

CLIFFORD BUNDLES, IMMERSIONS OF MANIFOLDS AND THE VECTOR FIELD PROBLEM

H. B. LAWSON, JR. & M. L. MICHELSON

TABLE OF CONTENTS

0. Introduction	237
1. The structure of Clifford algebras	239
2. The geometry of Clifford bundles and the Dirac operator	242
3. The fundamental theorems of immersions	247
4. Applications to projective spaces and algebraic varieties	257
5. The fundamental theorems for vector fields	261
6. Theorems on the geometric dimension	263

0. Introduction

The point of this article is to present a simple and unified approach to both the immersion problem and the vector field problem for manifolds by using the formalism of Clifford bundles. The fundamental constructions are based on the work of Atiyah [1] and involve the study of certain natural first order elliptic operators. The method not only applies to a broad spectrum of problems but also yields quite delicate results. It recaptures, for example, all the non-immersion and non-embedding theorems known to date concerning complex and quaternion projective spaces.

In §4 we will show that if a certain condition holds on these spaces, then our theorems will improve the old non-immersion results, and we conjecture that the resulting theorems would be sharp. Evidence to that effect comes from recent work of Davis and Mahowald.

In general outline, our approach is the following. To any riemannian manifold X there is naturally associated a bundle $Cl(T)$ whose fiber at a point x is the Clifford algebra of the tangent space $T_x(X)$. One now studies bundles of modules over this bundle of algebras. To any such bundle of modules M with an appropriate connection one can associate an elliptic, first order differential operator D which we call the Dirac operator of M . If X is

oriented and even-dimensional, there is a simple decomposition $M = M^+ \oplus M^-$, such that the restriction of D gives an elliptic operator $D^+ : \Gamma(M^+) \rightarrow \Gamma(M^-)$. The index of $D^+ = \dim(\ker D^+) - \dim(\text{coker } D^+)$ is a topological invariant which is easily computed via the Atiyah-Singer theorem [5]. (This construction is commonly employed in the study of spin manifolds.)

Suppose now that there is a smooth immersion $X^n \hookrightarrow S^{n+q}$ with normal bundle $N(X)$, and consider the bundle $Cl(T \oplus N)$ whose fiber at a point x is the Clifford algebra of $T_x(X) \oplus N_x(X)$. This is evidently a bundle of left and right modules over $Cl(T)$, in fact, over $Cl(T \oplus N)$. Using left module multiplication we decompose this bundle as above and obtain an operator $D^+ : \Gamma(Cl^+(T \oplus N)) \rightarrow \Gamma(Cl^-(T \oplus N))$. The subbundles $Cl^\pm(T \oplus N)$ are each modules by right multiplication over $Cl(T \oplus N)$. Since $T \oplus N$ is trivial, it admits $n + q$ pointwise orthonormal sections $\varepsilon_1, \dots, \varepsilon_{n+q}$. These sections generate a finite group in $\Gamma(Cl(T \oplus N))$. Averaging D^+ over this group produces a new operator \tilde{D}^+ with the same first order part and, therefore, the same index as D^+ . However, the kernel and cokernel of \tilde{D}^+ are now modules over Γ , i.e., they are modules for the Clifford algebra Cl_{n+q} , which is the group algebra of Γ . These modules are naturally \mathbf{Z}_2 -graded. Computing the index of D^+ , we obtain the following result. If $n \equiv 0 \pmod{4}$ and there exists a smooth immersion $X^n \hookrightarrow S^{n+q}$, then

$$2^q A(X) \equiv 0 \pmod{2a_{n+q}},$$

where $A(X)$ is the so-called A -genus of X (cf. [8]), and $2a_n$ is the dimension of an irreducible, real \mathbf{Z}_2 -graded module over Cl_{n+q} . (See §1.)

This is an exact analogue of Atiyah's proof [1] that if $n \equiv 0 \pmod{4}$, and X admits q everywhere linearly independent vector fields, then

$$L(X) \equiv 0 \pmod{2a_q},$$

where $L(X)$ is the signature of X .

The above construction is easily generalized by taking coefficients in a bundle E . One then obtains the following.

Theorem. *Let X^n be a compact oriented manifold of even dimension n . If there exists a smooth immersion $X^n \hookrightarrow S^{n+q}$, then*

$$2^q \{ch_2 E \cdot \mathbf{A}(X)\} [X] \equiv 0 \pmod{2b_{n+q}},$$

and if, on the other hand, X admits q linearly independent vector fields, then

$$\{ch_2 E \cdot \mathbf{L}(X)\} [X] \equiv O(2b_q),$$

for any complex vector bundle E over X .

Here $2b_k$ denotes the complex dimension of an irreducible complex \mathbf{Z}_2 -graded module over Cl_k ; $ch_2 E = \sum 2^k ch^k E$ where $ch E = \sum ch^k E$, $ch^k E \in H^{2k}(X; \mathbf{Q})$ is the Chern character of E ; and $A(X)$ and $L(X)$ denote the total A and L classes of X in the sense of Hirzebruch [8].

When q is even, the congruences in the theorem above can sometimes be improved by replacing 2^q with 2^{q-1} . In the case that E is real or quaternionic, further refinements of the theorem can be established by making careful use of the structure of Clifford algebras and their representations. (See §3 for precise statements.)

The theorems proved in §§3 and 5 recapture the results of a number of people including Atiyah and Hirzebruch [3], Mayer [11], Frank [7], and Schwarz [12]. The method employed here gives a certain new geometric insight into these results and has the advantage of being both conceptually and computationally simple.

The paper is organized as follows. In the first two sections we review some basic material concerning Clifford bundles and Dirac operators. In §§3 and 5 we prove the general theorems for immersions and vector fields. In §4 we compute specific results for immersions of CP^n and HP^n into euclidean space. In the last section we prove general results concerning the geometric dimension of the tangent bundle of a manifold. We then compute this bound precisely for CP^n and HP^n . We also give bounds for the geometric dimension of $\xi \oplus \dots \oplus \xi$ (N times) where ξ is the canonical (hyperplane) bundle over CP^n .

We would like to thank Michael Crabb for several valuable conversations related to this work.

1. The Structure of Clifford algebras

We shall present in this section a quick review of the theory of Clifford algebras and their real representations. For more details the reader is referred to the fundamental paper of Atiyah, Bott and Shapiro [2].

Let V be a real vector space with a quadratic form q . Associated to this pair is the Clifford algebra

$$Cl(V, q) = \mathfrak{T}(V)/\mathfrak{I}(q),$$

where $\mathfrak{T}(V) = \sum \otimes^r V$ is the tensor algebra of V , and where \mathfrak{I} is the ideal generated by the elements $v \otimes v + q(v)$ for $v \in V$. There is a canonical inclusion $i: V \hookrightarrow Cl(V, q)$ which comes from the degree 1 inclusion $V = \otimes^1 V \subset \mathfrak{T}(V)$. Any map $f: V \rightarrow \mathcal{A}$, where \mathcal{A} is an associative algebra with unit, and the property that $f(v) \cdot f(v) + q(v) = 0$ extends to a unique algebra homomorphism $\hat{f}: Cl(V, q) \rightarrow \mathcal{A}$.

In particular the map $v \rightarrow -v$, sending $V \rightarrow V \subset Cl(V, q)$, has this property. It therefore extends to an algebra automorphism $\alpha: Cl(V, q) \rightarrow Cl(V, q)$ with $\alpha^2 = 1$. Let $Cl^0(V, q)$ and $Cl^1(V, q)$ denote the 1 and -1 eigenspaces of α respectively. Then the decomposition

$$(1.1) \quad Cl(V, q) = Cl^0(V, q) \oplus Cl^1(V, q)$$

gives $Cl(V, q)$ the structure of a \mathbb{Z}_2 -graded algebra, that is, $Cl^i(V, q) \cdot Cl^j(V, q) \subset Cl^{i+j}(V, q)$ where the indices are taken mod 2.

We shall only be concerned with the case where V is finite dimensional and q is positive definite. Let e_1, \dots, e_n be an orthonormal basis for V with respect to the inner product determined by q . Then $Cl(V, q)$ is isomorphic to the associative algebra generated by e_1, \dots, e_n subject to the relations:

$$(1.2) \quad e_i e_j + e_j e_i = -2\delta_{ij}$$

for $1 \leq i, j \leq n$. We denote this algebra by Cl_n .

The algebras Cl_n have been determined up to isomorphism in [2]. They satisfy the periodicity relation $Cl_{n+8} = Cl_n \otimes Cl_8$. The first eight are given by the following table

n	1	2	3	4	5	6	7	8
Cl_n	\mathbb{C}	\mathbb{H}	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$

TABLE 1

Here $K(n)$ denotes the algebra of $n \times n$ matrices over the field K , and \mathbb{C} denotes the complex numbers and \mathbb{H} the quaternions. One has the relation $K(16n) = K(n) \otimes \mathbb{R}(16)$.

Note that given an orientation in V one can define a canonical *volume element*

$$(1.3) \quad \omega = e_1 \cdots e_n,$$

where (e_1, \dots, e_n) is any oriented orthonormal basis of V . This element has the following properties:

$$(1.4) \quad \omega^2 = (-1)^{n(n+1)/2},$$

$$(1.5) \quad v\omega = (-1)^{n-1}\omega v \quad \text{for all } v \in V.$$

If $n \equiv 3$ or $4 \pmod{4}$, then $\omega^2 = 1$, and we can decompose the Clifford algebra into 1 and -1 eigenspaces under left multiplication by ω . That is, setting $Cl(V) = Cl(V, q)$ for convenience, we have a decomposition

$$(1.6) \quad Cl(V) = Cl^+(V) \oplus Cl^-(V),$$

where

$$(1.7) \quad Cl^\pm(V) = (1 \pm \omega) \cdot Cl(V),$$

If $n \equiv 3 \pmod{4}$, then ω is central, and the spaces $Cl^\pm(V)$ are (simple) subalgebras. The decomposition (1.6) corresponds to the decomposition seen in Table 1.

If $n \equiv 0 \pmod{4}$, then $Cl^\pm(V)$ are not subalgebras. In fact, from (1.5) we see that $L_v: Cl^\pm(V) \rightarrow Cl^\mp(V)$ where L_v denotes left multiplication by v for any $v \in V$. Note, however, that $Cl^\pm(V)$ are still invariant under right multiplication by elements of $Cl(V)$.

We now consider the question of real modules over the algebras Cl_n . From the classification above and the simplicity of the matrix algebras, we see that for $n \not\equiv 3 \pmod{4}$ there is only one equivalence class of irreducible modules over Cl_n . If $n \equiv 3 \pmod{4}$, there are two such classes. They have the same dimension.

