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# KÄHLER SUBMERSIONS AND HOLOMORPHIC CONNECTIONS

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## 0. Introduction

In this paper we consider two related fundamental real-geometric structures transferred to a complex-geometric setting; connections in principal bundles and Riemannian submersions [5]. Both of these notions, more obviously so the former, are integral part of Riemannian geometry; furthermore, as we shall see they are quite interdependent. Here we place these structures in the framework of Kähler geometry; that is, all maps and spaces will be taken to be holomorphic and, where possible, all metrics will be Kähler.

These notions separately have already been studied in a complex-analytic category. M. Atiyah [1] has considered holomorphic connections on principal bundles over compact Kähler manifolds, and, rather than the fundamental notion that connections are in the $C^\infty$ category, in this case the very existence of a holomorphic connection is an extremely restrictive notion. In particular, if the structural group is semi-simple or $Gl(n, \mathbf{C})$, the characteristic algebra of the bundle must vanish. More recently, B. Watson [7] has shown that the existence of a Kähler submersion is similarly restrictive; in fact the horizontal distribution must be integrable.

The purpose of this paper is to present some indication of the relationship between the ridigity of these two transferred concepts, in particular, we are able to show

**Theorem (1.5)** (*Watson*). *Let $\pi$: $M \to B$ be a Kähler submersion, $\mathcal{V} = \ker \pi_*$ the vertical subbundle, and $\mathcal{H} = \mathcal{V}^\perp$ the orthogonal distribution. Then $\mathcal{H}$ is integrable; furthermore, the integral submanifolds are totally geodesic.*

This phenomenon seems to be produced not primarily by the submersion, but by the interaction of the metric and the complex-analytic structure, as is evidenced by

**Theorem (2.1).** *Let $\mathcal{V} \subset T_*(M)$ be any holomorphic distribution, $M$ a Kähler manifold, and $\mathcal{H} = \mathcal{V}^\perp$ the orthogonal distribution. Then $\mathcal{H}$ is also a*

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*holomorphic distribution if and only if* $\mathcal{V}$ *and* $\mathcal{H}$ *both are not only integrable, but are totally geodesic.*

Here a distribution $\mathcal{Q}$ is said to be *holomorphic* if the subbundle of $T_*(M)$ is a holomorphic subbundle. This is easily seen to be equivalent to the existence locally of a spanning set of holomorphic vector fields in $\mathcal{Q}$, where a *holomorphic vector field* is a holomorphic section of $T_*(M)$.

**Remark.** Thus, in the case of a Kähler submersion, the horizontal distribution $\mathcal{H}$, even though integrable, is not itself holomorphic unless the fibers of the projection are totally geodesic.

Consider now a holomorphic principal bundle $\pi: P \to M$ over a Kähler manifold $M$ with complex-analytic structure group $G$. Given a fixed left-invariant hermitian metric $\langle\, , \rangle_G$ on $G$ there is, for each connection $\mathcal{H}$ of $P$, a naturally-associated metric $\langle\, , \rangle_{\mathcal{H}}$ on $P$ such that $\pi: P \to M$ is a Riemannian submersion.

**Theorem (3.4).** $\langle\, , \rangle_{\mathcal{H}}$ *is hermitian if and only if* $\mathcal{H}_p \subset T_*(P, p)$ *is a complex subspace for all* $p \in P$. *Furthermore,* $\mathcal{H}$ *is holomorphic if and only if* $\langle\, , \rangle_{\mathcal{H}}$ *is hermitian, and* $\nabla_x J = 0$ *for* $x \in \mathcal{H}$, *where* $J$ *is the complex structure tensor of* $P$, *and* $\nabla$ *is the Riemannian connection of* $\langle\, , \rangle_{\mathcal{H}}$.

