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SOME REMARKS ON FOLIATIONS WITH MINIMAL LEAVES

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Let $\mathfrak{F}$ be a foliation on a manifold X of dimension $n = p + q$, the leaves being submanifolds of dimension p and codimension q. Everything will be assumed to be of class C^∞. The question of the existence of a riemannian metric on X for which all the leaves are minimal submanifolds has been discussed by Rummler [5] and Sullivan [7]. Assume for simplicity that the tangent bundle $T\mathfrak{F}$ of the leaves of the foliation is orientable and oriented. They prove the following criterion.

Theorem (*Rummler-Sullivan.*) *Let g_0 be a smooth scalar product on $T\mathfrak{F}$. It is induced by a riemannian metric g on X for which the leaves are minimal submanifolds iff the volume p-form ω_0 on the leaves defined by g_0 (and the orientation) is the restriction to the leaves of a p-form ω on X which is relatively closed, namely, $d\omega(\xi_1, \cdots, \xi_{p+1}) = 0$ if the first p vector fields ξ_i are tangent to the leaves.*

Using the above criterion Rummler and Sullivan proved the existence or the nonexistence of such a metric in many interesting cases. Our goal is to prove that for a compact X the above condition depends only on the transverse structure of $\mathfrak{F}$, and to deduce from this some consequences.

We first give a short proof of the Rummler-Sullivan criterion. Let ν be a vector field on a small open set U of X such that the local flow φ_t generated by ν maps leaves to leaves. Let $K_0 \subset U$ be a piece of a leaf and let $K_t = \varphi_t(K_0)$. Consider a p-form ω on X extending ω_0. Then for $t = 0$

$$\frac{d}{dt}(\text{volume } K_t) = \frac{d}{dt}\int_{K_t}\omega = \frac{d}{dt}\int_{K_0}\varphi_t^*\omega = \int_{K_0}\nu.\omega = \int_{K_0} di_\nu\omega + i_\nu d\omega,$$

where $\nu.\omega$ is the Lie derivative of ω in the direction of ν. Assume there is a metric g extending g_0 such that all the leaves are minimal. Define ω such that $i_\nu\omega$ vanishes on each leaf for ν orthogonal to the leaves. For such a ν, $(d/dt)(\text{volume } K_t) = 0$, hence the above formula shows that $di_\nu\omega$ vanishes on the leaves, so ω is relatively closed.

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Conversely, let ω be a relatively closed form extending ω_0. At a point $x \in X$, the vectors ξ such that $i_\xi \omega = 0$ form a vector subspace N_x of $T_x X$ complementary to the tangent space of the leaf through x. Consider any metric g extending g_0 and such that N_x is orthogonal to the tangent space of the leaf at x, for all $x \in X$. Then the above formula shows that for ν orthogonal to the leaves, the first variatiion (d/dt)(volume K_t) is zero for any piece K_0 in a leaf. So each leaf is a minimal submanifold.

1. FORMS AND CURRENTS ON THE TRANSVERSE STRUCTURE OF A FOLIATION

1.1. Morphisms of pseudogroups

Recall that a pseudogroup H of diffeomorphisms of a manifold T is a collection of diffeomorphisms of open sets of T on open sets of T, which contains the identity map of T and is closed under composition (whenever it is defined), inverses, restrictions to open sets, and unions.

Consider two pseudogroups H and H' of diffeomorphisms of T and T' respectively. A morphism $\Phi : H \to H'$ is a collection Φ of diffeomorphism of open sets of T on open sets of T' such that:

(i) the sources of the $\varphi \in \Phi$ cover T,

(ii) if $h \in H$ and $\varphi_1, \varphi_2 \in \Phi$, then $\varphi_1 h \varphi_2^{-1} \in H'$,

(iii) if $h \in H$, $h' \in H'$, $\varphi \in \Phi$, then $h'\varphi h \in \Phi$,

(iv) Φ is closed under unions.

Any collection Φ_0 such that

(a) the H-orbit of each point of T intersects the source of a $\varphi \in \Phi$,

(b) if $h \in H$, and $\varphi_1, \varphi_2 \in \Phi_0$, then $\varphi_1 h \varphi_2^{-1} \in H'$ can be uniquely completed as a collection Φ satisfying (i)–(iv) by considering all unions of elements of the form $h'\varphi h$, $\varphi \in \Phi$, $h \in H$, $h' \in H'$. Such a Φ_0 will be called an atlas generating the morphism Φ.

If Φ' is a morphism of H' in H'', then the collection of all $\varphi'.\varphi$, $\varphi \in \Phi$, $\varphi \in \Phi'$, generates a morphism of H in H''. Under this composition, morphisms form a category. Φ_0 generates an isomorphism (or an equivalence) of H on H' iff the union of the targets of the $\varphi \in \Phi_0$ intersects each orbit of H' and $\varphi_2^{-1} h' \varphi_1 \in H$ for any $\varphi_1, \varphi_2 \in \Phi_0$, $h' \in H$. In that case we say that H is equivalent to H'. For instance, let U be an open subset of T and let H_U be the pseudogroup of diffeomorphisms of U whose elements are the restriction to U of the elements of H. Then the inclusion of U in T generates a morphism of H_U in H, and is an isomorphism iff U meets each orbit of H. In the case where the space T/H of H-orbits is a differentiable manifold, the natural projection $p : T \to T/H$ being locally a diffeomorphism, H is equivalent to the trivial pseudogroup on T/H (generated by the identity).

