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MORSE-SMALE SINGULARITIES IN SIMPLE MECHANICAL SYSTEMS

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Among the simplest of smooth dynamical systems are the so-called gradient-like Morse-Smale flows. These systems feature only a finite number of equilibrium points, and all other orbits of the system tend toward and away from one of these points. As a consequence, there are no closed or nontrivially recurrent orbits. All the complicated or pathological behavior associated with many physical systems are absent from these systems, and we may take a gradient-like Morse-Smale system as being completely understood.

On the other hand, among the most complicated dynamical systems are the Hamiltonian systems which arise in various branches of mechanics. Here, all of the orbits of the system are recurrent (at least in the compact case), and one generally expects random and highly unstable behavior. Furthermore, in practice, such systems are often complicated by the existence of singularities–that is, points where the differential equation itself "blows up" or is otherwise undefined. A good example of such a singularity is a collision of the various particles in the Newtonian n-body problem. The behavior of solutions which lead to or even which come close to such a singularity is often quite erratic and is not very well understood.

Our goal in this paper is to study this complicated behavior of a mechanical system near a singularity by reducing the problem to the study of a gradient-like Morse-Smale system. More precisely, we shall replace the singularity by a smooth compact manifold which we call the singularity manifold. After a suitable change of time scale, the scaled vector field will extend smoothly to the singularity manifold. Orbits which previously ended or began at the singularity will now tend asymptotically toward or away from the singularity manifold. Furthermore, the behavior of orbits which previously came close to the singularity will now be governed by the induced flow on the singularity manifold. Thus an important problem is to understand the phase portrait of this extended flow. Our main result is that, in general, the induced flow is quite simple.

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Theorem A. *For an open and dense set of simple mechanical systems with an isolated singularity of order k at the origin in* $\mathbf{R}^n$, *the induced flow on the singularity manifold is a gradient-like Morse-Smale flow.*

We remark that the gradient-like Morse-Smale systems always have sinks and sources. In contrast, Hamiltonian systems never admit such solutions, since they are volume preserving. Hence our extended system is no longer Hamiltonian. However, away from the singularity manifold, the orbit structure of the extended system is identical to the original system (up to a change of time scale along orbits). Thus the singularity manifold consists entirely of fictitious orbits, introduced only as a device for understanding solutions which pass close to the singularity.

The construction of the singularity manifold is due to R. McGehee, who used this method in [7] to describe orbits which come close to triple collision in the Newtonian three-body problem. Our results owe much to the methods developed in this paper.

When the induced flow on the singularity manifold is gradient-like and Morse-Smale, we shall call the singularity itself Morse-Smale. An important problem thus becomes: how does one recognize when a singularity of a mechanical system is Morse-Smale. Below, we shall relate the qualitative features of the flow on the singularity manifold to the potential energy of the system. Since the potential energy of the system is always known, this gives an effective means of determining when the singularity is Morse-Smale.

Our results along this line may be summarized as follows. Let S denote the unit sphere in configuration space in the metric determined by the kinetic energy. Let f denote the restriction to S of the principal part of the potential energy (to be defined below). Then we have

Theorem B. *The flow on the singularity manifold is a gradient-like Morse-Smale flow if*

1. f *has* 0 *as a regular value,*

2. f *is a Morse function on the set* $\{s \in S \,|\, f(s) \leqslant 0\}$,

3. *the stable and unstable manifolds of the equilibrium points in the singularity manifold meet transversely.*

Conditions 1 and 2 above are immediately verifiable in any mechanical system. Unfortunately, condition 3 is not as obvious and must be checked in each individual case.

The construction of the singularity manifold below also allows us to determine the structure of the orbits which end or begin at the singularity. We call such orbits (forward or backward) collision orbits. In general, the set of such orbits can be fairly complicated. But, for Morse-Smale singularities, at least, we have

Theorem C. *Suppose a simple mechanical system has a Morse-Smale singularity at the origin in* $\mathbf{R}^n$. *Then the set of collision orbits consists of a union of m submanifolds of varying dimensions, where m is the number of critical points of f on S.*

Moreover, in §8 below, we shall relate the dimensions of these various manifolds to the Morse indices of the critical points of f.

The structure of this paper is as follows. In §§1–3, we describe the basic construction of the singularity manifold and the extension of the flow. In §4 we give a class of examples to which this construction applies: the central force problems. We also describe the flow on the singularity manifold in some detail in the special case of the Newtonian central force or Kepler problem.

In §§5–6 below, we relate the structure of the equilibrium solutions in the singularity manifold to the potential energy of the system. Theorem B is proved in §5 and Theorem A in §6.

The anisotropic Kepler problem is the main topic of §7. This is a nonintegrable classical mechanical system which is of some recent physical interest. We include this as a particularly illuminating example of a Morse-Smale singularity.

Finally, in §8, we discuss collision orbits in more detail. It is here that we complete the proof of Theorem C.

1. Basic definitions

In this paper, we shall be primarily concerned with the simple mechanical systems of the form $\mathbf{q}'' = A^{-1} \operatorname{grad} V(\mathbf{q})$, where A is an $n \times n$ positive definite symmetric matrix, and $\mathbf{q} = (q_1, \ldots, q_n)$ is a point in $\mathbf{R}^n$. Equivalently, one may introduce the momentum variable $\mathbf{p} = A\mathbf{q}'$, and write this system as a first order system of differential equations on $\mathbf{R}^{2n}$:

$$\mathbf{q}' = A^{-1}\mathbf{p}, \quad \mathbf{p}' = -\nabla(\mathbf{q}). \tag{1.1}$$

This is a Hamiltonian system on $\mathbf{R}^n$. That is, if one introduces the Hamiltonian or total energy function

$$H(\mathbf{q}, \mathbf{p}) = K(\mathbf{p}) + V(\mathbf{q}), \tag{1.2}$$

where $K(\mathbf{p}) = \frac{1}{2}\mathbf{p}^t A^{-1}\mathbf{p}$, then (1.1) may be written in the form

$$\mathbf{q}' = \frac{\partial H}{\partial \mathbf{p}}, \quad \mathbf{p}' = \frac{-\partial H}{\partial \mathbf{q}}. \tag{1.3}$$

$V(\mathbf{q})$ is called the potential energy of the system, and $K(\mathbf{p})$ the kinetic energy.

It is well known that the Hamiltonian is a first integral for systems of the form (1.3). That is, H is constant along any solution of the system. Thus, instead of studying the full $2n$-dimensional system, one may restrict attention to the codimension-one level sets of H. These level sets are invariant under the flow of (1.3), and are called energy surfaces. We denote $H^{-1}(e)$ by Σ_e. If e is a regular value of H, then it follows from the Implicit Function Theorem and Σ_e is a smooth submanifold of $\mathbf{R}^{2n}$. We shall restrict our attention to a single such energy surface in the sequel.

As we mentioned above, we wish to consider Hamiltonian systems which admit singularities in the equations of motion. Such singularities occur whenever the potential energy suffers a singularity. We assume throughout that V has an isolated singularity at $\mathbf{q} = \mathbf{0}$, that is, $\lim_{\mathbf{q}\to\mathbf{0}} V(\mathbf{q})$ fails to exist, while V is smooth in a punctured neighborhood of the origin.

This assumption excludes some of the most complicated singularities in mechanics–for example, k-fold collision in the n-body ($k \geqslant 3$) problem is a nonisolated singularity in the above sense. Nevertheless, many important systems do admit isolated singularities, and their structure is already complicated enough to warrant special attention.