**Theorem (3.5).** *Let the metric* $\langle\, , \rangle_G$ *be Kähler.*

(a) $\pi: P \to M$ *is Kähler submersion if and only if* $\mathcal{H}$ *is holomorphic and flat.*

(b) *If* $\mathcal{H}$ *is holomorphic, the extent to which* $\langle\, , \rangle_{\mathcal{H}}$ *is not Kähler is given by the curvature of* $\mathcal{H}$.

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## 1. Kähler submersions

A *Riemannian submersion* is a submersion $\pi: M \to B$ of Riemannian manifolds such that, if $\mathcal{V} = \ker \pi_*$, and $\mathcal{H} = \mathcal{V}^\perp$, then $\pi_*|_{\mathcal{H}}$ is an isometry at each point. B. O'Neill [5] has characterized the geometry of this situation in terms of tensors $T, A$ defined for $E, F \in \mathfrak{X}(M)$ by

$$T_E F = \mathcal{V}\, \nabla_{\mathcal{V}E} \mathcal{H}F + \mathcal{H}\, \nabla_{\mathcal{V}E} \mathcal{V}F, \quad A_E F = \mathcal{V}\, \nabla_{\mathcal{H}E} \mathcal{H}F + \mathcal{H}\, \nabla_{\mathcal{H}E} \mathcal{V}F,$$

where the symbols $\mathcal{V}$ and $\mathcal{H}$ refer also to the orthogonal projections on the subspaces of $T_*(M, p)$ indicated. In particular, the geometry of $M$ is described by that of $B$, the fibers of $\pi$, and these tensors.

**Definition (1.1).** $\pi: M \to B$ is a *Kähler submersion* if $M$, $B$ are Kähler manifolds, and $\pi$ holomorphic with $\pi: M \to B$ a Riemannian submersion. All other objects are defined as in [5]. In particular, a vector field in $\mathcal{V}$ is said to be *vertical*, and one in $\mathcal{H}$ is termed *horizontal*.

**Lemma (1.2).** *If $J$ is the complex structure tensor of $M$, we have, for $E$, $F \in \mathfrak{X}(M)$,*

(1) $$T_E(JF) = JT_EF,$$

(2) $$T_{JE}V = JT_EV \text{ for } V \text{ vertical},$$

(3) $$A_E(JF) = JA_EF,$$

(4) $$A_{JE}H = JA_EH, \text{ for } H \text{ horizontal}.$$

*Proof.* As $M$ is Kähler, $J\nabla = \nabla J$. Also, both projections $\mathcal{V}$, $\mathcal{H}$ at $p \in M$ are complex-linear, thus $\mathcal{V}J = J\mathcal{V}$, $\mathcal{H}J = J\mathcal{H}$, and so

$$A_EJF = \mathcal{V}\ \nabla_{\mathcal{H}E}(\mathcal{H}JF) + \mathcal{H}\ \nabla_{\mathcal{H}E}(\mathcal{V}JF) = JA_EF.$$

If $H$, $Z$ are both horizontal, $A_HZ = -A_ZH$, [5], and $A_EF = A_{\mathcal{H}E}F$ by definition, so $A_{JE}H = A_{\mathcal{H}JE}H = -A_H\mathcal{H}JE = -JA_H\mathcal{H}E = +JA_{\mathcal{H}E}H = JA_EH$.

Similarly we establish the relations for $T$.

These first are expected relations in $A$, $T$. The next result, Theorem (1.5), is much less expected. First we must dispose of some more standard notions.

**Definition (1.3).** A vector field $X \in \mathfrak{X}(M)$ is said to be *basic* if

(1) $X$ is horizontal,

(2) $X$ is $\pi$-related to some vector field $\bar{X}$ on $B$.

The following straightforward facts may be found in [5].