1.2. Forms and currents on T/H

Let $\Omega_c^p(T)$ be the vector space of smooth p-forms on T with compact support, and denote by $\Omega_c^p(T/H)$ the quotient of $\Omega_c^p(T)$ by the vector subspace generated by elements of the form $\alpha - h^*\alpha$, where $h \in H$, and α is a p-form with compact support in the range of h. On $\Omega_c^p(T/H)$ we consider the topology obtained by taking the quotient of the usual C^∞-topology on $\Omega_c^p(T)$. In general this topology is not Hausdorff (see examples below).

The exterior differential $d : \Omega_c^p(T) \to \Omega_c^{p+1}(T)$ induces a continuous differential

$$d : \Omega_c^p(T/H) \to \Omega_c^{p+1}(T/H).$$

Thus we associated to H a differential graded topological vector space $\Omega_c^*(T/H)$. We shall see below that it depends only on the equivalence class of H, and its dual is naturally isomorphic to the space of invariant currents on T. Indeed let $C_p(T)$ be the space of p-currents on T, namely, the vector space of continuous linear forms on $\Omega_c^p(T)$. A p-current c is invariant by H if for any $h \in H$ and any p-form α with support in the range of h, then $c(\alpha) = c(h^*\alpha)$. So it defines a continuous linear form on the quotient $\Omega_c^p(T/H)$. If $\alpha \in \Omega_c^p(T)$ is such that $c(\alpha) = 0$ for all invariant current c, the class of α in $\Omega_c^p(T/H)$ is not zero in general, but is in the closure of the kernel of the projection $\Omega_c^p(T) \to \Omega_c^p(T/H)$. For this reason, it is in general easier to describe the space $C^p(T)^H$ of invariant currents.

Proposition. *A morphism Φ of H in H' induces functorially a continuous morphism of differential graded vector spaces*

$$\Phi^* : \Omega_c^*(T/H) \to \Omega_c^*(T'/H').$$

Proof. We can express each $\alpha \in \Omega_c^p(T)$ as a finite sum

$$\alpha = \sum_{\varphi \in \Phi} \alpha_\varphi,$$

where α_φ is a p-form with compact support in the source U_φ of $\varphi \in \Phi$, and is zero except for a finite number of ϕ.

The map Φ^* associates to the class of α the class $\Sigma_\varphi(\varphi^{-1})^*\alpha_\varphi$ in $\Omega_c^p(T'/H')$.

We have to check that this definition is independent of the choice of the decomposition of α and the choice of α in its class. For the first part, it is sufficient to note that if λ_φ is a partition of unity subordinated to the covering

U_φ of T, then

$$\sum_{\varphi} (\varphi^{-1})^* \alpha_\varphi = \sum_{\varphi,\psi \in \Phi} (\varphi^{-1})^* \lambda_\psi \alpha_\varphi$$

is equivalent to

$$\sum_{\varphi,\psi} (\varphi\psi^{-1})^*(\varphi^{-1})^* \lambda_\psi \alpha_\varphi = \sum_{\varphi,\psi} (\psi^{-1})^* \lambda_\psi \alpha_\varphi = \sum_{\psi} (\psi^{-1})^* \lambda_\psi \alpha$$

because $\varphi\psi^{-1} \in H'$.

Assume now that $\alpha = \beta - h^*\beta$, where β has its support in the range of $h \in H$. We can express β as a finite sum $\Sigma\, \beta_\varphi$, where the support of β_φ is in the source of φ, and the support of $h^*\beta_\varphi$ is in the source of some $\psi \in \Phi$ (for this, it is sufficient to multiply β by a partition of unity subordinated to the covering of the range of h by the intersections $U_\varphi \cap h(U_\psi)$, φ, $\psi \in \Phi$). Then $\alpha = \Sigma_\varphi\, \beta_\varphi - h^*\beta_\varphi$ can be mapped on

$$\sum (\varphi^{-1})^*\beta_\varphi - \sum (\psi^{-1})^* h^* \beta_\varphi = \sum (\varphi^{-1})^*\beta_\varphi - \sum (\psi^{-1})^* h^* \varphi^* (\varphi^{-1})^* \beta_\varphi,$$

which is equivalent to zero because $\varphi h \psi^{-1} \in H'$.

It is straightforward to check that Φ^* commutes with d and is continuous.

Corollary. *An isomorphism of H on H' induces a topological isomorphism of $\Omega_c^*(T/H)$ on $\Omega_c^*(T'/H')$.*

In particular, if H acts on T in a properly discontinuous way, i.e., if the map $T \to T/H$ is locally a diffeomorphism, then $\Omega_c^p(T/H)$ is just isomorphic to the vector space of p-forms with compact support on T/H.

Also if each point x of T has a neighborhood V such that the restriction of H to V is generated by a finite group of diffeomorphisms of V, then T/H is a manifold in the sense of Satake, and $\Omega_c^p(T)$ is what is usually called the space of differential forms on T/H.

1.3. The holonomy pseudogroup of a foliation

Let $\mathcal{F}$ be a foliation of codimension q on a manifold X. A transversal submanifold T is a manifold of dimension q together with an immersion $t : T \to X$ which is transversal to the leaves.

