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THE EULER CYCLE OF A FOLIATION

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Introduction

The Poincaré dual of the Euler class of (the tangent bundle of) a codimension-one foliation is a one-dimensional homology class. For a sufficiently smooth foliation the Euler class may be calculated locally by choosing a Riemannian metric on the underlying manifold. This determines a closed form (for a 2-dimensional foliation this is the curvature form of the induced connection on the tangent bundle to the foliation) which when integrated over any closed leaf gives its Euler characteristic (Gauss-Bonnet Theorem) and which in fact represents the Euler class of the foliation.

We present here an analogous construction for the Poincaré dual of the Euler class, but combinatorial rather than differentiable. The foliation now need only be of class $C^{0,1}$, i.e., tangent to a continuous field of hyperplanes. For the choice of a Riemannian metric we substitute the choice of a smooth triangulation in general position (in Thurston's sense) with respect to the foliation. We prove that this choice determines a locally computable 1-cycle representing the dual class. This will be the Euler cycle. The analogy with the differentiable construction is more than formal: the coefficient of a 1-simplex transverse to the foliation is equal to the "combinatorial curvature" (defined following Banchoff in §2) of the leaves of the foliation at the points where it intersects them.

The search for such a combinatorial expression was motivated by the representation, conjectured by Stiefel [9, p. 342], proved and reproved by Whitney, Cheeger, and Halperin-Toledo [5] for the duals of the Stiefel-Whitney classes of a manifold.

Consider then a smooth oriented n-manifold M carrying a transversely oriented $(n-1)$-dimensional foliation $\mathcal{F}$. We assume as mentioned above

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that the tangent plane to the foliation at $x \in M$ varies continuously with x. A smooth triangulation T of M is made up of a coherent collection of smooth maps σ: $\Delta^n \to M$ where $\Delta^n \subset R^n$ is a linear simplex. Each σ induces a foliation $\mathfrak{F}(\sigma)$ on Δ^n. For $x \in \Delta^n$ let N_x be the unit normal to $\mathfrak{F}(\sigma)$ at x in the direction of the transversal orientation.

Definition. A smooth triangulation T of M is in *general position* with respect to $\mathfrak{F}$ if for each $\sigma \in T$, for each $x \in \Delta^n$ and for each pair of vertices $V_i \neq V_j$ of Δ^n the normal vector at x to the induced foliation has nonzero dot-product with $V_j - V_i$:

$$N_x \cdot (V_j - V_i) \neq 0.$$

Remark. This is a rephrasing of a definition made for arbitrary codimension by Thurston in [10]. He proves there that any smooth triangulation of M can be subdivided and jiggled so as to give one in general position with respect to $\mathfrak{F}$.

Theorem. *Given M, $\mathfrak{F}$ as above, the choice of a triangulation T of M in general position with respect to $\mathfrak{F}$ determines a refinement K of the* 2*-skeleton of T and in K a rational* 1*-cycle $e = e(\mathfrak{F}, T)$ with the following properties.*

1. *The definition of e is local in the sense that the coefficient attached to a* 1*-simplex is calculated in the closed star* (*in T*) *of that simplex.*

2. *The homology class of e is Poincaré dual to the Euler class of the tangent bundle of $\mathfrak{F}$.*

Outline of Proof. After digressions on general position (§1) and combinatorial curvature (§2) the chain e will be constructed (§3). Property 1 will be apparent from the construction. Property 2 and the fact that e is a cycle will be proved together in §4 and §5.

1. Combinatorial consequences of general position

Consider a smooth n-manifold M carrying a codimension-one foliation $\mathfrak{F}$ and a triangulation T in general position with respect to $\mathfrak{F}$.

As in the definition, we consider an n-simplex σ: $\Delta^n \to M$ of T and the foliation $\mathfrak{F}(\sigma)$ induced on the linear simplex $\Delta^n \subset R^n$. Let $V_0, \cdots, V_n$ be the vertices of Δ^n. It follows from general position that each edge V_iV_j is transverse to $\mathfrak{F}(\sigma)$, since the definition gives $N_x \cdot (V_j - V_i) \neq 0$ for $x \in V_iV_j$. In fact non-transversality of any m-face of Δ^n, $m = 1, \cdots, n-1$, at some point x would mean that the tangent space to the face, $V_0, \cdots, V_m$ for instance, was contained in the tangent space to $\mathfrak{F}(\sigma)$ at x. But $V_m - V_0$ is tangent to this linear simplex, and $N_x \cdot (V_m - V_0) \neq 0$ shows this is impossible.

