhood of A in $\mathbf{R}^k \times \mathbf{R}^l$ onto a neighborhood of B in $\mathbf{R}^k \times \mathbf{R}^l$.*

Remark. Proposition 4.1 remains valid for more general classes \mathcal{C} ; for instance, the following conditions are sufficient:

(4.1) \mathcal{C} contains all affine mappings.

(4.2) If $f_i: U_i \subset M_i \rightarrow N_i$ are in \mathcal{C} , $i = 1, 2$, then also the function $(x_1, x_2) \in U_1 \times U_2 \rightarrow (f(x_1), f(x_2)) \in N_1 \times N_2$ is in \mathcal{C} .

The following version of the inverse function theorem will be useful in the discussion of tangent bundle.

Proposition 4.2. *Let $Y \subset \mathbf{R}^m$, $f \in C^k(\mathbf{R}^m, \mathbf{R}^n) \cap \mathcal{N}_Y$, $k \geq 1$, and, for $\tilde{x} \in Y$, set $N_{\tilde{x}} = \{\tilde{x} + \xi; Df(\tilde{x}, \xi) = 0\}$ —an affine subspace of \mathbf{R}^m . Then there is a C^k -homeomorphism h of a neighborhood U of \tilde{x} into \mathbf{R}^m such that $h(Y \cap U) \subset N_{\tilde{x}}$.*

Proof. We can set $\tilde{x} = 0$. If $N_0 = \mathbf{R}^m$, we can take $h = I_{\mathbf{R}^m}$. Otherwise, note that $Df(0): N_0^\perp \rightarrow Df(0)(\mathbf{R}^m) \subset \mathbf{R}^n$ is one to one and onto; denote by T the linear mapping $Ty = Df(0)^{-1}y$ if $y \in Df(0)\mathbf{R}^m$, $Ty = 0$ if $y \in Df(0)(\mathbf{R}^m)^\perp$, and by P the orthogonal projection of \mathbf{R}^m onto N_0 . Define

$$(4.3) \quad h(x) = Px + Tf(x).$$

Then $h(x) \in N_0$ if $x \in Y$ and, by the definition of T , $Dh(0; \xi) = P\xi + TDf(0)\xi = P\xi + TDf(0)P\xi + (I - P)\xi = P\xi + (I - P)\xi = \xi$. It follows that $Dh(0)$ is invertible and the result follows.

Remark. Proposition 4.2 is valid if C^k is replaced by any class $\mathcal{C} \subset C^1$ satisfying (4.1), (4.2)—under these conditions it can be asserted that $h \in \mathcal{C}$, but not necessarily that $h^{-1} \in \mathcal{C}$. The latter is true if \mathcal{C} satisfies, in addition, the condition:

$$(4.4) \quad \text{If } h \text{ is a homeomorphism in } \mathcal{C} \text{ and } h^{-1} \in C^1, \text{ then } h^{-1} \in \mathcal{C}.$$

The content of Proposition 4.2 and of its corollaries remains also valid in the context of Banach spaces: in this case we have to assume that $N_{\bar{x}}$ is complemented, and the range of $Df(\bar{x})$ is closed and complemented.

We list now some consequences of Proposition 4.2.

Corollary 1. *The restriction $P|_Y$ of the projection P is 1-1 and can be extended to a C^k -homeomorphism of some neighborhood of \bar{x} .*

This is an obvious consequence of (4.3).

Proposition 4.3. *Let $Y \subset \mathbf{R}^m$, and consider Y as a subcartesian space with the C^k -structure induced by the inclusion mapping. Then $\dim_p Y = m$ if and only if $Df(p) = 0$ for every $f \in \mathcal{N}_Y$.*