For applications in topology it is useful to consider the notion of a \mathbb{Z}_2 -graded module over $Cl_n = Cl_n^0 \oplus Cl_n^1$. This is a module $M = M^0 \oplus M^1$ such that $Cl_n^i \cdot M^j \subset M^{i+j}$ where the indices are taken mod 2. There is a natural equivalence of the category of \mathbb{Z}_2 -graded modules over Cl_n with the category of ungraded modules over Cl_n^0 (cf. [2]). Furthermore, there is an algebra isomorphism

$$Cl_{n-1} \cong Cl_n^0$$

given by extending the map $\mathbb{R}^{n-1} \rightarrow Cl_n$ which sends $e_j \mapsto e_j e_n$ for $j = 1, \dots, n-1$. This means that the dimension $2a_n$ of an irreducible real \mathbb{Z}_2 -graded module over Cl_n is exactly twice the dimension a_n of an irreducible ungraded module over Cl_{n-1} . The same statement applies to the complex dimension $2b_n(2c_n)$ of an irreducible complex (quaternionic) \mathbb{Z}_2 -graded module over Cl_n . These numbers can be read directly from Table 1. For any n , $a_{n+8} = 16a_n$, $b_{n+8} = 16b_n$ and $c_{n+8} = 16c_n$. For $n \leq 8$, the numbers are given by the following table.

n	1	2	3	4	5	6	7	8
a_n	1	2	4	4	8	8	8	8
b_n	1	1	2	2	4	4	8	8
c_n	2	2	2	2	4	8	16	16

TABLE 2

In particular we have that:

$$(1.8) \quad a_{8n+r} = 2^{4n} a_r = 2^{4n+\nu},$$

$$(1.9) \quad c_{8n+r} = 2^{4n} c_r = 2^{4n+\mu},$$

where ν_r and μ_r for $1 \leq r \leq 8$ can be read from Table 2. We also see that

$$(1.10) \quad b_n = 2^{\lfloor (n-1)/2 \rfloor},$$

for all n .

Remark 1.1. For future use we make the following observation. Let $n \equiv 0 \pmod{4}$ and consider the subspaces Cl_n^\pm given by (1.7). Since $\omega \in Cl_n^0$, each of these spaces carries a \mathbb{Z}_2 -grading:

$$(1.11) \quad Cl_n^\pm = (1 \pm \omega) \cdot Cl^0 \oplus (1 \pm \omega) \cdot Cl^1.$$

Under right multiplication these form \mathbb{Z}_2 -graded Cl_n -modules.

2. The geometry of Clifford bundles and the Dirac operator

We shall now briefly review the notions of Clifford structures in riemannian geometry. For a detailed exposition of this subject the reader is referred to [10].

The first observation of this section is that any functorial construction for vector spaces with positive quadratic forms carries over naturally to the category of vector bundles with inner products. In particular, suppose E is an n -dimensional real vector bundle with a riemannian metric over a space X . Then one can naturally form the *Clifford bundle*

$$Cl(E) = \mathfrak{T}(E)/\mathfrak{I}(E),$$

where $\mathfrak{T}(E)$ is the bundle of tensor algebras of E , and $\mathfrak{I}(E)$ is the bundle of ideals generated by the elements $e \otimes e + \|e\|^2$ for $e \in E$. The fiber $Cl_x(E)$ at $x \in X$ is just the Clifford algebra of the fiber E_x . There is a natural inclusion $E \subset Cl(E)$. The bundle map $e \rightarrow -e$ sending $E \rightarrow E \subset Cl(E)$ extends to a bundle automorphism $\alpha: Cl(E) \rightarrow Cl(E)$ with $\alpha^2 = 1$. This gives a decomposition

$$(2.1) \quad Cl(E) = Cl^0(E) \oplus Cl^1(E)$$

into the 1 and -1 eigenbundles of α respectively. Under fiberwise multiplication we have $Cl^i(E)Cl^j(E) \subset Cl^{i+j}(E)$ where the indices are taken mod 2.

If E is oriented, there is an invariant and therefore globally defined *volume form*

$$(2.2) \quad \omega = e_1 \cdot \dots \cdot e_n,$$

where at $x \in X$, (e_1, \dots, e_n) is any oriented orthonormal basis of E_x . This form satisfies the relations (1.4) and (1.5). In particular, if $n \equiv 3$ or $4 \pmod{4}$, then there is a decomposition

$$(2.3) \quad Cl(E) = Cl^+(E) \oplus Cl^-(E)$$

into the 1 and -1 eigenbundles of left multiplication L_ω by ω . Again we have that

$$(2.4) \quad Cl^\pm(E) = (1 \pm L_\omega) \cdot Cl(E)$$

If $n \equiv 3 \pmod{4}$, then $\alpha: Cl^+(E) \rightarrow Cl^-(E)$ is a vector bundle isomorphism. If $n \equiv 0 \pmod{4}$ and e is a nowhere vanishing section of E , then left multiplication by e , $L_e: Cl^+(E) \rightarrow Cl^-(E)$ is a vector bundle isomorphism.

Suppose, more generally, that $M \rightarrow X$ is a bundle of left modules over the bundle of algebras $Cl(E)$. (For example, $Cl(E)$ is itself such a bundle. If E carries a spin structure, then the fundamental $Spin_n$ representations gives rise to such bundles.) Then if $n \equiv 3$ or $4 \pmod{4}$ we have a decomposition

$$(2.5) \quad M = M^+ \oplus M^-$$

into 1 and -1 eigenbundles for L_ω , where

$$(2.6) \quad M^\pm = (1 \pm L_\omega)M.$$

If $n \equiv 0 \pmod{4}$, then by (1.5) we see that at each $x \in X$, left multiplication gives an isomorphism

$$(2.7) \quad L_e: M_x^\pm \rightarrow M_x^\mp$$

for all nonzero $e \in E_x$.

We now suppose that X is a smooth manifold and that all vector bundles under discussion are smooth. Recall that a *connection* on a vector bundle E over X is a linear map

$$\nabla: \Gamma(E) \rightarrow \Gamma(T^* \otimes E),$$

where T^* denotes the cotangent bundle of X such that

$$(2.9) \quad \nabla(fe) = df \otimes e + f\nabla e$$

for all functions $f \in C^\infty(X)$ and all smooth sections $e \in \Gamma(E)$. This means that to any smooth vector field V on X we have assigned a differential operator $\nabla_V: \Gamma(E) \rightarrow \Gamma(E)$ satisfying $\nabla_V(fe) = (Vf)e + f\nabla_V e$ for f and e as above. The value of $\nabla_V e$ at a point $x \in X$ depends only on V_x and the first order jet of e at x . If E has an inner product $\langle \cdot, \cdot \rangle$, we say ∇ is *riemannian* if

$$(2.10) \quad d\langle e_1, e_2 \rangle = \langle \nabla e_1, e_2 \rangle + \langle e_1, \nabla e_2 \rangle$$

for all $e_1, e_2 \in \Gamma(E)$. It is not difficult to show that riemannian connections always exist.

Let E carry a riemannian connection ∇ . Then there is a unique extension of ∇ to a riemannian connection on $Cl(E)$ with the property that

$$(2.11) \quad \nabla(\varphi \cdot \psi) = (\nabla\varphi) \cdot \psi + \varphi \cdot (\nabla\psi)$$

for all $\varphi, \psi \in \Gamma(Cl(E))$. Furthermore, if M is any riemannian bundle of modules over $Cl(E)$, i.e., Clifford multiplication by unit vectors in E is orthogonal on the fibers of M , then there is a riemannian connection ∇ on M such that

$$(2.12) \quad \nabla(\varphi \cdot \sigma) = \nabla(\varphi) \cdot \sigma + \varphi \cdot (\nabla\sigma)$$

for all $\varphi \in \Gamma(Cl(E))$ and all $\sigma \in \Gamma(M)$. See [10] for details. In the cases considered in this paper we shall construct these connections explicitly.

Recall that if E is oriented, there is a globally defined volume form ω (cf. (2.2)). It is a straightforward computation to see that ω is parallel, that is,

$$(2.13) \quad \nabla\omega = 0$$

for any riemannian connection ∇ on E , extended canonically by derivations to $Cl(E)$.

Suppose now that X is a compact riemannian n -manifold. Let T denote its tangent bundle, and let ∇ be the canonical riemannian connection on T . Let M be a bundle of modules over $Cl(T)$, and suppose M carries a compatible riemannian metric with a riemannian connection also denoted ∇ .

Under these general hypotheses one can define an elliptic, first order differential operator

$$D: \Gamma(M) \rightarrow \Gamma(M)$$

by setting

$$(2.14) \quad D\sigma = \sum_{j=1}^n e_j \cdot (\nabla_{e_j} \sigma),$$

where at the point x in question (e_1, \dots, e_n) represents any orthonormal basis of T_x . Locally on X we may choose (e_1, \dots, e_n) to be a smooth frame field, so it is clear that D maps smooth sections to smooth sections. It is easy to see that D is elliptic. In fact for any $\xi \in T_x^* \cong T_x$ the symbol $\sigma_\xi: M_x \rightarrow M_x$ is just given by Clifford multiplication: $\sigma_\xi(\varphi) = \xi \cdot \varphi$. Since $\xi \cdot \xi \cdot \varphi = \|\xi\|^2 \varphi$, this map is an isomorphism for all $\xi \neq 0$.

We shall always assume that the connection on M has property (2.12). In this case D is a self-adjoint operator.

Suppose now that $n \equiv 0 \pmod{4}$, and consider the decomposition (2.5) of M under L_ω . From the derivation property (2.12) and the fact that $\nabla\omega = 0$ it follows that

$$(2.15) \quad D \circ L_\omega = -L_\omega \circ D.$$

Therefore by restriction we get an operator

$$(2.16) \quad D^+: \Gamma(M^+) \rightarrow \Gamma(M^-),$$

which is elliptic since at a cotangent vector $\xi \in T_x^* \cong T_x$ its symbol $\sigma_\xi^+ : M_x^+ \rightarrow M_x^-$ is again Clifford multiplication by ξ . The adjoint of D^+ is the operator $D^- : \Gamma(M^-) \rightarrow \Gamma(M^+)$ obtained also by restriction. Consequently D^+ has a well defined *analytic index*

$$(2.17) \quad \begin{aligned} i(D^+) &= \dim(\ker D^+) - \dim(\operatorname{coker} D^+) \\ &= \dim(\ker D^+) - \dim(\ker D^-). \end{aligned}$$

The main result used in this paper is the following case of the Atiyah-Singer Index Theorem [4], [5]. Let $\pi : T \rightarrow X$ be the bundle projection, and consider the pullback bundles π^*M^\pm . Since D^+ is elliptic, its symbol gives an isomorphism of π^*M^+ with π^*M^- in the complement of the zero-section. Consequently $\pi^*M^+ - \pi^*M^-$ is a well-defined element in the K -theory of T with compact support. Let $ch : K_{cpt.}(T) \rightarrow H_{cpt.}^*(T)$ denote the Chern character, and $\Psi : H^*(X) \rightarrow H_{cpt.}^*(T)$ the Thom isomorphism.

Theorem 2.1. (*Atiyah and Singer*). *Let X be a compact oriented manifold, and suppose $D^+ : \Gamma(M^+) \rightarrow \Gamma(M^-)$ is an elliptic operator. Then*

$$(2.18) \quad i(D^+) = \{ \Psi^{-1}ch(\pi^*M^+ - \pi^*M^-) \cdot \mathcal{G}(X) \} [X],$$

where $\mathcal{G}(X)$ denotes the total Todd class of the bundle $T \otimes \mathbb{C}$.

Note that this formula holds for any operator $\tilde{D}^+ : \Gamma(M^+) \rightarrow \Gamma(M^-)$ having the same symbol as D^+ .

The Todd class is given by the multiplicative sequence of Chern classes associated to the power series $p(x) = x/(1 - e^x)$. Hence, for $\dim(X) = n$ even, we have that

$$(2.19) \quad \mathcal{G}(X) = \prod_{j=1}^{n/2} \frac{x_j}{1 - e^{-x_j}} \cdot \frac{-x_j}{1 - e^{-x_j}} = 1 + \mathcal{G}_1(p_1) + \mathcal{G}_2(p_1, p_2) + \dots,$$

where $\mathcal{G}_k(p_1, \dots, p_k) \in H^{4k}(X)$, and $p_k \in H^{4k}(X)$ denotes the k th Pontryagin class of X . The p_k 's are computed formally in terms of the x_j 's by the formula

$$(2.20) \quad p_k = \sigma_k(x_1^2, \dots, x_{n/2}^2),$$

where σ_k denotes the k th elementary symmetric function.

