J. DIFFERENTIAL GEOMETRY
14 (1979) 589–591

COMPLEX ANALYTIC PROJECTIVE CONNECTIONS

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Let M be a complex analytic manifold of dimension $n \geqslant 2$. *A complex analytic projective connection* on M is defined with respect to a coordinate covering (U, z^i) of M by its components (U, Γ^i_{jk}), which are complex analytic functions satisfying the compatibility relations

$$\frac{\partial z^i}{\partial z'^\alpha} \cdot \frac{\partial^2 z'^\alpha}{\partial z^j \partial z^k} - \frac{1}{n+1} \delta^i_j \frac{\partial \log \Delta}{\partial z^k} - \frac{1}{n+1} \delta^i_k \frac{\partial \log \Delta}{\partial z^j}$$
$$= \Gamma^i_{jk} - \Gamma'^\alpha_{\rho\gamma} \frac{\partial z^i}{\partial z'^\alpha} \cdot \frac{\partial z'^\beta}{\partial z^j} \cdot \frac{\partial z'^\gamma}{\partial z^k},$$

whenever $U \cap U' \neq \varnothing$, where $\Delta = \det(\partial z'^i / \partial z^j)$, [2, p. 99].
The left side determines a class $h(M) \in H^1(M, \mathbf{T}^* \otimes \mathbf{T} \otimes \mathbf{T}^*)$, where $\mathbf{T}$ and $\mathbf{T}^*$ denote the sheaves of germs of cross sections of the tangent bundle T and the cotangent bundle T^* of M this is the obstruction to the existence of complex analytic projective connections on M.

One has

$$h(M) = a(T) - \frac{1}{n+1} I \cup a(\Lambda^n T) - \frac{1}{n+1} a(\Lambda^n T) \cup I^*,$$

where $a(T) \in H^1(M, \mathbf{T}^* \otimes \mathbf{T} \otimes \mathbf{T}^*)$, $a(\Lambda^n T) \in H^1(M, \mathbf{T}^*)$ are the Atiyah classes [1, p. 188], and $I \in H^0(M, \mathbf{T}^* \otimes \mathbf{T})$, $I^* \in H^0(M, \mathbf{T} \otimes \mathbf{T}^*)$ are the identity endomorphisms, and "$\cup$" denotes the cup product.

The corresponding class in the differential case is always zero. The same is true if M is a Stein manifold.

If M is a compact Kähler manifold, $a(T)$ generates under the operation of the invariant polynomials of $GL_n(\mathbf{C})$, the characteristic cohomology ring of M (with complex coefficients) [1, Theorem 3]. Similarly, $h(M)$ will generate a ring which we will call *the projective characteristic ring of M*.

Received November 21, 1977.

Theorem. *Let M be a compact Kähler manifold of dimension $n \geqslant 2$. Then the projective characteristic ring of M is generated by the following classes*:

$$h_0(M) = 1,$$

$$(*) \qquad h_j(M) = \sum_{k=0}^{j} \frac{(-1)^k}{k!\,(n+1)^k} ch_{j-k}(M) ch_1^k(M) + \frac{(-1)^j}{j!\,(n+1)^j} ch_1^j(M), \quad 2 \leqslant j \leqslant n,$$

where $ch_j(M)$ are the components of the Chern character of M.

Proof. Let Γ^i_{jk} be the components of the canonical linear connection of the tangent bundle T associated to the hermitian structure of M. The forms $R^i{}_{jkl}\, dz^k \wedge d\bar{z}^l$, $R_{kl}\, dz^k \wedge d\bar{z}^l$, where $R^i{}_{jkl} = \partial\Gamma^i_{jk}/\partial z^l$, and $R_{kl} = R^i{}_{ikl}$, represent the Atiyah classes $a(T)$ and $a(\Lambda^n T)$ by the Serre-Dolbeault isomorphism. Also

$$R^i{}_{jkl}\, dz^k \wedge d\bar{z}^l - \frac{1}{n+1} \delta^i_j R_{kl}\, dz^k \wedge d\bar{z}^l - \frac{1}{n+1} \delta^i_k R_{jl}\, dz^k \wedge d\bar{z}^l$$

represents the class $h(M)$.

It is well-known that the ring of invariant polynomials of $GL_n(\mathbf{C})$ is generated by $(1/j!)\operatorname{tr}(A^j)$. Consequently, the projective characteristic ring of M is generated by the classes $h_j(M)$ represented by the forms

$$\frac{1}{j!}\Big(R^{l_1}{}_{l_2k_1k_2}\, dz^{k_1} \wedge d\bar{z}^{k_2} - \frac{1}{n+1}\delta^{l_1}_{l_2} R_{k_1k_2}\, dz^{k_1} \wedge d\bar{z}^{k_2} - \frac{1}{n+1}\delta^{l_1}_{k_1} R_{l_2k_2}\, dz^{k_1} \vee d\bar{z}^{k_2}\Big) \wedge \cdots \wedge \Big(R^{l_j}{}_{l_1k_{2j-1}k_{2j}}\, dz^{k_{2j-1}} \wedge d\bar{z}^{k_{2j}} - \frac{1}{n+1}\delta^{l_j}_{l_1} R_{k_{2j-1}k_{2j}}\, dz^{k_{2j-1}} \wedge d\bar{z}^{k_{2j}} - \frac{1}{n+1}\delta^{l_j}_{k_{2j-1}} R_{l_1k_{2j}}\, dz^{k_{2j-1}} \wedge d\bar{z}^{k_{2j}}\Big).$$

We recall that in the Kählerian case $R^i{}_{jkl} = R^i{}_{kjl}$. Then the formulas $(*)$ follows by standard calculations, since the form

$$\frac{(\sqrt{-1}\,)^j}{j!\,(2\pi)^j} R^{l_1}{}_{l_2k_1k_2} \cdots R^{l_j}{}_{l_1k_{2j-1}k_{2j}}\, dz^{k_1} \wedge d\bar{z}^{k_2} \wedge \cdots \wedge dz^{k_{2j-1}} \wedge d\bar{z}^{k_{2j}}$$

represents the j-the component of the Chern character of M.

Corollary. *If M admits a complex analytic projective connection, then $h_j(M) = 0$, $2 \leqslant j \leqslant n$.*

For example, there is not such a connection on the product $\mathbf{P}^1\mathbf{C} \times \cdots \times \mathbf{P}^1\mathbf{C}$ of $n \geqslant 2$ projective lines.

References

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