Proof. The necessity is immediate from Corollary 1—if $Df(p) \neq 0$ for some $f \in \mathcal{N}_Y$, then $\dim\{\xi; Df(p, \xi) = 0\} < m$, and the corollary provides a chart at p of dimension lower than m . To prove sufficiency assume that the condition of the proposition is satisfied, but $\dim_p Y = n < m$. Then there are a chart $\varphi: Y \rightarrow \mathbf{R}^n$ (we replace Y by $Y \cap U$ where U is a neighborhood of p) and a C^k -homeomorphism h in \mathbf{R}^N , $N \geq m$ such that $h|_Y = \varphi$. If $h(x) = (h_1(x), \dots, h_N(x))$, then $h_{n+1}|_Y = \dots = h_N|_Y = 0$ implying that $Dh_{n+1}(p; \xi) = \dots = Dh_N(p; \xi) = 0$ for every $\xi \in \mathbf{R}^m$ and $\dim Dh(p)\mathbf{R}^m < m$. On the other hand, since h is a C^k -homeomorphism, $\text{rank } Dh(p) = N - a$ contradiction. q.e.d.

We pass now to the notion of the tangent spaces to a subcartesian space of class C^k . Several equivalent definitions are possible; we choose to begin with one which seems to be the most expedient at the moment. The considerations remain valid for more general structures as indicated in the remark to Proposition 4.2.

Let X be a subcartesian space with a C^k -structure given by a maximal atlas Φ and $p \in X$. We first define the representatives of the tangent space to X at p in charts at p . If φ is a tangential chart at p , $\varphi(U_\varphi) \in \mathbf{R}^m$, then

$$T_p^\varphi X = \mathbf{R}^m.$$

Suppose that ψ is any chart at p , $\psi(U_\psi) \subset \mathbf{R}^l$, φ is as above, and h is any

C^k -homeomorphism extending $\psi \circ \varphi^{-1}$ to a neighborhood of $\varphi(p)$ in \mathbf{R}^n , $n \geq l$, then we set

$$(4.5) \quad T_p^\psi X = Dh(\varphi(p))\mathbf{R}^m.$$

We observe that $Dh(\varphi(p))\mathbf{R}^m \subset \mathbf{R}^l$, and $Dh(\varphi(p))\xi$, $\xi \in \mathbf{R}^m$, is independent of the choice of the extension h of $\psi \circ \varphi^{-1}$ at $\varphi(p)$. Both observations are immediate consequences of Proposition 4.2.

Also, directly from the definition, $T_p^\psi X = Dh(\varphi(p))T_p^\varphi X$ for any charts $\varphi, \psi \in \Phi$ at p , φ not necessary tangential and any h as above.

This allows us to define the equivalence relation: If $\xi \in T_p^\varphi X$, $\eta \in T_p^\psi X$ where $\varphi, \psi \in \Phi$ are any charts at p , then $\xi \sim \eta$ if and only if $\eta = Dh(\varphi(p); \xi)$ for some (and therefore every) extension h of $\psi \circ \varphi^{-1}$ to a C^k -homeomorphism of a neighborhood of $\varphi(p)$ in some \mathbf{R}^n . We define now the tangent space to X at p as the space of equivalence classes:

$$(4.6) \quad T_p X = \cup \{ T_p^\varphi X; \varphi \in \Phi, p \in U_\varphi \} / \sim.$$

$T_p X$ is clearly a vector space, and

$$(4.7) \quad \dim T_p X = \dim_p X.$$

We list now some properties of the tangent spaces.

For a set $Y \subset \mathbf{R}^m$, $y \in Y$ and a class \mathcal{C} of differentiable functions satisfying e.g. (1.8) (1.9) (1.10) define (see [9])

$$(4.8) \quad T_y Y = \{ \xi \in \mathbf{R}^m; Df(y; \xi) = 0 \text{ for all } f \in \mathcal{N}_{Y \cap U_y}, \\ U_f \text{-a neighborhood of } y \}.$$

If h is a \mathcal{C} -homeomorphism in \mathbf{R}^m , then it is immediate to check that $T_{h(y)}h(Y) = Dh(y)T_y Y$. Also if \mathcal{C} satisfies (4.1), then the definition of $T_y Y$ is independent of the choice of m such that $Y \subset \mathbf{R}^m$.