It follows that each leaf L of $\mathcal{F}(\sigma)$, which does not contain any vertices of Δ^n, inherits from Δ^n a stratification into submanifolds which fit together like the faces of a simplex. But we can be more precise.

Proposition 1.1. *Let L be a leaf of $\mathcal{F}(\sigma)$ which contains no vertices of Δ^n. Then*

(a) *L divides the vertices of Δ^n into 2 sets: those below L (in the sense of the transverse orientation), say $V_0, \cdots, V_k$, and those above L, say $V_{k+1}, \cdots, V_n$.*

(b) *With k as in part* (a), *L is combinatorially isomorphic to $\Delta^k \times \Delta^{n-k-1}$. (It follows that each face of L is also of the combinatorial type simplex $\times$ simplex). See Fig.* 2.

Before beginning the proof of this proposition, let us investigate more in detail the possible intersections of L with a 2-face of Δ^n.

Lemma 1.2. *Given a 2-face $V_iV_jV_k$ of Δ^n, either $L \cap V_iV_jV_k = \varnothing$ or L intersects two of the edges each exactly once.*

Proof. Suppose $(V_k - V_i)$, $(V_k - V_j)$ and $(V_j - V_i)$ all have positive dot-product with the normals of $\mathcal{F}(\sigma)$. Then their relative position with respect to the foliation will be as in Fig. 1a. Suppose L intersects $V_iV_jV_k$ at a point x. The intersection will be a curve through x. The normal vector to L has positive dot-product with the 3 edges, and so its projection in the 2-face must lie in the shaded open sector of Fig. 1b. It follows that the tangent vector to the intersection curve must lie in the orthogonal open sector bounded by the lines parallel to V_iV_j and V_jV_k and shown in Fig. 1a.

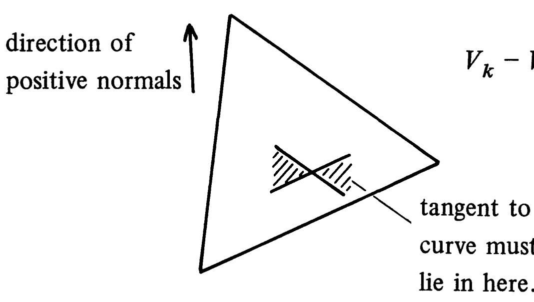

FIG. 1A

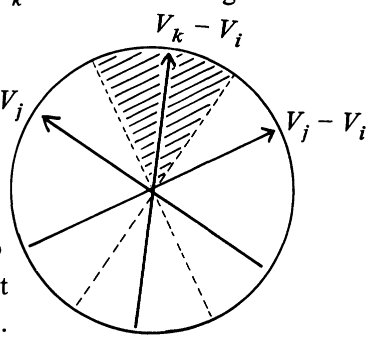

FIG. 1B

Since this holds true at every intersection point, it follows that the curve must meet V_iV_k and one of V_iV_j, V_jV_k.

If L had more than exactly two intersections with the edges of $V_iV_jV_k$, the same reasoning would show that there would be at least two intersections with the long edge V_iV_k, say p_1 and p_2. Every 2-plane P in R^n containing V_iV_k is transverse to $\mathcal{F}(\sigma)$ and therefore intersects L in a set of curves passing through p_1 and p_2. As P runs through all possible such planes, the union of these curves is L. So in at least one plane P_0 the points p_1 and p_2 would have to be joining by a curve in L, otherwise L would be disconnected. But in that plane P_0 a curve joining p_1 and p_2 would have to be parallel at some point to the segment p_1p_2; at that point the normal to L would be normal to V_iV_k; and this is forbidden by general position.

Proof of Proposition 1.1.