Remark 2.2. The index theorem above applies to both real and complex operators. In the case that M^\pm are complex bundles and D^+ is complex linear, the index of D^+ is defined as $\dim_{\mathbb{C}}(\ker D^+) - \dim_{\mathbb{C}}(\operatorname{coker} D^+)$. If M^\pm are real bundles, then the index is given as $\dim_{\mathbb{R}}(\ker D^+) - \dim_{\mathbb{R}}(\operatorname{coker} D^+)$, and $ch M^\pm$ is taken to mean $ch(M^\pm \otimes \mathbb{C})$. The real case can be deduced immediately from the complex one by complexifying the bundles M^\pm .

Suppose now that M^+ and M^- are associated to the tangent frame bundle $P(X)$ by linear representations of SO_n . Then M^\pm are induced from bundles \hat{M}^\pm over BSO_n by the classifying map f_T for T . Atiyah and Singer [5, §2] show that in $H^*(BSO_n; \mathbf{Q})$ there is a factorization $ch M^+ - ch M^- = \mathcal{Q} \cdot e$, where $e = x_1 \cdots x_n$ denotes the universal Euler Class, and that the first factor appearing in (2.18) is given by

$$\Psi^{-1}ch(\pi^*M^+ - \pi^*M^-) = f_T^*\mathcal{Q} \equiv \frac{ch M^+ - ch M^-}{e}.$$

Consequently, when M^\pm are associated to the tangent frame bundle of X , we have that

$$(2.21) \quad i(D^+) = \left\{ \frac{ch M^+ - ch M^-}{e} \cdot \mathcal{G}(X) \right\} [X].$$

Example. The simplest case of an operator of type (2.16) on a manifold of dimension $n \equiv 0 \pmod{4}$ is given by choosing $M^\pm = Cl^\pm(T)$. (See (2.3) above). In this case

$$ch M^+ - ch M^- = \prod_{j=1}^{n/2} (e^{x_j} - e^{-x_j}),$$

where the x_j 's have the same meaning as above, that is, the Pontryagin classes of X are given formally as the elementary symmetric functions in the x_j^2 . For a proof of this fact see the discussion following (3.11) below. Now using formula (2.19) and the fact that $e = x_1 \cdots x_{n/2}$, we see that

$$(2.22) \quad \frac{ch M^+ - ch M^-}{e} \cdot \mathcal{G}(X) = \prod_{j=1}^{n/2} \frac{x_j}{\tanh(x_j/2)}.$$

If, more generally, we assume that $M^\pm = Cl^\pm(T) \otimes E$ for some vector bundle E over X , then

$$(2.23) \quad \frac{ch M^+ - ch M^-}{e} \cdot \mathcal{G}(X) = ch E \cdot \prod_{j=1}^{n/2} \frac{x_j}{\tanh(x_j/2)}.$$

This formula will be useful in later computations.

We conclude this section with some useful technical comments. Let E_1 and E_2 be two riemannian vector bundles over X with riemannian connections ∇^1 and ∇^2 respectively.

Remark 2.3. There is a natural riemannian connection $\nabla^1 \oplus \nabla^2$ defined on the (orthogonal) Whitney sum $E_1 \oplus E_2$ by setting $(\nabla^1 \oplus \nabla^2)(e_1 \oplus e_2) = \nabla^1 e_1 \oplus \nabla^2 e_2$. This is called the *direct sum* connection.

E_1 and E_2 are both oriented, and ω_1 and ω_2 are the corresponding volume forms (cf. (2.2)), then $\nabla^j \omega_j = 0$ for $j = 1, 2$. Therefore *both ω_1 and ω_2 are parallel in the direct sum connection.*

Remark 2.4. There is a natural riemannian connection $\nabla^1 \otimes \nabla^2$ defined on the tensor product $E_1 \otimes E_2$ by setting $(\nabla^1 \otimes \nabla^2)(e_1 \otimes e_2) = (\nabla^1 e_1) \otimes e_2 + e_1 \otimes (\nabla^2 e_2)$. This is called the *tensor product connection*.

Hence the operations \oplus and \otimes have a natural meaning in the category of riemannian bundles over X with riemannian connection.

Remark 2.5. Any riemannian connection ∇ on $E_1 \oplus E_2$ induces riemannian connections ∇^j on E^j by setting $\nabla^j_\nu e = \pi^j(\nabla_\nu e)$ where $\pi^j: E \rightarrow E_j$ is the orthogonal bundle projection. In this way we produce a new connection $\nabla^1 \oplus \nabla^2$ on $E_1 \oplus E_2$ which we call *the projection of ∇* .

Remark 2.6. There is a natural vector bundle isomorphism

$$Cl(E_1 \oplus E_2) \cong Cl(E_1) \otimes Cl(E_2).$$

In fact, if multiplication on $Cl(E_1) \otimes Cl(E_2)$ is defined in the \mathbf{Z}_2 -graded sense of Atiyah, Bott and Shapiro [2, §1], then this becomes an algebra bundle isomorphism.

If one takes a direct sum connection on $E_1 \oplus E_2$, then the connection induced on $Cl(E_1) \otimes Cl(E_2)$ is the tensor product of the connections induced separately on the bundles $Cl(E_1)$ and $Cl(E_2)$.

Remark 2.7. If M is any bundle of modules over $Cl(E_1)$, then so is $M \otimes E_2$. If M carries a connection with property (2.12), then the tensor product connection on $M \otimes E_2$ also has this property.

The bundle $Cl(E_1) \otimes E_2$ is a bundle of left and right modules over $Cl(E_1)$. This follows directly from Remark 2.6 by noting that there is a natural containment $Cl(E_1) \otimes E_2 \subset Cl(E_1) \otimes Cl(E_2)$ which is stable under multiplication by $Cl(E_1) \cong Cl(E_1) \otimes 1$.

The verification of the remarks above is straightforward and is left to the reader.

3. The fundamental theorems for immersions

In this section we shall be concerned with the following question: given a compact differentiable n -manifold X^n , when does there exist a smooth immersion $X^n \hookrightarrow \mathbf{R}^{n+q}$ for $q < n$? Such an immersion always exists for $q = n - 1$. The point of this article will be to give lower bounds for q in terms of certain characteristic classes on X^n .

Before stating the main results we recall the notion of the total A -class of a manifold. This is a multiplicative sequence of Pontrjagin classes associated to the power series $p(z) = 2\sqrt{z} / \sinh(2\sqrt{z})$ (cf. [8]). In terms of the notation of

the last section,

$$A(X) = 1 + A_1(p_1) + A_2(p_1, p_2) + \cdots = \prod_{j=1}^{[n/2]} \frac{2x_j}{\sinh(2x_j)},$$

where $A_k(p_1, \dots, p_k) \in H^{4k}(X)$, and $p_k \in H^{4k}(X)$ is the k th Pontrjagin class computed formally as the k th elementary symmetric function of $x_1^2, \dots, x_{[n/2]}^2$. The sequence begins

$$A_1 = -\frac{2}{3}p_1,$$

$$A_2 = -\frac{2}{45}(7p_1^2 - 4p_2),$$

$$A_3 = -\frac{4}{315}(31p_1^3 - 44p_1p_2 + 16p_3).$$

If X is orientable and of dimension $n = 4k$, then the A -genus of X is defined to be the characteristic number $A(X) = A_k(X)[X] \in \mathbf{Q}$. This number is always an integer. (It is related to the so-called \hat{A} -genus by the formula $A(X) = 2^{4k}\hat{A}(X)$. The \hat{A} -genus is an integer if X is a spin manifold.) This number, of course, depends only on the oriented rational cobordism class of X .

We can now state the main results. For clarity of exposition we present these results and their proofs as a sequence of theorems of increasing generality. The first theorem embodies the basic construction. The subsequent theorems, although much stronger, are simply refinements obtained by using general coefficients and employing a detailed analysis of the representations of Clifford algebras.

Let $\{a_k\}_{k=1}^\infty$ be the sequence of integers defined at the end of §1. (See (1.8).)

Theorem 3.1. *Let X^n be a compact oriented manifold of dimension $n \equiv 0 \pmod{4}$. If there exists an immersion $X^n \looparrowright S^{n+q}$, then*

$$2^{q-1}A(X) \equiv 0 \pmod{a_{n+q}}.$$

If furthermore, $q \equiv 0 \pmod{4}$ and the normal Euler class $e_N = 0$, then

$$2^{q-2}A(X) \equiv 0 \pmod{a_{n+q}}.$$

The proof of this theorem gives the main construction. It can be strengthened by taking coefficients in an arbitrary bundle E over X . To state the stronger version we need the following definition. Let $chE = ch^0E + ch^1E + \cdots$, $ch^kE \in H^{2k}(X; \mathbf{Q})$, be the Chern character of E . Then for $t \in \mathbf{R}$ we define

$$(3.1) \quad ch_t E = \sum_k (ch^k E)t^k.$$

Let $\{b_k\}_{k=1}^\infty$ be the sequence of integers defined at the end of §1, and note from Table 2 that

$$2b_k = \begin{cases} 2a_k & \text{if } k \equiv 1, 0 \text{ or } -1 \pmod{8}, \\ a_k & \text{otherwise.} \end{cases}$$

Set $\mathfrak{E} = \mathfrak{E}(X, N, E) \equiv \{e_N \cdot ch E \cdot A^2[X]\}[X]$ where e_N is the normal Euler class.

Theorem 3.2. *Let X^n be a compact oriented manifold of dimension n , and suppose there exists an immersion $X^n \hookrightarrow S^{n+q}$. Then for any complex vector bundle E over X ,*

$$2^{q-1}\{ch_2 E \cdot A(X)\}[X] \equiv 0 \pmod{b_{n+q}},$$

and if, furthermore, both q and \mathfrak{E} are even, then

$$2^{q-2}\{ch_2 E \cdot A(X)\}[X] \equiv 0 \pmod{b_{n+q}}.$$

The statements of Theorem 3.2 can be refined if one restricts to real or quaternionic bundles. Let $\{c_k\}_{k=1}^\infty$ be the sequence of numbers given at the end of §1, and note from Table 2 that:

$$2c_k = \begin{cases} 4a_k & \text{if } k \equiv 1, 0 \text{ or } -1 \pmod{8}, \\ 2a_k & \text{if } k \equiv 2 \text{ or } 6 \pmod{8}, \\ a_k & \text{otherwise.} \end{cases}$$

Theorem 3.3. *Let X^n be a compact oriented manifold of dimension $n \equiv 0 \pmod{4}$ and suppose there exists an immersion $X^n \hookrightarrow S^{n+q}$.*

(i) *If $E = E \otimes_{\mathbf{R}} \mathbf{C}$ for some real bundle E over X , then*

$$2^{q-1}\{ch_2 E \cdot A(X)\}[X] \equiv 0 \pmod{a_{n+q}},$$

and also if, $q \equiv 0 \pmod{4}$ and \mathfrak{E} is even and divisible by 4 when $n + q \equiv 4 \pmod{8}$ then

$$2^{q-2}\{ch_2 E \cdot A(X)\}[X] \equiv 0 \pmod{a_{n+q}}.$$

(ii) *If E is a quaternionic bundle over X , then*

$$2^{q-1}\{ch_2 E \cdot A(X)\}[X] \equiv 0 \pmod{c_{n+q}},$$

and also if, $q \equiv 0 \pmod{4}$ and \mathfrak{E} is even and divisible by 4 when $n + q \equiv 0 \pmod{8}$, then

$$2^{q-2}\{ch_2 E \cdot A(X)\}[X] \equiv 0 \pmod{c_{n+q}}.$$

Note that one recaptures Theorem 3.1 from part (i) above by taking E to be the trivial line bundle over X .