We shall say that V has a singularity or pole of order k at the origin if

$$V(\mathbf{q}) = F(\mathbf{q}) + N(\mathbf{q}), \tag{1.4}$$

where F is a homogeneous function of degree $-k$, and N contains higher order terms, i.e., $r^k N(\mathbf{q})$ extends smoothly over $\mathbf{0}$, assuming the value 0 there. Equivalently, if we choose "polar coordinates" $(r, \mathbf{s})$ on $\mathbf{R}^n$, where $r \in [0, \infty)$, $\mathbf{s} \in S^{n-1}$, then we may write $r^k V(r\mathbf{s}) = F(\mathbf{s}) + O(r)$ where F is now regarded as a smooth function on the $(n - 1)$-sphere in $\mathbf{R}^n$. We call F the *principal part* of the potential energy.

We finally list several important mechanical systems to which our analysis below applies. Each of these examples admits an isolated singularity or order k at the origin.

1. Central force problems. Here the Hamiltonian assumes the form

$$\frac{1}{2}|\mathbf{p}|^2 - \frac{1}{k|\mathbf{q}|^k},$$

where $|\cdot|$ denotes the standard norm in $\mathbf{R}^n$. The structure of these systems near the singularity is quite different depending on whether $k > 2$ or $k < 2$.

2. The anisotropic Kepler problem. Here the Hamiltonian is given by $(1/2)\mathbf{p}^t A^{-1}\mathbf{p} - 1/|\mathbf{q}|$, where A is an $n \times n$ diagonal matrix with positive entries. If A is the identity matrix, we have the usual (Newtonian) central force or Kepler problem. If all of the entries along the diagonal are distinct, however, then the orbit structure changes quite dramatically. Our analysis below gives a good picture of how orbits behave near the origin.

2. The singularity manifold

The goal of this section is to replace a singularity or order k by a smooth compact manifold. In the next section, we shall show how to extend the system over this manifold to get a well-defined vector field without singularities.

Let $r = (\mathbf{q}^t A\mathbf{q})^{1/2}$. Let S denote the unit sphere in $\mathbf{R}^n$ in the metric defined by A, i.e.,

$$S = \{\mathbf{s} \in \mathbf{R}^n | \mathbf{s}^t A\mathbf{s} = 1\}. \tag{2.1}$$

We think of r, $\mathbf{s}$ as defining polar coordinates in the configuration space of the system. We also introduce the variables

$$y = \mathbf{s}^t\mathbf{p}, \quad \mathbf{x} = A^{-1}\mathbf{p} - y\mathbf{s}. \tag{2.2}$$

Since $\mathbf{s}^t A\mathbf{x} = 0$, $\mathbf{x}$ may be regarded as the component of momentum tangent to S, and y as the radial component of momentum. We also think of the pair $(\mathbf{s}, \mathbf{x})$ as defining a point in the tangent bundle to S.

In these coordinates, the system (1.1) assumes the form

$$\begin{aligned} r' &= y, \\ y' &= \frac{1}{r}\mathbf{x}^t A\mathbf{x} - \mathbf{s}^t \nabla V(r\mathbf{s}), \\ \mathbf{s}' &= \frac{1}{r}\mathbf{x}, \\ \mathbf{x}' &= -\frac{1}{r}\left[y\mathbf{x} + (\mathbf{x}^t A\mathbf{x})\mathbf{s}\right] - A^{-1}\nabla V(r\mathbf{s}) + \left[\mathbf{s}^t \nabla V(r\mathbf{s})\right]\mathbf{s}. \end{aligned} \tag{2.3}$$

We regard (2.3) as a vector field on the open manifold $(0, \infty) \times \mathbf{R} \times TS$, where TS denotes the tangent bundle to S. This system still has singularities, but the singularity set has been enlarged. Now, the system is singular along the entire boundary $r = 0$.

We now scale the momentum variables by a factor of $r^{k/2}$. That is, let

$$\mathbf{u} = r^{k/2}\mathbf{x}, \quad v = r^{k/2}y. \tag{2.4}$$

The differential equation (2.3) goes over to

$$\begin{aligned} r' &= r^{-k/2}v, \\ v' &= r^{-1-k/2}\left[\mathbf{u}^t A\mathbf{u} + (k/2)v^2\right] - r^{k/2}\left[\mathbf{s}^t \nabla V(r\mathbf{s})\right], \\ s' &= r^{-1-k/2}\mathbf{u}, \\ \mathbf{u}' &= r^{-1-k/2}\left[-(\mathbf{u}^t A\mathbf{u})\mathbf{s} + (-1 + k/2)v\mathbf{u}\right] \\ &\quad + r^{k/2}\left[-A^{-1}\nabla V(r\mathbf{s}) + (\mathbf{s}\nabla V(r\mathbf{s})\mathbf{s})\right], \end{aligned} \tag{2.5}$$

where the energy relation becomes

$$e = r^{-k}\left[(1/2)\mathbf{u}^t A\mathbf{u} + (1/2)v^2\right] + V(r\mathbf{s}). \tag{2.6}$$

The singularity at $r = 0$ may now be removed by the change of time scale

$$dt = r^{1+k/2}\, d\tau. \tag{2.7}$$

If we let a dot indicate differentiation with respect to τ, the scaled system becomes

$$\begin{aligned}
\dot r &= rv,\\
\dot v &= (k/2)v^2 + \mathbf{u}^t A\mathbf{u} - r^{k+1}\mathbf{s}^t\nabla V(r\mathbf{s}),\\
\dot{\mathbf{s}} &= \mathbf{u}\\
\dot{\mathbf{u}} &= (-1 + k/2)v\mathbf{u} - (\mathbf{u}^t A\mathbf{u})\mathbf{s} + r^{k+1}\left[\mathbf{s}^t\nabla V(r\mathbf{s})\mathbf{s} - A^{-1}\nabla V(r\mathbf{s})\right].
\end{aligned} \tag{2.8}$$

Using (1.4) we may write

$$\nabla V(r\mathbf{s}) = r^{-k-1}\nabla F(\mathbf{s}) + \nabla N(r\mathbf{s}),$$

and using Euler's formula also gives

$$\mathbf{s}^t\nabla V(r\mathbf{s}) = r^{-k-1}\left[-kF(\mathbf{s}) + \mathbf{s}^t\nabla N(r\mathbf{s})\right].$$

Hence the system (2.8) may be written more simply as

$$\begin{aligned}
\dot r &= rv,\\
\dot v &= (k/2)v^2 + \mathbf{u}^t A\mathbf{u} + kF(\mathbf{s}) + O(r),\\
\dot{\mathbf{s}} &= \mathbf{u},\\
\dot{\mathbf{u}} &= (-1 + k/2)v\mathbf{u} - (\mathbf{u}^t A\mathbf{u})\mathbf{s} - kF(\mathbf{s})\mathbf{s} - A^{-1}\nabla F(\mathbf{s}) + O(r).
\end{aligned} \tag{2.9}$$

The system (2.9) has no singularity at $r = 0$. In fact, (2.9) is well-defined on $r = 0$, and moreover, since $\dot r = rv$, it follows that $r = 0$ is invariant under the flow induced by (2.9). Hence this new system may be regarded as a smooth vector field on all of the manifold with boundary $[0, \infty) \times \mathbf{R} \times TS$.