**Lemma (1.4).**

(1) *If $X$, $Y$ are horizontal, $A_XY = \frac{1}{2}\mathcal{V}[X, Y]$.*

(2) *If $X$ is basic, and $V$ is vertical, then $[X, V]$ is vertical.*

**Theorem (1.5)** (*Watson*). *If $\pi$: $M \to B$ is a Kähler submersion, the horizontal distribution $\mathcal{H}$ is integrable and totally geodesic.*

*Proof.* Were the horizontal distribution integrable, in general $A_XY$ would be the second fundamental form of the integral submanifolds for horizontal vector fields $X$, $Y$; thus, we have only to show $A = 0$. By Lemma (1.4) it suffices to show $\mathcal{V}[X, Y] = 0$ for $X$, $Y$ basic. Let $V$ be vertical, and let $\phi$ be the Kähler form of $M$. As $M$ is Kähler,

$$0 = d\phi(X, Y, V) = X\langle Y, JV\rangle - Y\langle X, JV\rangle + V\langle X, JY\rangle$$
$$-\langle[X, Y], JV\rangle + \langle[X, V], JY\rangle - \langle[Y, V], JX\rangle.$$

By Lemma (1.4) and the definitions involved, the first and last two terms vanish. We will be done once we show

**Lemma (1.6).** *$V$, $X$, $Y$ as above, then*

$$V\langle X, Y\rangle = 0.$$

Note that $JY$ is basic if and only if $Y$ is.

*Proof.* As $X$, $Y$ are basic, $\langle X, Y\rangle_p = \langle \bar{X}, \bar{Y}\rangle_{\pi(p)}$, that is, $\langle X, Y\rangle = \pi^*(\langle \bar{X}, \bar{Y}\rangle)$. As $V \in \ker \pi_*$, $V\langle X, Y\rangle = 0$.

At first glance this appears to show only that $A_X Y = 0$ for $X$, $Y$ horizontal. However, an easy unwinding of the definitions shows $A = 0$ from this.

## 2. Foliations in Kähler manifolds

In the interest not only of generality, but to determine where the rigidity occurs in the notion of a Kähler submersion, we weaken our assumptions considerably in this section, only to find that the rigidity of the situation remains basically intact. Consider a Kähler manifold $M$ with a holomorphic distribution $\mathcal{V}$. $\ker \pi_*$ from §1 will serve as a prime example; however, we do not *a priori* assume that $\mathcal{V}$ is integrable. As before, however, we see that under apparently very mild conditions involving $\mathcal{V}$ we again have very strong restrictions.

**Theorem (2.1).** *Let $\mathcal{V} \subset T_*(M)$ be a holomorphic distribution of codimension k, M a Kähler manifold, and $\mathcal{H} = \mathcal{V}^\perp$ the orthogonal distribution. Then $\mathcal{H}$ is also a holomorphic distribution if and only if $\mathcal{V}$ and $\mathcal{H}$ both are not only integrable, but totally geodesic.*

**Remark.** Thus, if $\mathcal{H}$ is holomorphic, both distributions are parallel; equivalently, both of the orthogonal foliations are totally geodesic, or in O'Neill's terminology, the tensors $A$ and $T$ both vanish. (These tensors may be defined for any foliation, and appear to yield similar information about the geometry of $M$ as in the submersion case.) In [3] it is shown that this implies $M$ is locally isometric to a Riemannian product; the reader may also refer to a forthcoming paper by the author and L. B. Whitt, *Totally geodesic foliations*, to appear in the next issue of J. Differential Geometry.

*Proof.* Assume first that $\mathcal{V}$, $\mathcal{H}$ are totally geodesic. Let $U$ be a neighborhood of $M$ such that $\mathcal{V}|_U$ is trivial. Then there is an analytic submersion $\pi$: $U \to \mathbf{C}^k$ (not a Riemannian submersion) such that $\mathcal{V}|_U = \ker \pi_*$. Let $X$ be a horizontal vector field (i.e., in $\mathcal{H}$) such that $X$ is $\pi$-related to a vector field $\bar{X}$ on $\mathbf{C}^k$; assume moreover that $\bar{X}$ is holomorphic. It suffices to show that $X$ is holomorphic, as an easy argument will then show that $\mathcal{H}$ may be locally spanned by holomorphic vector fields.