The theorems stated above contain results due to Atiyah and Hirzebruch [3] and to Mayer [11]. Note that all conditions involving $\mathcal{G}(X, N, E)$ are trivially satisfied if $e_N = 0$, e.g., if the immersion is in fact an embedding.

Proof of Theorem 3.1. Suppose there is an immersion $f: X^n \hookrightarrow S^{n+q}$, and let N denote the normal bundle to F . There is a natural metric with riemannian connection on $T \oplus N$ induced from S^{n+q} by f . Let ∇ denote the associated projected connection (cf. Remark 2.3), and extend ∇ canonically to $Cl(T \oplus N)$ by derivations, that is, so that (2.11) is satisfied.

We now consider $Cl(T \oplus N)$ as a bundle of left modules over $Cl(T)$, and let D be the associated Dirac operator. Note that the projected connection is an extension of a riemannian connection on T . Hence the derivation property (2.11) for $Cl(T \oplus N)$ as a bundle of algebras implies the derivation property (2.12) for $Cl(T \oplus N)$ as a bundle of modules over $Cl(T)$. It now follows from the discussion in §2 that since X is oriented and of dimension $n \equiv 0 \pmod{4}$, the restriction of D defines an elliptic operator $D^\pm: \Gamma(M^\pm) \rightarrow \Gamma(M^\mp)$ where

$$(3.2) \quad M^\pm = (1 \pm L_\omega) \cdot Cl(T \oplus N),$$

and ω is the unit volume form for T .

It is clear from formula (3.2) that M^+ and M^- are both invariant under right multiplication by elements of $Cl(T \oplus N)$. Therefore if φ is any section of the bundle $Cl(T \oplus N)^\times$ of units in $Cl(T \oplus N)$, then the operator $R_\varphi^{-1} \circ D \circ R_\varphi$ (where R_φ denotes right Clifford multiplication by φ) maps $\Gamma(M^\pm)$ to $\Gamma(M^\mp)$. Furthermore, one can see easily from the derivation property (2.11) for ∇ that D and $R_\varphi^{-1} \circ D \circ R_\varphi$ have the same first order part, i.e., they differ by a zero-order operator.

Now the bundle $T \oplus N$ is, of course, trivial. Hence we can find a set $\varepsilon_1, \dots, \varepsilon_{n+q}$ of pointwise orthonormal sections of $T \oplus N$, and define a new first order operator on $Cl(T \oplus N)$ by setting

$$(3.3) \quad \tilde{D} = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} R_\gamma^{-1} \circ D \circ R_\gamma,$$

where Γ is the finite (multiplicative) subgroup of $\Gamma(Cl(T \oplus N))$ generated by $\varepsilon_1, \dots, \varepsilon_{n+q}$. This operator is self-adjoint, differs from D by a zero-order operator, and maps $\Gamma(M^\pm)$ to $\Gamma(M^\mp)$. Hence by restriction we get an operator $\tilde{D}^+: \Gamma(M^+) \rightarrow \Gamma(M^-)$ having the same index as D^+ . Its adjoint is the operator $\tilde{D}^-: \Gamma(M^-) \rightarrow \Gamma(M^+)$ also given by restricting \tilde{D} . Thus we have that

$$(3.4) \quad i(D^+) = i(\tilde{D}^+) = \dim(\text{Ker } \tilde{D}^+) - \dim(\text{Ker } \tilde{D}^-).$$

We now observe that since $\tilde{D} \circ R_\gamma = R_\gamma \circ \tilde{D}$ for $\gamma \in \Gamma$, the spaces $\text{Ker } \tilde{D}^+$ and $\text{Ker } \tilde{D}^-$ are invariant under right Clifford multiplication by elements of

Γ . This obviously makes these spaces into modules for the Clifford algebra Cl_{n+q} . In fact, they are in a natural way \mathbf{Z}_2 -graded modules over Cl_{n+q} . The grading on M^+ and M^- is given (as in Remark (1.9)) by taking the 1 and -1 eigenbundles of the bundle map α . Since $\nabla\alpha = 0$, α preserves the subspaces $\ker \tilde{D}^+$ and $\ker \tilde{D}^-$ and gives the grading there. We conclude that

$$(3.5) \quad i(D^+) \equiv 0 \pmod{2a_{n+q}}.$$

It remains only to compute $i(D^+)$ using Theorem 2.1. We begin by observing that under the natural isomorphism $Cl(T \oplus N) = Cl(T) \otimes Cl(N)$ (cf. Remark 2.5) equation (3.2) becomes

$$M^\pm = [(1 \pm L_\omega)Cl(T)] \otimes Cl(N) = Cl^\pm(T) \otimes Cl(N).$$

To apply formula (2.23) we must compute $ch(Cl(N) \otimes \mathbf{C})$. To do this, note that $T \oplus N = (n + q)\theta$ where θ denotes the trivial line bundle. Hence $Cl(T) \otimes Cl(N) = 2^{n+q}\theta$. It follows that

$$ch(Cl(N) \otimes \mathbf{C}) = 2^{n+q}ch(Cl(T) \otimes \mathbf{C})^{-1}.$$

Since $Cl(T) \cong \Lambda^*(T)$, we have that (cf. [5])

$$\begin{aligned} ch(Cl(T) \otimes \mathbf{C}) &= \prod_{j=1}^{n/2} (1 + e^{-x_j})(1 + e^{x_j}) \\ &= 2^n \prod_{j=1}^{n/2} \cosh^2(x_j/2), \end{aligned}$$

and so

$$ch(Cl(N) \otimes \mathbf{C}) = 2^q \prod_{j=1}^{n/2} \cosh^{-2}(x_j/2).$$

Consequently, using (2.20) and formula (2.23) we have that

$$\begin{aligned} (3.6) \quad i(D^+) &= \left\{ 2^q \prod_{j=1}^{n/2} \frac{x_j}{\cosh(x_j/2)\sinh(x_j/2)} \right\} [X] \\ &= 2^{n+q} \left\{ \prod_{j=1}^{n/2} \frac{x_j}{e^{x_j} - e^{-x_j}} \right\} [X] \\ &= 2^{n/2+q} \left\{ \prod_{j=1}^{n/2} \frac{2x_j}{e^{2x_j} - e^{-2x_j}} \right\} [X] \\ &= 2^q \left\{ \prod_{j=1}^{n/2} \frac{2x_j}{\sinh(2x_j)} \right\} [X] = 2^q A(X). \end{aligned}$$

In the third line we use the fact that the terms which are not zero when evaluated on the fundamental class of X are homogeneous of degree $n/2$ in $x = (x_1, \dots, x_{n/2})$. Combining (3.5) and (3.6) now gives the first part of the theorem.

Suppose now that $q \equiv 0 \pmod{4}$, and let ω_N denote the unit normal volume form. Since we are using the projected connection, $\nabla\omega_N = 0$. It follows that $D \circ L_{\omega_N} = L_{\omega_N} \circ D$, and so restriction of D gives a new elliptic complex $D^{++}: \Gamma(M^{++}) \rightarrow \Gamma(M^{-+})$ where

$$(3.7) \quad M^{\pm\pm} = (1 \pm L_{\omega})(1 \pm L_{\omega_N})Cl(T \oplus N) = Cl^{\pm}(T) \otimes Cl^{\pm}(N).$$

Note that $L_{\omega} \circ L_{\omega_N} = L_{\omega_N} \circ L_{\omega}$, and also that the bundles $M^{\pm\pm}$ continue to be \mathbb{Z}_2 -graded modules over $Cl(T \oplus N)$ under right Clifford multiplication. It follows as above that $i(D^{++}) \equiv i(D^{-+}) \equiv 0 \pmod{2a_{n+q}}$.

It is easy to see that $i(D^+) = i(D^{++}) + i(D^{-+})$. Furthermore, a straightforward computation shows that

$$(3.8) \quad i(D^{++}) - i(D^{-+}) = \kappa\{e_N \cdot \mathcal{G}(X)\}[X] = \kappa\{e_N \cdot \hat{A}^2(X)\}[X],$$

where $\kappa = 2^{(1/2)(n+q)}$, and e_N denotes the Euler class of the normal bundle. (The key to this calculation is the following observation. Let E be a real oriented $2m$ -dimensional bundle with Pontrjagin classes $p_k(E) = \sigma_k(x_1^2, \dots, x_m^2)$. Then $ch(Cl^+(E) - Cl^-(E)) = \Pi(e^x - e^{-x}) = 2^m e_E \cdot \alpha(E)$, where $e_E = x_1 \cdots x_m$ is the Euler class of E , and $\alpha(E)$ is a multiplicative sequence). It now follows immediately that if $e_N = 0$, then $i(D^+) = 2i(D^{++}) \equiv 0 \pmod{4a_{n+q}}$. This completes the proof of Theorem 3.1.

Remark 3.4. Michael Crabb has made the following observations. The second conclusion of Theorem 3.1 will hold in general if one can show that

$$(3.9) \quad \{e_N \cdot \hat{A}^2(X)A(x)\}[X] \equiv 0 \begin{cases} \pmod{2} & \text{when } n + q \equiv 0 \pmod{8}, \\ \pmod{4} & \text{when } n + q \equiv 4 \pmod{8}. \end{cases}$$

This number has several geometric interpretations. For example, let σ be any section of the normal bundle N , which is transversal to zero, and consider the oriented $(n - q)$ -dimensional submanifold $Z \subset X$ given by the zeros of σ . $[Z]$ is the Poincaré dual of e_N . Then

$$\hat{A}(Z) = \{e_N \cdot \hat{A}^2(X)\}[X].$$

Since the normal bundle of Z is $N \oplus N$, Z is a spin manifold, and so $2\hat{A}(Z)$ satisfies the conditions above. Unfortunately the condition that the normal bundle of Z is a “square” does not in general imply that $\hat{A}(Z)$ satisfies these conditions.

There are similar interpretations of $\{e_N \cdot \mathcal{G}(X)\}[X]$ in terms of the self-intersection locus of the immersion and also in terms of the Gauss map. Of course, this number always vanishes for embeddings.

Remark 3.5. Recall that if $q \equiv 3 \pmod{4}$, then again $\omega_N^2 = 1$ and we obtain subbundles $M^{\pm\pm} = Cl^\pm(T) \otimes Cl^\pm(N)$. In this case $Cl^+(N)$ and $Cl^-(N)$ are canonically isomorphic under the automorphism α_N (extending -1 on N). By restriction of D we get an operator $D^{++}: \Gamma(M^{++}) \rightarrow \Gamma(M^{--})$. The bundles $M^{\pm\pm}$ are again right Cl_{n+q} -modules. However, *they are not \mathbb{Z}_2 -graded modules*. Hence the factor of two gained in one place is lost in another, and we are left with a different proof of the same result.