Now the energy integral (2.6) also extends to the boundary, and gives

$$0 = r^k e = \tfrac{1}{2}\mathbf{u}^t A\mathbf{u} + \tfrac{1}{2}v^2 + F(\mathbf{s}) + O(r). \tag{2.10}$$

If the right-hand side of this expression has 0 as a regular value, then the relation (2.10) defines a smooth codimension-one submanifold of the boundary $r = 0$. In this case, we denote this submanifold by Λ; Λ is called the *singularity manifold* of the system.

The following proposition gives a criterion for Λ to be a smooth submanifold in terms of the potential energy of the system.

Proposition 2.1. *If* 0 *is a regular value of the restriction of* F *to* S, *then* Λ *is a smooth submanifold of* $r = 0$.

Proof. Let $g: \mathbf{R} \times TS \to \mathbf{R}$ be given by

$$g(v, \mathbf{s}, \mathbf{u}) = \tfrac{1}{2}(v^2 + \mathbf{u}^t A\mathbf{u}) + F(\mathbf{s}). \tag{2.11}$$

Note that $g^{-1}(0) = \Lambda$. So Λ is a submanifold of $r = 0$ if 0 is a regular value of g. We shall show that if 0 is a regular value of the restriction of F to S, then 0 is also a regular value for g.

To see this, assume that g has a critical point at $(v_0, \mathbf{s}_0, \mathbf{u}_0)$ and $g(v_0, \mathbf{s}_0, \mathbf{u}_0) = 0$. Differentiating (2.11) shows that $dF(\mathbf{s}_0) = 0$, and also that $\mathbf{u}_0 = 0$, $v_0 = 0$. Hence we must have $F(\mathbf{s}_0) = 0$. If 0 is a regular value of F, then we must have $dF(\mathbf{s}_0) \neq 0$. This contradiction proves that 0 is a regular value of g. q.e.d.

In case 0 is a regular value of F on S, we say that the singularity at the origin is *nondegenerate*. It is well known that this condition holds for an open and dense subset of the smooth functions on S, and so we have shown that most singularities of order k are nondegenerate.

In case the singularity is nondegenerate, the topology of Λ is easy to describe. Recall first that $TS \times \mathbf{R}$ is diffeomorphic to $S \times \mathbf{R}^n$ in a natural way. If $F(\mathbf{s})$ is strictly negative on S, then the energy relation (2.10) gives

$$\tfrac{1}{2}(\mathbf{u}^t A\mathbf{u} + v^2) = -F(\mathbf{s}) \tag{2.12}$$

in $r = 0$. This defines an ellipsoid in $\mathbf{R}^n$ for each fixed $\mathbf{s} \in S$. Hence Λ is diffeomorphic to $S^{n-1} \times S^{n-1}$ in this case.

If $F(\mathbf{s})$ is not strictly negative, then the topology of Λ is more complicated. Since 0 is a regular value of the restriction of F to S, it follows that $M = \{\mathbf{s} \in S \,|\, F(\mathbf{s}) \leqslant 0\}$ is a smooth submanifold with boundary in S. Denote the boundary of M by ∂M. Then, as before, (2.12) defines an ellipsoid in $\mathbf{R}^n$ as long as $\mathbf{s} \in M - \partial M$. If $\mathbf{s} \in \partial M$, then only $\mathbf{u} = 0$, $v = 0$ satisfies (2.12). Hence we may regard Λ in this case as a pinched or reduced sphere bundle over M: the fibers of this bundle are spheres, except over ∂M, where the fibers degenerate into single points. We remark that the homology groups of Λ may be easily related to the homology of M and ∂M by using a Mayer-Vietoris sequence.

We also remark that Λ is independent of the energy level e. Consequently, Λ may be regarded as the invariant boundary of each fixed energy level. Orbits of the original system which previously reached the singularity set in finite time now tend asymptotically toward Λ. And orbits which previously came close to $\mathbf{0}$ now behave like orbits on Λ itself. In the next section we discuss this flow in some detail.

Finally, we remark that Proposition 2.1 is the prototype of the theorems we have for Λ: the question of the nondegeneracy of the singularity is reduced to a question about the principal part of the potential energy. This question is always easy to answer, given the potential energy of the system.

3. The flow on the singularity manifold

In this section, we assume that the origin is a nondegenerate singularity of (1.1). By Proposition 2.1, it follows that Λ is a submanifold of $r = 0$ having dimension $2n - 2$.

The restriction of (2.9) to the entire boundary $r = 0$ is given by

$$\begin{aligned} \dot{v} &= (k/2)v^2 + \mathbf{u}^t A\mathbf{u} + kF(\mathbf{s}), \\ \dot{\mathbf{s}} &= \mathbf{u}, \\ \dot{\mathbf{u}} &= (-1 + k/2)v\mathbf{u} - (\mathbf{u}^t A\mathbf{u})\mathbf{s} - kF(\mathbf{s})\mathbf{s} - A^{-1}\nabla F(\mathbf{s}). \end{aligned} \tag{3.1}$$

This is a smooth vector field on $TS \times \mathbf{R}$. Using the energy relation (2.10), the flow on the submanifold Λ is given by the simpler system

$$\begin{aligned} \dot{v} &= (1 - k/2)\mathbf{u}^t A\mathbf{u}, \\ \dot{\mathbf{s}} &= \mathbf{u}, \\ \dot{\mathbf{u}} &= (-1 + k/2)v\mathbf{u} - (\mathbf{u}^t A\mathbf{u})\mathbf{s} - kF(\mathbf{s})\mathbf{s} - A^{-1}\nabla F(\mathbf{s}), \end{aligned} \tag{3.2}$$

where the point $(v, \mathbf{s}, \mathbf{u}) \in \Lambda$.

In this section, our goal is to show that the flow induced by (3.2) is relatively simple–generally, (3.2) is a gradient-like Morse-Smale system on Λ.

Our first observation is that, if $k \neq 2$, then (3.2) is always gradient-like. Recall that a vector field is called *gradient-like* if there is a smooth function which increases along all nonequilibrium orbits. A gradient-like vector field is relatively simple in the sense that there are no nontrivially recurrent or periodic solutions: the only nonwandering points are the equilibrium solutions.

To see that (3.2) is gradient-like, we first observe that the equilibrium solutions on Λ consist of all points of the form $(v_0, \mathbf{s}_0, \mathbf{u}_0)$ where

$$\begin{aligned} \mathbf{u}_0 &= \mathbf{0}, \\ kF(\mathbf{s}_0) + A^{-1}\nabla F(\mathbf{s}_0) &= 0, \\ (1/2)v^2 + F(\mathbf{s}_0) &= 0. \end{aligned} \tag{3.3}$$

When $k < 2$, we consider the function $g: \Lambda \to \mathbf{R}$ given by $g(v, \mathbf{s}, \mathbf{u}) = v$ (if $k > 2$, take $g = -v$ and the argument is similar). Let $\dot{g}$ denote the derivative of g in the direction of the flow. One computes easily that

$$\dot{g} = (1 - k/2k)\mathbf{u}^t A\mathbf{u},$$

so that $\dot{g} > 0$ if $\mathbf{u} \neq \mathbf{0}$. Hence g increases along any orbit segment which misses $\mathbf{u} = \mathbf{0}$.