**Remark.** As noted in the introduction, we consider a vector field $X$ to be *holomorphic* if it is a holomorphic section of $T_*(M)$, contrary to common practice in complex geometry, which deals with the complexified tangent bundle.

The following lemma is well-known [4].

**Lemma (2.2).** *$X \in \mathfrak{X}(M)$ is holomorphic if and only if $[X, JY] = J[X, Y]$ for all $Y \in \mathfrak{X}(M)$, that is, if and only if $X$ is an infinitesimal automorphism of the complex structure.*

Note that this condition is tensorial in $Y$.

Now consider the Kähler form $\phi$ of $M$, and let $V$, $W$ be holomorphic vertical (i.e., in $\mathcal{V}$) vector fields. We have

$$\begin{aligned} 0 &= d\phi(W, JV, X) \\ &= X\langle W, -V\rangle + \langle [W, X], -V\rangle - \langle [JV, X], JW\rangle \\ &= -\langle \nabla_X W, V\rangle - \langle \nabla_X V, W\rangle - \langle \nabla_W X, V\rangle + \langle \nabla_X W, V\rangle \\ &\quad -\langle [JV, X], JW\rangle \\ &= -\langle \nabla_V X + [X, V], W\rangle + \langle X, \nabla_W V\rangle - \langle [JV, X], JW\rangle. \end{aligned}$$

The second term vanishes as $\mathcal{V}$ is totally geodesic, similarly for the first part of the first term. We see therefore that $[X, JV] - J[X, V]$ is horizontal. However, $V$, being vertical, is $\pi$-related to 0, and $X$ is $\pi$-related to $\overline{X}$, thus $[X, V]$ is $\pi$-related to 0. This says that both terms above are vertical, so they are equal. That $[X, JY] = J[X, Y]$ for $Y$ horizontal follows from the fact that $\mathcal{H}$ is integrable by taking $Y$ $\pi$-related (locally) to $\overline{Y}$ a vector field on $\mathbf{C}^k$, as in the construction of $X$. This shows that $X$ is holomorphic, thus $\mathcal{H}$ is.

Now assume that $\mathcal{H}$ is holomorphic. We will be done if we show $\mathcal{H}$ is totally geodesic, for then we may switch the rôles of $\mathcal{V}$ and $\mathcal{H}$ ($\mathcal{V}$ is holomorphic). Let $X$, $Y$ be holomorphic sections of $\mathcal{H}$. Note that we need not assume integrability. Then $[X, JY] = J[X, Y]$, hence $\langle J[X, Y], V\rangle = \langle [X, JY], V\rangle$ for $V$ vertical and holomorphic. But the right-hand side

$$\begin{aligned} &= \langle \nabla_X JY - \nabla_{JY} X, V\rangle \\ &= \langle J\nabla_X Y, V\rangle + \langle X, \nabla_{JY} V\rangle \\ &= \langle J\nabla_X Y, V\rangle + \langle X, \nabla_V JY + [JY, V]\rangle \\ &= -\langle \nabla_X Y, JV\rangle + \langle X, J(\nabla_V Y + [Y, V])\rangle \\ &= -\langle \nabla_X Y, JV\rangle + \langle X, \nabla_Y JV\rangle \\ &= -\langle \nabla_X Y, JV\rangle - \langle \nabla_Y X, JV\rangle. \end{aligned}$$

Meanwhile, the left-hand side $= -\langle \nabla_X Y - \nabla_Y X, JV\rangle$, so $\langle \nabla_Y X, V\rangle = 0$, and $\mathcal{H}$ is totally geodesic.

This theorem implies the surprising

**Corollary (2.3).** *Let $\pi\colon M \to B$ be a Kähler submersion and $\mathcal{V} = \ker \pi_*$ the tangents to the fibers. Then $\mathcal{H} = \mathcal{V}^\perp$ is not a holomorphic distribution, even though the integral submanifolds of $\mathcal{H}$ are complex submanifolds, unless $T = 0$.*

We also derive the following corollary on Kähler metrics of product manifolds.