Proof of Theorem 3.2. This argument is essentially the same as the one given above with one addition. Let E be any complex vector bundle over X with an inner product for which multiplication by i is fiberwise an isometry. We can choose a riemannian connection ∇ on E with respect to which multiplication by i is parallel, i.e., $\nabla(ie) = i\nabla e$ for all $e \in \Gamma(E)$. We then consider the bundle $Cl(T \oplus N) \otimes_{\mathbb{R}} E$ endowed with the tensor product connection (cf. Remark 2.4). This is naturally a bundle of complex right and left modules over $Cl(T \oplus N)$. We may assume $n = 2m$, and consider the element $w = i^m e_1 \cdots e_{2m}$ where as before $e_1 \cdots e_{2m}$ denotes the oriented volume element of T . Again $w^2 = 1$ and $we = -ew$ for any $e \in T$. Hence proceeding as before we can define subbundles $M^\pm = (1 \pm L_w)Cl(T \oplus N) \otimes E$ and averaged Dirac operators $\tilde{D}^\pm: \Gamma(M^\pm) \rightarrow \Gamma(M^\mp)$ with the property that the spaces $K^\pm = \ker(\tilde{D}^\pm)$ are complex \mathbb{Z}_2 -graded modules for the algebra Cl_{n+q} . It follows immediately that

$$(3.10) \quad i(\tilde{D}^+) \equiv 0 \quad (2b_{n+q}),$$

and it remains to compute the index of this operator.

Remark 3.6. We should point out at this time that in previous computations we have dealt with real operators on real bundles and have computed the index in terms of the real dimensions of the kernel and cokernel. This is of course equivalent to complexifying the bundles and computing the complex dimension of the kernel and cokernel, since these spaces will be the complexifications of the former ones. However, in this case the coefficient bundle E has induced a natural complex structure on M^+ and M^- with respect to which \tilde{D}^+ is complex linear since multiplication by i is parallel. Consequently, in this case we have applied the index theorem in complex form.

To compute the index of \tilde{D}^+ we first note that

$$Cl(T \oplus N) \otimes_{\mathbb{R}} E \cong Cl(T \oplus N)_c \otimes_{\mathbb{C}} E \cong Cl(T)_c \otimes_{\mathbb{C}} Cl(N)_c \otimes_{\mathbb{C}} E,$$

where the subscript $()_c$ denotes complexification. It follows immediately that $M^\pm \cong Cl^\pm(T)_c \otimes_{\mathbb{C}} Cl(N)_c \otimes_{\mathbb{C}} E$ where $Cl^\pm(T)_c = (1 \pm L_w)Cl(T)_c$. We

now assert that

$$(3.11) \quad ch(Cl^+(T)_c - Cl^-(T)_c) = \prod_{j=1}^m (e^{x_j} - e^{-x_j}),$$

where $n = 2m$, and the x_j 's are as before. By the splitting principle we may assume for the purposes of computation that $T = \xi_1 \oplus \dots \oplus \xi_m$ where ξ_j is an oriented 2-plane bundle with Euler class x_j . Then $Cl(T)_c \cong Cl(\xi_1)_c \otimes \dots \otimes Cl(\xi_m)_c$. Moreover, $w = w_1 \cdot \dots \cdot w_n$ where $w_j = ie_j \cdot f_j$ for an oriented orthonormal basis (e_j, f_j) of ξ_j . Consequently $Cl^+(T)_c = \sum Cl^\pm(\xi_1)_c \otimes \dots \otimes Cl^\pm(\xi_m)_c$ with the sum taken over all strings of + and - with an even number of -'s appearing. $Cl^-(T)_c$ is represented similarly with an odd number of -'s appearing. It follows immediately that $Cl^+(T)_c - Cl^-(T)_c = \prod(Cl^+(\xi_j)_c - Cl^-(\xi_j)_c)$. Now one can easily check that for an oriented 2-plane bundle ξ , $Cl^+(\xi)_c - Cl^-(\xi)_c = \xi - \bar{\xi}$. This proves (3.11).

As a result of (3.11) we see that the computation of the index of \tilde{D}^+ proceeds formally in the same way as before with the exception that one carries a multiplicative factor $ch(E)$. The result is (cf. (3.6)) that

$$i(D^+) = 2^{n/2+q} \left\{ ch E \cdot \prod_{j=1}^{n/2} \frac{x_j}{\sinh(x_j)} \right\} [X].$$

The term involving the x_j 's can be reexpressed as $A_{1/2}(X)$ where by definition

$$A_t(X) = \sum_{k=0}^{n/4} A_k(p_1, \dots, p_k) t^{2k}$$

for $t \in \mathbf{R}$.

Lemma 3.7. For all $s, t \in \mathbf{R}^+$,

$$\{ch_t E \cdot A_s(X)\} [X] = (st)^{n/2} \{ch_{1/s} E \cdot A_{1/t}(X)\} [X].$$

The proof of this lemma is easy and is left to the reader. It now follows directly that

$$i(\tilde{D}^+) = 2^q \{ch_2 E \cdot A(X)\} [X].$$

Combining this with (3.10) establishes the first part of Theorem 3.2.

Suppose now that $q = 2p$, and consider the element $w_N = i^p \omega_N$ where ω_N is the oriented volume form for N . Then $Cl(N)_c = Cl^+(N)_c \oplus Cl^-(N)_c$ where $Cl^\pm(N) = (1 \pm L_{w_N})Cl(N)_c$. As in the proof of Theorem 3.1 we consider the subbundles

$$\begin{aligned} M^{\pm\pm} &= (1 \pm L_w)(1 \pm L_{w_N})Cl(T \oplus N) \otimes_{\mathbf{R}} E \\ &= Cl^\pm(T)_c \otimes_{\mathbf{C}} Cl^\pm(N)_c \otimes_{\mathbf{C}} E, \end{aligned}$$

each of which is a bundle of complex \mathbb{Z}_2 -graded modules under right multiplication by $Cl(T \oplus N)$. Restriction of the Dirac operator \tilde{D}^+ gives operators $\tilde{D}^{\pm\pm}: \Gamma(M^{\pm\pm}) \rightarrow \Gamma(M^{\mp\mp})$ which are adjoints of one another. These operators commute with right multiplication by elements of Γ , hence the spaces $K^\pm = \ker(\tilde{D}^{\pm\pm})$ are complex \mathbb{Z}_2 -graded Cl_{n+q} modules. It follows that

$$i(\tilde{D}^{++}) \equiv i(\tilde{D}^{+-}) \equiv 0 \pmod{2b_{n+q}}.$$

It is clear that $i(\tilde{D}^+) = i(\tilde{D}^{++}) + i(\tilde{D}^{+-})$. Furthermore, one has that

$$(3.12) \quad i(\tilde{D}^{++}) - i(\tilde{D}^{+-}) = \kappa\{e_N \cdot ch E \cdot \hat{A}^2(X)\}[X],$$

where $\kappa = 2^{\frac{1}{2}(n+q)}$ and where e_N denotes the normal Euler class. Hence, if $e_N = 0$, then $i(\tilde{D}^+) = 2i(\tilde{D}^{++}) \equiv 0 \pmod{4b_{n+q}}$.

We note that it suffices for this second part of the theorem to know that

$$(3.13) \quad \{e_N \cdot ch E \cdot \hat{A}^2(X)\}[X] \equiv 0 \pmod{2}.$$

This number has interpretations analogous to those mentioned in Remark 3.4. In particular, if $Z \subset X$ is a submanifold dual to e_N as before, then

$$\{e_N \cdot ch E \cdot \mathcal{G}(X)\} = \{ch E \cdot \hat{A}(Z)\}[Z].$$

Proof of Theorem 3.3. Suppose that $E = E_0 \otimes_{\mathbb{R}} \mathbb{C}$ for some real bundle E_0 over X . Then we can run through the argument given for Theorem 3.2 with E replaced by E_0 . We insist at each step that all bundles in question be real bundles. For this reason we must require that $n \equiv 0 \pmod{4}$, and for the second half of the argument that $q \equiv 0 \pmod{4}$. The spaces K^\pm will in this case be real \mathbb{Z}_2 -graded modules over Cl_{n+q} . Applying the index theorem for real operators (See Remark 3.7), we see that $i(\tilde{D}^+) = \dim_{\mathbb{R}} K^+ - \dim_{\mathbb{R}} K^- \equiv 0 \pmod{2a_{n+q}}$. However $i(\tilde{D}^+) = 2^q \{ch(E_0 \otimes \mathbb{C}) \cdot \mathbf{A}(X)\}[X]$. This proves part (i).

Suppose now that E is an \mathbb{H} -bundle, and introduce on E a riemannian connection with the property that scalar multiplication is parallel. Repeat the construction of Theorem 3.2. In this case the spaces K^\pm become quaternionic \mathbb{Z}_2 -graded modules over Cl_{n+q} . This gives the first part of (ii). (Note that it is necessary that we have $n \equiv 0 \pmod{4}$ for this argument to work.) If $q \equiv 0 \pmod{4}$, the same arguments as above yield the second part of (ii), and the proof is complete.

We conclude this section with some remarks concerning Theorem 3.2. This theorem is significantly more general than Theorem 3.1; among other things it gives nontrivial statements for manifolds of dimension $n \equiv 2 \pmod{4}$. The trick of taking coefficients in a bundle E may seem a bit formal; however, it has a fairly concrete geometric interpretation. Note that Theorem 3.1 gives a

condition only in terms of the cobordism class of the manifold X . The more elaborate statement of Theorem 3.2 extends this condition to the cobordism group of X .

This is most concretely seen in the case of oriented bundles E of real dimension 2. Such bundles are in natural one-to-one correspondence with elements of $H^2(X; \mathbf{Z})$. Hence Theorem 3.2 implies the following.

Corollary 3.8. *Let X^n be a compact oriented manifold of dimension $n \equiv 2 \pmod{4}$, and suppose there exists an immersion $X^n \hookrightarrow S^{n+q}$. Then*

$$2^{q-1}\{e^{2\alpha}\mathbf{A}(X)\}[X] \equiv 0 \pmod{b_{n+q}}$$

for all $\alpha \in H^2(X; \mathbf{Z})$. If, furthermore, q and $\{e^\alpha \cdot e_N \cdot \hat{\mathbf{A}}^2(X)\}[X]$ are even, then

$$2^{q-2}\{e^{2\alpha}\mathbf{A}(X)\}[X] \equiv 0 \pmod{b_{n+q}}.$$

We recall now that $H^2(X; \mathbf{Z}) \cong [X, \mathbf{CP}^\infty]$. Therefore, given $\alpha \in H^2(X; \mathbf{Z})$, we choose a smooth map $f_\alpha: X^n \rightarrow \mathbf{CP}^N$ ($N \gg n$) representing α . By Sard's Theorem for families there is a hyperplane $\mathbf{CP}^{N-1} \subset \mathbf{CP}^N$ to which f_α is transversal. The counterimage $Y_\alpha = f_\alpha^{-1}(\mathbf{CP}^{N-1})$ is an oriented codimension-2 submanifold whose homology class represents the Poincaré dual of α . If E is the bundle on X with Euler class α , then $E|_{Y_\alpha}$ is the normal bundle of Y_α in X . Hence

$$\begin{aligned} A(Y_\alpha) &= \{\mathbf{A}(E)^{-1} \cdot \mathbf{A}(X)\}[Y_\alpha] \\ &= \left\{ \frac{\sinh 2\alpha}{2\alpha} \cdot \mathbf{A}(X) \right\}[Y_\alpha] \\ &= \frac{1}{2} \{\sinh 2\alpha \cdot \mathbf{A}(X)\}[X]. \end{aligned}$$

This gives the following conclusion.