Given two points x_1, x_2 in T such that $t(x_1)$ and $t(x_2)$ are in the same leaf L, then a homotopy class of paths γ joining $t(x_1)$ to $t(x_2)$ in L determines a germ at x_1 of a diffeomorphism h of a neighborhood of x_1 on a neighborhood of x_2, called the holonomy defined by the path γ: if x is close to x_1, there is a path close to γ and contained in a leaf, joining $t(x)$ to $t(hx)$. *The holonomy pseudogroup induced by $\mathcal{F}$ on T* is the pseudogroup whose elements are local diffeomorphisms of T whose germs at each point are determined in this way.

The transversal submanifold $t : T \to X$ will be said to be *complete* if $t(T)$ cuts every leaf of $\mathfrak{F}$. If $t' : T' \to X$ is another complete transversal submanifold, then the holonomy pseudogroup H' induced on T' is canonically equivalent to H. Indeed the set Φ of elements of the holonomy pseudogroup induced on the disjoint union of T and T' with source in T and range in T' is a morphism of H in H'. Hence to each foliation $\mathfrak{F}$ we can associate a well defined equivalence class of pseudogroups, namely, the class of any holonomy pseudogroup H induced by $\mathfrak{F}$ on a complete transversal submanifold T. By abuse of language, such an H will be called *the (transverse) holonomy pseudogroup of* $\mathfrak{F}$.

Definition. *We shall denote by* $\Omega_c^*(Tr\ \mathfrak{F})$ *the topological differential graded vector space of forms on* T/H, where H is the holonomy pseudogroup induced on a complete transversal submanifold T. This definition is independent of the choice of the transversal T, because if H' is the holonomy pseudogroup induced on a complete transversal submanifold T', then $\Omega_c^*(T/H)$ is canonically isomorphic to $\Omega_c^*(T'/H')$.

A continuous linear form on $\Omega_c^k(\mathrm{Tc}\ \mathfrak{F})$ *will be called an holonomy invariant* k*-current*. In other words, it is a k-current defined on every transversal submanifold, and is invariant by holonomy. The vector space of invariant k-currents will be denoted by $C_k(\mathrm{Tr}\ \mathfrak{F})$. This is the natural generalization of the concept of holonomy invariant measure (cf. [2]). An invariant o-current will also be called an invariant distribution.

Let $f : X' \to X$ be a differentiable map transverse to $\mathfrak{F}$, and let $\mathfrak{F}' = f^{-1}(\mathfrak{F})$ be the foliation on X' inverse image by f of $\mathfrak{F}$. An immersion $t : T \to X'$ is a transversal submanifold to $\mathfrak{F}'$ iff $f \circ t$ is a transversal submanifold to $\mathfrak{F}$. One has a well-defined morphism of the holonomy pseudogroup induced by $\mathfrak{F}'$ on T in the holonomy pseudogroup induced by $\mathfrak{F}$ on $f \circ T$, hence a functorial morphism

$$\Omega_c^*(\mathrm{Tr}\, f^{-1}\mathfrak{F}) \to \Omega_c^*(\mathrm{Tr}\ \mathfrak{F}).$$

A regular covering of $\mathfrak{F}$ will be a covering of X by open sets U_i such that:

(i) The space of leaves of the foliation $\mathfrak{F}_i$ induced by $\mathfrak{F}$ on U_i is a q-manifold T_i, the natural projection $f_i : U_i \to T_i$ being a submersion. The inverse images $f^{-1}(y), y \in T_i$, are the plaques in U_i.

(ii) Each plaque $f_i^{-1}(y_i)$ in U_i meets at most one plaque $f_y^{-1}(y_j)$.

Let h_{ji} be the diffeomorphism mapping y_i on y_j; it is a diffeomorphism of an open set of T_i on an open set of T_j. Let T be the disjoint union of the T_i, and let H be the pseudogroup generated by the h_{ij}. It is easy to see that it is equivalent to the holonomy pseudogroup of $\mathfrak{F}$; it will be called *the holonomy pseudogroup associated to the regular covering* $\{U_i\}$.

2. EXAMPLES

2.1. Foliations given by closed 1-forms

Let T be the circle R/Z, and let H be the pseudogroup generated by a rotation $x \mapsto x + \rho$, where ρ is an irrational number. The Lebesgue measure is invariant by H, and any invariant distribution (or o-current) is a multiple of this measure. Any invariant 1-current is a multiple of the current defined by integration on H. Hence $C_0(T)^H$ and $C_1(T)^H$ are 1-dimensional.

Suppose that ρ *satisfies a diophantine condition*: namely, there are positive numbers s and c such that

$$|m\rho + n| \geqslant \frac{c}{(1 + m^2)^s},$$

for any integers $m, n \neq (0, 0)$. Then $\Omega^0(T/H)$ and $\Omega^1(T/H)$ are isomorphic to R. *Otherwise,* ρ *is called a Liouville number*; then $\Omega^0(T/H)$ and $\Omega^1(T/H)$ are not Hausdorff, but their quotient by the closure of 0 is still isomorphic to R.

The proof of these facts is a standard argument using Fourier series expansion. A function f on T with Fourier series $\Sigma_m f_m e^{2i\pi mx}$ is C^∞ iff for each positive integer k, there is a constant c such that

$$|f_m| < \frac{c}{(1 + m^2)^k}.$$

f is 0 in $\Omega^0(T/H)$ iff there is a C^∞-function g such that $f(x) = g(x) - g(x + \rho)$. A necessary condition is $\int_T f(x)\, dx = f_0 = 0$, and the Fourier coefficients g_m, $m \neq 0$, are uniquely defined (ρ is irrational). If ρ satisfies a diophantine condition, g_m will be the Fourier coefficients of a C^∞-function g; if ρ is a Liouville number, this will not be the case for a general f.