**Corollary (2.4).** *If $M = M_1 \times M_2$ is a complex product manifold, where $M_1$ and $M_2$ are Kähler, then the only Kähler metrics on $M$ such that the subbundles $\pi_i^*(T_*(M_i))$ are orthogonal are product metrics, where $\pi_i\colon M \to M_i$ are the product projections.*

## 3. Holomorphic connections

Let $\pi\colon P \to M$ be a complex-analytic principal bundle, with complex Lie group $G$ as structure group and fiber, and $\mathfrak{g}$ the Lie algebra. Let $\mathcal{V} = \ker \pi_*$ be the holomorphic subbundle of $T_*(P)$ consisting of tangents to the fibers. A connection $\mathcal{H}$ is *holomorphic* if the distribution $\mathcal{H}$ is a holomorphic subbundle of $T_*(P)$ (cf. §0). As $G$ is complex, $G$ has a left-invariant hermitian metric $\langle\ ,\ \rangle_G$ given by a hermitian inner product on $\mathfrak{g}$ also denoted by $\langle\ ,\ \rangle_G$, which we will consider to be fixed in the sequel.

Given any smooth connection $\mathcal{H}$ on $P$ we can construct a unique Riemannian metric on $P$ compatible with both the connection and the metric on the base (in the sense that $\pi\colon P \to M$ is a Riemannian submersion). Such a metric is clearly equivariant under the action of $G$ on $P$. Conversely, given such an equivariant metric on $P$, the horizontal subspaces are a connection on $P$ whose associated metric is the given one. Precisely; we have

**Definition (3.1).** The *Riemannian metric* $\langle\ ,\ \rangle_{\mathcal{H}}$ *on $P$ associated to* $\mathcal{H}$ is defined by, for $X$, $Y$ tangent vectors on $P$ at $p \in P$,

(1) if $X \in \mathcal{V}_p$, and $Y \in \mathcal{H}_p$, then $\langle X, Y\rangle_{\mathcal{H}} = 0$,

(2) if both $X$, $Y \in \mathcal{V}_p$, then $\langle X, Y\rangle_{\mathcal{H}} = \langle \omega(X), \omega(Y)\rangle_G$, where $\omega$ is the 1-form of the connection $\mathcal{H}$,

(3) if both $X$, $Y \in \mathcal{H}$, then $\langle X, Y\rangle_{\mathcal{H}} = \langle \pi_* X, \pi_* Y\rangle$.

As above, we say $X$ is *horizontal* (*vertical*) providing $X \in \mathcal{H}_p$ (resp., $\mathcal{V}_p$). With this metric clearly $\pi\colon P \to M$ is a Riemannian submersion.

There is a natural subalgebra of $\mathfrak{X}(P)$, denoted by $\bar{\mathfrak{g}}$, defined by, for $X \in \mathfrak{g}$, any connection 1-form $\omega$ (this is independent of choice of connection), $\bar{X} \in \bar{\mathfrak{g}}$ is the vertical vector field such that $\omega(\bar{X}) = X$ [2]. Vector fields $\bar{X} \in \bar{\mathfrak{g}}$ are said to be *fundamental* in [2].

**Remark.** The metric defined above is equivariant in the following sense: Given $g \in G$, $R_{g*}$ the induced right action of $g$ on $T_*(P)$, if $X$, $Y$ are horizontal, then $\langle R_{g*}(X), R_{g*}(Y)\rangle = \langle X, Y\rangle$; if $X$, $Y$ are vertical, extend to fundamental vector fields $\bar{X}$, $\bar{Y}$, then $\langle R_{g*}(\bar{X}), R_{g*}(\bar{Y})\rangle = \langle \overline{(\mathrm{Ad}_{g^{-1}}X)}, \overline{(\mathrm{Ad}_{g^{-1}}Y)}\rangle$, where $X$, $Y \in \mathfrak{g}$ correspond to $\bar{X}$, $\bar{Y} \in \mathfrak{g}$. Conversely, given any Riemannian metric on $P$ which is equivariant in this sense, if $\mathcal{H} = \mathcal{V}^{\perp}$ then $\mathcal{H}$ is a connection.