Corollary 3.9. *Let X^n be a compact oriented submanifold of dimension $n \equiv 2 \pmod{4}$, and suppose there exists an immersion $X^n \hookrightarrow S^{n+q}$. Then for every compact oriented submanifold Y of codimension 2 in X ,*

$$2^q A(Y) \equiv 0 \pmod{b_{n+q}}.$$

Furthermore if q and $\{e^\alpha \cdot e_N \cdot \hat{\mathbf{A}}^2(X)\}[X]$ are even where $\alpha \in H^2(X; \mathbf{Z})$ is dual to $[Y]$, then

$$2^{q-1} A(Y) \equiv 0 \pmod{b_{n+q}}.$$

This condition is slightly stronger than the condition one obtains from Theorem 3.1 by observing that Y immerses in codimension $q + 2$. The additional information comes from the fact that the normal bundle to Y carries a 2-plane field.

4. Applications to complex and quaternionic projective varieties

In this section we shall apply the theorems of the last section to get non-immersion and non-embedding theorems for complex and quaternionic projective spaces CP^n and HP^n , for complex hypersurfaces $H^n(d)$ of CP^{n+1} of odd degree d , and for $4m$ -dimensional manifolds cobordant to CP^{2m} and $H^{2m}(d)$, d odd. These theorems are not new, or at least they can be retrieved from the work of Mayer [11], however, they recapture in a very simple and conceptual way all previously known theorems on the subject; cf. James survey article [9]. At the end of this section we will conjecture “best possible” non-immersion theorems for CP^n and HP^n . If we let $\alpha(n)$ denote the number of 1’s in the dyadic expansion of n , our first result is the following: The symbol $\not\hookrightarrow$ means “does not immerse”, and the symbol $\not\hookrightarrow$ means “does not embed”.

Theorem 4.1. *Let M^{2n} be CP^n or $H^n(d)$, d odd, or for $n = 2m$ let M^{2n} be any manifold cobordant to CP^{2m} or $H^{2m}(d)$. Then*

$$M^{2n} \not\hookrightarrow S^{4n-2\alpha(n)-1},$$

$$M^{2n} \not\hookrightarrow S^{4n-2\alpha(n)},$$

and, for even n ,

$$\alpha(n) \equiv 1 \pmod{4} \Rightarrow M^{2n} \not\hookrightarrow S^{4n-2\alpha(n)},$$

$$\alpha(n) \equiv 2 \text{ or } 3 \pmod{4} \Rightarrow M^{2n} \not\hookrightarrow S^{4n-2\alpha(n)+1},$$

$$\alpha(n) \equiv 3 \pmod{4} \Rightarrow M^{2n} \not\hookrightarrow S^{4n-2\alpha(n)+2}.$$

We also prove

Theorem 4.2. *Let HP^n be quaternionic projective n -space. Then for all n*

$$HP^n \not\hookrightarrow S^{8n-2\alpha(n)-3},$$

$$HP^n \not\hookrightarrow S^{8n-2\alpha(n)-2},$$

and

$$\alpha(n) \equiv 2 \pmod{4} \Rightarrow HP^n \not\hookrightarrow S^{8n-2\alpha(n)-2},$$

$$\alpha(n) \equiv 0 \text{ or } 3 \pmod{4} \Rightarrow HP^n \not\hookrightarrow S^{8n-2\alpha(n)-1},$$

$$\alpha(n) \equiv 0 \pmod{4} \Rightarrow HP^n \not\hookrightarrow S^{8n-2\alpha(n)}.$$

Proof of Theorem 4.1. We begin with the case n even. Consider first CP^{2m} , and suppose that $CP^{2m} \hookrightarrow S^{4m+q}$. It is known [8] that $A(CP^{2m}) = (-1)^m \binom{2m}{m}$. Thus it follows from Theorem 3.1 that

$$(4.1) \quad 2^{q-1} \binom{2m}{m} \equiv 0 \pmod{a_{4m+q}}.$$

If K is an integer, let $\nu(K)$ denote the highest power of 2 which divides K . Now $\nu(\binom{2m}{m}) = \alpha(m)$, and we denoted (§1) $\nu(a_r)$ by ν_r . Thus we have

$$(4.2) \quad \nu_{4m+q} \leq \alpha(m) + q - 1.$$

If we write $q = 8a + b$, $1 \leq b \leq 8$, and suppose for the moment that m is even, then $\nu_{4m+q} = 2m + 4a + \nu_b$, and (4.2) becomes

$$(4.3) \quad 4m - 2\alpha(m) + 2\nu_b - b + 2 \leq 8a + b = q.$$

Since $4m \equiv 0 \pmod{8}$, the congruency class of $4m - 2\alpha(m) \pmod{8}$ is determined by $\alpha(m) \pmod{4}$. We make the table:

(4.4)

b	1	2	3	4	5	6	7	8
$2\nu_b - b + 2$	1	2	3	2	3	2	1	0

Since $2\nu_b - b + 2 \geq 0$, we always have $q \geq 4m - 2\alpha(m)$. In fact we can read off the rest of the non-immersion results in this case. For example, if $\alpha(n) \equiv 1 \pmod{4}$, then $q \geq 4m - 2\alpha(m) + 1$, because if $q = 4m - 2\alpha(m)$ then $b = 8$ but also $q \equiv 6 \pmod{8}$. If $CP^{2m} \hookrightarrow S^{4m+q}$, (4.3) becomes $4m - 2\alpha(m) + 2\nu_b - b + 4 \leq 8a + b = q$ when $b = 4$ or 8 , and the results follow. If $m = 2t + 1$, then $\nu_{4m+q} = 4t + 4a + \nu_{4+b}$. Instead of (4.3) we have $4m - 2\alpha(m) + 2\nu_{4+b} - b - 2 \leq 8a + b = q$, and when there is an embedding and $b = 4$ or 8 , we have $4m - 2\alpha(m) + 2\nu_{4+b} - b \leq 8a + b$. The same results follow.

We note that our argument depends only on the cobordism class of CP^{2m} since it uses only the A -genus of CP^{2m} .

We now show that $\nu(A(\mathbf{H}^{2m}(d))) = \nu(A(CP^{2m}))$, so that the same argument yields the theorem for M^{2n} cobordant to $\mathbf{H}^{2m}(d)$. Let l denote the canonical line bundle over CP^{2m+1} ($c_1(l) = \omega$). Then setting $\bar{T} = T(CP^{2m+1})|_{\mathbf{H}^{2m}(d)}$ and $T = T(\mathbf{H}^{2m}(d))$, we get the stable equation

$$T \oplus l^d = \bar{T} \cong (2m + 2)l.$$

Thus we have

$$A(T) = A(l)^{2m+2} / A(l^d),$$

and therefore

$$A(\mathbf{H}^{2m}(d)) = \left\{ \left(\frac{2\omega}{\sinh 2\omega} \right)^{2m+2} \frac{\sinh 2d\omega}{2d\omega} \right\} [\mathbf{H}^{2m}(d)].$$

By observing that this is d times a polynomial in d^2 of degree $2m$ over d , which has zeroes at $d = 2, 4, \dots, 2m$, and using $w_2(\mathbf{H}^{2m}(\text{even})) = 0$ and the

fact that a spin manifold with positive scalar curvature has a vanishing A -genus, we are able to write

$$A(\mathbf{H}^{2m}(d)) = c_m \cdot d \cdot \prod_{j=1}^n (d^2 - (2j)^2).$$

If d is odd, then $d \cdot \prod_{j=1}^n (d^2 - 2j)^2$ is odd so

$$\nu(A(\mathbf{H}^{2m}(d))) = \nu(c_m) = \nu(A(\mathbf{H}^{2m}(1))) = \alpha(n),$$

since $\mathbf{H}^{2m}(1) = \mathbf{C}P^{2m}$. In fact, it is easy to compute from $A(\mathbf{C}P^{2m}) = (-1)^m \binom{2m}{m}$ that $c_m = 2^{2m}/(2m + 1)!$. So we have the theorem for M^{2n} equal to or cobordant to $\mathbf{H}^{2m}(d)$.

We now consider the case $n = 2t + 1$. If $M^{2n} = \mathbf{C}P^n$ and $M^{2n} \hookrightarrow S^{2n+q}$, then $\mathbf{C}P^{2t}$ is a codimension-2 submanifold, and Corollary 3.9 tells us

$$2^{q-\varepsilon} A(\mathbf{C}P^{2t}) \equiv 0 \pmod{b_{2n+q}},$$

where $\varepsilon = 1$ if q is even and $\mathbf{C}P^n \hookrightarrow S^{2n+q}$, and $\varepsilon = 0$ otherwise. Then $\alpha(t) + q - \varepsilon \geq \nu(b_{n+q})$. If q is odd or $\mathbf{C}P^n \hookrightarrow S^{2n+q}$, we can write $q = 2s + 1 - \varepsilon$ and $\nu(b_{2n+q}) = n + s - \varepsilon$. A little arithmetic with $\alpha(n) = \alpha(t) + 1$ gives

$$q \geq 2n - 2\alpha(n) + \varepsilon + 1.$$

If $\mathbf{C}P^n$ only immerses in S^{2n+q} and $q = 2s$, then $\nu(b_{2n+q}) = n + s - 1$ and

$$q \geq 2n - 2\alpha(n).$$

The theorem for $\mathbf{C}P^n$, n odd, follows.

The same argument works for $\mathbf{H}^n(d)$, $n = 2t + 1$, since a hyperplane section of $\mathbf{H}^n(d)$ is a degree- d hypersurface of $\mathbf{C}P^n$, that is, $\mathbf{H}^{2t}(d)$. Thus the theorem is proved.

Proof of Theorem 4.2. We use Theorem 3.3 letting $E = \xi$, the canonical \mathbf{H} line bundle over $\mathbf{H}P^n$ (the hyperplane bundle). Let x be a generator for $H^4(\mathbf{H}P^n; \mathbf{Z})$, and let $w = \sqrt{x}$ formally, (or one can pull x back to $\mathbf{C}P^{2n+1}$ and take $w = \sqrt{x}$ there). Now

$$\begin{aligned} ch \xi &= e^{2w} + e^{-2w} = 2 \cosh 2w, \\ \mathbf{A}(\mathbf{H}P^n) &= \left(\frac{2w}{\sinh 2w} \right)^{2n+2} \frac{\sinh 4w}{4w}. \end{aligned}$$

We want the coefficient of w^{2n} in this power series. That is,

$$\begin{aligned} & \{ch\ 2\xi \cdot \mathbf{A}(\mathbf{HP}^n)\}[X] \\ &= \text{coeff of } w^{2n} \text{ in } \left\{ 2 \cosh 2w \left(\frac{2w}{\sinh 2w} \right)^{2n+2} \frac{\sinh 4w}{4w} \right\} \\ &= \frac{1}{2\pi i} 2^{2n+1} \int_{|t|=\epsilon} \frac{\cosh^2 t}{\sinh^{2n+1} t} dt = \frac{1}{2\pi i} 2^{2n+1} \int \frac{\sqrt{1+z}}{z^{n+1}} dz \\ &= 2^{2n+1} \frac{1}{n!} \frac{1}{2} \left(-\frac{1}{2}\right) \cdots \left(\frac{-2n+3}{2}\right) \binom{2n-1}{2n-1} \\ &= (-1)^{n-1} 2 \binom{2n}{n} \frac{1}{2n-1}. \end{aligned}$$

Now if we suppose \mathbf{HP}^n immerses in S^{4n+q} , then Theorem 3.3 gives us

$$\nu(c_{4n+q}) \leq q + \alpha(n).$$

Thus if we suppose to begin with that $n = 2m$ and we write $q = 8a + b$, we have

$$(4.5) \quad 4m - 2\alpha(n) + 2\nu(c_b) - b \leq 8a + b = q.$$

Again we make a table:

b	1	2	3	4	5	6	7	8
$2\nu(c_b) - b$	1	0	-1	-2	-1	0	1	0

as we did for \mathbf{CP}^n we read off the claimed non-immersion results. If $\mathbf{HP}^n \hookrightarrow S^{4n+q}$, then (4.5) becomes $4m - 2\alpha(n) + 2\nu(c_b) - b + 2 \leq 8a + b = q$, and the non-embedding results follow. When m is odd, one gets similar tables with the right and left halves interchanged, and the same results can be read off.

