J. DIFFERENTIAL GEOMETRY
15 (1980) 107–133

NEW ATTRACTORS IN HYPERBOLIC DYNAMICS

F. T. FARRELL & L. E. JONES

0. Introduction

Recall that a hyperbolic set for a diffeomorphism f of a smooth compact manifold M is a compact subset Λ of M satisfying

(a) $f : \Lambda \to \Lambda$ is a homeomorphism;

(b) the tangent bundle of M splits over Λ, $T(M)|\Lambda = \xi_u \oplus \xi_s$, as the Whitney sum of continuous subbundles each of which is left invariant by df (the differential of f);

(c) for some Riemannian metric $| \ |$ on M, $|df(X)| > |X|$ for each nonzero $X \in \xi_u$ and $|df(X)| < |X|$ for each nonzero $X \in \xi_s$.

The hyperbolic set Λ is an *attractor* for $f : M \to M$ provided there exists a compact neighborhood U for Λ satisfying:

$f(U)$ is contained in the interior of U,

$$\bigcap_{n=1}^{\infty} f^n(U) = \Lambda,$$

and each point of Λ is nonwandering.

A natural problem is to classify all hyperbolic attractors $f : \Lambda \to \Lambda$.

Towards solving this problem, R. F. Williams [10], [13], has shown that every attractor $f : \Lambda \to \Lambda$, for which the dimension of the fiber in ξ_u equals the topological dimension of Λ, (these are called *expanding attractors*), is obtained by a simple construction from an expanding endomorphism on a compact smooth branched manifold (without boundary). (He makes the extra assumption that the stable foliation is C^1.) Conversely, every expanding endomorphism on a compact smooth branched manifold which satisfies three properties (cf. [13, 3.0]) yields an expanding attractor. Recall an *expanding endomorphism* on a compact smooth manifold N is a smooth map $h : N \to N$ such that

$$|dh(X)| > |X|$$

Communicated by S. Smale, June 12, 1978. Both authors were partially supported by grants from the National Science Foundation.

for all nonzero vectors $X \in T(N)$ with respect to some Riemann metric $|\ |$ on M.

More generally, a self-map $h : K \to K$ of a compact smooth branched manifold K is an *expanding endomorphism* if h is a smooth immersion, and

$$|dh(X)| > |X|$$

for all nonzero vectors X tangent to K with respect to some Riemann metric on K. (Smooth branched manifolds and smooth immersions of branched manifolds are defined in [13, §1].)

The main result of this paper (Theorem 0.1) is the construction of an expanding endomorphism on a 2-dimensional, simply connected, compact branched manifold (without boundary); applying Williams' theory to this example, we obtain (Corollary 0.2) an expanding attractor $f : \Lambda \to \Lambda$ such that $\check{H}_1(\Lambda, \mathbf{Z}) = 0$. (These are the first examples of this type.) We now state the results precisely.

Theorem 0.1. *There is a smooth immersion $G : K \to K$, where K is a simply connected compact 2-dimensional branched manifold satisfying*

(a) $|dG(X)| > |X|$ *for each nonzero vector X tangent to K (relative to some Riemann metric on K),*

(b) *G has a dense orbit, and*

(c) *for each point $x \in K$, there is a neighborhood N of x such that $G(N)$ is contained in a a 2-cell which is smoothly embedded in K.*

(Conditions (a), (b) and (c) imply Williams' properties 1, 2 and 3^+ of [13, §3].)

Corollary 0.2. *There exists a 2-dimensional expanding attractor $f : \Lambda \to \Lambda$ satisfying $\check{H}_0(\Lambda) = \mathbf{Z}$ and $\check{H}_1(\Lambda) = 0$, where $\check{H}_i(\ )$ denotes the i-dimensional integral Čech homology functor.*

In [13], Williams proved that every expanding attractor Λ, with stable foliation C^1, is locally homeomorphic to the product of a Cantor set and a u-disc where $u = \dim \Lambda$; in [4], Robinson and Williams removed the assumption that the stable foliation is C^1. This motivated the conjecture [12, Conjecture J], [13, Conjecture, p. 171] that when Λ is orientable, it is the total space of a fiber bundle with base space a manifold and a Cantor set for fiber. Williams [11] verified this conjecture when $\dim \Lambda = 1$. We give a counterexample to it when $\dim \Lambda = 2$.

Corollary 0.3. *There exists an (orientable) expanding attractor Λ with* $\dim \Lambda = 2$ *such that Λ is not the total space of a fiber bundle with a manifold for base space and a Cantor set for fiber.*

Associated to an expanding endomorphism $g : K \to K$ of a n-dimensional branched manifold K satisfying properties (1), (2) and (3^+) from [12, p. 949], Williams defined a space Σ to be the inverse limit of the sequence

$$K \underset{g}{\leftarrow} K \underset{g}{\leftarrow} K \underset{g}{\leftarrow} \cdots$$

and $h : \Sigma \to \Sigma$ to be the coordinate shift

$$h(x_0, x_1, \cdots) = (gx_0, gx_1, gx_2, \cdots).$$

The pair (Σ, h) is called the *n-solenoid presented* by (K, g). In [12, p. 950, Question], it is asked if

$$g_{\#} : \pi_1(K) \to \pi_1(K)$$

determines (Σ, h) up to topological conjugacy; when $\dim K = 1$, Williams [11] proves it does. We show it does not in general.

Corollary 0.4. *There exists a 2-dimensional branched manifold K together with expanding endomorphisms $f, g : K \to K$ satisfying properties* (1), (2) *and* (3^+) *from* [12, *p.* 949] *such that*

(i) *f and g induce the same map on $\pi_1(K)$; but*

(ii) *the 2-solenoid presented by (K, f) is not topologically conjugate to the one presented by (K, g).*

We pose the following problem as a possible generalization of Theorem 0.1.

Question 0.5. Let M denote an n-dimensional compact connected Riemannian manifold, and $g : M \to M$ a continuous map which is covered by a linear bundle map $\hat{g} : T(M) \to T(M)$ (monic on fibers). Given $\varepsilon > 0$ and $\beta > 1$, does there exist a cell complex structure (for its definition see [8, p. 100]) C for M and a homotopy $g_t : M \to M$ with $g_0 = g$, satisfying:

(a) $d(g_t(x), g(x)) < \varepsilon$ for all $x \in M$ and $t \in [0, 1]$, where $d(\ ,\ )$ is the distance function determined by the Riemannian metric;

(b) the $(n-1)$-dimensional skeleton of C, denoted by C^{n-1}, is a smooth branched submanifold of M;

(c) g_1 is an expanding endomorphism when restricted to C^{n-1}; in particular, $|dg_1(X)| \geqslant \beta|X|$ for all vectors X tangent to C^{n-1}, and g_1 leaves C^{n-1} invariant;

(d) if condition (a) is dropped, then g_1 can be chosen to have an orbit dense in C^{n-1};

(e) for any point $x \in C^{n-1}$, there is a neighborhood N of x in C^{n-1} such that $g_1(N)$ is contained in an $(n-1)$-cell smoothly embedded in C^{n-1}?

Remark 0.6. When M is a codimension-zero submanifold of $\mathbf{R}^3$, and g is either the constant map or the identity function, the answer to Question 0.5 is affirmative. (In a preliminary version of this paper, we verified this when g is the identity map, and it is easier to verify when g is constant.)

The following problem is related to Question 0.5.

Question 0.7. If $h : K \to K$ is an expanding endomorphism of a compact smooth branched manifold (without boundary), do the real Pontryagin classes of the tangent bundle of K vanish?

Remark 0.8. When K is a manifold, the answer to Question 0.7 is yes; in fact,

$$h^* : H^i(K, \mathbf{R}) \to H^i(K, \mathbf{R})$$

is expanding (for $i > 0$); in particular, 1 is not an eigenvalue of h^*.

Remark 0.9. Questions 0.5 and 0.7 conflict. In particular, choose M in Question 0.5 to have nonzero first real Pontryagin class (dimenson $M > 4$) and $g : M \to M$ to be the identity map. If Question 0.5 has an affirmative answer in this case, then the posited map $g_1 : C^{n-1} \to C^{n-1}$ answers Question 0.7 negatively (where $h = g_1$ and $K = C^{n-1}$) since the first real Pontryagin class of C^{n-1} must also be nonzero (note $T(C^{n-1}) \oplus \theta^1 = T(M)|C^{n-1}$).

The proof of Theorem 0.1 bares a superficial resemblance to the construction of structurally stable diffeomorphisms by Smale [6], Shub and Sullivan [5]. But the details are as different as the results. For example, the diffeomorphisms constructured in [5], [6] all have zero-dimensional hyperbolic sets; whereas, any diffeomorphism constructed from Theorem 0.1 using Williams′ theory [13] has a 2-dimensional hyperbolic set.

We now outline the proof of Theorem 0.1. In §1 a cell structure $\mathbf{B}_1$ is constructed for $\mathbf{R}^3$ which is invariant under the group Γ of all translations (of $\mathbf{R}^3$) by vectors with integral entries. The 2-skeleton B_1 of $\mathbf{B}_1$ is a branched 2-manifold smoothly embedded in $\mathbf{R}^3$. Its orbit space $Y = B_1|\Gamma$ is the 2-skeleton of a "natural" cell structure $\mathbf{B}_1|\Gamma$ for the 3-torus T^3; $J : B_1 \to Y$ denotes the canonical quotient map. (Note B_1 is simply connected.) In §2 we construct an immersion I of Y into B_1; this construction is motivated by the following consequence of the Smale-Hirsch immersion theory; namely, T^3 with a single point removed immerses in $\mathbf{R}^3$. Immersions $I_n : B_1 \to (n)^{-1}B_1$ are constructed in §3 such that

$$|dI_n(X)| \geqslant \beta|X|$$

for all vectors X tangent to B_1, where $\beta > 0$ is a number independent of n and X. Proposition 3.3 is the main result needed to construct I_n; its proof is quite long and is postponed until §6; this result is motivated by Hirsch's smoothing theory [2]. Consider the following diagram

$$\begin{tikzcd} \frac{1}{n}B_1 \arrow[r, "\rho_n"] & B_1 \arrow[d, "J"] \\ B_1 \arrow[u, "I_n"] & Y \arrow[l, "I"] \end{tikzcd}$$

where $(n)^{-1}B_1 = \{n^{-1}x | x \in B_1\}$ and $\rho_n(x) = nx$. The composite $IJ\rho_n I_n : B_1 \to B_1$ is an expanding endomorphism for all n sufficiently large. It is easy to construct a simply connected, compact, branched submanifold K of B_1 containing $I(Y)$; this is the branched manifold posited in Theorem 0.1 and $IJ\rho_n I_n | K$ is an approximation to G in that it satisfies properties (a) and (c) of Theorem 0.1. In §4 we modify this immersion to one with a dense orbit which still satisfies conditions (a) and (c) of Theorem 0.1.