When $\mathbf{u} = \mathbf{0}$, however, $\dot{g} = 0$. Nevertheless, we claim that g still increases along any nonstationary orbit that meets $\mathbf{u} = \mathbf{0}$. This can be seen as follows. When $\mathbf{u} = \mathbf{0}$, we have $\dot{\mathbf{u}} = -kF(\mathbf{s})\mathbf{s} - A^{-1}\nabla F(\mathbf{s})$, and, from the energy relation, $v^2 = -2F(\mathbf{s})$. Hence $|\dot{\mathbf{u}}| = 0$ iff the point $(v, \mathbf{s}, \mathbf{u})$ is an equilibirum solution. If $|\dot{\mathbf{u}}| > 0$, then the orbit leaves the submanifold $\mathbf{u} = \mathbf{0}$ immediately, and g increases everywhere by continuity.

In the special case when $k = 2$, the function g defined above is an integral for the flow on Λ. This follows immediately from the fact that $\dot{g} = (1 - k/2)(\mathbf{u}^t A \mathbf{u}) = 0$ in this case. In fact, one may check that the flow on Λ is itself Hamiltonian, with g as the Hamiltonian function.

We summarize these facts in a proposition.

Proposition 3.1. *Suppose the system* (1.1) *has a nondegenerate singularity of order k at the origin. If* $k < 2$, *the flow on* Λ *is gradient-like with respect to* $g(v, \mathbf{s}, \mathbf{u}) = v$. *If* $k > 2$, *the flow on* λ *is gradient-like with respect to* $g(v, \mathbf{s}, \mathbf{u}) = -v$. *Moreover, if* $k = 2$, *then the flow has an integral given by* $g(v, \mathbf{s}, \mathbf{u}) = v$.

We turn our attention now to the equilibrium solutions of the flow on the singularity manifold. Let $f(\mathbf{s})$ denote the restriction of $F(\mathbf{s})$ to S. Also, let $\langle\ ,\ \rangle$ denote the Riemannian metric on S defined by $\langle \xi, \eta \rangle = \xi^t A \eta$ where ξ, η are tangent to S at $\mathbf{s}$. Recall that $\xi \in \mathbf{R}^n$ is tangent to S at $\mathbf{s}$ iff $\xi^t A \mathbf{s} = 0$.

The *gradient vector field* associated to $f(\mathbf{s})$ is the vector field grad f defined by the requirement that

$$\langle \operatorname{grad} f(\mathbf{s}), \xi \rangle = df_{\mathbf{S}}(\xi). \tag{3.4}$$

The following proposition gives a formula for computing grad f on our coordinates.

Proposition 3.2. $\operatorname{grad} f(\mathbf{s}) = kF(\mathbf{s})\mathbf{s} + A^{-1}\nabla F(\mathbf{s})$.

Proof. First observe that $kF(\mathbf{s})\mathbf{s} + A^{-1}\nabla F(\mathbf{s})$ is indeed tangent to S at $\mathbf{s}$. This follows since

$$\mathbf{s}^t A\big(kF(\mathbf{s})\mathbf{s} + A^{-1}\nabla F(\mathbf{s})\big) = kF(\mathbf{s}) + \mathbf{s}^t \nabla F(\mathbf{s}) = 0$$

via Euler's formula. For any vector ξ tangent to S at $\mathbf{s}$, we also have

$$\xi^t A\big(kF(\mathbf{s})\mathbf{s} + A^{-1}\nabla F(\mathbf{s})\big) = \xi^t \nabla F(\mathbf{s}) = df_{\mathbf{S}}(\xi) = df_{\mathbf{s}}(\xi).$$

Hence, $\operatorname{grad} f(\mathbf{s}) = kF(\mathbf{s})\mathbf{s} + A^{-1}\nabla F(\mathbf{s})$. q.e.d.

This proposition immediately gives the relationship between equilibrium points for the flow on Λ and critical points of $f(\mathbf{s})$:

Corollary 3.3. *The flow on* Λ *has an equilibrium point at* $(v_0, \mathbf{s}_0, \mathbf{u}_0)$ *iff*

1. $\mathbf{u}_0 = \mathbf{0}$,
2. $\mathbf{s}_0$ *is a critical point of* f,
3. $v_0 = \pm(-2f(\mathbf{s}_0))^{1/2}$.

Proof. The proof follows immediately from (3.3) and Proposition 3.2, together with the fact that grad f vanishes precisely at the critical points of f.

4. Central force problems

As an example of the previous construction, and as motivation for what follows, we digress in this section to discuss a physically important class of examples for such we may construct singularity manifolds. These examples also serve to illustrate the fact that the flow on the singularity manifold is relatively easy to describe.

For each $k > 0$, we consider the central force problem in $\mathbf{R}^n$ defined by the Hamiltonian

$$H(\mathbf{q}, \mathbf{p}) = \tfrac{1}{2}|\mathbf{p}|^2 - \frac{1}{k|\mathbf{q}|^k}.$$

Here $|\cdot|$ denotes the usual norm in $\mathbf{R}^n$. The case $k = 1$ is particularly important in classical mechanics; this is the familiar Kepler or Newtonian central force problem. The other cases correspond to different non-Newtonian laws of attraction; there are many applications for these systems in various branches of physics and astronomy.

For each k, the potential energy of the system is homogeneous of degree $-k$ and has an isolated singularity at the origin. The restriction of F to S is therefore identically $-1/k$, so the singularity is nondegenerate. Thus we may introduce a singularity manifold at the origin. This is accomplished via the changes of coordinates in §2. The resulting system is given by

$$\begin{aligned} \dot{r} &= rv, \\ \dot{v} &= (k/2)v^2 + |\mathbf{u}|^2 - 1, \\ \dot{\mathbf{s}} &= \mathbf{u}, \\ \dot{\mathbf{u}} &= (k/2 - 1)v\mathbf{u} - |\mathbf{u}|^2\mathbf{s} \end{aligned} \tag{4.1}$$

with the energy relation

$$r^k e = \tfrac{1}{2}(|\mathbf{u}|^2 + v^2) - 1/k. \tag{4.2}$$

The singularity manifold is therefore defined by

$$r = 0, \quad \tfrac{1}{2}(|\mathbf{u}|^2 + v^2) = 1/k.$$

So Λ is diffeomorphic to $S^{n-1} \times S^{n-1}$ for each k. The flow on Λ depends on k and is given by

$$\begin{aligned} \dot{v} &= (1 - k/2)|\mathbf{u}|^2, \\ \dot{\mathbf{s}} &= \mathbf{u}, \\ \dot{\mathbf{u}} &= (-1 + k/2)v\mathbf{u} - |\mathbf{u}|^2\mathbf{s}. \end{aligned}$$

We wish to discuss the phase portrait of this flow in some detail.

Observe first that this system has an equilibrium point at each point of the form $\mathbf{u} = \mathbf{0}$, $v = \pm\sqrt{2/k}$. Hence there are two $(n-1)$-spheres of equilibria, which we denote by Z^+ and Z^-, the choice of sign depending on the sign of v. When $k < 2$, v increases along nonequilibrium orbits, so orbits in Λ travel from Z^- to Z^+. Just the opposite is true for $k > 2$, since $-v$ increases along orbits in this case. For $k \neq 2$, each orbit in Λ is both forward and backward asymptotic to a unique equilibrium point in $Z^{\pm}$; we refer to [8] for more details on how this behavior depends on k.

We also wish to consider the behavior of the flow in the direction transverse to Λ. Since $\dot{r} = rv$, it follows that each equilibrium point in Z^- has a one-dimensional stable manifold tangent to the r-direction, while each point in Z^+ has a one-dimensional unstable manifold in the same direction. This gives us completely different behavior for the flow near Λ in the two cases $k > 2$ and $k < 2$.