As in §1, a vector field $X$ on $P$ is said to be *basic* if $X$ is horizontal and $\pi$-related to a vector field $\bar{X}$ on $M$. Note that this differs from the terminology of [2]; in particular, their basic vector fields are not $\pi$-related to a vector field on the base.

**Lemma (3.2).** *If $X$ is basic, and $V$ is fundamental, then $[V, X] = 0$.*

*Proof.* [2] shows, as $V$ fundamental and $X$ horizontal, that $[V, X]$ is horizontal, while Lemma (1.4) shows, as $V$ vertical and $X$ basic, that $[V, X]$ is vertical.

Let $\nabla$ be the covariant derivative operator of the Riemannian connection associated to the metric $\langle\ ,\ \rangle_{\mathcal{H}}$ on $P$ for a given connection $\mathcal{H}$.

**Proposition (3.3).** *The fibers are totally geodesic.*

*Proof.* The horizontal part $\mathcal{H}\ \nabla_V W$ of $\nabla_V W$ (notation as in §1) for vertical vector fields $V$ and $W$ is $T_V W$. $T_V W$ is tensorial, so we may assume that $V$, $W$ are fundamental. We have, for $X$ horizontal,

$$2\langle\nabla_V W, X\rangle = V\langle W, X\rangle + W\langle V, X\rangle - X\langle V, W\rangle + \langle[V, W], X\rangle$$
$$+\langle[X, V], W\rangle + \langle V, [X, W]\rangle.$$

$X\langle V, W\rangle = 0$ as $\langle V, W\rangle$ is constant. The rest are easily seen to be zero as $\mathcal{V}$ integrable and by Lemma (3.2), thus $\nabla_V W$ is vertical.

**Theorem (3.4).** *$\langle\ ,\ \rangle_{\mathcal{H}}$ is hermitian if and only if $\mathcal{H}_p$ is a complex subspace of $T_*(P, p)$ at each point. Furthermore, $\mathcal{H}$ is holomorphic if and only if $\langle\ ,\ \rangle_{\mathcal{H}}$ is hermitian and $\nabla_x(J) = 0$ for $x \in \mathcal{H}$, where $J$ is the complex structure tensor of $P$.*

*Proof.* The first statement is elementary. Now assume that $\mathcal{H}$ is holomorphic. Extend $x$ to a basic vector field $X$, and let $Y$ be another basic field. $\mathcal{H}\ \nabla_X JY$ is also basic, being $\pi$-related to $\overline{\nabla}_{\pi_* X}\pi_*(JY)$ ($\overline{\nabla}$= Riemannian connection on $M$) [5]. As $M$ is Kähler, $\overline{\nabla}_{\pi_* X}\pi_*(JY) = J\ \overline{\nabla}_{\pi_* X}\pi_* Y$. But $\mathcal{H}J\ \nabla_X Y$ is also basic, being $\pi$-related to $J\ \overline{\nabla}_{\pi_* X}\pi_* Y$. As $\pi_*|_{\mathcal{H}}$ is an isometry, we have

$$\mathcal{H}\ \nabla_x JY = \mathcal{H}\ \nabla_x JY.$$

Now change the extension of $x$ to a *holomorphic* vector field $X$. Then, if $Y$ is basic as before, $\mathcal{V}\ \nabla_X JY = \mathcal{V}\frac{1}{2}[X, JY] = \mathcal{V}J\frac{1}{2}[X, Y] = J\mathcal{V}\ \nabla_X Y$. Thus, if $Y$ is basic, $\nabla_x JY = J\ \nabla_x Y$. Now, if $Y$ is any horizontal vector field, then $Y = a_i Y_i$, $Y_i$ basic, and a simple calculation yields