We note that (K, G) is shift equivalent to (Y, G') where G' is a self immersion of Y.

Finally in §5, Corollaries 0.2, 0.3 and 0.4 are verified.

It is a great pleasure to thank Bob Williams for his constructive criticism of an earlier version of this paper which was very useful in preparing the present one.

1. Cell structures and thickenings

In this section two interelated sequences A_n and $\mathbf{B}_n$, $n \geqslant 1$, of regular cell structures for $\mathbf{R}^3$ are described. We also associate to each subcomplex C of A_n a "thickened" subcomplex $B(C, n)$ in $\mathbf{B}_n$.

Use A_1 to denote the partitioning of $\mathbf{R}^3$ into unit cubes having integral lattice points of $\mathbf{R}^3$ for vertices. The intersections of the cubes generate the 2-, 1- and 0-dimensional cells of A_1. The 0-dimensional cells of A_1 are called *vertices* and denoted by $v, u, w \in A_1$. The 1-dimensional cells of A_1 are called *edges* and denoted by $e \in A_1$. The 2-dimensional cells of A_1 are faces and denoted by $f \in A_2$. The 3-dimensional cells are the cubes.

There are four types of 3-dimensional cells in $\mathbf{B}_1$: those associated to the vertices, edges, faces and cubes of A_1. These will be termed *balls, tubes, solid plates* and *volumes* of $\mathbf{B}_1$, respectively.

Ball of $\mathbf{B}_1$. To each vertex $v \in A_1$ we associate the ball $b(v)$ of radius $(10)^{-1}$ centered at v.

Tubes of $\mathbf{B}_1$. To each edge $e \in A_1$ we associate a tube $t(e)$ as follows. Let D_1, D_2 denote discs of radius $(10)^{-1}$ centered at (0, 0, 0) and (1, 0, 0) and lying in the (x, y)-plane of $\mathbf{R}^3$. Pick a smooth function $g : [0, 1] \to \mathbf{R}$ satisfying

$$\text{(a)}\quad \operatorname{image}(g) \subset [.001, .1],$$

$$(1.1)\qquad \text{(b)}\quad g(x) = \begin{cases} (.01 - x^2)^{\frac{1}{2}}, & x \in [0, .09], \\ (.01 - (x-1)^2)^{\frac{1}{2}}, & x \in [.91, 1], \end{cases}$$

$$\text{(c)}\quad g(x) > \begin{cases} (.01 - x^2)^{\frac{1}{2}}, & x \in (.09, .1], \\ (.01 - (x-1)^2)^{\frac{1}{2}} & x \in [.9, .91). \end{cases}$$

The union of the graph of g with boundaries of D_1 and D_2 is illustrated below

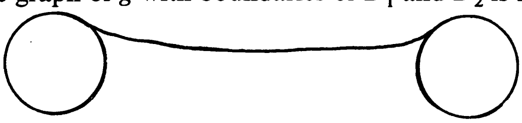

FIG. 1.1.

Rotate this set about the x-axis in $\mathbf{R}^3$ to obtain a branched surface of rotation S which bounds a solid V. Note that V is the union of balls b_1, b_2 of radius $(10)^{-1}$ centered at $(0, 0, 0)$ and $(1, 0, 0)$, and a solid tube T that connects b_1 to b_2. Let $r : \mathbf{R}^3 \to \mathbf{R}^3$ be any rigid motion sending $(0, 0, 0) \to v_1$ and $(1, 0, 0) \to v_2$, where v_1, v_2 are the end points of the edge e; define $t(e)$ to be $r(T)$.

Solid plates of $\mathbf{B}_1$. To any face f of A_1 we associate a solid plate $p(f)$ as follows. Let $B(\partial f)$ denote the union of all balls and tubes in $\mathbf{B}_1$ associated to vertices and edges of A_1 which lie in the boundary ∂f of f. Let p denote the union of all lines of length $(10)^{-4}$ which intersect f perpendicularly at their center points. Set $p' = \text{closure}(p - B(\partial f))$. Note that the surface $\partial(B(\partial f) \cup p')$ is differentiable except at the two "circles" $\partial(B(\partial f) \cap p')$. These "corners" can be rounded away by gradually shrinking the diameters of the lines wich form p as their center points travel to the center of f. This process shrinks p' to $p(f)$ so that $p(f) \cap B(\partial f) = p' \cap B(\partial f)$, and $\partial(p(f) \cup B(\partial f))$ becomes a smooth surface.

Volumes of $\mathbf{B}_1$. Let X denote the union of all balls, tubes, and solid plates in $\mathbf{B}_1$. Note that closure $(\mathbf{R}^3 - X)$ is a collection of disjoint closed 3-dimensional cells in $\mathbf{R}^3$, with exactly one lying in the interior of each cube of A_1. These are the volume cells of $\mathbf{B}_1$.

The lower dimensional cells of $\mathbf{B}_1$ are generated by intersecting the 3-dimensional cells. In the rest of this paper we shall denote the 2-skeleton of $\mathbf{B}_1$ by B_1.

There are various smooth surfaces (some with boundary) embedded as subcomplexes of B_1, which shall be referred to frequently in later chapters. We introduce these surfaces with names and symbols now.

(1.2)

(i) For each vertex v of A_1, the boundary of $b(v)$ is a sphere $s(v)$ of B_1.

(ii) To each edge e of A_1 is associated a cylinder $b(v)$ of B_1 where $c(e)$ is the closure of $\partial t(e) - (b(v_1) \cup b(v_2))$; v_1 and v_2 are the endpoints of e.

(iii) To each face f of A_1 is associated two plates $p^+(f)$ and $p^-(f)$ of B_1 which are the connected components of the closure of $\partial p(f) - B(\partial f)$. Note each plate is a 2-cell.

For the rest of this paper, we assume the choice of balls, tubes and solid plates has been made so that $\mathbf{B}_1$ is invariant under translation by all vectors having integer valued coordinates. The following result is obvious.

Lemma 1.1. *The set B_1 is a smooth branched submanifold of $\mathbf{R}^3$.*

For any positive integer n, the cell structures A_n, $\mathbf{B}_n$ are just the image of A_1, $\mathbf{B}_1$ under multiplication by $1/n$ mapping $\mathbf{R}^3 \to \mathbf{R}^3$. Balls, tubes, solid plates of $\mathbf{B}_1$ and all the surfaces listed in (1.1) are sent to the objects with the same names in $\mathbf{B}_n$. The notation set up for $\mathbf{B}_1$ is also used for $\mathbf{B}_n$. For example, $p^+(f)$, $f \in A_n$, denotes a plate in $\mathbf{B}_n$ associated to the face of A_n, and B_n denotes the 2-skeleton of $\mathbf{B}_n$.

For any closed subcomplex M of A_m, $B(M, m)$ will denote the union of M with all balls, tubes, and solid plates of $\mathbf{B}_m$ associated to vertices, edges, and faces of A_m which lie in M. Note that M is also a subcomplex of A_{mk} for all positive integers k, so $B(M, mk)$ is defined and is a subcomplex of $\mathbf{B}_{mk}$. The next result is geometrically clear; its proof is left as an exercise.

Lemma 1.2. *If M is a 3-dimensional manifold with boundary which is a subcomplex of A_m, then $B(M, m)$ is a smooth codimension-0 submanifold with boundary of $\mathbf{R}^3$.*

Let π: $\mathbf{R}^3 \to T^3$ denote the standard covering projection of $\mathbf{R}^3$ onto the 3-dimensional torus. Set

$$Y = \pi(B_1), J = \pi|B_1; \tag{1.3}$$

Y is the 2-skeleton of a finite cell structure for T^3 and is equipped with a smooth branched manifold structure with respect to which $J : B_1 \to Y$ is a smooth immersion. There are the following smooth surfaces (some have boundary) smoothly embedded in Y:

$$s = J(s(0)), \quad c_i = J(c(e_i)), \quad p_i^{\pm} = J\big(p^{\pm}(f_i)\big), \quad (i = 1, 2, 3). \tag{1.4}$$

Here 0 denotes the vertex (0, 0, 0) of A_1; e_1, e_2 and e_3 are the edges starting at 0 and ending at (1, 0, 0), (0, 1, 0) and (0, 0, 1), respectively; f_i is the face of A_1 perpendicular to e_i and containing the other two edges in $\{e_1, e_2, e_3\}$. So s, c_i and $p_i^{\pm}$ are just the diffeomorphic images (under J) of the spheres, cylinders and plates of B_1; we will call them the sphere, cylinders and plates of Y.

2. An immersion

This section is devoted to proving the following result.

Theorem 2.1. *There exists an immersion $I : Y \to B_1$.*

This result is motivated by the fact that T^3 with a point deleted immerses in $\mathbf{R}^3$, which follows from Smale-Hirsch immersion theory.

It is easy to define $I|Y_1$ where

(2.1) $$Y_1 = s \cup c_1 \cup c_2 \cup c_3,$$

(note that Y_1 is a branched submanifold of Y). To do this, map s diffeomorphically to $s((0, 0, 0))$ by letting $I|s$ be the inverse of $J|s((0, 0, 0))$, and map each $c_i (i = 1, 2, 3)$ diffeomorphically to certain "cylinders" C_i which we now construct. (See Fig. 2.1. below.)