When $k > 2$, the sphere Z^- is a global attractor for the flow, while Z^+ is a global repellor. As a consequence, it follows easily that any orbit of the system which comes close enough to the origin must in fact eventually collide with the origin.

When $k < 2$, the situation is quite different. Only an $(n-1)$-dimensional submanifold tends to collision with the singularity in each time direction, so most orbits in an energy surface do not experience a collision. Orbits which come close to a collision tend to behave like orbits on the singularity manifold. That is, nearby noncollision orbits approach Λ near one of the stable manifolds tending to Z^-, then follow the flow on Λ toward Z^+, and finally leave a neighborhood of the singularity near one of the unstable manifolds emanating from Z^+. We examine this behavior in more detail below for the special case of the Kepler problem, i.e., when $k = 1$.

Using (4.1), the flow of the Kepler problem is given by the system

$$\begin{aligned} \dot{r} &= rv, \\ \dot{v} &= \tfrac{1}{2}(v^2 + |\mathbf{u}|^2) - 1, \\ \dot{\mathbf{s}} &= \mathbf{u}, \\ \dot{\mathbf{u}} &= -\tfrac{1}{2}v\mathbf{u} - |\mathbf{u}|^2\mathbf{s}. \end{aligned} \tag{4.3}$$

Define E: $[0, \infty) \times \mathbf{R} \times TS \to \mathbf{R}^n$ by

$$E(r, v, \mathbf{s}, \mathbf{u}) = (1 - |\mathbf{u}|^2)\mathbf{s} + v\mathbf{u}.$$

E is a generalization of the so-called eccentric axis of the system [11].

Proposition 4.1. *E is a constant of the motion for the Kepler problem.*

Proof. We differentiate E in the direction of the flow:

$$\begin{aligned}\dot{E} &= -2(\mathbf{u}^t\dot{\mathbf{u}})\mathbf{s} + (1 - |\mathbf{u}|^2)\mathbf{s} - \dot{v}\mathbf{u} - v\dot{\mathbf{u}} \\ &= v|\mathbf{u}|^2\mathbf{s} - |\mathbf{u}|^2\mathbf{u} + \mathbf{u} + \left(\tfrac{1}{2}v^2 + |\mathbf{u}|^2 - 1\right) + v\left(-\tfrac{1}{2}v\mathbf{u} - |\mathbf{u}|^2\mathbf{s}\right) \\ &= 0. \quad \text{q.e.d.}\end{aligned}$$

On Λ, we have $E^tE = 1$, so that E takes its values in S for points on the singularity manifold. Note that at an equilibrium point $(\pm\sqrt{2}, \mathbf{s}_0, 0)$ we have $E = \mathbf{s}_0$, so E is an injection on both Z^+ and Z^-. Hence, to determine the ultimate behavior of a point $(\mathrm{v}, \mathbf{s}, \mathbf{u}) \in \Lambda$ under the flow, one simply computes $E(\mathrm{v}, \mathbf{s}, \mathbf{u})$. Then the orbit of $(\mathrm{v}, \mathbf{s}, \mathbf{u})$ must be forward (resp. backward) asymptotic to $(\sqrt{2}, E(\mathrm{v}, \mathbf{s}, \mathbf{u}), \mathbf{0})$ (resp. $(-\sqrt{2}, E, \mathbf{0})$). As a consequence, we note that each orbit in Λ begins and ends with the same s-coordinate. This is not in general true for the other central force potentials.

This fact has the following geometric consequence. An orbit in the Kepler problem which begins or ends in collision with the origin does so with a specific limiting direction in configuration space; that is, such an orbit tends in forward time to a unique equilibrium point in Z^-, say $(-\sqrt{2}, \mathbf{s}_0, \mathbf{0})$. The s-coordinate gives this limiting direction.

Now orbits which are close to a collision orbit but which miss the origin, eventually must leave a neighborhood of $\mathbf{0}$. How these orbits leave is governed by the flow on Λ. For the Kepler problem, an orbit which approaches collision near $(-\sqrt{2}, \mathbf{s}_0, \mathbf{0})$ follows the flow on Λ to a neighborhood of $(+\sqrt{2}, \mathbf{s}_0, \mathbf{0})$ and then leaves near the collision orbit emanating from this equilibrium point. That is, since both equilibria project to the same point in S, a nearby noncollision orbit leaves a neighborhood of the origin in the same direction as it approached. It is this idea which has been exploited by Easton in his geometric regularization of the Kepler problem [4].

5. Equilibrium solutions on Λ

In this section we continue the discussion of the flow on Λ. Our main goal is to compute the characteristic exponents of the equilibrium solutions. As before, we shall relate these dynamical properties of the flow on Λ to the properties of the principal part of the potential energy of the system.

Recall that $f(\mathbf{s})$ denotes the restriction of this principal part to the unit sphere S. We say that f has a nondegenerate critical point at $\mathbf{s}$ iff $df_{\mathbf{s}} = \mathbf{0}$ and the Hessian $d^2f_{\mathbf{s}}$ is a nonsingular bilinear form. In the case of a nondegenerate critical point, the index of f at $\mathbf{s}$ is defined to be the maximal dimension of a subspace on which $d^2f_{\mathbf{s}}$ is negative definite. If f has a nondegenerate critical point at $\mathbf{s}$, then it follows easily that the vector field grad f has a hyperbolic equilibrium point at $\mathbf{s}$ and that, moreover, the dimension of the stable manifold of grad f at $\mathbf{s}$ is equal to the index of f at $\mathbf{s}$. If f has only nondegenerate critical points, then f is called a *Morse function*. It is well known that Morse functions are open and dense in the set of all smooth functions on a compact manifold. For more details on elementary Morse theory, we refer to Milnor's book [9].

Corollary 3.3 shows that each critical point of f generates two equilibrium points for the flow on Λ; the proposition below relates the nondegeneracy of these critical points to the hyperbolicity of the equilibria.

Proposition 5.1. *Suppose $k \neq 2$. If f has a nondegenerate critical point at $\mathbf{s}_0$, then the corresponding equilibrium solution of (2.9) at $(0, v_0, \mathbf{s}_0, \mathbf{0})$ is hyperbolic, where $v_0 = \pm\sqrt{2f(\mathbf{s}_0)}$.*

Proof. The characteristic exponents at the equilibrium point may be computed in $(r, v, \mathbf{s}, \mathbf{u})$ coordinates as follows. The linearization of (2.9) at $(0, v_0, \mathbf{s}_0, \mathbf{0})$ is given by the $2n \times 2n$ matrix

$$\begin{bmatrix} v_0 & 0 & 0 & 0 \\ * & kv_0 & 0 & 0 \\ 0 & 0 & 0 & I \\ * & 0 & -B & (k/2-1)v_0 I \end{bmatrix} \tag{5.1}$$

where B is the $(n-1)\times(n-1)$ matrix giving the linearization of grad f on S at $\mathbf{s}_0$.

Clearly, there are two eigenvalues of (5.1) which are given by v_0 and kv_0. Since 0 is a regular value of f, and $f(\mathbf{s}_0) \leqslant 0$, it follows that $v_0 = \pm\sqrt{-2f(\mathbf{s}_0)} \neq 0$. The remaining $2n-2$ characteristic exponents are the eigenvalues of the submatrix

$$\begin{pmatrix} 0 & I \\ -B & (k/2-1)Iv_0 \end{pmatrix}. \tag{5.2}$$

These may be computed by using the following lemma.