$$\nabla_x JY = J\ \nabla_x Y.$$

We have only to check $\nabla_x JV = J\nabla_x V$ for $V$ vertical. If $H$ is a horizontal field, $\langle\nabla_x JV, H\rangle = -\langle JV, \nabla_x H\rangle = +\langle V, \nabla_x JH\rangle = \langle J\ \nabla_x V, H\rangle$. If $W$ is another vertical vector field, extend $x$ to $X$ horizontal and holomorphic, as before. Then

$$\langle\nabla_X JV, W\rangle = \langle\nabla_{JV} X, W\rangle + \langle[X, JV], W\rangle$$
$$= -\langle X, \nabla_{JV} W\rangle + \langle J[X, V], W\rangle.$$

By Proposition (3.3) the first term vanishes, so the above

$$\begin{aligned} &= +\langle X, \nabla_V JW\rangle + \langle J[X, V], W\rangle \\ &= \langle J\nabla_V X, W\rangle + \langle J[X, V], W\rangle \\ &= \langle J\,\nabla_X V, W\rangle. \end{aligned}$$

Thus we have $\nabla_x(J) = 0$.

Now assume that $\langle\ ,\ \rangle_{\mathcal{H}}$ is hermitian and that $J$ is parallel in horizontal directions. Let $\bar{X}_j$ be holomorphic vector fields on $M$ which span $T_*(M)$ in a neighborhood, and let $X_j$ be the unique basic lifts, which span $\mathcal{H}$ locally. It suffices to show that these are holomorphic.

First, let $Y$ be basic. As before, $\mathcal{H}[X_j, JY]$ is basic, being $\pi$-related to $[\bar{X}_j, \pi_* JY] = J[\bar{X}_j, \pi_* Y]$. But $\mathcal{H}J[X_j, Y]$ is basic, and is $\pi$-related to $J[\bar{X}_j, \pi_* Y]$ as well, so

$$\mathcal{H}\left[X_j, \pi_* JY\right] = \mathcal{H}J\left[X_j, \pi_* Y\right].$$

On the other hand, $\mathcal{V}[X_j, JY] = 2\mathcal{V}\,\nabla_{X_j} JY = 2\mathcal{V}J\,\nabla_{X_j} Y = \mathcal{V}J[X_j, Y]$, by the parallelism of $J$. Thus $[X_j, JY] = J[X_j, Y]$ if $Y$ is basic. Now, if $Y$ is an arbitrary horizontal field, then $Y = a_i Y_i$, $Y_i$ basic, and again we see $[X_j, Ja_i Y_i] = a_i[X_j, JY_i] + X_j(a_i)JY_i = Ja_i[X_j, Y_i] + J(X_j(a_i))Y_i = J[X_j, Y]$.

Now let $V$ be a fundamental field. As $X_j$ are basic, Lemma (3.2) implies $[X_j, JV] = 0 = J[X_j, V]$. We may, as before, easily extend this to show $[X_j, JV] = J[X_j, V]$ for $V$ vertical, which completes the proof of Theorem (3.4).

**Theorem (3.5).** *Let $\langle\ ,\ \rangle_G$ be chosen to be Kähler.*

(a) *$P$ is Kähler, and so $\pi\colon P \to M$ is a Kähler submersion, if and only if $\mathcal{H}$ is holomorphic and flat.*

(b) *If $\mathcal{H}$ is holomorphic, the extent to which $\langle\ ,\ \rangle_{\mathcal{H}}$ is not Kähler is given by the curvature $\Omega$ of $\mathcal{H}$, that is, for $X$, $Y$ horizontal, $V$ vertical,*

$$\langle \nabla_V JX - J\,\nabla_V X, Y\rangle = \langle J\,\overline{\Omega(X, Y)}, V\rangle.$$

*Proof.* Part (a) is immediate from Theorems (1.5) and (2.1) along with Proposition (3.3).