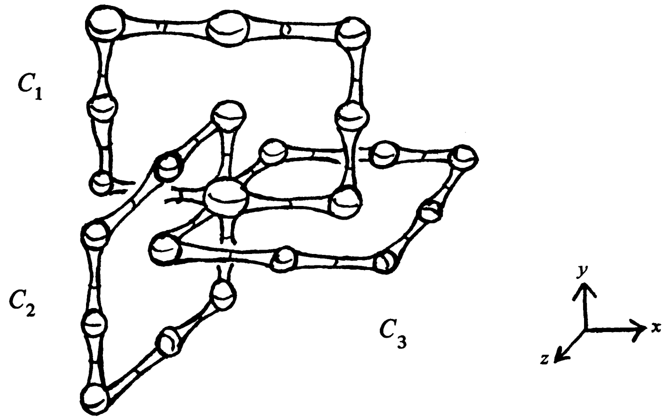

FIG. 2.1

Let α_1 be the polygonal arc (in A_1) which is the boundary of the square $\mathbb{S}_1$ with vertices $(-1, 0, 0)$, $(1, 0, 0)$, $(1, 2, 0)$ and $(-1, 2, 0)$; likewise, α_2 and α_3 are the boundaries of the squares $\mathbb{S}_2$ and $\mathbb{S}_3$ with vertices $(0, \pm 1, 0)$, $(0, \pm 1, 2)$ and $(0, 0, \pm 1)$, $(2, 0, \pm 1)$, respectively. We define $C_i (i = 1, 2, 3)$ to be the closure of the boundary of $B(\alpha_i, 1) - b((0, 0, 0))$.

We require that $I|c_1$ should have no unnecessary twists; i.e., $I(J(f_3) \cap c_1)$ should be a subset of P_{xy} (the (x, y)-plane in $\mathbf{R}^3$) and $I(J(f_2) \cap c_1)$ should be disjoint from P_{xy}; more precisely, the following should be true

(2.2) $$I(p_3^{\pm} \cap c_1) \subset B(P_{xy}, 1), I(p_2^{\pm} \cap c_1) \cap B(P_{xy}, 1) = \phi.$$

Similarly, $I|c_2$ and $I|c_3$ should have no unnecessary twists; i.e.,

(2.3) $$I(p_2^{\pm} \cap c_3) \subset B(P_{xz}, 1), \quad I(p_1^{\pm} \cap c_2) \subset B(P_{yz}, 1),$$

$$I(p_1^{\pm} \cap c_3) \cap B(p_{xz}, 1) = \phi, \quad I(p_3^{\pm} \cap c_2) \cap B(P_{yz}, 1) = \phi,$$

where P_{xz} and P_{yz} denote the (x, z) and (y, z) planes, respectively.

It remains to define $I|p_i^{\pm}$, $i = 1, 2, 3$; we do this only for $i = 2$ since the other cases are analogous. Let $\gamma^{\pm}$ denote the boundaries of $p_2^{\pm}$. Then both γ^+ and γ^- are simple closed arcs as are $I(\gamma^+)$ and $I(\gamma^-)$. (Note $\gamma^+ \cup \gamma^- \subset Y_1$.) See the picture below illustrating $I(\gamma^-)$; the part of $I(\gamma^+)$ contained in C_3 would be "above" $I(\gamma^-)$, and the part of $I(\gamma^+)$ in C_1 would be "inside" of $I(\gamma^-)$.

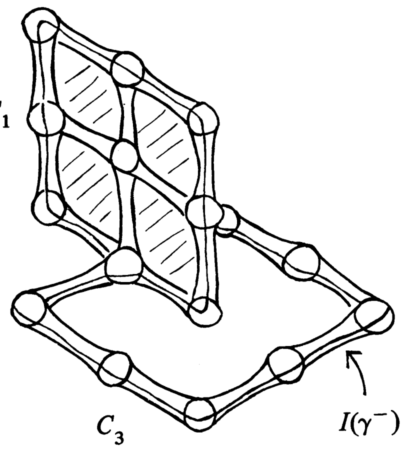

FIG. 2.2

To extend the definition of I to p^-, it suffices to find a smooth disc in B_1 which spans $I(\gamma^-)$. (This disc should not intersect the image under I of a short collar of γ^- in Y_1, in order to prevent I from folding along γ^-.) Examining Fig. 2.2, it is clear that γ^- spans such a disc inside the surface T where T is the boundary of $B(\mathbb{S}_1 \cup \alpha_3, 1)$. Note T has genus 1; below we given a simplified (but topologically correct) drawing of T in which the "bump" (hatched top lobe in Fig. 2.2) coming from the part of T in $B(\mathbb{S}_1, 1)$ has been flattened, and the spanning disc is indicated by hatching. (Inside the part of T intersecting $B(\alpha_3, 1)$, the disc lies on the "top" as pictured in Fig. 2.3.)

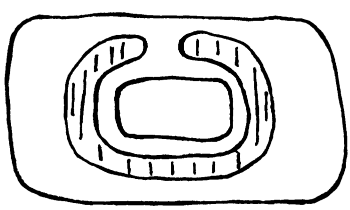

FIG. 2.3

To extend I to p_2^+, one constructs an appropriate disc in B_1 spanning $I(\gamma^+)$; this is slightly harder to visualize and is left as an exercise. (Hint: α_3 is the boundary of the "pan" $\mathbb{S}$ pictured below, and $I(\gamma^+)$ is spanned by a disc in the boundary of $B(\mathbb{S} \cup \alpha_1, 1)$.)

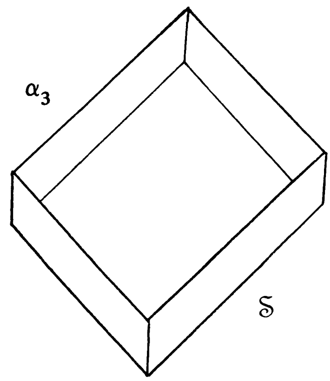

FIG. 2.4

3. The main argument

Section 3 is devoted to proving the followng result (modulo the results of §6).

Theorem 3.1. *There exists a number $\beta > 0$ and immersions $I_n : B_1 \to B_{n\alpha}$ (where $\alpha = 10^4$ and $n \geqslant 1$) such that*

$$|dI_n(X)| \geqslant \beta|X|$$

for all vectors X tangent to B_1 and all integers $n \geqslant 1$.

We start by constructing a cell structure $\mathcal{C}$ for $\mathbf{R}^3$ which is the rectilinear version of $\mathbf{B}_1$ and is obtained from A_1 in much the same way as $\mathbf{B}_1$. To be more explicit, the cell structure $\mathcal{C}$ is generated by requiring that certain rectilinear subsets $b'(v)$, $t'(e)$ and $p'(f)$ (defined below) be 3-cells of $\mathcal{C}$.

For each vertex v in A_1, $b'(v)$ denotes the cube in $\mathbf{R}^3$ centered at v, having edges of length $2(10)^{-1}$ parallel to the coordinate axes. For each edge e in A_1, consider the square S perpendicular to e and centered at the midpoint of e with edges parallel to coordinate axes having length $2(10)^{-2}$; $t'(e)$ is the solid parallelepiped generated by parallel translating S along e'–the subarc of e having the same midpoint (as e) and having total length equal to $1 - 2(10)^{-1}$. For each face f of A_1, $p'(f)$ denotes the union of all line segments which intersect f perpendicularly in their midpoints, having length $2(10)^{-3}$ and are not contained in the interior of any of the $b'(v)$, $t'(e)$ defined above. (There are also the analogues of the volumes of $\mathbf{B}_1$; we do not discuss these since they are not used.) The picture below illustrates these definitions.

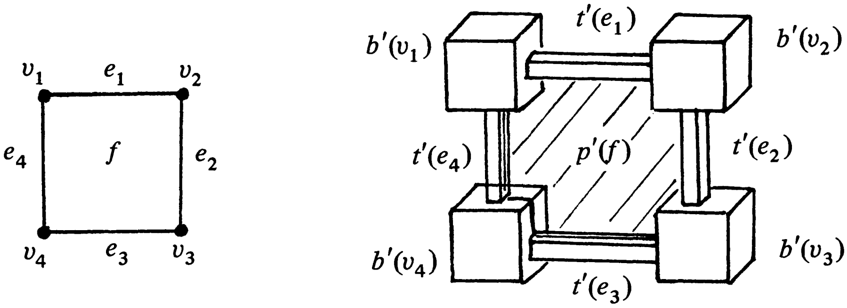

FIG. 3.1

Note $b'(v)$, $t'(e)$ and $p'(f)$ are subcomplexes of A_α where $\alpha = 10^4$.

Next, we construct codimension-0 branched submanifolds C_n of $B_{n\alpha}$; C_n is the union of the smooth 2-submanifolds $\hat{s}(v, n)$, $\hat{s}(e, n)$ and $\hat{s}(f, n)$ defined below, where v, e and f are vertices, edges and faces of A_1. For each vertex v of A_1, $\hat{s}(v, n)$ is the boundary of the closed 3-manifold $B(b'(v), n\alpha)$ (cf. Lemma 1.2). If e is an edge of A_1, $\hat{s}(e, n)$ is the boundary of the manifold

$$B(t'(e) \cup b'(v_0) \cup b'(v_1), n\alpha), \tag{3.1}$$

where v_0 and v_1 are the vertices of e. Likewise, if f is a face of A_1, then $\hat{s}(f, n)$ is the boundary of the manifold

$$B\left(p'(f) \cup \left(\bigcup_{i=1}^{4} t'(e_i)\right) \cup \left(\bigcup_{i=1}^{4} b'(v_i)\right), n\alpha\right), \tag{3.2}$$

where the e_i and v_i ($i = 1, \cdots, 4$) are the edges and vertices of f.

Now an immersion $\varphi : B_1 \to C_1$ is needed satisfying certain properties which we proceed to formulate. Define smooth submanifolds $s(e)$ and $s(f)$ of B_1 (the analogues of $\hat{s}(e, n)$ and $\hat{s}(f, n)$) by

$$s(e) = \partial B(e, 1), \quad s(f) = \partial B(f, 1), \tag{3.3}$$

where e and f are edges and faces, respectively, of A_1. (If M is a manifold, then ∂M denotes its boundary. Clearly, $B(e, 1)$ and $B(f, 1)$ are smooth 3-manifolds with boundary.) The first property is that

$$\varphi(s(v)) \subset \hat{s}(v, 1), \varphi(s(e)) \subset \hat{s}(e, 1), \quad \varphi(s(f)) \subset \hat{s}(f, 1) \tag{3.4}$$

for each vertex v, edge e and face f of A_1.