Lemma 5.2. *Let λ_i, $i = 1, \ldots, n-1$, denote the eigenvalues of B. Then, for each i,*

$$\zeta_i^{\pm} = \tfrac{1}{2}\left(\mu \pm \sqrt{\mu^2 - 4\lambda_i}\right)$$

are both eigenvalues of (5.2), where $\mu = (k/2-1)v_0$.

Proof. Let w be an eigenvector of B corresponding to the eigenvalue λ_i. Then the equation

$$\begin{pmatrix} 0 & I \\ -B & \mu I \end{pmatrix}\begin{pmatrix} w \\ \alpha w \end{pmatrix} = \begin{pmatrix} \alpha w \\ \alpha^2 w \end{pmatrix} = \alpha\begin{pmatrix} w \\ \alpha w \end{pmatrix}$$

has a solution iff

$$-Bw + \mu \alpha w = \alpha^2 w,$$

which happens iff

$$\alpha^2 - \mu\alpha + \lambda_i = 0.$$

But the roots of this polynomial are

$$\zeta_i^{\pm} = \tfrac{1}{2}\left(\mu \pm \sqrt{\mu^2 - 4\lambda_i}\,\right).$$

Since $\mu \neq 0$, it follows that $\zeta_i^{\pm} \neq 0$ for all i. This completes the proof. q.e.d.

The proof of Proposition 5.1 also enables us to compute the dimensions of the various invariant manifolds of the equilibrium points in Λ. These dimensions depend only on k, $\mathrm{sgn}(v_0)$, and the index of $\mathbf{s}_0$ as a critical point of f.

Let $W^s(p)$, $W^u(p)$, and $W^c(p)$ denote the stable, unstable, and center manifolds in Λ respectively of an equilibrium point p. These manifolds are tangent at p to the eigenspaces in $T_p\Lambda$ corresponding to eigenvalues with negative, positive, and zero real parts respectively.

Table 1 summarizes the dimensions of these invariant manifolds in Λ. We emphasize that these are the dimensions in Λ; one may also consider the stable and unstable manifolds in all of the energy surface. In fact, we shall do so in §8. In general, these manifolds have larger dimension.

To compute these dimensions, we first observe that only the eigenvalues ζ_i^+ given by Lemma 5.2 give eigenvectors tangent to Λ; the two remaining eigenvalues lie in the r, v-plane which is transverse to Λ at $(0, v_0, \mathbf{s}_0, \mathbf{0})$.

Now when $k = 2$, we have $k/2 - 1 = \mu = 0$, so that

$$\zeta_i^{\pm} = \pm\sqrt{-4\lambda_i}\,.$$

Hence we have $2(n - m)$ pure imaginary eigenvalues, where m is the index of $\mathbf{s}_0$ as a critical point of f. For each eigenvalue $\lambda_i < 0$, ζ_i^+ is positive while ζ_i^- is negative, so the stable and unstable manifolds in this case are both m-dimensional.

When $k < 2$ and v_0 is positive, we have that $\mu > 0$ and

$$\sqrt{\mu^2 - 4\lambda_i} > \mu,$$

provided $\lambda_i < 0$. Thus there are $n - 1 - m$ characteristic exponents ζ_i^+ which are positive. The remaining $n + m - 1$ characteristic exponents have negative real parts. In a similar fashion, one may verify the other entries in Table. 1.

Degree	Sgn v_0	dim W^s	dim W^u	dim W^c
$k < 2$	+	m	$2n - m$	0
$k < 2$	−	$m + 2$	$2n + m - 2$	0
$k > 2$	+	m	$2n - m$	0
$k > 2$	−	$2n - m$	m	0
$k = 2$	+	m	m	$2(n - m)$
$k = 2$	−	m	m	$2(n - m)$

TABLE 1. The dimensions of the invariant manifolds of equilibrium points in Λ. Here m denotes the index of s_0 as a critical point of f.

6. Morse-Smale singularities

Thus far, we have shown that a nondegenerate singularity of order k may be replaced in phase space by a smooth manifold over which the scaled flow extends smoothly. If the principal part of the potential energy restricts to a Morse function on S, then we also know that the resulting flow on Λ is gradient-like and possesses only hyperbolic equilibrium points when $k \neq 2$. In this section, we shall impose an additional restriction on the flow on Λ.

We shall say that a singularity of order $k \neq 2$ at the origin is *Morse-Smale* if

1. the restriction of the principal part of the potential energy to S is a Morse function,
2. 0 is a regular value of f,
3. the stable and unstable manifolds of all equilibrium points in Λ meet transversely.

Condition 2 implies that Λ is actually a manifold, while 1 and 3 imply that the flow on Λ is a gradient-like Morse-Smale flow. Also, conditions 1 and 2 are immediately computable in terms of the potential energy. It would be interesting to relate condition 3 to the potential energy in the same manner, but this seems rather difficult.

We have already observed that conditions 1 and 2 hold for an open dense subset of the set of potential energy functions. Here we use the Whitney C^1 topology on the set of potentials which have a singularity of order k at the origin in $\mathbf{R}^n$. Actually, one may allow perturbations of the potential energy, which change the asymptotic behavior at the singularity; only the restriction of the principal part of the potential energy to the unit sphere S matters for conditions 1 and 2.

The goal of this section is to show that condition 3 also holds for an open dense set of potential energy functions. This will complete the proof of Theorem A and will show that most singularities of order k are Morse-Smale.

The proof is essentially the same as the usual proof of the genericity of transverse intersection of stable and unstable manifolds (see, for example, Abraham-Robbin [1]) with some minor modifications due to the fact that we are considering a special class of vector fields. Recall that, restricted to $r = 0$, the system is given by

$$\begin{aligned} \dot{\mathbf{s}} &= \mathbf{u}, \\ \dot{v} &= (k/2)v^2 + \mathbf{u}^t A \mathbf{u} + kf(\mathbf{s}), \\ \dot{\mathbf{u}} &= (-+k/2)v\mathbf{u} - (\mathbf{u}^t A \mathbf{u})\mathbf{s} - \operatorname{grad} f(\mathbf{s}). \end{aligned} \tag{6.1}$$

We are only allowed to perturb f, so, roughly speaking, we may only perturb the system in the "fiber" or v, $\mathbf{u}$-directions. We shall show below that we still have enough leeway to make all stable and unstable manifolds meet transversely with such perturbations. We first need a lemma.

Lemma 6.1. *Let p be a hyperbolic equilibrium point for* (6.1). *Let Π denote the projection of $\mathbf{R} \times TS$ onto S, i.e., $\Pi(v, \mathbf{s}, \mathbf{u}) = \mathbf{s}$. Let $x \in W^s(p)$ (or $W^u(p)$). Then there is a point y on the orbit of x satisfying: that the restriction of $d\Pi_y$ to $W^s(p)$ (or $W^u(p)$) has maximal rank.*

Proof. Using the matrix of the linearization of the flow (5.1) at p, one checks easily that the projections of the stable and unstable eigenspaces at p to the tangent space of S have maximal rank. This implies that there are open neighborhoods of p in the stable and unstable manifolds having the same property. Since all orbits in $W^s(p)$ and $W^u(p)$ enter these neighborhoods, the proof is complete. q.e.d.

We now complete the proof of Theorem A. The openness of transverse intersection is immediate, hence we restrict ourselves to proving density. For this, we assume some familiarity with the methods in Abraham-Robbin [1].