Now if it can be shown that given the hypotheses of Theorem 4.1 that $\{e_N \cdot \hat{\mathbf{A}}^2(\mathbf{CP}^{2m})\}[\mathbf{CP}^{2m}]$ is even whenever $q \equiv 0 \pmod{4}$ and is divisible by 4 whenever $4m + q \equiv 4 \pmod{8}$ and that $\{e^w \cdot e_N \cdot \hat{\mathbf{A}}^2(\mathbf{CP}^{2m+1})\}[\mathbf{CP}^{2m+1}]$ is even whenever q is even, then the non-embedding results just proved for \mathbf{CP}^n would be non-immersion results. Moreover, recent immersion results for \mathbf{CP}^n of Davis and Mahowald [6] would show this to be sharp in the range where their work is most effective, that is, where $\alpha(n) \leq 7$ or $\alpha(n) = 8$ and n is odd. This leads us to make the following conjecture.

Conjecture 1. *If q is the smallest codimension for which $\mathbf{CP}^n \hookrightarrow S^{2n+q}$, then*

$$q = \begin{cases} 2n - 2\alpha(n) + 2 & \text{if } n \text{ is even and } \alpha(n) \equiv 2 \pmod{4}, \\ 2n - 2\alpha(n) + 3 & \text{if } n \text{ is even and } \alpha(n) \equiv 3 \pmod{4}, \\ 2n - 2\alpha(n) + 1 & \text{otherwise.} \end{cases}$$

In response to our conjecture Davis has been able to find likely candidates for obstructions from the Davis-Mahowald point of view. It appears, however, that it would be difficult to prove these candidates to actually be obstructions.

We similarly conjecture

Conjecture 2. *If q is the smallest codimension for which $\mathbf{HP}^n \looparrowright S^{4n+q}$, then*

$$q = \begin{cases} 4n - 2\alpha(n) + 1 & \text{if } \alpha(n) \equiv 0 \pmod{4}, \\ 4n - 2\alpha(n) & \text{if } \alpha(n) \equiv 3 \pmod{4}, \\ 4n - 2\alpha(n) - 1 & \text{otherwise.} \end{cases}$$

The negative side of this result would be proved if $\mathbf{HP}^n \looparrowright S^{4n+q}$ could be shown to imply that $\{e^w \cdot e_N \cdot \hat{A}^2(\mathbf{HP}^n)\}[\mathbf{HP}^n]$ is even whenever $q \equiv 0 \pmod{4}$ and is divisible by 4 whenever $4n + q \equiv 0 \pmod{8}$. For some interpretations of this and the previously mentioned requirements see Remark 3.4 of §3.

5. The fundamental theorems for vector fields

In this section we shall be concerned with the following question: Given a compact differentiable manifold X^n , what is the largest number q such that there exist q everywhere linearly independent vector fields on X^n ? Before stating the main result we recall Hirzebruch's notion of the total L -class of a manifold. This is a multiplicative sequence of Pontrjagin classes associated to the power series $p(z) = \sqrt{z} / \tanh(\sqrt{z})$ (cf. [8]). In terms of the notation of chapters 2 and 3,

$$L(X) = 1 + L_1(p_1) + L_2(p_1, p_2) + \cdots = \prod_{j=1}^{[n/2]} \frac{x_j}{\tanh(x_j)},$$

where $L_k(p_1, \dots, p_k) \in H^{4k}(X)$, and where $p_k \in H^{4k}(X)$ is the k th Pontrjagin class of X computed formally as the k th elementary symmetric function of $x_1^2, \dots, x_{[n/2]}^2$. If X is oriented and of dimension $n = 4k$, then by the classical result of Hirzebruch, the Pontrjagin number $L(X) = L_k(p_1, \dots, p_k)[X]$ is the signature of X .

The following results are not essentially new. The case where E is trivial is contained in the work of Frank [7] and Atiyah [1].

The general version, although not explicitly stated, can be deduced from the work of Mayer [11]. The main point here is that the strongest known theorems can all be deduced, as in [1], from the local symmetries of the Dirac operator.

Theorem 5.1. *Let X be a compact oriented manifold, and suppose X admits q everywhere linearly independent vector fields. Then for any complex vector bundle E over X*

$$\{ch_2 E \cdot L(X)\}[X] \equiv 0 \pmod{2b_q}.$$

Furthermore, if q is even and $ch^q(E) = 0$, then

$$\{ch_2 EL(X)\}[X] \equiv 0 \pmod{4b_q}.$$

Theorem 5.2. *Let X be a compact oriented $4k$ -manifold which admits q everywhere linearly independent vector fields. Then*

$$(5.1) \quad \{ch_2 E \cdot L(X)\}[X] \equiv 0 \begin{cases} \pmod{2a_q} & \text{for all real bundles } E \text{ on } X, \\ \pmod{2c_q} & \text{for all quaternion bundles } E \text{ on } X. \end{cases}$$

Furthermore, if $q \equiv 0 \pmod{4}$ and $ch^q(E) = 0$, then statement (5.1) holds with $2a_q, 2c_q$ replaced by $4a_q$ and $4c_q$ respectively.

In particular, we conclude that if $q \equiv 0 \pmod{4}$, then $\text{sig}(X) \equiv 0 \pmod{4a_q}$.

Proof. The arguments here are entirely similar to the ones given in §3, so we shall only sketch the proof. We can suppose $n = 2m$, and consider the form $w = i^m \omega$ where ω is the oriented volume form for T . We then split $Cl(T)_c = Cl^+(T)_c \oplus Cl^-(T)_c$ where $Cl^\pm(T)_c = (1 \pm L_w)Cl(T)_c$, and consider the bundles $M^\pm = Cl^\pm(T)_c \otimes_C E$ with appropriate connections. Let $\varepsilon_1, \dots, \varepsilon_q$ be q pointwise orthonormal vector fields on X , and let $\tilde{D}^+ : \Gamma(M^+) \rightarrow \Gamma(M^-)$ be the operator obtained by averaging the Dirac operator over the finite group generated by right Clifford multiplication by $\varepsilon_1, \dots, \varepsilon_q$. Then $i(\tilde{D}^+) \equiv 0 \pmod{2b_q}$, and it remains only to compute this index. That $i(\tilde{D}^+) = \{ch_2 E \cdot L(X)\}$ follows easily from formula (2.23) and the obvious analogue of Lemma 3.7.

Suppose now that q is even. Note that T has an orthogonal splitting $T = T_0 \oplus T_1$, where T_1 is the span of the vector fields $\varepsilon_1, \dots, \varepsilon_q$, and where T_0 has dimension $2k$ for some integer k . Let $w_0 = i^k \omega_0$ where ω_0 is the unit volume form for T_0 . Then $w_0^2 = 1$ and w_0 commutes with each ε_j . We average \tilde{D} over the group of order 2 generated by R_{w_0} (right Clifford multiplication by w_0). We then set $M^{\pm\pm} = (1 \pm L_w)(1 \pm R_{w_0})Cl(T)_c \otimes E$, and observe that restriction of \tilde{D} gives operators $\tilde{D}^{\pm\pm} : \Gamma(M^{\pm\pm}) \rightarrow \Gamma(M^{\pm\pm})$ such that: $i(\tilde{D}^{++}) \equiv i(\tilde{D}^{+-}) \equiv 0 \pmod{2b_q}$ and $i(\tilde{D}^{++}) + i(\tilde{D}^{+-}) = i(\tilde{D}^+)$. A straightforward computation shows that

$$i(\tilde{D}^{++}) - i(\tilde{D}^{+-}) = 2^{q/2} \{ch E \cdot e_{T_0}\}[X],$$

where e_{T_0} is the Euler class of T_0 . Hence, if $ch^q(E) = 0$, we have $i(\tilde{D}^+) = 2i(\tilde{D}^{++}) \equiv 0 \pmod{4b_q}$. This completes the proof of Theorem 5.1. The arguments for Theorem 5.2 are entirely similar.

6. Theorems on the geometric dimension

Given a real bundle E over a space X one can ask for the smallest dimension k of bundles E' which are stably equivalent to E . This number is called the (stable) *geometric dimension* of E . For the tangent bundle T of a manifold X this amounts to asking how many everywhere linearly independent sections one can find for the stable tangent bundle $T \oplus \theta$. The methods of the preceding sections apply to give bounds for this number. Our main result is the following.

Theorem 6.1. *Let X be a compact oriented manifold of dimension n , and suppose that the tangent bundle $T(X)$ has geometric dimension $\leq k$. Then*

$$\{ch_2 E \cdot L(X)\}[X] \equiv 0 \pmod{b_{n+2-k}}$$

for all complex vector bundles E over X . Furthermore, if E is a quaternionic bundle over X , and if $\dim X \equiv 0 \pmod{4}$, then

$$\{ch_2 E \cdot L(X)\}[X] \equiv 0 \pmod{c_{n+4-k}}.$$

Finally, if $E = E_0 \otimes_{\mathbb{R}} \mathbb{C}$ for some real bundle E_0 over X , and if $n \equiv 0 \pmod{4}$, then

$$\{ch_2 E \cdot L(X)\}[X] \equiv 0 \pmod{a_{n+1-k}}.$$

Proof. Consider the bundle $T \oplus 2\theta$ with a product metric and a direct sum connection. Our assumption is that this bundle admits $q = n + 2 - k$ pointwise orthonormal sections $\varepsilon_1, \dots, \varepsilon_q$. We then consider the bundle $Cl(T \oplus 2\theta) = Cl(T) \oplus Cl(2\theta)$ and observe that left multiplication by the volume element for 2θ gives a parallel almost complex structure on this bundle. Hence $Cl(T \oplus 2\theta) = Cl(T) \oplus_{\mathbb{R}} 2\mathbb{C} = 2Cl(T)_c$. We now decompose $Cl(T)_c = Cl^+(T)_c \otimes Cl^-(T)_c$, and consider the bundles $M^{\pm} = 2Cl^{\pm}(T)_c \otimes_{\mathbb{C}} E$ with appropriate connection as we did before. These are bundles of complex, \mathbb{Z}_2 -graded modules under right multiplication by elements of $Cl(T \oplus 2\theta)$. Hence one can construct, as before, an averaged Dirac operator $\tilde{D}^+ : \Gamma(M^+) \rightarrow \Gamma(M^-)$ whose kernel and cokernel are complex \mathbb{Z}_2 -graded modules for Cl_q . Therefore $i(\tilde{D}^+) \equiv 0(2b_q)$. Evidently, the index of this operator is twice the index of the operator considered in §5. This proves the first part of the theorem.