For each edge e of A_1, let $t''(e)$ denote the union of all closed cubes of A_{100} which intersect $t'(e)$ but are not contained in $b'(v_0)$ or $b'(v_1)$ where v_0 and v_1 are the vertices of e. And define a smooth 2-manifold $s^*(e)$ by the formula

$$s^*(e) = \partial B(t''(e) \cup b'(v_0) \cup b'(v_1), 10^4). \tag{3.5}$$

The second property is that

$$\varphi(s(v) \cap s(e)) \subset \hat{s}(v, 1) \cap s^*(e) \tag{3.6}$$

for each pair e, v where e is an edge of A_1 and v is a vertex of e. The picture below illustrates this property.

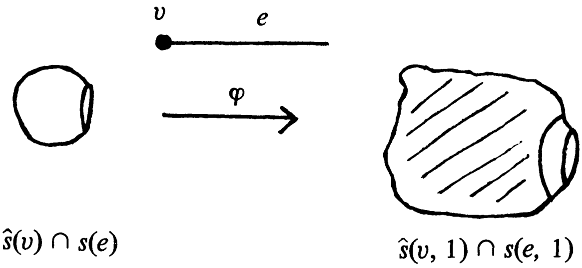

FIG. 3.2

The shaded region is $\hat{s}(v, 1) \cap s^*(e)$ and has been geometrically simplified in this drawing. (Note $\hat{s}(v, 1) \cap s^*(e) \subset \hat{s}(v, 1) \cap \hat{s}(e,1)$.)

For each face f of A_1, let $p''(f)$ denote the union of all closed cubes in A_{1000} which intersect $p'(f)$ but are not contained in

$$\left(\bigcup_{i=1}^{4} b'(v_i)\right) \cup \left(\bigcup_{i=1}^{4} t'(e_i)\right), \tag{3.7}$$

where v_i and e_i $(i = 1, \cdots, 4)$ are the vertices and edges of f. For each edge e of f, define a smooth 2-manifold $s^*(f, e)$ by the formula

$$s^*(f, e) = \partial B\left(p''(f) \cup t'(e) \cup \left(\bigcup_{i \neq j} t''(e_i)\right) \cup \left(\bigcup_{i=1}^{4} b'(v_i)\right), 10^4\right), \tag{3.8}$$

where $e = e_j$. The third property is that

$$\varphi(s(e) \cap s(f)) \subset \hat{s}(e, 1) \cap s^*(f, e) \tag{3.9}$$

for each such pair f, e. The picture below illustrates this property.

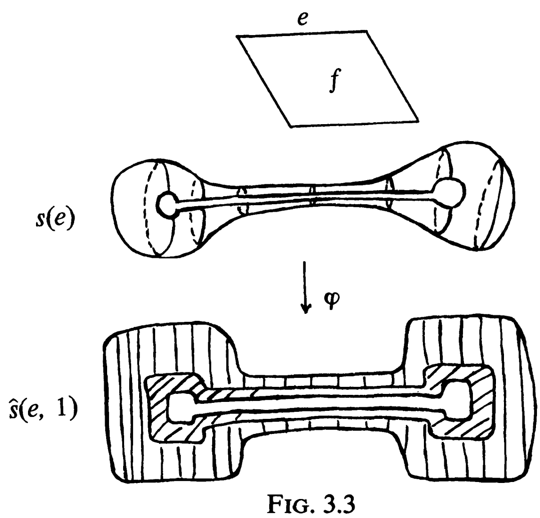

FIG. 3.3

The shaded region in $s(e)$ is $s(e) \cap s(f)$. The vertically shaded region in $\hat{s}(e, 1)$ represents $\hat{s}(e, 1) \cap s^*(f, e)$; the union of this region and the diagonally shaded region is $\hat{s}(e, 1) \cap \hat{s}(f, 1)$. (Note $\hat{s}(e, 1)$ is geometrically distorted in Fig. 3.3 to facilitate drawing it.)

Let Γ denote the group of all translations of $\mathbf{R}^3$ by vectors with integral entries.

Lemma 3.2. *There exists a Γ-equivariant immersion $\varphi : B_1 \to C_1$ satisfying properties* (3.4), (3.6) *and* (3.9).

This result is geometrically clear but messy to prove; it is left as an exercise.

The immersions I_n (of Theorem 3.1) are composites of φ with certain other maps posited below.

Proposition 3.3. *There exists a number $\gamma > 0$ and diffeomorphisms*

$$\psi_{v,n} : \hat{s}(v, 1) \to \hat{s}(v, n),$$
$$\psi_{e,n} : \hat{s}(e, 1) \to \hat{s}(e, n),$$
$$\psi_{f,n} : \hat{s}(f, n) \to \hat{s}(f, n)$$

such that

$$|d\psi_{\sigma,n}(X)| \geqslant \gamma|X|$$

for all vectors X tangent to $s(\sigma, 1)$ (where $\sigma = v$, e or f) and all integers $n \geqslant 1$. Furthermore, if v is a vertex of e, then $\psi_{v,n}$ and $\psi_{e,n}$ agree when restricted to $\hat{s}(v, 1) \cap s^*(e)$. Also, if e is an edge of f, then $\psi_{e,n}$ and $\psi_{f,n}$ agree when restricted to $\hat{s}(e, 1) \cap s^*(f, e)$.

This result is proven in §6. We now define the immersions I_n by the following formula

$$I_n(x) = \begin{cases} \psi_{n,v}\varphi(x) & \text{if } x \in \hat{s}(v, 1), \\ \psi_{n,e}\varphi(x) & \text{if } x \in \hat{s}(e, 1), \\ \psi_{n,f}\varphi(x) & \text{if } x \in \hat{s}(f, 1); \end{cases} \tag{3.10}$$

these immersions are well defined by properties (3.6), (3.9) of φ and the last two sentences of Proposition 3.3. Since φ is Γ-invariant, $|d\varphi(X)|$ is bounded away from zero as X varies over all vectors of length one tangent to B_1; this together with Proposition 3.3 shows dI_n satisfies the metric property posited in Theorem 3.1. Because $\hat{s}(v, n)$, $\hat{s}(e, n)$ and $\hat{s}(f, n)$ are all submanifolds of $B_{n\alpha}$, $I_n : B_1 \to B_{n\alpha}$; this completes the proof of Theorem 3.1 modulo proving Proposition 3.3 which is done in §6.

4. Proof of Theorem 0.1

In this section we prove Theorem 0.1; as a first approximation, we have the following result. Let $\mathcal{K}$ be the cube in $\mathbf{R}^3$ centered at (0, 0, 0) with sides parallel to the coordinate axes and having length 4. Regarding $\mathcal{K}$ as a subcomplex of A_1, let $\mathcal{K}^2$ denote its 2-skeleton and $K = B(\mathcal{K}^2, 1) \cap B_1$.

Proposition 4.1. *Given any number $\alpha > 0$, there exists an immersion $g : K \to K$ satisfying property* (c) *of Theorem* 0.1 *and such that*

$$|dg(X)| \geqslant \alpha |X|$$

for all vectors X tangent to K.

Proof. Consider the composites of the maps in the following diagram

$$\text{(4.1)} \qquad \begin{array}{ccc} K & \xrightarrow{I_n|K} & B_n \\ {\scriptstyle I}\uparrow & & \downarrow{\scriptstyle \rho_n} \\ Y & \xleftarrow[J]{} & B_1 \end{array}$$

where $\rho_n(x) = nx$ for $x \in B_n$. (Examining the construction of $I : Y \to B_1$ (cf. Theorem 2.1), it is clear that its image is contained in K.) When n is large enough, these self maps of K are sufficiently expanding (because of Theorem 3.1) to satisfy the metric condition of Proposition 4.1. Also by their construction, the immersions I_n are flattening; i.e., each $x \in B_1$ has a neighborhood which is mapped by I_n into a smooth 2-cell. Hence all of the above composites satisfy property (c) of Theorem 0.1, so that g can be chosen to be one of them.

To complete the proof of Theorem 0.1, we begin with an expanding immersion $g : K \to K$ provided by Proposition 4.1, and modify it so it has a dense orbit (retaining properties (a) and (c) of Theorem 0.1). This modification procedure depends on Lemma 4.2; we first state this result, then complete the proof of Theorem 0.1 and finally verify Lemma 4.2.

Lemma 4.2. *There exists an immersion $\lambda : K \to K$ such that $\lambda(S) = K$ when S is any of the following subset of K : $s(v)$, $c(e)$ or $p^{\pm}(f)$ where v, e and f denote an arbitrary vertex, edge or face, respectively, of $\mathcal{K}$.*

We now complete the proof of Theorem 0.1. Let G be the composite of the maps λ and g posited in Lemma 4.2 and Proposition 4.1, respectively, with α chosen sufficiently large so that $G = \lambda g$ satisfies property (a) of Theorem 0.1. (Since g satisfies property (c) of Theorem 0.1, G must also.) It remains to show G has a dense orbit; as is well-known, it suffices to show, for each open set U in K, there exists an integer n such that $G^n(U) = K$. If $gG^{n-1}(U)$ contains a set of the form $s(v)$, $c(e)$ or $p^{\pm}(f)$, then $G^n(U) = K$ by Lemma 4.2. An elementary, but slightly complicated, covering space (curve lifting) type argument shows gG^{n-1} must contain a set of this type when n is sufficiently large. (The details of this argument are left as an exercise.) This completes the verification of Theorem 0.1.

It remains to prove Lemma 4.2. Define smooth 2-cells $D(v, e)$ by the identity

$$D(v, e) = s(v) \cap t(e), \tag{4.2}$$

where e is an arbitrary edge of A_1, and v is a vertex of e. (See §1 for the definition of $t(e)$.) Our proof depends on constructing an immersion $\eta : K \to K$ such that

$$\eta(D(v, e)) = K \tag{4.3}$$

for each pair v, e where v is in the interior of $\mathcal{K}$. Given such an immersion, let λ be the composite of the maps of the following diagram

$$\begin{tikzcd} K \arrow[rr, "J|K"] & & Y \arrow[dl, "I"] \\ & K \arrow[ul, "\eta"] & \end{tikzcd} \tag{4.4}$$

i.e., $\lambda(x) = \eta IJ(x)$ for $x \in K$. Referring back to the construction of I in §2, it is clear that this composite has the property posited in Lemma 4.2.