First suppose that $W^u(p)$ and $W^s(q)$ have a point x of nontransverse intersection, and that the dimension of one of these manifolds, say $W^u(p)$, is greater than or equal to $n - 1$. By Lemma 1 we may choose a point y along the orbit of x so that $d\Pi_y | W^u(p)$ has maximal rank. In particular, because $\dim W^u(p) \geqslant n - 1$, it follows that $W^u(p)$ is transverse to the fiber $\Pi^{-1}(\Pi(y))$ at y. We may assume that this fiber is diffeomorphic to an $(n - 1)$-dimensional sphere.

Now (6.1) may be perturbed in any direction tangent to these fibers. By the Perturbation Lemma of [1, p. 107], we may therefore perturb $W^u(p)$ in j independent directions transverse to $W^u(p)$, where j is the codimension of

$W^u(p)$ in the energy surface. If follows that $W^u(p)$ may be perturbed to be transverse to $W^s(q)$.

In case both $W^s(q)$ and $W^u(p)$ have dimensions $< n - 1$, the above proof does not work. However, we may adapt the proof using the following trick. Lemma 6.1 still applies to $W^u(p)$, yielding a point y in $W^u(p) \cap W^s(q)$ where $d\Pi_y | W^u(p)$ has maximal rank. This implies that $\Pi(W^u(p))$ is locally a submanifold of S near $\Pi(y)$. Let V be an open neighborhood of $\Pi(y)$ in this submanifold of S. Let $W = \Pi^{-1}(V)$. Then W is a submanifold of Λ with dimension $n - 1 + \dim W^u(p)$. We now perturb $W^u(p)$ by adding vector fields whose support in $W^u(p)$ is tangent to W. This means that the unstable manifold for the perturbed flow remains tangent to W over V. As before, we can now perburb $W^u(p)$ in $n - 1$ transverse directions in W. Hence we may guarantee that $W^u(p)$ misses $W^s(q)$, at least over V. This again implies that the perturbed stable and unstable manifolds are transverse to each other. This completes the proof of Theorem A.

7. The anisotropic Kepler problem

The anisotropic Kepler problem is a one-parameter family of Hamiltonian systems recently introduced by Gutzwiller [5] to model certain quantum mechanical systems. When the parameter $\mu = 1$, we have the ordinary (planar) Kepler problem considered in §4. When $\mu > 1$, the kinetic energy of the system becomes anisotopic. This destroys the integrability of the problem and changes the orbit structure of the system dramatically. Our concern here is to show that the singularity of this system is a Morse-Smale singularity of order 1, at least for most values of the parameter.

Let $\mathbf{q}, \mathbf{p} \in \mathbf{R}^2$. The potential energy of the system is given, as in the Kepler problem, by $V(\mathbf{q}) = -1/|\mathbf{q}|$, where $|\cdot|$ denotes the usual norm in $\mathbf{R}^2$. The kinetic energy of the system depends on a real parameter $\mu \geqslant 1$ and is given by $K(\mathbf{p}) = \frac{1}{2}\mathbf{p}^t A \mathbf{p}$, where A^{-1} is the 2×2 matrix

$$\begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix}. \tag{7.1}$$

The Hamiltonian is then $H = K + V$, with the resulting differential equation

$$\mathbf{q}' = A^{-1}\mathbf{p}, \quad \mathbf{p}' = -\mathbf{q}|\mathbf{q}|^3. \tag{7.2}$$

This sytem has a nondegenerate singularity of order 1 at the origin. Hence we may introduce a singularity manifold at the origin via the changes of coordinates in §2. The resulting system is given by

$$\begin{aligned} \dot{r} &= rv, \\ \dot{v} &= \tfrac{1}{2}v^2 + \mathbf{u}^t A\mathbf{u} - 1/|\mathbf{s}|, \\ \dot{\mathbf{s}} &= \mathbf{u}, \\ \dot{\mathbf{u}} &= -\tfrac{1}{2}v\mathbf{u} - (\mathbf{u}^t A\mathbf{u})\mathbf{s} + \mathbf{s}/|\mathbf{s}| - A^{-1}\mathbf{s}/|\mathbf{s}|^3. \end{aligned} \tag{7.3}$$

Recall that S is defined by $\mathbf{s}^t A\mathbf{s} = 1$, so the expression for $\dot{\mathbf{u}}$ cannot be simplified unless $\mu = 1$.

The singularity manifold is defined by the energy relation

$$0 = \tfrac{1}{2}(v^2 + \mathbf{u}^t A\mathbf{u}) - 1/|\mathbf{s}|. \tag{7.4}$$

Clearly, Λ is a two-dimensional torus. The flow on Λ is given by

$$\begin{aligned} \dot{v} &= \tfrac{1}{2}\mathbf{u}^t A\mathbf{u}, \\ \dot{s} &= \mathbf{u}, \\ \dot{\mathbf{u}} &= -\tfrac{1}{2}v\mathbf{u} - (\mathbf{u}^t A\mathbf{u})\mathbf{s} + \mathbf{s}/|\mathbf{s}| - A^{-1}\mathbf{s}/|\mathbf{s}|^3. \end{aligned} \tag{7.5}$$

This system is much more complicated than the corresponding system (4.3) for the Kepler problem. Below we sketch some of the highlights of the phase portrait. For proofs and further details, we refer to [2], [3].

When $\mu = 1$, there are two circles of equilibria in Λ, as we showed in §4. When $\mu > 1$, each of these circles break up into four isolated equilibrium points. One may check easily that each circle degenerates into two saddle points, a sink, and a source. Equivalently, one may check that the real valued function $-1/|\mathbf{s}|$ is a Morse function on S with eight critical points. A sketch of the phase portrait of the flow on Λ is provided in Figure 1.

To show that the singularity is a Morse-Smale singularity, it suffices to check that all of the stable and unstable manifolds meet transversely. This has been verified for an open and dense set of parameter values by the author [2]. Also, Gutzwiller has computed the phase portrait of this flow numerically. See [5].

Orbits of (7.2) which either begin or end at the singularity are called *collision orbits*. They play an important role in the entire phase portrait of the anisotropic Kepler problem. By the changes of variables in §2, such orbits are slowed down so that they tend asymptotically toward or away from one of the equilibrium points in Λ. Using the linearization of (7.3) about each of these equilibria, one may check that for each $\mu > 1$:

1. There is a unique collision orbit tending toward each source in Λ.
2. There is a unique collision orbit tending away from each sink in Λ.