Let $\mathcal{F}$ be a foliation given on a compact manifold by a closed 1-form ω; the cohomology class of ω defines a homomorphism of $H_1(X, Z)$ in R whose image is called the group P of periods of ω. The holonomy pseudogroup is equivalent to the pseudogroup of T generated by the rotations $x \to x + \alpha/\alpha_0$, where α_0 is a fixed nonzero period and $\alpha \in P$. The rank of P is at least one and is larger than one iff every leaf is dense.

More generally, suppose that $\mathcal{F}$ is given by q independent closed 1-forms. They define a homomorphism of $H_1(X, Z)$ in $\mathbf{R}^q$ whose image P (the group of periods) is of rank q over $\mathbf{R}$. If X is compact, the holonomy pseudogroup is equivalent to the pseudogroup of transformations of $\mathbf{R}^q$ generated by the translations belonging to P. For everywhere-dense leaves, this is equivalent to the existence of periods $\alpha, \beta_1, \cdots, \beta_q \in P$ such that $\beta_1, \cdots, \beta_q$ are linearly independent over $\mathbf{R}$, and $\alpha = a_1\beta_1 + \cdots + a_q\beta_q$, where the real numbers

$1, a_1, \cdots, a_q$ are linearly independent over the rationals Q. The space $C_k(\mathrm{Tr}\ \mathfrak{F})$ of k-invariant currents is isomorphic to the space of $(q-k)$-forms on R^q invariant by all translations. The quotient of $\Omega^k(\mathrm{Tr}\ \mathfrak{F})$ by the closure of 0 is isomorphic to the invariant k-forms on $\mathbf{R}^q$, namely, to the dual of the k-exterior power $\Lambda^k\mathbf{R}^q$ of $\mathbf{R}^q$. However, if $(a_1, \cdots, a_q)$ satisfies a diophantine condition (cf. for instance Hermann [1]), then $\Omega^k(\mathrm{Tr}\ \mathfrak{F})$ is actually isomorphic to the dual of $\Lambda^k R^q$.

The previous examples are particular cases of transversely homogeneous foliations. Let G/H be a homogeneous space, where H is a closed subgroup of the Lie group G. We assume that G acts effectively on G/H and that G/H is simply connected. A transversely homogeneous foliation $\mathfrak{F}$ on X is given by an open covering $\{U_i\}$ and local submersions $f_i : U_i \to G/H$ such that the transition diffeomorphisms h_{ij} are restrictions of translations of G/H by elements of G. To such a foliation is associated a homomorphism

$$\Phi : \pi_1(X, x) \to G$$

whose image Γ is called the global holonomy group of $\mathfrak{F}$. On the covering $\tilde{X}$ of X corresponding to the kernel of Φ, the induced foliation $\tilde{\mathfrak{F}}$ is given by a submersion $f : \tilde{X} \to G/H$ which is Γ-equivariant, Γ acting on X by covering translations (cf. Haefliger, Comment. Math. Helv. **32** (1958) 280–281).

If X and H are compact, it is easy to see that f is a fiber map with connected fibers. Then it follows that the *holonomy pseudogroup of $\mathfrak{F}$ is generated by Γ acting on G/H.*

In general, for an homogeneous space G/H of dimension n (for which the action of G preserves an orientation), the k-currents invariant by G are given by the G-invariant $(n-k)$-forms on G/H (such a form α defines the current c associating to a k-form ω with compact support on G/H the number $\int \alpha \wedge \omega$).

If X and H are compact, $\mathfrak{F}$ has an everywhere-dense leaf iff Γ is dense in G. In that case the holonomy invariant currents are precisely the G-invariant forms on G/H.

2.2. Reeb component

Let R be the solid torus $S^1 \times D^2$ with a Reeb foliation such that the infinitesimal holonomy group of ∂R is nontrivial. Then the holonomy pseudogroup is equivalent to the pseudogroup H of transformations of $T = [0, \infty[$ generated by $h : x \to \lambda x$, where $0 < \lambda < 1$. Thus $\Omega_c^0(T/H)$ is isomorphic to the space of h-invariant C^∞ functions on $]0, \infty[$ (which is itself isomorphic to space of C^∞ functions on the circle). The isomorphism maps the class of

$f \in \Omega_c^0(T)$ on the function on $]0, \infty[$ given by

$$x \to \sum_{m=-\infty}^{+\infty} \lambda^m x f'(\lambda^m x).$$

Similarly $\Omega_c^1(T/H)$ is isomorphic to the space of h-invariant 1-forms on $]0, \infty[$.

3. INTEGRATION ALONG THE LEAVES

3.1. Theorem. *$\mathfrak{F}$ be a foliation on X with leaves of dimension p, and assume that the tangent bundle to the leaves is oriented. Then there is a continuous open surjective linear map*

$$\int_{\mathfrak{F}} : \Omega_c^{p+k}(X) \to \Omega_c^k(\mathrm{Tr}\ \mathfrak{F})$$

which commutes with d.

Proof. The construction is directly inspired by the construction of the Ruelle-Sullivan current associated to an invariant measure [4].