We define η to be the composite of immersions $\eta_{v,e}$ (constructed below) having the properties

$$\begin{aligned} &\text{(i)} \quad \eta_{v,e}(D(v, e)) = K, \\ &\text{(ii)} \quad \eta_{v,e}(x) = x \quad \text{for } x \in K - D(v, e), \end{aligned} \tag{4.5}$$

where v is an arbitrary vertex of A_1 in the interior of $\mathcal{K}$, and e is any edge incident to v. Since $\{D(v, e)\}$ is a disjoint collection, η satisfies (4.3). (Also, the order of composition is immaterial; i.e., (4.3) is satisfied regardless of it.)

To construct $\eta_{v,e}$, consider all surfaces S in B_1 of the followng types (1, 2, 3 and 4) starting with those of type 1 defined as follows

$$S = (s(v) - (D(v, e_1) \cup D(v, e_2))) \cup c'(e_1) \cup c'(e_2), \tag{4.6}$$

where $c'(e_1)$ denotes the "half" of $c(e_i)$ adjoining $s(v)$. (Here v is a vertex of both e_1, e_2 and $e_1 \neq e_2$.) More precisely, the plane perpendicular to e_i at its midpoint bisects $c(e_i)$ into two halves; $c'(e_i)$ is the half adjoining $s(v)$. The surfaces of type 2 are those of the form

$$S = (\partial B(f, 1) - (D(v_1, e_1) \cup D(v_2, e_2))) \cup c'(e_1) \cup c'(e_2), \tag{4.7}$$

where f is a face of A_1, v_1 and v_2 are vertices of f, e_i ($i = 1, 2$) is incident to v_i but not an edge of f, and $e_1 \neq e_2$. Those of type 3 have the form

$$S = c'(e), \tag{4.8}$$

where $c'(e)$ is a half cylinder in B_1. Finally, S is of type 4 if

$$S = (s(v) - D(v, e)) \cup c'(e), \tag{4.9}$$

where v is a vertex of e. We illustrate these surfaces below.

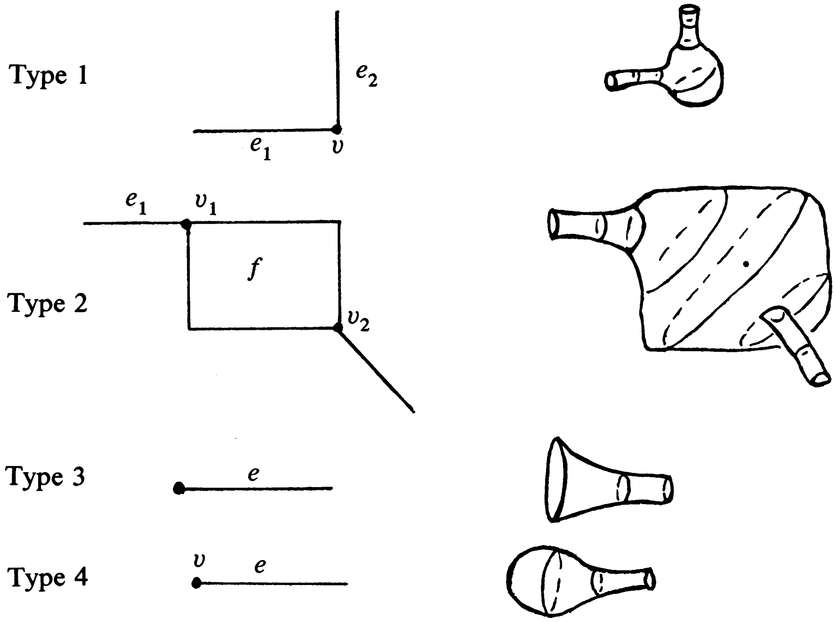

FIG. 4.1

Note that the surfaces of types 1, 2 and 3 are all diffeomorphic to $S^1 \times \mathbf{R}$, and those to type 4 to $\mathbf{D}^2$, where $\mathbf{D}^2$ is the unit disc in $\mathbf{R}^2$, and S^1 is its boundary-the circle.

Let v be a vertex of A_1 in the interior of $\mathcal{K}$, and e an edge incident to v. We claim that there is a finite sequence of surfaces S_j (where $1 \leqslant j \leqslant k$) in K of types 1, 2, 3 or 4 satisfying

(4.10)

(i) S_k is of type 4 and the others are not,

(ii) $S_1 = c'(e)$ adjoining $s(v)$,

(iii) $S_j \cap S_{j+1}$ is diffeomorphic to S^1 (when $1 \leqslant j < k$),

(iv) $S_j \cup S_{j+1}$ is a smooth surface and

(v) $\bigcup_{j=1}^{k} S_j = K$.

We use this claim (before verifying it) to construct $\eta_{v,e}$. Pick a filtration of $D(v, e)$

(4.11) $$D(v, e) = D_1 \supset D_2 \supset D_3 \supset \cdots \supset D_k$$

such that each D_i is diffeomorphic to $\mathbf{D}^2$, and D_{j+1} is contained in the interior of D_j (for $1 \leqslant j < k$), and let S_j' be the closure of $D_j - D_{j+1}$. Using (4.10), it is easy to construct an immersion $\eta_{v,e}' : D(v, e) \to K$ such that

$$(4.12)\qquad \begin{array}{ll} \text{(i)} & \eta'_{v,e}(x) = x \quad \text{for } x \in \partial D(v, e), \\ \text{(ii)} & \eta'_{v,e} \text{ maps } S'_j \text{ diffeomorphically onto } S_j (1 \leqslant j \leqslant k). \end{array}$$

Then define $\eta_{v,e}$ by the formula

$$(4.13)\qquad \eta_{v,e}(x) = \begin{cases} \eta'_{v,e}(x) & \text{if } x \in D(v, e), \\ x & \text{if } x \in K - D(v, e). \end{cases}$$

Finally, we construct surfaces S_i satisfying (4.10). Let $T_1, T_2, \cdots, T_l$ be a list of all the surfaces of types 1 and 2 in K; clearly

$$(4.14)\qquad \bigcup_{j=1}^{l} T_j = K.$$

To each surface T_j, associate two edges e_j^-, e_j^+ and two vertices v_j^-, v_j^+ defined as follows; if T_j is a surface of type 1, then $v_j^- = v_j^+ = v$ in formula (4.6) while (making an arbitrary choice) $e_j^- = e_1$ and $e_j^+ = e_2$ in (4.6); if T_j is a surface of type 2, then $e_j^- = e_1$, $v_j^- = v_1$, $e_j^+ = e_2$ and $v_j^+ = v_2$ in formula (4.7). Select "immersed" polygonal paths connecting e to e_1^- and e_j^+ to e_{j+1}^- (for $1 \leqslant j < l$); i.e., sequences of edges $e(j, 1), e(j, 2), \cdots, e(j, k_j)$ and vertices $v(j, 0)$, $v(j, 1), \cdots, v(j, k_j)$ (where $0 \leqslant j < l$) satisfying

$$(4.15)\qquad \begin{array}{ll} \text{(i)} & v(0, 0) = v, \quad v(j, 0) = v_j^+ \quad (1 \leqslant j < l), \\ \text{(ii)} & v(j, k_j) = v_{j+1}^- \quad (0 \leqslant j < l), \\ \text{(iii)} & v(j, i-1) \text{ and } v(j, i) \text{ are the vertices of } e(j, i), \\ \text{(iv)} & e(0, 1) = e, \quad e(j, 1) = e_j^+ \quad (0 \leqslant j < l), \\ \text{(v)} & e(j, k_j) = e_{j+1}^- \quad (0 \leqslant j < l), \\ \text{(vi)} & e(j, i) \neq e(j, i+1) \quad (1 \leqslant i < k_j). \end{array}$$

Let $\sigma(j) = 1 + k_0 + k_1 + \cdots + k_{j-1}$ (for $1 \leqslant j \leqslant l$), $\sigma(0) = 1$ and $k = \sigma(l) + 1$; define the surfaces S_i by the following formulas

$$(4.16)\qquad \begin{array}{ll} \text{(i)} & S_{\sigma(j)} = T_j \text{ for } 1 \leqslant j \leqslant l; \\ \text{(ii)} & S_1 = c'(e) \text{ the surface of type 3 adjoining } s(v); \\ \text{(iii)} & S_k = (s(v') - D(v', e_l^+)) \cup c'(e_l^+) \text{ the} \\ & \text{surface of type 4 where } v' \text{ and } v_l^+ \text{ are} \\ & \text{the vertices of } e_l^+\text{; and} \\ \text{(iv)} & S_{\sigma(j)+p} \text{ (where } 1 \leqslant p < k_j, 0 \leqslant j < l\text{) is} \\ & \text{the surface of type 1 determined by replacing} \\ & v, e_1 \text{ and } e_2 \text{ in formula (4.6) by } v(j, p), \\ & e(j, p) \text{ and } e(j, p+1)\text{, respectively.} \end{array}$$

Clearly, these surfaces satisfy (4.10).

5. Proof of the corollaries

In this section, Corollaries 0.2, 0.3 and 0.4 are deduced from Theorem 0.1.

Proof of Corollary **0.2.** Let $G : K \to K$ be the immersion posited in Theorem 0.1; by [13, §3], it is a presentation for a 2-solenoid $h : \Sigma \to \Sigma$ where Σ is the inverse limit of

$$K \xleftarrow{G} K \xleftarrow{G} K \xleftarrow{G} \cdots . \tag{5.1}$$

Applying $\check{H}(\ )$ to (5.1), we observe $\check{H}_i(\Sigma)$ is the inverse limit (cf. [1, Theorem 3.1, p. 261]) of

$$H_i(K) \xleftarrow{G_*} H_i(K) \xleftarrow{G_*} H_i(K) \xleftarrow{G_*} \cdots ; \tag{5.2}$$

hence $\check{H}_0(\Sigma) = \mathbf{Z}$ and $\check{H}_1(\Sigma) = 0$ since K is simply connected. By [13, Theorem B], $h : \Sigma \to \Sigma$ is conjugate to an expanding attractor.