(a) Say that V_i and V_j are *on the same side* of L if $L \cap V_iV_j = \emptyset$. This relation is clearly symmetric and reflexive; Lemma 1.2 implies that it is transitive. Also, if V_i, V_j, V_k were in different equivalence classes, then L would intersect all three edges of $V_iV_jV_k$; this is impossible, again by Lemma 1.2. Choosing V_i from one equivalence class and V_j from the other, we call the first one "below" and the second "above" if $N_x \cdot (V_j - V_i) > 0$ for some $x \in \Delta^n$, and vice-versa otherwise. (The lemma may be used to show that this is independent of the choices.)

(b) With barycentric coordinates $(t_0, \cdots, t_n) = \Sigma_{i=0}^n t_iV_i$ for Δ^n, consider the hyperplane section H given by

$$t_0 + \cdots + t_k = t_{k+1} + \cdots + t_n \left(= \tfrac{1}{2}\right).$$

The section H also separates $V_0 \cdots V_k$ from $V_{k+1} \cdots V_n$. It is a convex linear $(n-1)$-cell of combinatorial type $\Delta^k \times \Delta^{n-k-1}$ as can be seen from the map

$$(r_0, \cdots, r_k), (s_0, \cdots, s_{n-k-1}) \to \left(\tfrac{1}{2}r_0, \cdots, \tfrac{1}{2}r_k, \tfrac{1}{2}s_0, \cdots, \tfrac{1}{2}s_{n-k-1}\right),$$

which sends $\Delta^k \times \Delta^{n-k-1}$ facewise onto H. The proof of the proposition will be completed by the construction of a face-preserving homeomorphism: $L \to H$. Let $\Delta^k \subset \Delta^n$ be the k-face spanned by $V_0, \cdots, V_k$, and Δ^{n-k-1} the $(n-k-1)$-face spanned by $V_{k+1}, \cdots, V_n$. The simplex Δ^n is the join of these two faces: the segments VW, $V \in \Delta^k$, $W \in \Delta^{n-k-1}$ fill out all Δ^n. Each of these segments intersects H exactly once, in fact in its midpoint, as is easy to check. We will prove that each VW intersects L exactly once (so the desired homeomorphism may be obtained by sliding L to H along the VW). Since the VW lying in edges of Δ^n have exactly one intersection with L as proved in Lemma 1.2, extra intersections or no intersections would imply the existence of points where a VW was tangent to L. But this is impossible: write V as $\Sigma_{i=0}^k r_iV_i$, and W as $\Sigma_{i=0}^{n-k-1} s_iW_i$, where $W_0 = V_{k+1}$, etc. We know that $N_x \cdot (W_j - V_i) > 0$ for any $x \in L$, for $i = 0, \cdots, k$ and for $j = 0, \cdots, n-k-1$. Then

$$N_x \cdot (W - V) = \sum_{i=0}^{k} \sum_{j=0}^{n-k-1} s_jr_iN_x \cdot (W_j - V_i) > 0.$$

2. Combinatorial curvature

Let γ^k be a convex linear cell in R^k, v a vertex of γ. Thinking of the points of R^k as vectors, let N be the set of unit vectors $n \in S^{k-1}$ such that $n \cdot (x - v) \leqslant 0$ for all $x \in \gamma$. Using the usual $(k-1)$-volume on S^{k-1}, the (normalized) *exterior angle* of γ at v is defined by Banchoff [2] to be

$$\mathcal{E}(\gamma, v) = \frac{\text{vol } N}{\text{vol } S^{k-1}}.$$

Examples. For $k = 0$, $\mathcal{E}(\gamma, v) = 1$; for $k = 1$, $\mathcal{E}(\gamma, v) = \frac{1}{2}$; for $k = 2$, $\mathcal{E}(\gamma, v) = (2\pi)^{-1}$ (π–interior angle of γ at v). It is easily seen that $\sum_{v\in\gamma} \mathcal{E}(\gamma, v) = 1$.

Let $\Gamma \subset R^n$ be a p-dimensional complex made up of convex linear cells, and let v be a vertex of Γ. Following Banchoff [2] we define the *polyhedral Gaussian curvature* of Γ at v to be

$$K(v) = \sum_{i=0}^{p} \sum_{v\in\gamma^i\in\Gamma} (-1)^i \mathcal{E}(\gamma^i, v).$$

Example. For $p = 2$ this gives $(2\pi)^{-1}$ (2π–sum of the face angles at v).