First recall that if $f : X \to Y$ is a submersion of a $(p + q)$-manifold X in a q-manifold Y, the fibers $f^{-1}(y)$ being coherently oriented, there is a continuous map

$$\int_{\mathfrak{F}} : \Omega_c^{p+k}(X) \to \Omega_c^k(Y)$$

commuting with d. If ω has its support in a coordinate neighborhood where f is expressed as the linear projection

$$f(x', \cdots, x^p, y', \cdots, y^q) = (y', \cdots, y^q),$$

$$\omega = \sum_J a_J(x, y)\, dy^J \wedge dx^i \wedge \cdots \wedge dx^p$$

$$+ \text{terms of degree} < p \text{ in the } x^i,$$

then

$$\int_f \omega = \left(\sum \int a_J(x, y)\, dx^1 \cdots dx^p \right) dy^J.$$

Let $\{U_i\}$ be a regular covering of X for $\mathfrak{F}$, with projections $f_i : U_i \to T_i$. Let T be the disjoint union of the T_i, and H the induced holonomy pseudogroup generated by the h_{ij} (cf. §1.3.). Given $\omega \in \Omega_c^{p+k}(X)$, we can express it as a finite sum $\omega = \sum \omega_i$, where the support of ω_i is in U_i. $\int_{\mathfrak{F}} \omega$ will be defined as the class in $\Omega_c^k(T/H) = \Omega_c^k(\mathrm{Tr}\ \mathfrak{F})$ of $\sum \bar{\omega}_i$, where $\bar{\omega}_i = \int_{f_i} \omega$. The class of $\sum \bar{\omega}_i$ is independent of the decomposition of ω. Indeed, if $\{\lambda_i\}$ is a partition of unity subordinated to $\{U_i\}$, then

$$\sum_i \int_{f_i} \omega_i = \sum_{i,j} \int_{f_i} \lambda_j \omega_i$$

is equivalent to

$$\sum_{i,j} \int_{f_j} \lambda_j \omega_i = \sum_j \int_{f_j} \lambda_j \omega,$$

because if the support of α is in $U_i \cap U_j$, then

$$\int_{f_i} \alpha = h_{ij}^* \int_{f_j} \alpha.$$

It is obvious that this map is continuous and commutes with d. One easily shows that it is independent of the choice of the regular covering (by passing to common refinements).

Corollary. *The transpose of $\int_{\mathfrak{F}}$ gives a linear map*

$$C_k(\mathrm{Tr}\ \mathfrak{F}) \to C_{k+p}(X)$$

of the space of holonomy invariant k-currents in the space of $(p + k)$-currents on X. This map commutes with the boundary operator δ.

This is a straightforward generalization of the construction of Ruelle-Sullivan [4] associating to an invariant measure a p-current on X.

To see an example of a p-current on X arising from a holonomy invariant distribution which is not a measure, consider a Reeb foliation like in Example 2.2. Let L be a noncompact leaf, and ξ a vector field along L invariant by holonomy (i.e., projectable with respect to local projections). Let ω be a 2-form on X, and denote by $\xi.\omega$ its derivative in the direction of ξ (restricted to L). Then $\int_L \xi.\omega$ is finite and defines a 2-current on X which arises from a holonomy invariant distribution of order one.

3.2. The kernel of $\int_{\mathfrak{F}}$

Following the terminology of [5], a $(p + k)$-form is $\mathfrak{F}$-trivial if for any sequence $\xi_1, \cdots, \xi_{p+k}$ of vector fields such that p of them are tangent to $\mathfrak{F}$, then $\omega(\xi_1, \cdots, \xi_{p+k}) = 0$.

Theorem. *The kernel of $\int_{\mathfrak{F}}$ is the vector subspace generated by $\mathfrak{F}$-trivial forms and differential of $\mathfrak{F}$-trivial forms.*

Proof. We first prove the assertion in the particular case of the foliation given by the natural linear submersion $f: R^q \times R^p \to R^q$, where $f(x, y) = x$. Any $(p + k)$-form ω with compact support can be written as $\omega = \alpha + \beta$, where β is $\mathcal{F}$-trivial, and

$$\alpha = \sum a_I \, dx^I \wedge dy^1 \wedge \cdots \wedge dy^p,$$

where $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, $1 \leqslant i_1 < \cdots < i_k \leqslant q$. By assumption, for each I, $\int_{\mathbf{R}^p} a_I(x, y)\, dy^1 \wedge \cdots \wedge dy^p = 0$. Hence there are smooth $(p-1)$-forms γ_I on R^p depending smoothly on the parameter x, such that $d\gamma_I = a_i dy^1 \wedge \cdots \wedge dy^p$ (cf. [3] where a smooth homotopy operator is constructed). Let $\gamma = (-1)^k \Sigma\, dx^i \wedge \gamma_I$. Then $d\gamma = \alpha + \beta'$, where β' is $\mathcal{F}$-trivial. Hence $\omega = d\gamma - \beta' + \beta$ where γ, β and β' are $\mathcal{F}$-trivial.