If, for every k , $\mathcal{C}|_{\mathbf{R}^k}$ is a group with pointwise addition, then for every $f, g \in \mathcal{C}|_{\mathbf{R}^m}$ such that $f|_{Y \cap U_y} = g|_{Y \cap U_y}$ for some neighborhood of y , $Df(y, \xi) = Dg(y; \xi)$ for every $\xi \in T_y Y$.

As an immediate consequence of Proposition 4.3, we get with $\mathcal{C} = C^k$:

Proposition 4.4. *If X is a subcartesian space (with C^k -structure), $\varphi: U \rightarrow \mathbf{R}^m$ is a chart, $p \in U$, then $T_p^\varphi X = T_{\varphi(p)}\varphi(U)$.*

Proposition 4.4 shows that the space T_p^φ depends only on the image $\varphi(U_\varphi)$ and the class C^k defining the structure. Thus we can and will from now on assume that X is a subset of \mathbf{R}^n with the C^k -structure determined by the inclusion.

Proposition 4.5. *Let $X \subset \mathbf{R}^n$ be as above, $0 \in X$, $\dim_0 X = m$; denote by $Q: \mathbf{R}^n \rightarrow \mathbf{R}^n$ a rotation such that $QT_0 X = \mathbf{R}^m$, and by P the orthogonal*

projection $P: \mathbf{R}^n \rightarrow T_0X$. Then $QP: U \cap X \rightarrow \mathbf{R}^m$ is a tangential chart at 0 for some neighborhood U of 0 in \mathbf{R}^n .

Proof. (4.7) and Proposition 4.4 imply that if P is a chart at p (compatible with the inclusion chart), then P is tangential at p . Thus it is sufficient to show that $P|_{X \cap U}$ can be extended to a C^k -homeomorphism of a neighborhood U of p .

We can assume that $p = 0$ and $T_0X = \mathbf{R}^m \subset \mathbf{R}^n$. Let φ be any tangential chart at 0, and h be a C^k -homeomorphism extending φ^{-1} to a neighborhood V of $\varphi(0)$ in \mathbf{R}^N , $N \geq n$.

We can assume that $\varphi(0) = 0$. Write $h(x) = (h_1(x), \dots, h_N(x))$ and for $x \in \mathbf{R}^N$, $x = (x', x'', x''')$, $x' \in \mathbf{R}^m$, $x'' \in \mathbf{R}^{n-m}$, $x''' \in \mathbf{R}^{N-n}$. By (4.5) we have $Dh(0)\mathbf{R}^m = \mathbf{R}^m$, and it follows that $x' \rightarrow h'(x', 0, 0) \equiv h'(x')$ is a C^k -homeomorphism of a neighborhood V' of 0 in \mathbf{R}^m onto another such neighborhood U' . Define now for $y \in U = P^{-1}(U') \cap h(V) \subset \mathbf{R}^n$: $g(y) = y' + y'' - h''(h^{-1}(y'))$. Clearly g is a C^k -homeomorphism of U . Also if $y \in X \cap U$, then $y = h(x') = h'(x') + h''(x')$ for some $x' \in V \cap \mathbf{R}^m$, $y' = h'(x')$, $y'' = h''(x') = h''(h^{-1}(y'))$ and $g(y) = y' = Py$. q.e.d.

A similar argument gives rise to the following proposition.

Proposition 4.6. *If X is a sub- \mathbf{R} space with C^k -structure, φ, ψ are two charts of the atlas, $p \in U = U_\varphi \cap U_\psi$, $\varphi(U) \subset \mathbf{R}^n$, $\psi(U) \subset \mathbf{R}^l$, then there is a C^k -homeomorphism g extending $\psi \circ \varphi^{-1}$ to a neighborhood of $\varphi(p)$ in \mathbf{R}^N , $N = \max(l, n)$.*