Proof of Corollary **0.3.** As in the above argument, let (Σ, h) be the 2-solenoid presented by (K, G). Then Σ is homeomorphic to an (orientable) expanding attractor Λ [13, Theorem B]. In fact, there is a nested sequence of compact manifolds K_i (where $K_i \supset K_{i+1}$) such that $\cap K_i = \Lambda$, and each K_i is homotopically equivalent to K. Note the following three properties.

(5.3)
(a) each K_i is simply connected,
(b) $\check{H}^2(\Lambda) \neq 0$,
(c) Λ is connected.

(Property (b) follows from [9].)

We proceed via proof by contradiction assuming a fiber bundle

$$F \to \Lambda \xrightarrow{p} B \tag{5.4}$$

where F is a Cantor set, and B is a 2-dimensional manifold. B cannot be the 2-sphere since this would necessitate $\Lambda = B \times F$ contradicting (5.3)(c). (Recall the structure group for (5.4) is totally disconnected.) A similar argument shows B is not the projective plane; hence B must be aspherical. Since B is an ANR, p extends to a map $p' : K_i \to B$ for some index i; but, by (5.3)(a), p', and hence p, is homotopic to a constant map. By the covering homotopy theorem for (5.4) and property (5.3)(c), the identity map of Λ is homotopic to a constant map; in particular, $\check{H}^2(\Lambda) = 0$ contradicting (5.3)(b).

Proof of Corollary **0.4.** Choose (K, f) and (K, g) to be (K, G) and (K, G^2), respectively, where (K, G) is the immersion posited in Theorem 0.1. Let (Σ, h) be the 2-solenoid presented by (K, f). Then (K, g) is a presentation for (Σ, h^2). Note that property (i) of Corollary 0.4 is true because K is simply connected. By [9], $\check{H}^2(\Sigma, \mathbf{R})$ is a finite dimensional $\mathbf{R}$-vector space, and the maximum eigenvalue of

$$h^* : \check{H}^2(\Sigma, \mathbf{R}) \to \check{H}^2(\Sigma, \mathbf{R}) \tag{5.5}$$

is a real number larger than 1. Hence this linear transformation and

$$(h^2)^* : H^2(\Sigma, \mathbf{R}) \to H^2(\Sigma, \mathbf{R}) \tag{5.6}$$

are not conjugate. Consequently (Σ, h) and (Σ, h^2) are not topologically conjugate; i.e., property (ii) of Corollary 0.4 is also satisfied.

6. Proof of Proposition 3.3.

We first outline the proof of Proposition 3.3. Let σ be either a vertex v, edge e or face f of A_1, and associate to it a subcomplex E_σ of A_α (where $\alpha = 10^4$) defined by the following formulas

(6.1)

(i) $E_v = b'(v)$,

(ii) $E_e = t'(e) \cup b'(v_0) \cup b'(v_1)$ where v_0 and v_1 are the vertices of e,

(iii) $E_f = p'(f) \cup \left(\bigcup_{i=1}^{4} t'(e_i) \right) \cup \left(\bigcup_{i=1}^{4} b'(v_i) \right)$ where e_i, v_i are the edges and vertices of f.

(See §3 for the definitions of $b'(\ )$, $t'(\ )$ and $p'(\ )$.) If n is a positive integer, and S is a subset of $\mathbf{R}^3$, then nS denotes the image of S under multiplication by n; i.e.,

$$nS = \{nx | x \in S\}. \tag{6.2}$$

Throughout this section, we use E to denote any set of the form nE_σ; each E is a compact 3-manifold with boundary and a subcomplex of A_α. For any positive integer m, $B(E, m\alpha)$ is the subcomplex of $\mathbf{B}_{m\alpha}$ defined in §1. We will construct smooth flows $\varphi_m(\ , t)$ defined in the closure of $B(E, m\alpha) - E$ which will be transverse to both $\partial B(E, m\alpha)$ and ∂E, and flow each point in $\partial B(E, m\alpha)$ to a unique point of ∂E defining a homeomorphism

$$g_m : \partial B(E, m\alpha) \to \partial E. \tag{6.3}$$

Any two of these flows $\varphi_k(\ , t)$, $\varphi_m(\ , t)$ will be equal in a sufficiently small neighborhood of ∂E; thus all of the composites

$$g_m^{-1} g_k : \partial B(E, k\alpha) \to \partial B(E, m\alpha) \tag{6.4}$$

are diffeomorphisms. When E is E_v, E_e and E_f, respectively, and $k = 1$, $n = m$, these composites (6.4) are the diffeomorphisms posited in Proposition 3.3.

We now begin to fill in the details by defining a slightly different cell structure on ∂E than the one induced from A_α: namely, we amalgamate some faces. To explicitly describe this cell structure, consider the figure below showing 5 faces f_i of A_α all sharing the vertex (0, 0, 0) with f_1, f_2, f_3 in the (x, y)-plane, f_4 in the (x, z)-plane and f_5 in the (y, z)-plane;

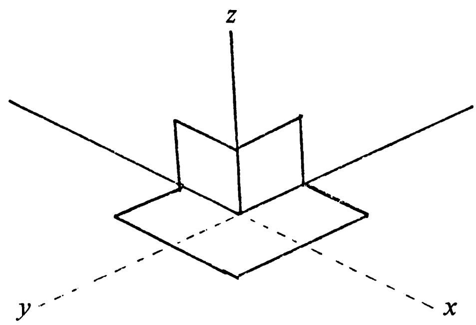

FIG. 6.1

amalgamate f_1, f_2 and f_3 into a single 2-cell $f_1 \cup f_2 \cup f_3$. If $f_i'(i = 1, \cdots, 5)$ are five faces of A_α in ∂E which can be mapped by a rigid motion of $\mathbf{R}^3$ to f_i $(i = 1, \ldots, 5)$, respectively, then amalgamate f_1', f_2' and f_3' into a single 2-cell which we call a *special* 2-cell in ∂E. The other faces of A_α in ∂E (not of type f_1', f_2', f_3') represent the remaining 2-cells in ∂E. The 0-cells are just the vertices of A_α in ∂E, and the 1-cells are those edges of A_α in ∂E which do not meet the interior of a special 2-cell.

Next we define a finite valued vector field $F(\ )$, whose domain is the set of all cells in ∂E, which will approximate $(d/dt)\varphi_1(x, 0)$. If σ is a 2-cell in ∂E, $F(\sigma)$ is the unit vector perpendicular to σ and pointing into E. If τ is a 0 or 1-cell in ∂E, and $\{\sigma_i\}$ is the set of all 2-cells (in ∂E) containing v, then $F(\tau)$ is the unit vector in the same direction as $\Sigma F(\sigma_i)$.

Let $\mathcal{E}_m$ denote the closure of $B(E, m\alpha) - E$ and define a smooth vector field $V_1(x)$ (for $x \in \mathcal{E}_1$) by

$$V_1(x) = \Sigma_v \eta(x) F(v) + \Sigma_e \eta_e(x) F(e) + \Sigma_\sigma \eta_\sigma(x) F(x), \tag{6.5}$$

where v, e and σ vary over all 0-cells v, 1-cells e and 2-cells σ in ∂E; η_v, η_e and η_σ are smooth $\mathbf{R}$-valued functions associated to v, e and σ, respectively, which will be described after a short digression. We will eventually integrate V_1 to construct the flow φ_1.

Let $d'(x, y)$ denote the cubical metric on $\mathbf{R}^3$; namely, for $x, y \in \mathbf{R}^3$,

$$d'(x, y) = \max_i |x_i - y_i|, \tag{6.6}$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. If S is a subset of $\mathbf{R}^3$, and $\varepsilon > 0$, then $S(\varepsilon)$ is defined by

$$S(\varepsilon) = \{x \in \mathbf{R}^3 | d'(x, S) \leqslant \varepsilon\}. \tag{6.7}$$

In particular, we are interested in $E(\varepsilon)$ when $\varepsilon < 2^{-1}\alpha$.

We next associate to the cells v, e, σ of ∂E thickenings v', e', σ' in $\mathfrak{S}_1$. To each 0-cell v in ∂E, associate the 1-cell v' (in $\mathfrak{S}_1$) defined by

$$v' = \{v - rF(v) | 0 \leqslant r \leqslant \alpha\}. \tag{6.8}$$

If e is a 1-cell in ∂E with vertices v_0 and v_1, define e' to be the 2-cell in the plane containing $e \cup \{x + F(e) | x \in e\}$ such that

$$\begin{aligned} &\text{(i)} \quad e \cup v_0' \cup v_1' \subset \partial e', \\ &\text{(ii)} \quad \partial e' - (e \cup v_0' \cup v_1') \subset \partial B(E, \alpha). \end{aligned} \tag{6.9}$$

If σ is a 2-cell in ∂E with $\{e_i\}$ denoting the set of all edges contained in $\partial\sigma$, define σ' to be the 3-cell (in $\mathfrak{S}_1$) such that

$$\begin{aligned} &\text{(i)} \quad \left(\bigcup_i e_i'\right) \subset \partial\sigma', \text{ and} \\ &\text{(ii)} \quad \partial\sigma' - \left(\left(\bigcup_i e_i'\right) \cup \sigma\right) \subset \partial B(E, \alpha). \end{aligned} \tag{6.10}$$

If D is a subcomplex of ∂E, define D' by

$$D' = \bigcup_\tau \tau', \tag{6.10.1}$$

where τ varies over all the cells in D. The following statement is easily verified.