We now consider the general case. To construct $\int_{\mathcal{F}}$ we use as before a regular covering of X such that each $f_i : U_i \to T_i$ is diffeomorphic to a linear projection as above. If ω is $\mathcal{F}$-trivial, then using a partition of unity we can express it as a finite sum of $\mathcal{F}$-trivial forms ω_i with support in U_i. Thus it is clear that

$$\int_{f_i} \omega_i = 0, \int_{f_i} d\omega_i = 0.$$

Conversely, assume that $\int_{\mathcal{F}} \omega = 0$. This means that there are k-forms β_{ji} with compact support in T_i such that

$$\sum_i \int_{f_i} \omega_i = \sum_{i,j} h_{ij}^*(\beta_{ji}) - \beta_{ji}.$$

Hence

$$\int_{f_i} \omega_i = \sum_j h_{ji}^*(\beta_{ij}) - \beta_{ji}.$$

Let α_{ji} be $(p + k)$-forms with compact support in $U_i \cap U_j$ such that

$$\int_{f_i} \alpha_{ji} = \beta_{ji}.$$

Note that

$$\int_{f_i} \alpha_{ji} = h_{ij}^*(\beta_{ji}),$$

hence

$$\int_{f_i} \tilde{\omega}_i = 0,$$

where

$$\tilde{\omega}_i = \omega_i - \sum_j (\alpha_{ij} - \alpha_{ji}).$$

It follows from the particular case that each $\tilde{\omega}_i$ is the sum of a $\mathcal{F}$-trivial form and the differential of a $\mathcal{F}$-trivial form. But this is also true for ω because $\omega = \sum \tilde{\omega}_i$.

3.3. Interpretation of $\Omega_c^0(\mathrm{tr}\ \mathcal{F})$

Let $\Omega_c^r(\mathcal{F})$ be the vector space of smooth r-forms along the leaves (namely, the smooth sections of the rth exterior power of the cotangent bundle of the leaves). The differential $\Omega_c^r(\mathcal{F}) \to \Omega_c^{r+1}(\mathcal{F})$ along the leaves will be denoted by d_0. If we denote by $X_{\mathcal{F}}$ the set X which is the union of the leaves of $\mathcal{F}$ and considered as a manifold of dimension p, then the identity map $j : X_{\mathcal{F}} \to X$ is an immersion. $\Omega_c^r(\mathcal{F})$ is the image in $\Omega^r(X_{\mathcal{F}})$ by j^* of $\Omega_c^r(X)$, and d_0 is the restriction to $\Omega_c^r(\mathcal{F})$ of the differential in $\Omega^r(X_{\mathcal{F}})$. Let $H^r(\mathcal{F})$ be the r-th cohomology group of $\Omega_c^*(\mathcal{F})$. This is (almost by definition) the r-th cohomology group of X with value in the sheaf of germs of smooth functions which are constant on the leaves.

Corollary. $\Omega^0(\mathrm{Tr}\ \mathcal{F})$ *is canonically isomorphic to* $H_c^p(\mathcal{F})$, *where* $p = \dim \mathcal{F}$.

Indeed, $\Omega_c^p(\mathcal{F})$ is just the quotient of $\Omega_c^p(X)$ by $\mathcal{F}$-trivial forms. Also $j^* d\Omega_c^{p-1}(X) = d_0 \Omega_c^{p-1}(\mathcal{F})$.

4. APPLICATIONS TO FOLIATIONS BY MINIMAL LEAVES

Throughout this section we assume X to be compact and $\mathcal{F}$ oriented. The following theorem is a direct consequence of the preceding section.

4.1. Theorem. *A p-form ω_0 along $\mathcal{F}$ with compact support is the restriction of a relatively closed form ω with compact support if $d \int_{\mathcal{F}} \omega_0 = 0$ in $\Omega_c^1(\mathrm{Tr}\ \mathcal{F})$.*

Proof. Let $\tilde{\omega}$ be a p-form with compact support in X such that $\omega_0 = j^*\tilde{\omega}$. As $\int_{\mathcal{F}} \omega_0 = \int_{\mathcal{F}} \tilde{\omega}$ and $d \int_{\mathcal{F}} \tilde{\omega} = \int_{\mathcal{F}} d\tilde{\omega} = 0$, by §3.2 there is a p-form $\alpha \in \Omega_c^p(X)$ which is $\mathcal{F}$-trivial (i.e., $j^*\alpha = 0$) such that $d\tilde{\omega} - d\alpha$ is $\mathcal{F}$-trivial. Then $\omega = \tilde{\omega} - \alpha$ is relatively closed and $j^*\omega = \omega_0$.

Corollary. *Let $\mathcal{F}$ be an oriented foliation on a compact manifold X. Let g_0 be a smooth riemannian metric along the leaves and let ω_0 be the volume form along the leaves defined by g_0 and the orientation of $\mathcal{F}$. Then there is a riemannian metric g on X inducing g_0 on the leaves and for which the leaves are minimal submanifolds iff $d \int_{\mathcal{F}} \omega_0 = 0$.*

This follows from the above theorem and the theorem of Rummler-Sullivan mentioned in the introduction (cf. [5] and [6]).

Corollary 2 (*Rummler* [6]). *Suppose that the foliation is a generalized Seifert bundle. Then the metric* g_0 *along the leaves extends to a riemannian metric* g *on* X *for which all the leaves are minimal iff the volume of each generic leaf* L *is constant.*

Corollary 3. *Let* $\mathcal{F}$ *be a foliation on a compact manifold* X *given by a closed* 1*-form* ω, *and assume that there are at least two* Q*-independent periods of* ω. *Then any riemannian metric on the leaves can be aproximated in the* C^∞*-topology by a metric which is the restriction to the leaves of a riemannian metric for which the leaves are minimal. If there are two periods whose ratio satisfies a diophantine condition, then any smooth metric on the leaves is the restriction of a metric on* X *for which the leaves are minimal.*

This follows from the considerations in Example 2.1.