Theorem 2.1 (*Banchoff's polyhedral Gauss-Bonnet theorem* [2]). $\sum_{v\in\Gamma} K(v) = X(\Gamma)$, *the Euler characteristic of* Γ.

Historical note. This theorem for $p = 2$ is fairly classical [1], [6], [7] and, in fact, for Γ the boundary of a convex polyhedron, it was known to Descartes[1] that, in our notation, $\sum K(v) = 2$.

The definition of combinatorial Gaussian curvature at a vertex x of an abstract regular cell complex C ideally would proceed by replacing each $c \in C$ by a canonically chosen, combinatorially identical, convex linear cell $\gamma(c)$ and taking the polyhedral Gaussian curvature of the resulting convex-linear-cell complex. Unfortunately such a substitution process is not possible in general (see Grünbaum [4, §11.5]). Fortunately the cells we shall encounter in our analysis of foliations are the generic intersections of leaves with m-simplexes, $m = 1, \cdots, n$, of a triangulation in general position with respect to the foliation, and it follows from Proposition 1.1 that each such $(m-1)$-cell is combinatorially equivalent to some $\Delta^k \times \Delta^{m-k-1}$, $k = 0, \cdots, m-1$, and therefore has a canonical realization as the orthogonal product $\gamma = \Delta^k \times \Delta^{m-k-1}$ of unit simplexes.

In this case $\mathcal{E}(\gamma, v)$ is the same at all vertices of γ. Since there are

[1] "Si quatuor anguli plani recti ducantur per numerum angulorum solidorum & ex producto tollantur 8 anguli recti plani, remanet aggregatum ex omnibus angulis planis qui in superficie talis corporis solidi existent," [3, p. 265]. According to the editors of [3], a copy in Leibnitz' hand of this unpublished work, which they date 1619–1621, lay forgotten among Leibnitz' papers until its rediscovery in 1860. The original manuscript is lost.

$(k+1)(m-k)$ vertices, it follows that $\mathfrak{S}(\gamma, v) = 1/[(k+1)(m-k)]$.

Example. For $m = 3$ typical sections are shown in Fig. 2.

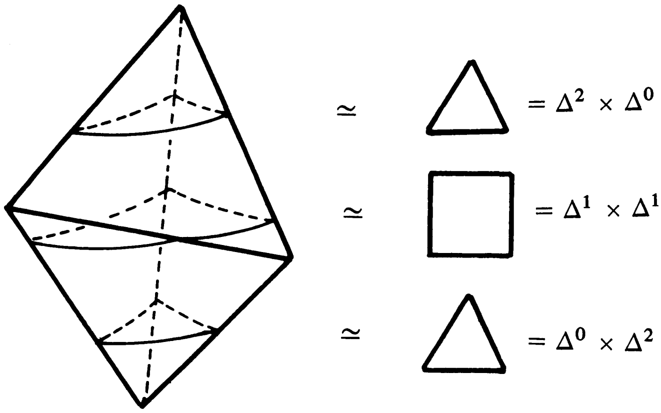

FIG. 2

Definition. If c^{m-1} is combinatorially equivalent to $\Delta^k \times \Delta^{m-k-1}$, then the (normalized) *combinatorial exterior angle* at a vertex x of c^{m-1} is defined to be

$$E(c, x) = \frac{1}{(k+1)(m-k)}.$$

Definition. If all the cells of a regular p-dimensional cell complex C have the combinatorial type simplex $\times$ simplex, then the *combinatorial Gaussian curvature* at a vertex $x \in C$ is defined by

$$k(C, x) = \sum_{i=0}^{p} \sum_{x \in c^i \in C} (-1)^i E(c, x) = \sum_{l,m} (-1)^{l+m} \frac{N(l, m)}{(l+1)(m+1)},$$

where $N(l, m)$ is the number of $c \in C$ such that $x \in c$ and $c \simeq \Delta^l \times \Delta^m$.

Theorem 2.2. *If all the cells of a regular cell complex C have the combinatorial type simplex $\times$ simplex, then*

$$\sum k(C, x) = X(C),$$

the sum being taken over all vertices $x \in C$.