Suppose now that E is quaternionic, and consider the bundle $T \oplus 4\theta$ which admits $n + 4 - k$ orthonormal sections. Choose a connection on

$T \oplus 4\theta$ which is the direct sum of a riemannian connection on E and the canonical flat connection on 4θ . Let e_0, \dots, e_3 be parallel orthonormal sections of 4θ , and consider the bundle $M_0 = (1 + L_\omega)Cl(T \oplus 4\theta) = Cl(T) \otimes Cl^+(4\theta)$ where $\omega = e_0 \cdot \dots \cdot e_3$. This bundle carries a parallel \mathbf{H} -structure given by setting $i = L_{e_0 e_1}, j = L_{e_0 e_2}$ and $k = L_{e_0 e_3}$. Note that scalar multiplication by i, j and k commutes with the Dirac operator and preserves the \mathbf{Z}_2 -grading. We now have that $M_0 = Cl(T) \otimes 2\mathbf{H}$, and as usual we split $M_0 = M_0^+ \oplus M_0^-$ where $M_0^\pm = 2Cl^\pm(T) \otimes_{\mathbf{R}} \mathbf{H}$. We then introduce on E a quaternionic (i.e., ‘‘symplectic’’) connection, and consider the bundles $M^\pm = M_0^\pm \otimes_{\mathbf{H}} E \cong 2Cl^\pm(T) \otimes E$. From here the argument proceeds as before.

In the case that $E = E_0 \otimes_{\mathbf{R}} \mathbf{C}$ and $n \equiv 0 \pmod{4}$, one repeats the above construction with $T \oplus \theta$ and with E replaced by E_0 . The bundles and the operator are now real. Applying the index theorem in real form completes the proof.

Example. Let $X = \mathbf{C}P^{2n-1}$ and let $E = \xi$, the canonical complex line bundle (i.e., the ‘‘hyperplane’’ bundle), over $\mathbf{C}P^{2n-1}$. Then letting ω denote the generator of $H^2(\mathbf{C}P^{2n-1}; \mathbf{Z})$, we have that

$$\begin{aligned} & \{ch_2 \xi \cdot \mathbf{L}(\mathbf{C}P^{2n-1})\}[\mathbf{C}P^{2n-1}] \\ &= \left\{ e^{2\omega} \cdot \left(\frac{\omega}{\tanh \omega} \right)^{2n} \right\} [\mathbf{C}P^{2n-1}] \\ &= \text{the coefficient of } \omega^{2n-1} \text{ in } \left\{ e^{2\omega} \left(\frac{\omega}{\tanh \omega} \right)^{2n} \right\} \\ &= \text{the coefficient of } \omega^{2n-1} \text{ in } \left\{ \sinh 2\omega \left(\frac{\omega}{\tanh \omega} \right)^{2n} \right\} \\ &= \frac{1}{2\pi i} \int_{|t|=\varepsilon} \frac{\sinh 2t}{(\tanh t)^{2n}} dt = 2n. \end{aligned}$$

It follows that if the tangent bundle of $\mathbf{C}P^{2n-1}$ has geometric dimension k , then $2n$ is a multiple of b_q where $q = 4n - k$.

Conversely, suppose that $2n$ is a multiple of b_q . Then there exists a complex \mathbf{Z}_2 -graded Cl_q -module $\mathfrak{N} = \mathfrak{N}^0 \oplus \mathfrak{N}^1$ where $\dim_{\mathbf{C}}(\mathfrak{N}^0) = \dim_{\mathbf{C}}(\mathfrak{N}^1) = 2n$. Let $\varepsilon_1, \dots, \varepsilon_q$ denote canonical generators for Cl_q (i.e., $\varepsilon_i \cdot \varepsilon_j + \varepsilon_j \cdot \varepsilon_i = -2\delta_{ij}$). Then multiplication by ε_j gives an isomorphism $\mathfrak{N}^0 \rightarrow \mathfrak{N}^1$ such that for all nonzero $z \in \mathfrak{N}^0$, the vectors $\varepsilon_1 z, \dots, \varepsilon_q z$ are linearly independent over \mathbf{R} . (To see this note that every $v \in \text{span}\{\varepsilon_1, \dots, \varepsilon_n\}$ satisfies $v^2 = -\|v\|^2 \cdot 1$. Hence Clifford multiplication by any $v \neq 0$ is an isomorphism). This means precisely that there are q everywhere linearly independent sections of $2n\xi = T \oplus 2\theta$. Hence we have proved the following result of Steer [13].

Theorem 6.2. *The geometric dimension of the tangent bundle of CP^{n-1} is precisely $2n - q$ where*

$$q = \max\{q': b_q | n\}.$$

Note. Since the Stiefel-Whitney class w_{4m} of CP^{2m} is nonzero, one knows that the geometric dimension of its tangent bundle is $4m$.

The first few cases of the above result for CP^{odd} run as follows:

$$\begin{aligned} g.d.(TCP^1) &= 0, & g.d.(TCP^3) &= 2, & g.d.(TCP^5) &= 8, \\ g.d.(TCP^7) &= 8, & g.d.(TCP^9) &= 16. \end{aligned}$$

In general, $g.d.(TCP^{2^n-1}) = 2(2^n - n - 1)$.

Example. Let $X = HP^{2n-1}$ and let $E = \xi$, the canonical quaternionic line bundle over HP^{2n-1} . Then letting x denote the generator of $H^4(HP^{2n-1}; \mathbb{Z})$ and setting $\omega = \sqrt{x}$ (formally), we have that

$$\begin{aligned} &\{ch_2\xi \cdot L(HP^{2n-1})\}[HP^{2n-1}] \\ &= \left\{ 2 \cosh 2\sqrt{x} \left(\frac{\sqrt{x}}{\tanh\sqrt{x}} \right)^{4n} \frac{\tanh(2\sqrt{x})}{2\sqrt{x}} \right\} [HP^{2n-1}] \\ &= \text{the coefficient of } \omega^{4n-2} \text{ in } \left\{ \frac{\omega^{4n-1} \sinh(2\omega)}{(\tanh \omega)^{4n}} \right\} \\ &= \frac{1}{2\pi i} \int_{|t|=\epsilon} \frac{\sinh(2t)}{(\tanh t)^{4n}} dt = 4n. \end{aligned}$$

Consequently, if the tangent bundle of HP^{2n-1} has geometric dimension $\leq k$, then $4n$ is divisible by c_q where $q = 8n - k$.

Theorem 6.3. *The geometric dimension of the tangent bundle of HP^{n-1} is $\geq 4n - q$ where*

$$q = \max\{q': c_q | 2n\}.$$

Note. Since w_{8m} of HP^{2m} is nonzero we know that the geometric dimension of the tangent bundle of HP^{2m} is $8m$.

Remark. It seems probable that the bound given in Theorem 6.3 is sharp. However the argument given above for CP^n fails in this case since the tangent bundle of HP^n is not stably equivalent to $n + 1$ copies of the canonical quaternion line bundle.

As a final application of our method we consider the problem of computing lower bounds for the geometric dimension of $N\xi$ over CP^n .

Theorem 6.4. *Let ξ denote the canonical complex line bundle over CP^n , and suppose the geometric dimension of $N\xi$ is $\leq k$. Then for all complex vector bundles E over CP^n ,*

$$\left\{ ch_2 E \cdot (1 + e^{2\omega})^{N-n-1} \cdot \left(\frac{\omega}{\tanh \omega} \right)^{n+1} \right\} [CP^n] \equiv 0 \pmod{b_{2N-k}},$$

where ω is the generator of $H^2(CP^n; \mathbf{Z})$. In particular,

$$\binom{N}{n-r} 2^{N-n+r} \equiv 0 \pmod{2b_{2N-k}},$$

for $r = 0, 1, 2, \dots$

Proof. Consider $N\xi = T \oplus 2\theta \oplus (N - n - 1)\xi$, and introduce on this bundle a connection which is a direct sum of a riemannian connection on T , the canonical flat connection on 2θ and the standard unitary connection on ξ . Our assumption is that this bundle admits $2n - k$ pointwise orthonormal sections.

Consider the bundle $Cl(N\xi) = Cl(T) \otimes Cl(2\theta) \otimes Cl(\xi)^{N-n-1}$. Let ω_0 denote the oriented volume form for 2θ , and ω_j the oriented volume form for the j th copy of $\xi, j = 1, \dots, N - n - 1$. Left multiplication by ω_0 gives a parallel complex structure on this bundle which commutes with the Dirac operator and preserves the \mathbf{Z}_2 -grading. Hence

$$Cl(N\xi) \cong Cl(T) \otimes 2C \otimes Cl(\xi)^{N-n-1} = (2Cl T)_c \otimes_c Cl(\xi)_c^{N-n-1}.$$

We then consider the bundles $M^\pm = 2Cl^\pm(T)_c \otimes_c Cl^+(\xi)_c^{N-n-1} \otimes_c E$, where $Cl^+(\xi)_c = (1 + iL_\omega)Cl(\xi)_c \cong \theta_c \oplus \xi$ and ω is the volume element for ξ . In lengthier terms, we have

$$\begin{aligned} Cl^+(\xi)_c^{N-n-1} &= (1 + iL_{\omega_1}) \cdots (1 + iL_{\omega_{N-n-1}}) \cdot \\ &\quad \cdot Cl(\xi_1)_c \otimes \cdots \otimes Cl(\xi_{N-n-1})_c \\ &\cong (1 + \xi_1) \otimes \cdots \otimes (1 + \xi_{N-n-1}). \end{aligned}$$

Taking the averaged Dirac operator and preceding as usual establishes the general formula.

Setting $E = \xi^{k+1}$ and evaluating the integral one finds that:

$$\frac{1}{2} \{ \text{The coefficient of } z^n \text{ in } (1 + z)^k (2 + z)^N \} \equiv 0 \pmod{b_{2N-k}}.$$

Looking at these conditions successively for $k = 0, 1, 2, \dots$ gives the result.

References

[1] M. F. Atiyah, *Vector fields on manifolds*, Arbeitsgemeinschaft für Förschung des Landes Nordrhein-Westfalen, Heft 200.
 [2] M. F. Atiyah, R. Bott & A. Shapiro, *Clifford modules*, Topology Vol. 3, Suppl 1, (1964) 3-38.

- [3] M. F. Atiyah & F. Hirzebruch, *Quelques théorèmes de non-plongement pour les variétés différentiables*, Bull. Soc. Math. France **87** (1959) 383–396.
- [4] M. F. Atiyah & I. Singer, *The index of elliptic operators. I*, Ann. of Math. **87** (1968) 484–530.
- [5] _____, *The index of elliptic operators. III*, Ann. of Math. **87** (1968) 546–604.
- [6] D. Davis & M. E. Mahowald, *Immersions of complex projective spaces and the generalized vector field problem*, Proc. Lond. Math. Soc. (3) **35** (1977) 333–344.
- [7] D. Frank,
- [8] F. Hirzebruch, *Topological methods in algebraic geometry*, Springer, New York, 1966.
- [9] I. M. James, *Two problems studied by Heinz Hopf*, Lecture Notes in Math. Vol. 279, Springer, Berlin, 1972, 134–174.
- [10] H. B. Lawson & M. L. Michelsohn, *Spinors and Dirac operators in topology and geometry*,
- [11] K. H. Mayer, *Elliptische Differentialoperatoren und Ganzahligkeitssätze für Charakteristische Zahlen*, Topology **4** (1965) 295–313.
- [12] W. Schwartz, *Spezielle G -äquivariante elliptische Differentialoperatoren, ein Charaktersatz und Anwendungen*, Bonn. Math. Schr. **59** (1972)
- [13] B. Steer, *Une interprétation géométrique des nombres de Radon-Hurwitz*, Ann. Inst. Fourier (Grenoble) **17** (1967) 209–218.

UNIVERSITY OF CALIFORNIA, BERKELEY
INSTITUT DES HAUTES ETUDES SCIENTIFIQUES