Corollary 4. *Assume there is no holonomy invariant distribution. Then any riemannian metric* g_0 *on the leaves is close in the* C^∞*-topology to a metric which is the restriction of a riemannian metric on* X *for which the leaves are minimal.*

Proof. Let $\omega_0 \in \Omega^p(\mathcal{F})$ be the volume form of g_0. In any neighborhood of ω_0 there is a form $\bar{\omega}_0$ such that $\int \bar{\omega}_0 = 0$, because the map $\int_{\mathcal{F}}$ is open and, by assumption, 0 is dense in $\Omega_c^0(\mathrm{Tr}\ \mathcal{F})$. Now $\bar{\omega}_0$ is the volume form of a riemannian metric on the leaves close to g_0. So we can apply Corollary 1.

Remark. More generally, the conclusion of the first part of Corollary 3 is still valid for a transversely G/H-homogeneous foliation $\mathcal{F}$ on a compact manifold X with an everywhere dense leaf, assuming G compact connected. In that case, it follows from §2.1 that the space of holonomy invariant distributions is isomorphic to $\mathbf{R}$. Thus the quotient of $\Omega^0(\mathrm{Tr}\ \mathcal{F})$ by the closure of zero is isomorphic to $\mathbf{R}$, representative for its element being constant functions on G/H (which is compact by assumption). Hence, if ω_0 is a volume form on the leaves, then there is a constant c such that $\int \omega_0 - c$ is adherent to zero. So we can replace ω_0 as above by an arbitrary close form $\bar{\omega}_0$ such that $\int \bar{\omega}_0$ is equivalent to the constant c, and hence has zero differential.

Corollary 5. *Let* g_0 *be a riemannian metric on the leaves of an oriented foliation* $\mathcal{F}$ *on a compact manifold* X. *A necessary condition for* g_0 *to be arbitrarily close to the restriction to the leaves of a metric for which the leaves are minimal is that*

$$\left\langle c, d\int_{\mathcal{F}} \omega_0 \right\rangle = 0$$

for each holonomy invariant 1*-current* c, *where* ω_0 *is the volume form on the leaves defined by* g_0. *This condition is also sufficient if* $\mathcal{F}$ *is transversely oriented and of codimension* 1.

Proof. The necessity follows from Corollary 1, and the sufficiency is implied by the following assertion.

Claim. Let H be a pseudogroup of orientation-preserving local diffeomorphisms of a 1-dimensional manifold T. Assume that T has a finite number of connected components, and let f be a smooth function with compact support on T such that $\langle c, df \rangle = 0$ for each H-invariant 1-current c. Then arbitrarily close to f in the C^∞-topology, there is a smooth function g such that $dg = 0$ in $\Omega^1_c(T/H)$.

To prove this we can assume that H is irreducible in the following sense: we can order the connected components T_i of T so that for each i there is $h_i \in H$ with source an open set in $\bigcup_{j<i} T_j$ and target in T_i. By assumption, there is a sequence $\alpha_n \in \Omega^1_c(T)$ such that α_n converges to df in the C^∞-topology and $\alpha_n = 0$ in $\Omega^1_c(T/H)$. This implies that $\int_T \alpha_n = 0$, because integration on T gives an invariant current. If the integral of α_n on each T_i would be zero, then α_n would be the differential of a function f_n with compact support on T, and the sequence f_n (modified by suitable constants on the compact components T_i) would converge to f.

To achieve this condition, we argue by descending induction on r. Assume that $\int_{T_i} \alpha_n = 0$ for each $i > r$. Then one can find a sequence α'_n such that α'_n converges to df, α'_n is zero in $\Omega^1_c(T/H)$ and $\int_{T_i} \alpha_n = 0$ for $i > r - 1$. Indeed, choose a 1-form γ with compact support in the target of h_r such that $\int_{T_r} \gamma = 1$. Then we define

$$\alpha'_n = \alpha_n - c_n\gamma + h_r(c_n\gamma),$$

where $c_n = \int_{T_r} \alpha_n$. Note that c_n tends to zero because $\int_{T_r} \alpha_n$ converges to $\int_T df = 0$.

Remark. Corollary 5 implies the following. Let $\mathcal{F}$ be an oriented and transversely oriented foliation of codimension one. Then any metric on the leaves is arbitrarily close to the restriction of a metric on X for which all the leaves are minimal if and only if $\partial C_1(\mathrm{Tr}\ \mathcal{F}) = 0$, where $\partial : C_1(\mathrm{Tr}\ \mathcal{F}) \to C_0(\mathrm{Tr}\ \mathcal{F})$ is the dual of d.

As an example (besides the one given in Corollary 3), assume that the holonomy pseudogroup of $\mathcal{F}$ is equivalent to the pseudogroup generated by a cocompact subgroup Γ of $PSl_2(\mathbf{R})$ acting as usual on S^1 identified with the boundary of the Poincaré disk D. (For instance, $\mathcal{F}$ might be the Anosov foliation associated to the geodesic flow on a compact riemann surface with constant negative curvature). The only Γ-invariant 1-current on S^1 are the multiple of the current defined by integration on S^1. Indeed any 1-current c on S^1 is the restriction to S^1 of a harmonic function f on D, and if c is Γ-invariant, then f is also Γ-invariant, and hence constant because $\Gamma \setminus D$ is compact. So any Γ-invariant 1-current has a trivial boundary.