Proof. Follows immediately from Theorem 2.1 and comparison of C with the combinatorially identical convex-linear-cell complex $\Gamma = \bigcup_{c \in C} \gamma(c)$.

3. Construction of the chain e

With M, $\mathfrak{F}$, T as in §1, the 1-simplexes of T are oriented by the positive normals to the foliation. By Lemma 1.2, a typical 2-simplex $A_1A_2A_3$ of T inherits from $\mathfrak{F}$ a foliation as in Fig. 3. If we agree that the symbol AB

represents the edge oriented from A to B, then the edges of this simplex, as oriented by the foliation, are A_1A_2, A_1A_3 and A_3A_2.

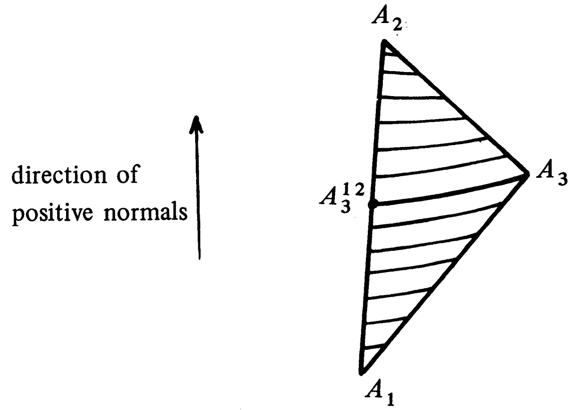

FIG. 3

Let K be the 2-dimensional complex formed by splitting each 2-simplex of T along the leaf containing its middle vertex. This introduces a new vertex A_i^{12} for each vertex $A_i \in \operatorname{link}_T(A_1A_2)$ such that A_1A_i and A_2A_i get opposite orientations from the foliation, as well as a new "horizontal" 1-simplex $A_iA_i^{12}$. The other 1-simplexes of K, which are all transverse to $\mathfrak{F}$, will be referred to as "vertical" in what follows. We shall define e as a 1-chain of K.

Determination of coefficients. Let V be a vertical 1-simplex of K. If x is a point in the interior of V, then $\operatorname{star}_T(x)$ cuts out a cell complex $C(x)$ on the leaf through x. The combinatorial type of this complex does not depend on $x \in V$, and is appropriate (Proposition 1.1) for the definition of combinatorial curvature given in §2. We assign to V the coefficient $k(C(x), x)$.

Let $H = A_iA_i^{jk}$ be a horizontal 1-simplex of K. Pick $x \in A_jA_k$ slightly above A_i^{jk} and y slightly below. The triangle $A_iA_jA_k$ determines edges $H(x) \in C(x)$, $H(y) \in C(y)$. The coefficient of H in e is defined to be

$$\sum_{i=1}^{n-1} \sum_{H(x)<c^i \in C(x)} (-1)^i E(C(x), x) - \sum_{i=1}^{n-1} \sum_{H(y)<c^i \in C(y)} (-1)^i E(C(y), y),$$

where $H < c$ means that H is a face of c.

Remark. If L is a compact leaf of $\mathfrak{F}$ containing no vertices of T, then L inherits from T the structure of a cell complex C where every cell is of simplex $\times$ simplex type; the chain e intersects L transversely at the vertices of C, and the intersection number of L and e is $\sum_{x\in C} k(C, x)$ which by Theorem 2.2 is the Euler characteristic $X(L)$.

The next two sections prove that, more generally, for any $(n-1)$-cycle Z, the intersection number of Z and e is well-defined and is the Euler number $X(T\mathfrak{F})[Z]$ of the tangent bundle to $\mathfrak{F}$ restricted to Z.

4. The chain α

The remaining properties of e will be established by comparison with the singular 1-chain α defined as follows.

For each m-simplex σ^m of T, $m = 1, \cdots, n$, let $\alpha(\sigma^m)$: $[0, 1] \rightarrow M$ be a singular 1-simplex running from the lowest vertex of σ^m to the highest (these relative heights as before determined by the transverse orientation), interior to σ^m except at the endpoints, and everywhere transverse to $\mathfrak{F}$.