For (b), first note that both

$$\nabla_x(J) = 0 \text{ and } \nabla_V JW = J\,\nabla_V W$$

for $x \in \mathcal{H}$, $V$, $W$ vertical, thus the extend to which $\langle\ ,\ \rangle_{\mathcal{H}}$ is not Kähler, that is, $\nabla_v(J)$ is given by $\nabla_v JX - J\,\nabla_v X$ for $v \in \mathcal{V}$, $X$ horizontal. As $\langle \nabla_v X, W\rangle = -\langle X, \nabla_v W\rangle = 0$ for $W$ vertical by Proposition (3.3), $\nabla_v JX - J\,\nabla_v X$ is horizontal. For $X$, $Y$ basic and holomorphic, $V$ vertical and holomorphic,

$$\begin{aligned}\langle \nabla_V JX - J\,\nabla_V X,\, Y\rangle &= \langle \nabla_{JX} V,\, Y\rangle + \langle [V, JX], Y\rangle \\ &\quad + \langle \nabla_X V, JY\rangle + \langle [V, X], JY\rangle \\ &= -\langle V, \nabla_{JX} Y + J\,\nabla_X Y\rangle = -2\langle V, J\,\nabla_X Y\rangle.\end{aligned}$$

Now, $\mathcal{V}\,\nabla_X Y = \mathcal{V}\frac{1}{2}[X, Y]$ by Lemma (1.4), so the above $= -\langle V, J[X, Y]\rangle$ $= -\langle V, J\overline{\Omega(X, Y)}\rangle$. As both sides of the equation

$$\langle \nabla_V JX - J\,\nabla_V X,\, Y\rangle = -\langle V, J\,\overline{\Omega(X, Y)}\,\rangle$$

are tensorial, this completes the proof.

**Corollary (3.6).** *Let $M$ be a Riemann surface. $\mathcal{H}$ is holomorphic if and only if $\langle\ ,\ \rangle_{\mathcal{H}}$ is Kähler.*

*Proof.* It is enough to show that if $\mathcal{H}$ is holomorphic, it must be flat. But, for a nonzero $X$ holomorphic and basic, $X$ and $JX$ span $\mathcal{H}$ locally, and $\mathcal{V}[X, JX] = \mathcal{V}J[X, X] = 0$, so $\mathcal{H}$ must be integrable.

**Corollary (3.7).** *Suppose that the bundle $P$ is the bundle of complex bases of $T_*(M)$, and $\mathcal{H}$ is the Riemannian connection of the Kähler metric on $M$. Then $\mathcal{H}$ is holomorphic if and only if the associated metric is Kähler.*

*Proof.* Again, it suffices to show that if $\mathcal{H}$ is holomorphic, then $\mathcal{H}$ is flat. The argument of the preceding corollary shows that $\Omega(X, JX) = 0$. However, $\Omega$ is the Riemannian curvature of $M$, thus, by standard Kähler geometry [4], as $\Omega(X, JX) = 0$, $\Omega = 0$.

**Remark.** The fundamental property of $\Omega$ used here is that $\Omega(X, Y) = \Omega(JX, JY)$. Any such curvature form is of course determined by its values on $(X, JX)$. Thus the result of Corollary (3.7) may be extended to a larger class of bundles and connections, described in [6].

## References


[1] M. Atiyah, *Complex analytic connections in fiber bundles*, Trans. Amer. Math. Soc. **85** (1957) 181–207.

[2] R. L. Bishop & R. J. Crittendon, *Geometry of manifolds*, Academic Press, New York, 1964.

[3] S. Kashiwabara, *The structure of a Riemannian manifold admitting a parallel field of tangent vector subspaces*, Tôhoku Math. J. **12** (1960) 102–119.

[4] S. Kobayaski & K. Nomizu, *Foundations of differential geometry*, Vol. II, Interscience, New York, 1969.

[5] B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J. **13** (1966) 459–469.

[6] I. M. Singer, *Geometric interpretation of a special connection*, Pacific J. Math. **9** (1959) 585–590.

[7] B. Watson, *Almost Hermitian submersions*, J. Differential Geometry **11** (1976) 147–165.


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