4.1. Theorem. *On the compact manifold X there is a metric such that the leaves of $\mathcal{F}$ are minimal submanifolds iff for a representative H of the holonomy pseudogroup acting on a q-manifold T, there is a smooth positive function f with compact support, which is strictly positive on a set intersecting each orbit, and satisfies that $df = 0$ in $\Omega_c^1(T/H)$.*

Before giving the proof of this theorem, we state two corollaries.

Corollary 1. *The existence of a riemannian metric for which the leaves are minimal depends only on the holonomy pseudogroup of $\mathcal{F}$.*

Corollary 2. *If there is a representative H for the holonomy pseudogroup acting on a compact manifold T, then there is a metric for which the leaves are minimal.*

Indeed we can choose $f \equiv 1$. For instance this is the case if the holonomy pseudogroup is generated by a discrete subgroup of a Lie group acting on a compact manifold. Such an example is given by a foliation defined by q independent closed 1-forms.

Proof of the theorem. First we note that the existence of such an f is independent of the representative for the holonomy pseudogroup.

More precisely, let H' be a pseudogroup acting on T' which is equivalent to H by an isomorphism $\Phi : H' \to H$. Let K' be a compact set intersecting each orbit of H'. Then there is a positive smooth function f' with compact support equivalent to f and which is strictly positive on K'. To see that, we choose a finite number of $\varphi_i \in \Phi$, $i = 1, \cdots, r$, whose domains U_i cover K' such that f is strictly positive on $\varphi_i(U_i)$. One can find a covering of K by compact sets $K_i \subset K' \cap U_i$. Let φ_j, $r < j \leqslant s$ be elements of Φ such that the ranges of the φ_k, $1 \leqslant k \leqslant s$, cover the support S of f. Choose a partition of unity λ_k subordinated to the covering of T by the ranges of the φ_k, $1 \leqslant k \leqslant s$ (and also the complement of S). We can choose the λ_i strictly positive on the $\varphi_i(K_i)$ for $1 \leqslant i \leqslant r$. Then

$$f' = \sum_1^s \varphi_i^*(\lambda_i f)$$

is the desired function.

Let $\{U_i\}$ be a finite regular covering of X for $\mathcal{F}$ with local projections f_i: $U_i \to T_i$. We can assume that the f_i are diffeomorphic to natural projections $U_i = T_i \times R^q \to T_i$. Let $\{V_i\}$ be a covering of X by compact sets V_i contained in U_i. In each U_i we can construct a closed p-form α_i, whose restriction to each plaque P of U_i has compact support, is strictly positive on $P \cap V_i$, and satisfies $\int_P \alpha_i = 1$. Let H be the holonomy pseudogroup induced on T = union of T_i. By hypothesis and the preceding considerations, we can find a smooth positive function f with compact support on T, which is strictly positive on each $K_i = f_i(V_i)$ and satisfies that $df = 0$ in $\Omega^1(T/H)$. Let g_i be the restriction of f to T_i.

Then $\omega = \Sigma\, \omega_i$. where $\omega_i = f_i^*(g_i)\alpha_i$ is a p-form on X, which is positive on the leaves and whose integral over $\mathfrak{F}$ is equivalent to f. Then we can apply Corollary 1 of Theorem 4.1.

4.3. Examples of foliations having no riemannian metric for which the leaves are minimal

This will be in particular the case for a foliation $\mathfrak{F}$ having a positive holonomy invariant measure which is the boundary of an invariant 1-current (cf. [7]). Indeed in this case, for any p-form ω_0 positive on the leaves, we have

$$c\left(d\int_F \omega_0\right) = \partial c\left(\int_F \omega_0\right) > 0.$$

For instance in the case of codimension 1, let R be a Reeb component with boundary δR; a transversal curve entering R cannot cross the boundary again. The 1-current defined by the integral on positively oriented transversal curves is an invariant current whose boundary is the Dirac measure corresponding to δR.

In the case of the horocycle flow (cf. Sullivan [6]), one has on the transverse submanifold a positive invariant 2-form which is the exterior differential of an invariant 1-form (defining an invariant 1-current). This example can be generalized as follows. Let G be a semisimple Lie group acting on a manifold M of dimension n so that the induced action on the space T_0^*M of nonzero cotangent vectors is transtive. For instance, G might be the conformal group $0(n+1, 1)$ acting on the n-sphere S^n or the linear group $Sl(n+1, R)$ acting on S^n identified to the rays in R^{n+1}.

On T^*M, one has the canonical 1-form ω which is invariant by the differential of any diffeomorphism of M, and whose exterior differential $d\omega$ is the canonical symplectic form. Then $(d\omega)^n$ is a volume form on T_0^*M, which is the differential of $\omega \wedge (d\omega)^{n-1}$. This form defines a 1-current invariant by the differential of any diffeomorphism, and its boundary is the invariant measure defined by $(d\omega)^n$.

Let Γ be a discrete subgroup of G such that $\Gamma \setminus G$ is compact. Let H be the subgroup of G leaving a given covector fixed. Then the cosets gH are the leaves of a foliation on G parametrized by the space T_0^*M. This foliation is invariant by the left action of Γ on G. So we get on $\Gamma \setminus G$ a foliation whose transverse structure is T_0^*M, the holonomy pseudogroup being generated by Γ.

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