Let α be the singular 1-chain

$$\alpha = \sum_{m=1}^{n} \sum_{\sigma^m \in T} (-1)^{m+1}\alpha(\sigma^m).$$

The Stiefel process [9, p. 340], [8, p. 202] gives a continuous vector field V on M tangent to $\mathfrak{F}$ and nonzero except on $|\alpha|$ = the support of α (Fig. 4). This vector field is generic except at the vertices of T. At each interior point of an $\alpha(\sigma^{\text{odd}})$, V has index $+1$; at each interior point of an $\alpha(\sigma^{\text{even}})$, V has index-1.

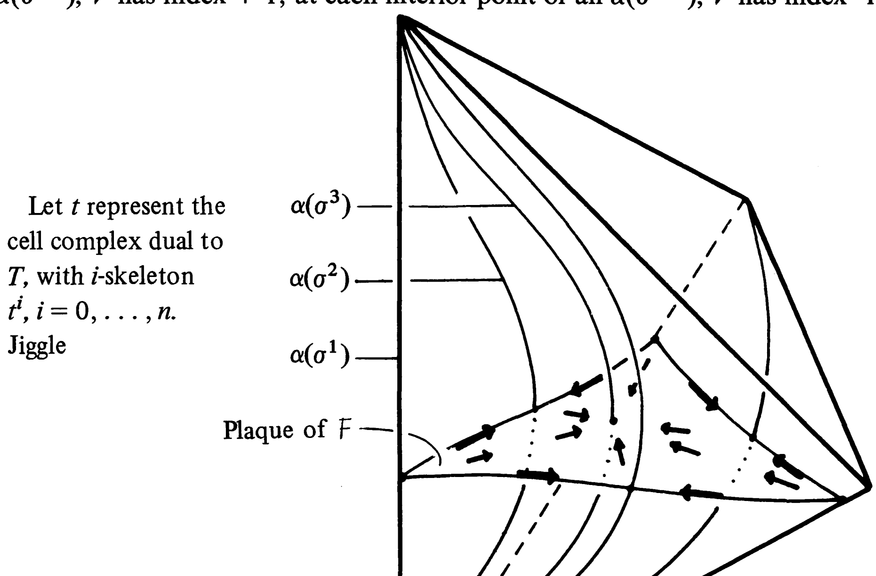

FIG. 4

t slightly to put it in general position with respect to $|\alpha|$ (this will mean that t^{n-2} does not intersect any $|\alpha(\sigma^i)|$ and that all intersections of t^{n-1} with $|\alpha(\sigma^i)|$ are transverse) while preserving the duality between t^{n-1} and T^1: each $(n-1)$-cell $s \in t^{n-1}$ intersects exactly one $|\alpha(\sigma^1)|$; in particular, $t^{n-1} \cap T^0 = \varnothing$.

The vector field V defines a nonzero section in $T\mathfrak{F}|t^{n-2}$, since $t^{n-2} \cap |\alpha| = \varnothing$, where $T\mathfrak{F}$ is the $(n-1)$-plane bundle tangent to $\mathfrak{F}$.

Proposition 4.1. *The obstruction to extending this nonzero section over any $(n-1)$-cell $s \in t^{n-1}$ is the intersection number of s and α.*

Corollary 4.2. *This means that "intersecting with α" is equal to the $(n-1)$-cochain obstructing the extension of $V|t^{n-2}$ to a nonzero section over t^{n-1}. Since this cochain is a cocycle* [8, *p.* 166] *representing the Euler class $X(T\mathfrak{F})$, α is a cycle and the homology class of α is Poincaré dual to $X(T\mathfrak{F})$.*

Proof of proposition. Choose a trivialization of $T\mathfrak{F}|s$ which respects the orientation of $\mathfrak{F}$. This puts sections of $T\mathfrak{F}|s$ in 1-1 correspondence with maps of s into R^{n-1}, and the vector field V defines a map $\bar{V}: s - \{p_1, \cdots, p_k\} \to R^{n-1} - \{0\}$ where $\{p_1, \cdots, p_k\} = s \cap |\alpha|$. Let us orient s by its unique intersection with T^1. Then the degree of this map at p_j is a well-defined integer. To calculate it, write p_j as $s \cap |\alpha(\sigma^i)|$. Now since $|\alpha(\sigma^i)|$ intersects both s and $\mathfrak{F}$ transversely at p_j, projection along $|\alpha(\sigma^i)|$ will map the leaf L through p_j onto s nonsingularly near p_j. The degree of this projection is equal to the intersection number ε_j of $\alpha(\sigma^i)$ with s at p_j ($\varepsilon_j = \pm 1$), since $\alpha(\sigma^i)$ is oriented positively with respect to L. Then the degree of $\bar{V}$ at p_j is ε_j times the degree of V (in L) at p_j, i.e., $(-1)^{i+1}\varepsilon_j$; this is equal to the intersection number of s and α at p_j (see Fig. 5).