Proof. Assume that $l \geq n$, and also following φ and ψ by suitable affine isomorphisms that $\varphi(p) = 0$, $\psi(p) = 0$, and $T_p^\varphi X = \mathbf{R}^m = T_p^\psi X$. Let h be a C^k -homeomorphism extending $\psi \circ \varphi^{-1}$ to a neighborhood of 0 in \mathbf{R}^N , $N \geq \max(l, n)$. Then, as in the preceding proof, $h'(x') = Ph(x')$, $x' = Px$ is a C^k -homeomorphism of a neighborhood of 0 in \mathbf{R}^m , and P denotes the projection $P: \mathbf{R}^N \rightarrow \mathbf{R}^m$. By Proposition 4.5 the restrictions $P: \varphi(U) \subset \mathbf{R}^n \subset \mathbf{R}^l \rightarrow \mathbf{R}^m$, $P: \psi(U) \subset \mathbf{R}^l \rightarrow \mathbf{R}^m$ can be extended to C^k -homeomorphisms of neighborhoods of 0 in \mathbf{R}^l , which we denote by g_1, g_2 . The homeomorphism h' can be extended in \mathbf{R}^l by the formula $g_3(x', x'') = h'(x') + x''$, $x'' \in \mathbf{R}^{n-l}$. It is obvious now that $g = g_2^{-1} \circ g_3 \circ g_1$ has the desired properties. q.e.d.

We remark that the result of Proposition 4.6 is not valid for classes \mathcal{C} which either do not satisfy conditions (4.1), (4.2) or are not differentiable. The example illustrating the failure of the proposition because of the second reason is Example 3 of §3.

An example of a differentiable structure in which the proposition is not true can be obtained as follows.

Let $X = \mathbf{R}^1$, and suppose that the atlas Φ consists of two charts φ, ψ such

that $U_\varphi = (-a, \infty)$, $U_\psi = (-\infty, a)$, $a > 0$ and $\varphi: U_\varphi \rightarrow \{(x_1, x_2) \in \mathbf{R}^2; x_1 = x_2 > -a/2\}$ $\psi: U_\psi \rightarrow \{(x_1, x_2) \in \mathbf{R}^2, x_1 = -x_2 < a/2\}$ are natural isometries. Consider the least type \mathcal{H} containing the rotations in \mathbf{R}^3 about the lines $\{x_1 = x_2, x_3 = 0\}$ and $\{x_1 = -x_2, x_3 = 0\}$ by $\pi/2$. Then Φ defines on X a structure of type \mathcal{H} ; both charts φ and ψ are tangential, but there is no extension of $\psi \circ \varphi^{-1}$ to an \mathcal{H} -homeomorphism in \mathbf{R}^2 .

We end this section with the definition and some remarks about the tangent bundle to a subcartesian space with a C^k -structure. Again, the definition and the remarks remain valid for more general differentiable structures, in particular, for those satisfying (4.1), (4.2), (4.4).

Let X be a subcartesian space with a C^k -structure given by a maximal atlas Φ . For every $p \in X$, $u \in T_p X$ (see (4.6)), and every chart $\varphi \in \Phi$ at p , we define $D\varphi(p; u) = \xi$, where ξ is the element of T_p^φ in the equivalence class u . Clearly $D\varphi(p): T_p X \rightarrow T_p^\varphi$ is a linear isomorphism.

The tangent bundle TX of X is the set

$$TX = \cup \{(p, u); p \in X, u \in T_p X\} = \bigcup_{p \in X} \{p\} \times T_p$$

with the topology defined by the atlas $\Phi_* = \{\varphi_*; \varphi \in \Phi\}$ where φ_* is the chart with the domain $\cup \{\{p\} \times T_p X; p \in U_\varphi\} = TU_\varphi$; and if $\varphi: U_\varphi \rightarrow \mathbf{R}^m$, then $\varphi_*: TU_\varphi \rightarrow \mathbf{R}^m \times \mathbf{R}^m$ with $\varphi_*(q, v) = (\varphi(q), D\varphi(q)v)$ for $q \in U_\varphi$, $v \in T_p X$.