3. Two of the saddle points admit two-dimensional manifolds of orbits tending toward the equilibrium.

4. The other two saddles admit two-dimensional manifolds of orbits tending away from the equilibrium.

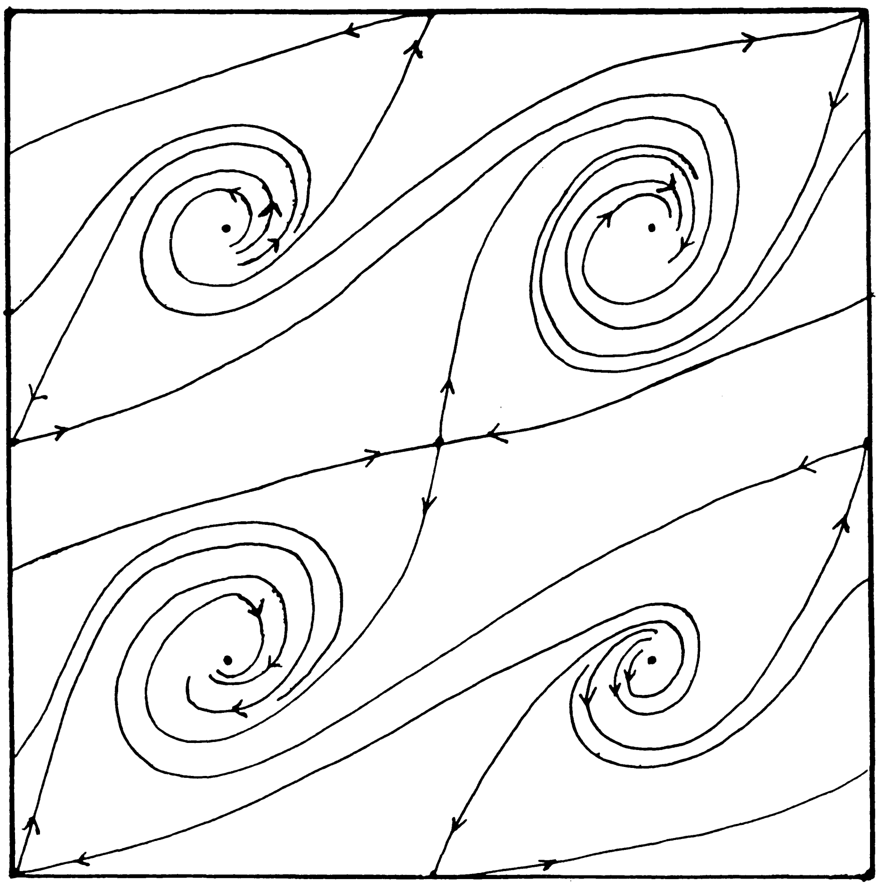

FIG. 1. The phase portrait on the singularity manifold for the anisotropic Kepler problem.

As a consequence of these facts, we have the following proposition.

Proposition 7.1. *For all* $\mu > 1$, *the set of collision orbits in the anisotropic Kepler problem consists of a finite union of submanifolds of phase space.*

In the following section, we shall prove a more general theorem which holds for all Morse-Smale singularities.

We wish finally to mention one application which follows easily from the considerations above. The singularity at the origin is a so-called nonregularizable singularity. Roughly speaking, this means that certain orbits which are close to each other when they approach collision leave a neighborhood of collision far apart. In the anisotropic Kepler problem, such behavior can be verified as follows. Consider one of the collision orbits which tends to one of the saddle points. In any neighborhood of this orbit, there are noncollision orbits. These orbits come close to the saddle point and then leave, following one of the two branches of the unstable manifold of the saddle point in Λ. See Fig. 1. These branches die in distinct sinks. Hence the nearby noncollision orbits tend to leave a neighborhood of collision near the collision orbits emanating from the two sinks. Since these are far apart, this proves the nonregularizability of the problem. For further details, we refer to [3].

8. Collision orbits

In this section we complete the proof of Theorem C. Let C^s denote the set of points on $\Sigma_e - \Lambda$ whose forward orbits end in collision with the singularity. Similarly, let C^u denote the set of points which collide with the origin in backward time. Our goal is to show that, for a Morse-Smale singularity, both C^s and C^u consist of a union of j submanifolds of varying codimension in Σ_e, where j is the number of critical points of f.

To prove this, we first observe that if $k \neq 2$, then any orbit of (2.9) which is asymptotic to Λ must in fact tend toward one of the equilibrium points in Λ. This is a consequence of the gradient-like structure of the flow on Λ. Let $C^s(p)$ (resp. $C^u(p)$) denote the set of collision orbits tending to p in forward (resp. backward) time. by Proposition 5.1, p is hyperbolic, so $C^s(p)$ and $C^u(p)$ are contained in the stable and unstable manifolds respectively of p. In fact, we have $C^s(p) = W^s(p) \cap (\Sigma_e - \Lambda)$ and $C^u(p) = W^u(p) \cap (\Sigma_e - \Lambda)$. Since $\Sigma_e - \Lambda$ is open, it follows that both $C^s(p)$ and $C^u(p)$ are open submanifolds of $W^s(p)$ and $W^u(p)$, and hence both are manifolds themselves.

Now suppose p has coordinates $(v_0, \mathbf{s}_0, \mathbf{0})$ in Λ. We claim that, depending on the sign of v_0, one of $C^s(p)$ or $C^u(p)$ is empty. To see that we recall that the characteristic exponents at p are given by $\zeta_i^{\pm}$ for $i = 1, \ldots, n-1$, together with v_0, kv_0. The characteristic vectors associated to the $\zeta_i^{\pm}$ are all tangent to Λ, whereas the remaining characteristic vector tangent to the energy surface lies in either the v_0 or kv_0 eigenspace. In any event, the eigenvalue associated to this vector has the same sign as v_0. Now only this characteristic vector is not tangent to Λ, so the sign of v_0 determines whether $C^s(p)$ or $C^u(p)$ is empty. We have

Proposition 8.1. *Let* $(v_0, \mathbf{s}_0, \mathbf{0})$ *be an equilibrium point for the flow on* Λ. *If* $v_0 < 0$, *then* $C^u(v_0, \mathbf{s}_0, \mathbf{0})$ *is empty. If* $v_0 > 0$, *then* $C^s(v_0, \mathbf{s}_0, \mathbf{0})$ *is empty.*

Consequently, corresponding to each critical point s_0 of f, we find a unique submanifold of collision orbits tending toward and away from one of the equilibrium points determined by s_0. This completes the proof of Theorem C.

We remark that, using Table 1 one may easily compute the dimensions of these various submanifolds in C^s and C^u. We summarize the results below.

Proposition 8.2. *Let* s_0 *be a nondegenerate critical point for* f *of index* m. *Let* $v_0^{\pm} = \pm\sqrt{-2f(s_0)}$. *If* $k < 2$, *then*

$$\dim C^s(v_0^-, s_0, \mathbf{0}) = n - m = \dim C^u(v_0^+, s_0, \mathbf{0}).$$

If $k > 2$, *then*

$$\dim C^s(v_0^-, s_0, \mathbf{0}) = n + m = \dim C^u j(v_0^+, s_0, \mathbf{0}).$$

References

[1] R. Abraham & J. Robbin, *Transversal mappings and flows*, Benjamin, New York, 1967.

[2] R. Devaney, *Collision orbits in the anisotropic Kepler problem*, Invent. Math. **45** (1978) 221–251.

[3] ______, *Non-regularizability of the anisotropic Kepler problem*, J. Differential Equations, **29** (1978) 253–268.

[4] R. Easton, *Regularization of vector fields by surgery*, J. Differential Equations **10** (1971) 92–99.

[5] M. C. Gutzwiller, *The anisotropic Kepler problem in two dimensions*, J. Math. Phys. **14** (1973) 139–152.

[6] ______, *Bernoulli sequences and trajectories in the anisotropic Kepler problem*, J. Math. Phys. **18** (1977) 806–823.

[7] R. McGehee, *Triple collision in the collinear three-body problem*, Invent. Math. **27** (1974) 191–227.

[8] ______, *Double collisions for non-Newtonian potentials*, to appear.

[9] J. Milnor, *Morse theory*, Annals of Math. Studies No. 51, Princeton University Press, Princeton, 1963.

[10] J. Moser, *Stable and random motions in dynamical systems*, Annals of Math. Studies, No. 77, Princeton University Press, Princeton, 1973.

[11] H. Pollard, *Celestial mechanics*, Carus Math. Monographs, No. 18, Math. Assoc. Amer., 1976.

[12] C. L. Siegel & J. Moser, *Lectures on celestial mechanics*, Springer, New York, 1971.

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