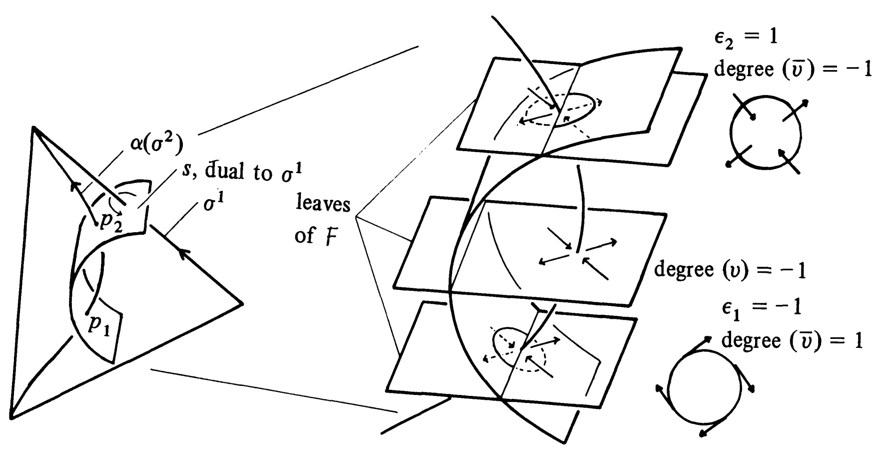

FIG. 5

So the sum of the degrees of $\bar{V}$ at the p_j, which is the obstruction to extending V to a nonzero vector field over s, is equal to the intersection number of s with α.

5. Comparison of e and α

This section will show that the chains e and α are homologous. It will then follow from Corollary 4.2 that e is also a cycle and that e has property 2, as claimed; so this fact, when established, will complete the proof of the theorem.

The definition of a certain 1-chain β will be convenient for this comparison: a simple calculation will show that β is homologous to α. On the other hand, β will turn out to be equal to e.

First some notation. Consider an m-simplex $\sigma \in T$, and suppose for simplicity that its vertices are labelled $A_0, \cdots, A_m$ with $A_0 < A_1 < \cdots < A_m$ in the sense that the edges A_jA_k with $j < k$ are positively oriented by the transversal orientation of $\mathcal{F}$. Let the plaque through A_i be denoted "level i", and let $K(\sigma)$ be the subdivision of σ inherited from K. Every vertical 1-simplex V of $K(\sigma)$ runs from level i to level $(i + 1)$ for some $i = 0, \cdots, m - 1$, and every horizontal edge lies in some level i, $i = 1, \cdots, m - 1$, and therefore runs from A_i to some A_i^{jk} with $j < i < k$.

Now for $\sigma \in T$ we define a simplical 1-chain $\beta(\sigma)$ in $K(\sigma)$ thus: If V is a vertical 1-simplex running from level i to level $(i + 1)$, then its coefficient is $[(i + 1)(m - i)]^{-1}$. If H is horizontal in level i, then its coefficient is $[(i + 1)(m - i)]^{-1} - [i(m - i + 1)]^{-1}$.