The atlas Φ_* defines on TX a coupled structure of a rather special kind. The net of model spaces is the sequence $\{\mathbf{R}^n \times \mathbf{R}^n\}$, each space being considered with its cartesian product structure. For $l > 0$ consider the class of all the homeomorphisms \tilde{h} of the form:

$$\tilde{h}(x, \xi) = (h(x), H(x)\xi)$$

where for some $n = n(\tilde{h})$, \tilde{h} is a C^l -homeomorphism of an open set $\Omega \subset \mathbf{R}^n$ into \mathbf{R}^n , and $H \in C^l(\Omega, GL(\mathbf{R}^n))$. It is clear that this class of homeomorphisms is a type. We refer to it as $C^l L$ -type (or type $C^l L$).

A C^k -atlas Φ , as above, defines on TX (via Φ_*) a structure of type $C^{k-1} L$. If $\varphi, \psi \in \Phi$, $U_\varphi \cap U_\psi = U \neq \emptyset$, and h is a C^k -homeomorphism extending $\psi \circ \varphi^{-1}$ to a neighborhood Ω in \mathbf{R}^n of a point in $\varphi(U)$, then $(h, Dh): \Omega \times \mathbf{R}^n \rightarrow \mathbf{R}^n \times \mathbf{R}^n$ is a $C^{k-1} L$ -homeomorphism extending locally $\psi_* \circ \varphi_*^{-1}$.

The name tangent bundle collides somewhat with the standard usage—the fibers $T_p X$ of TX are not, in general, of constant dimension. In cases of possible confusion the word pseudobundle is more suitable, [5].

As already noted in §1, it is useful to consider the type $C^l L$ in a more general setting. The net of model spaces is $\mathbf{R}^m \times \mathbf{R}^n$, and homeomorphisms \tilde{h}

are of the form $\tilde{h} = (h, H)$ where $h: \Omega \subset \mathbf{R}^m \rightarrow \mathbf{R}^m$ is a C^l -homeomorphism, Ω open, and $H \in C^l(\Omega, GL(\mathbf{R}^n))$, $m = m(\tilde{h})$, $n = n(\tilde{h})$.

Associated with the type C^lL is the class C^lL consisting of functions f defined on sets of the form $\Omega \times \mathbf{R}^n$, Ω open in \mathbf{R}^m with values in \mathbf{R}^k , $m = m(f)$, $n = n(f)$, $k = k(f)$, which are linear in the second variable and of class C^l in the first.

If $F \subset \mathbf{R}^m \times \mathbf{R}^n$ is the set of the form $F = \cup_{x \in Y} \{x\} \times F_x$ where Y is a subset of \mathbf{R}^m and F_x is a subspace of \mathbf{R}^n for every $x \in Y$, then by $\mathcal{N}_F (= \mathcal{N}_F \cap C^lL)$ we denote the class of all functions f in C^lL defined on sets of the form $\Omega \times \mathbf{R}^n$ where $\Omega = \Omega(f)$ is a neighborhood of Y and such that $f|_F = 0$.

The sets of the above form occur in particular as images of charts of Φ_* :

$$\varphi_*TU_\varphi = T^\varphi\varphi(U_\varphi) = \cup \{ \{ \varphi(p) \} \times T_p^{(\varphi)}; p \in U_\varphi \}.$$

With the above notations we state now the upper semicontinuity property of the bundle TX :

$$(4.9) \quad \text{If } f(\varphi(p), \xi) = 0 \text{ for all } f \in \mathcal{N}_{T^\varphi\varphi(U_\varphi)}, \text{ then } \xi \in T_p^\varphi.$$

This is an immediate consequence of Proposition 4.4. (4.8) implies that for every sequence $p_r \in X, p_r \xrightarrow{r \rightarrow \infty} p$ and every chart $\varphi \in \Phi$ at p we have

$$(4.10) \quad \xi_r \in T_{p_r}^\varphi, \lim_{r \rightarrow \infty} \xi_r = \xi \text{ imply } \xi \in T_p^\varphi.$$

(4.10) implies in turn that

$$(4.11) \quad \begin{aligned} &\text{the function } p \rightarrow \dim T_pX \text{ is upper semicontinuous which} \\ &\text{also is an immediate consequence of (4.7) and Proposition 3.1.} \end{aligned}$$

A more detailed discussion of the bundle TX will be given in another paper devoted to the calculus on subcartesian spaces with differentiable structures.

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