Remark 6.1. *There is a homeomorphism $h : \mathfrak{S}_1 \to \partial E \times [0, 1]$ satisfying $h(D') = D \times [0, 1]$ for each subcomplex D of ∂E.*

Fix numbers ε_i $(i = 1, 2, 3)$ such that $\alpha \gg \varepsilon_1 \gg \varepsilon_2 \gg \varepsilon_3 > 0$ (for instance, $\varepsilon_i = 10^{-i-4}$). We choose the functions η_v, η_e and η_σ in formula (6.5) to satisfy the following properties

$$\begin{aligned} &\text{(i)} \quad 1 \geqslant \eta_\tau(x) \geqslant 0 \text{ for each cell } \tau \text{ in } \partial E, \\ &\text{(ii)} \quad \eta_v(v) = \begin{cases} 1 & \text{if } d(x, v') \leqslant 2^{-1}\varepsilon_1, \\ 0 & \text{if } d(x, v') \geqslant \varepsilon_1, \end{cases} \\ &\text{(iii)} \quad \eta_e(x) + \eta_u(x) + \eta_v(x) = 1 \text{ if } d(x, e') < 2^{-1}\varepsilon_2 \\ &\qquad \text{where } u \text{ and } v \text{ are the vertices of } e, \\ &\text{(iv)} \quad \eta_e(x) = 0 \text{ if } d(x, e') \geqslant \varepsilon_2, \text{ and } \eta_\sigma(x) = 0 \text{ if} \\ &\qquad d(x, \sigma') \geqslant \varepsilon_3, \\ &\text{(v)} \quad \Sigma_\tau\, \eta_\tau(x) = 1 \text{ for all } x \in \mathfrak{S}_1 \text{ where } \tau \text{ varies} \\ &\qquad \text{over all cells in } \partial E. \end{aligned} \tag{6.11}$$

Let $\hat{E}$ be a second set of the form nE_σ (cf. (6.1) and (6.2)) with $\{\eta'_\tau\}$ its associated partition of unity, we additionally require the following congruences: if T is a translation of $\mathbf{R}^3$, σ a cell in ∂E, and U a neighborhood of σ in $\mathbf{R}^3$ such that $T\sigma$ is a cell in $\partial\hat{E}$ and $(TU) \cap \hat{E} = T(U \cap E)$, then

$$(6.12) \qquad \eta_\sigma(x) = \eta'_{T\sigma}(T(x))$$

for all $x \in \mathbf{R}^3$. (Families of functions satisfying properties (6.11) and (6.12) exist.)

By integrating $V_1(x)$ (cf. (6.5)) inside $\mathcal{E}_1$, we construct the flow $\varphi_1(x, t)$. To define the flows $\varphi_m(\lambda, t)$, set $\hat{E} = mE$, and let $V'_1(x)$ be the vector field defined on the closure $\hat{\mathcal{E}}_1$ of $B(\hat{E}, \alpha) - \hat{E}$ by formula (6.5). Then define a vector field $W_m(x)$ on $\mathcal{E}_m$ by

$$(6.13) \qquad W_m(x) = V'_1(mx).$$

(Note $m\mathcal{E}_m = \hat{\mathcal{E}}_1$.) Choose $\alpha_1 > 0$ sufficiently small (independent of E) so that $E(\alpha_1) \subset B(E, \alpha)$ for all sets E of the type nE_σ; cf. (6.1) and (6.2). Consequently, $E(m^{-1}\alpha_1) \subset B(E, m\alpha)$ for every positive integer m. For each pair E and m, select a smooth Urysohn function ψ_m on $\mathbf{R}^3$ (i.e., $\psi_m(\mathbf{R}^3) \subseteq [0, 1]$) satisfying

$$(6.14) \qquad \begin{aligned} &\text{(i)} \quad \psi_m(x) = \begin{cases} 1 & \text{if } x \in \mathbf{R}^3 - E(m^{-1}\alpha_1), \\ 0 & \text{if } x \in E((2m)^{-1}\alpha_1), \end{cases} \\ &\text{(ii)} \quad \left| \frac{\partial \psi_m}{\partial x_i} \right| < m\alpha_2 \text{ for } i = 1, 2, 3, \end{aligned}$$

where $\alpha_2 > 0$ is a real number independent of E and m. If $\hat{E}$ is a second set of type nE_σ (cf. (6.1) and (6.2)), and ψ'_m is its associated Urysohn function, then we require the following additional property: for each translation T of $\mathbf{R}^3$, cell σ in ∂E and neighborhood U of σ in $\mathbf{R}^3$ such that $T\sigma$ is a cell in $\partial\hat{E}$ and $(TU) = \cap \hat{E} = T(U \cap E)$,

$$(6.14.1) \qquad \psi'_m(T(x)) = \psi_m(x) \text{ for all } x \in \sigma^{(m)}.$$

(Here $\sigma^{(m)} = (m)^{-1}(m\sigma)'$ where $(m\sigma)'$ is defined by formula (6.10.1) using $\hat{E} = mE$ in place of E.) We proceed to construct these Urysohn functions. Let c_0 denote the cube centered at (0, 0, 0) whose sides have length α and are parallel to the coordinate axes, and $\psi : \mathbf{R}^3 \to \mathbf{R}$ be a fixed smooth Urysohn function such that

$$(6.14.2) \qquad \psi(x) = \begin{cases} 1 & \text{if } d'(x, c_0) \geqslant \alpha_1, \\ 0 & \text{if } d'(x, c_0) \leqslant (2)^{-1}\alpha_1. \end{cases}$$

For each cube c in A_α, define the composite $\psi_c = \psi T$ where T is the unique translation such that $T(c) = c_0$; define ψ_1 by the following formula

(6.14.3) $$\psi_1(x) = \pi_c \psi_c(x);$$

i.e., the product over all cubes c of A_α which are inside E. For each positive integer m, define ψ_m by

(6.14.4) $$\psi_m(x) = \psi_1'(mx),$$

where ψ_1' is the Urysohn function defined by (6.14.3) for $\hat{E} = mE$.

Now define a vector field $V_m(x)$ on $\mathfrak{S}_m$ by

(6.15) $$V_m(x) = \psi_m(x) W_m(x) + (1 - \psi_m(x)) V_1(x);$$

the flow $\varphi_m(x, t)$ is defined by integrating $V_m(x)$.

Next we state several lemmas about the flows $\varphi_m(x, t)$, deduce Proposition 3.3 from them, and finally complete the paper by proving these lemmas.

Lemma 6.2. *For each $x \in \partial B(E, m\alpha)$, $V_m(x)$ is transverse to $\partial B(E, m\alpha)$ and points into $B(E, m\alpha)$. For each $x \in \partial E(\varepsilon) \cap \mathfrak{S}_m$ (where $\varepsilon \geqslant 0$), $V_m(x)$ is transverse to $\partial E(\varepsilon)$ and points into $E(\varepsilon)$.*

If D is a subcomplex of ∂E, define $D^{(m)}$ by

(6.16) $$D^{(m)} = \bigcup \tau^{(m)},$$

where τ varies over all cells of ∂E contained in D; cf. the sentence following (6.14.1).

Lemma 6.3. *Let D be any subcomplex of ∂E, and m a positive integer, then, for each $x \in D^{(m)}$, there exists a number $t_m(x) \geqslant 0$ (depending continuously on x) such that*

(i) $\varphi_m(x, t) \in D^{(m)}$ *for* $0 \leqslant t \leqslant t_m(x)$,

(ii) $\varphi_m(x, t_m(x)) \in \partial E$.

Let E, $\hat{E}$ be two sets of the form nE_σ (cf. (6.1) and (6.2)), and φ_m, φ_m' the associated flows on $\mathfrak{S}_m$, $\hat{\mathfrak{S}}_m$, respectively. Suppose D is a subcomplex of ∂E, U a neighborhood of D in $\mathbf{R}^3$, and $T : \mathbf{R}^3 \to \mathbf{R}^3$ a translation.

Lemma 6.4. *If $T(U) \cap \hat{E} = T(U \cap E)$, and $T(D)$ is a subcomplex of $\partial \hat{E}$, then*

$$T\varphi_m(x, t) = \varphi_m'(Tx, t)$$

for all $x \in D^{(m)}$ and all t such that $0 \leqslant t \leqslant t_m(x)$. (Recall $t_m(x)$ is described in Lemma 6.3 and $D^{(m)}$ in (6.16).)

Because of Lemma 6.3, we can define the homeomorphisms (6.3) by the formula

(6.17) $$g_m(x) = \varphi_m(x, t_m(x)) \quad \text{for all } x \in \partial B(E, m\alpha).$$

Let σ be a 2-cell in ∂E. Then $\sigma^{(m)}$ (cf. (6.14.1)) is a smooth 3-cell with corners, and $\partial \mathcal{E}_m \cap \sigma^{(m)}$ is a smooth 2-cell with corners which we denote by σ_m. By Lemma 6.3, the composite $g_m^{-1}g_1$ restricted to σ_1 is a homeomorphism of σ_1 to σ_m, which we denote by $g_{\sigma,m}$. But the composition

$$(6.18) \qquad g_n^{-1}g_1 : \partial B(E, \alpha) \to \partial B(E, n\alpha)$$

is always a diffeomorphism because the flows $\varphi_1(x, t)$ and $\varphi_n(x, t)$ agree for x sufficiently close to ∂E and $t \geqslant 0$ by (6.14) and (6.15). Hence each $g_{\sigma,m}$ is a diffeomorphism.

Lemma 6.5. *There exists a number $\alpha_3 > 0$ such that, for all E of type nE_σ (cf. Lemmas 6.1 and 6.2), any positive integer m and each 2-cell σ in ∂E,*

$$|dg_{\sigma,m}(X)| \geqslant \alpha_3 |X|$$

for all vectors X tangent to σ_1.

We now use these lemmas to complete the proof of Proposition 3.3. We choose the diffeomorphisms $\psi_{v,n}$, $\psi_{e,n}$ and $\psi_{f,n}$ (posited in Proposition 3.3) to be the composites $g_n^{-1}g_1$ (of (6.18)) where E is E_v, E_e and E_f, respectively. The inequality in Proposition 3.3 is implied by Lemma 6.5, and the last two sentences of the Proposition are satisfied because of Lemmas 6.3 and 6.4.

Our last task is to prove these lemmas.

Proof of Lemma **6.2.** For $m = 1$, the transversality statements follow directly from (6.5), (6.11) and the definition of $F(\ )$; for $m > 1$, they are a consequence of the case $m = 1$ by using (6.13) and (6.14)(i).