Lemma 5.1. *$\partial\beta(\sigma) = A_m - A_0$, and thus $\alpha(\sigma)$ is homologous to $\beta(\sigma)$ in $|K(\sigma)|$.*

Proof. The edges of $K(\sigma)$ arriving at A_m are all vertical, and they constitute the final portions of A_iA_m, $i = 0, \cdots, m - 1$. Thus there are m such edges, each running from level $(m - 1)$ to level m. Hence the coefficient of A_m in $\partial\beta(\sigma)$ is

$$\text{coefficient of } A_m = m\frac{1}{((m - 1) + 1)(m - (m - 1))} = 1.$$

Similarly,

$$\text{coefficient of } A_0 = -m\frac{1}{(0 + 1)(m - 0)} = -1.$$

At each A_i^{jk} there are only three edges of $K(\sigma)$:

(1) a vertical edge going up from level i,

(2) a horizontal edge coming from A_i to A_i^{jk} in level i,

(3) a vertical edge coming up from level $(i - 1)$. Consequently

$$\text{coefficient of } A_i^{jk} = -\frac{1}{(i+1)(m-i)} + \left[\frac{1}{(i+1)(m-i)} - \frac{1}{i(m-i+1)}\right] + \frac{1}{i(m-i+1)} = 0.$$

Finally at a vertex A_i, $0 < i < m$, there are in $K(\sigma)$

(1) $(m - i)$ vertical edges going up from level i, namely, the initial portions of A_iA_k, $k = i + 1, \cdots, m$,

(2) $i(m - i)$ horizontal edges leaving A_i, namely, the $A_iA_i^{jk}$ for $j = 0, \cdots, i - 1$ and $k = i + 1, \cdots, m$,

(3) i vertical edges coming up from level $(i - 1)$, namely, the final portions of A_jA_i, $j = 0, \cdots, i - 1$. Consequently

$$\begin{aligned}\text{coefficient of } A_i = &-(m-i)\frac{1}{(i+1)(m-i)} \\ &- i(m-1)\left[\frac{1}{(i+1)(m-i)} - \frac{1}{i(m-i+1)}\right] \\ &+ i\frac{1}{i(m-i+1)} = 0.\end{aligned}$$

Corollary 5.2. *$\alpha = \Sigma_{\sigma^m \in T}(-1)^{m+1}\alpha(\sigma^m)$ is homologous to*

$$\beta = \sum_{\sigma^m \in T} (-1)^{m+1}\beta(\sigma^m).$$

Lemma 5.3. $\beta = e$.

Proof. Let $\sigma = A_0 \cdots A_m$ as before be a simplex of T. Given V running from level i to level $(i + 1)$ in $K(\sigma)$, choose x in the interior of V and let $c(\sigma; x)$ represent the cell defined by the plaque through x. Then $c(\sigma; x)$ separates $A_0, \cdots, A_i$ (below) from $A_{i+1}, \cdots, A_m$ (above) and hence has the combinatorial type $\Delta^i \times \Delta^{m-i+1}$. The coefficient of V in $\beta(\sigma)$ is $[(i + 1) \cdot (m - i)]^{-1}$ which is equal to the combinatorial exterior angle $E(c(\sigma; x), x)$. So the coefficient of V in β is

$$\begin{aligned}&\sum_{m=1}^{n} \sum_{\dim \sigma = m,\ V<\sigma} (-1)^{m+1}E(c(\sigma; x), x) \\ &\qquad = \sum_{m=1}^{n} \sum_{\dim c(\sigma; x) = m-1} (-1)^{m+1}E(c(\sigma; x), x) \\ &\qquad = k(C(x), x) = \text{coefficient of } V \text{ in } e.\end{aligned}$$

Given H in $K(\sigma)$ running from A_i to A_i^{jk}, pick $x \in A_jA_k$ slightly above A_i^{jk} and y slightly below. Then σ determines cells $c(\sigma; x)$ and $c(\sigma; y)$, and as above the coefficient of H in $\beta(\sigma)$ is $E(c(\sigma; x), x) - E(c(\sigma; y), y)$. Hence the coefficient of H in β is

$$\sum_{m=2}^{n} \sum_{\dim \sigma = m,\, H \in K(\sigma)} (-1)^{m+1}\left[E(c(\sigma; x), x) - E(c(\sigma; y), y)\right]$$

$$= \sum_{m=2}^{n} \sum_{\dim c(\sigma; x) = m-1} (-1)^{m+1} E(c(\sigma; x), x)$$

$$- \sum_{m=2}^{n} \sum_{\dim c(\sigma; y) = m-1} (-1)^{m+1} E(c(\sigma; y), y)$$

$$= \text{coefficient of } H \text{ in } e.$$

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