Proof of Lemma **6.3.** For each 0-cell v of ∂E and $x \in v'$, $V_1(x)$ is parallel to the line segment v'; for each 1-cell e of ∂E and $x \in e'$, $V_1(x)$ is parallel to the plane containing e'. Because of (6.13) and (6.15), these properties persist when $V_1(x)$ is replaced by $V_m(x)$ for any positive integer m. Consequently, for each fixed point $x \in \mathcal{E}_m$, the $\mathbf{R}$-valued function $d'(\varphi_m(x, t), E)$ is differentiable (in t) provided $\varphi_m(x, t)$ is in the interior of $\mathcal{E}_m$. By the compactness of $\mathcal{E}_m$ together with Lemma 6.2, the derivatives of these functions are all strictly negative and bounded away from zero. Hence for each $x \in \mathcal{E}_m$ there exists a smallest number $t_m(x) \geqslant 0$ such that $d'(\varphi_m(x, t_m(x)), E) = 0$. In particular, $\varphi_m(x, t_m(x)) \in \partial E$, and $\varphi_m(x, t) \in \mathcal{E}_m$ for $0 \leqslant t \leqslant t_m(x)$. It is easily seen that $t_m(x)$ depends continuously on x. By the second sentence of this proof and Lemma 6.2, for each cell τ in ∂E and $x \in \tau^{(m)}$, $\varphi_m(x, t) \in \tau^{(m)}$ for $0 \leqslant t \leqslant t_m(x)$; this implies property (i) of Lemma 6.3.

Proof of Lemma **6.4.** Denote the vector fields (cf. (6.15)) determining φ_m and φ'_m by V_m and V'_m, respectively. Because of (6.5), (6.12), (6.13), (6.14.1) and (6.15), we have that $V_m(x) = V'_m(T(x))$ for all $x \in D^{(m)}$. By the uniqueness of solutions to ordinary differential equations, this implies the validity of Lemma 6.4.

Proof of Lemma **6.5.** We factor $g_{\sigma,m}$ as the composite of three maps

$$g_\sigma = g_1|\sigma_1 : \sigma_1 \to \sigma,$$
$$(6.19) \qquad h_{\sigma,m} : \sigma \to \sigma_m^+,$$
$$k_{\sigma,m} : \sigma_m^+ \to \sigma_m,$$

where $\sigma_m^+ = \sigma^{(m)} \cap \partial E(m^{-1}\alpha_1)$; cf. the sentence after (6.13) for the definition of α_1. Both $h_{\sigma,m}$ and $k_{\sigma,m}$ are induced by the flow φ_m. To be precise, for each $x \in \sigma_m$ there exists a unique number $t_m^+(x)$ such that

$$(6.20) \qquad \begin{aligned} &\text{(i)} \quad 0 < t_m^+(x) < t_m(x),\\ &\text{(ii)} \quad \varphi_m(x, t_m^+(x)) \in \sigma_m^+. \end{aligned}$$

The function $t_m^+(x)$ is continuous (in x); this is seen by the same argument used to prove Lemma 6.3. In fact, the functions $t_m(x)$ and $t_m^+(x)$ are smooth for $x \in \sigma_m$; in particular, g_σ is a diffeomorphism. The map (diffeomorphism) $k_{\sigma,m}$ is the inverse of the diffeomorphism

$$(6.21) \qquad x \to \varphi_m(x, t_m^+(x)) \text{ for } x \in \sigma_m.$$

Define a smooth $\mathbf{R}$-valued function $t_m^-(x)$ (for $y \in \sigma_m^+$) by the following formula

$$(6.22) \qquad t_m^-(y) = t_m(k_{\sigma,m}(y)) - t_m^+(k_{\sigma,m}(y))$$

and the map (diffeomorphism) $h_{\sigma,m}$ to be the inverse of the diffeomorphism

$$(6.23) \qquad y \to \varphi_m(y, t_m^-(y)) \text{ for } y \in \sigma_m^+.$$

With these preliminaries, Lemma 6.5 is an immediate consequence of the following assertion whose verification will complete this article.

Assertion 6.6. *There exist numbers $\gamma_i > 0$ (where $i = 1, 2, 3$) such that, for all E of type nE_σ (cf.* (6.1) *and* (6.2)), *any positive integer m and each 2-cell σ in ∂E,*

(i) $|dg_\sigma(X_1)| \geqslant \gamma_1|X_1|$,
(ii) $|dh_{\sigma,m}(X_2)| > \gamma_2|X_2|$,
(iii) $|dk_{\sigma,m}(X_3)| > \gamma_3|X_3|$,

where X_1, X_2 and X_3 are arbitrary vectors tangent to σ_1, σ and σ_m^+, respectively.

We first observe that inequality (i) follows from Lemma 6.4 with $m = 1$, since there are only finitely many equivalence classes of 2-cells σ if we proclaim as equivalent 2-cells σ and $\hat\sigma$ in ∂E and $\partial\hat E$, respectively, provided there exist a neighborhood U of σ (in $\mathbf{R}^3$) and a translation T such that $T(\sigma) = \hat\sigma$ and $T(U \cap E) = T(U) \cap \hat E$.

To verify inequality (iii), let $\hat E = mE$, and let $\rho : \mathbf{R}^3 \to \mathbf{R}^3$ denote multiplication by m. X_3 is tangent to σ_m^+ at some point $x \in \sigma_m^+$ and let τ be a 2-cell in $\partial\hat E$ such that $\tau \subset m\sigma$ and $mx \in \tau_1^+$; cf. (6.19). Consider the following commutative diagram

$$(6.24)\qquad \begin{array}{ccc} \tau_1^+ & \xrightarrow{k_{\tau,1}} & \tau_1 \\ \rho\uparrow & & \uparrow\rho \\ \sigma_m^+ \supset m^{-1}\tau_1^+ & \xrightarrow[k_{\sigma,m}]{} & m^{-1}\tau_1 \subset \sigma_m \end{array}$$

(cf. (6.13), (6.14), (6.15) and (6.21)). Since $|d\rho(X)| = m|X|$ for all vectors X tangent to $\mathbf{R}^3$, diagram (6.24) shows that it suffices to demonstrate inequality (iii) when $m = 1$. But this follows now from Lemma 6.4 in the same way that inequality (i) did; i.e., there are only finitely many maps $k_{\tau,1}$ up to translation.

To verify inequality (ii), we need an extra ingredient not needed above; namely, we use a basic result from the elementary qualitative theory of ordinary differential equations which gives a Lipschitz constant for the solutions in terms of the Lipschitz constant for the equation. Because of (6.5), (6.13), (6.14) and (6.15), there exists a number β_1 (independent of E, m and σ) such that for each $x \in \sigma^{(m)}$

(6.25) the angle between $V_m(x)$ and the plane through x parallel to σ is $\geqslant \beta_1$.

Also, there exists a number β_2 (independent of E, and m and σ) such that (for $x \in \sigma^{(m)}$)

$$(6.26)\qquad \beta_2 \geqslant |V_m(x)| \geqslant (\beta_2)^{-1}.$$

Arguing as in the verification of inequality (i) by using formulas (6.5) and (6.12), there exists a constant β_3' (independent of E and σ) such that

$$(6.27)\qquad \left|\frac{\partial V_1}{\partial x_i}\right| \leqslant \beta_3' \quad \text{for } i = 1, 2, 3;$$

and arguing as in the verification of inequality (ii) using formulas (6.13) and (6.27), we obtain that

$$(6.28)\qquad \left|\frac{\partial W_m}{\partial x_i}\right| \leqslant m\beta_3' \quad \text{for } i = 1, 2, 3.$$

Combining (6.27), (6.28), (6.14) and (6.15), there exists a constant β_3 (independent of E, m and σ) such that, for $x \in \sigma^{(m)}$,

$$(6.29)\qquad \left|\frac{\partial V_m}{\partial x_i}(x)\right| \leqslant m\beta_3 \quad \text{for } i = 1, 2, 3.$$

Inequality (iii) now follows from (6.25), (6.26), (6.29) and the theorem about ordinary differential equations referred to above. (See [3, p. 169] for the exact statement of this result.) This theorem is applied to φ_m; to be precise, it is applied to the normalized vector field $(V_m(x) \cdot F(\sigma))^{-1} V_m(x)$ which has the same integral curves as φ_m; cf. the paragraph preceding (6.5) for the definition of $F(\sigma)$.

Added in Proof. We have recently answered Question 0.5; see F. T. Farrell and L. E. Jones, *Expanding immersions on bounded manifolds*, to appear in Amer. J. Math. This paper also contains a negative answer to Question 0.7; see Remark 0.9.

References

[1] S. Eilenberg & N. Steenrod, *Foundations of algebraic topology*, Princeton University Press, Princeton, 1952.
[2] M. Hirsch, *On combinatorial submanifolds of differentiable manifolds*, Comment. Math. Helv. **36** (1961) 103–111.
[3] M. Hirsch & S. Smale, *Differential equations, dynamical system, and linear algebra*, Academic Press, New York, 1974.
[4] C. Robinson & R. F. Williams, *Classification of expanding attractors: an example*, Topology **15** (1976) 321–323.
[5] M. Shub & D. Sullivan, *Homology theory and dynamical systems*, Topology **14** (1975) 109–132.
[6] S. Smale, *Stability and isotopy in discrete dynamical systems*, Proc. Sympos. Dynamical Systems, Salvador, 1971.
[7] ______, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967) 747–817.
[8] N. Steenrod, *The topology of fiber bundles*, Princeton University Press, Princeton, 1951.
[9] D. Sullivan & R. F. Williams, *On the homology of attractors*, Topology **15** (1976) 259–262.
[10] R. F. Williams, *One-dimensional non-wandering sets*, Topology **6** (1967) 473–487.
[11] ______, *Classifications of 1-dimensional attractors*, Proc. Sympos. Pure Math. Vol. 14, Amer. Math. Soc., 1970, 341–361.
[12] ______, *The structure of attractors*, Proc. Internat. Congress Math., Nice, 1970, 947–951.
[13] ______, *Expanding attractors*, Inst. Hautes Études Sci. Publ. Math. No. 43 (1973) 169–203.

UNIVERSITY OF MICHIGAN
STATE UNIVERSITY OF NEW YORK, STONY BROOK