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DEFORMATION OF COMPLEX STRUCTURES ON MANIFOLDS WITH BOUNDARY II: FAMILIES OF NON-COERCIVE BOUNDARY VALUE PROBLEMS

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In this paper we develop the machinery necessary for the theorems in Deformations of Complex Structures on Manifolds with Boundary I. However these results hold in very general circumstances. We have relied heavily on the important paper [5] by J. J. Kohn and L. Nirenberg. The main work in this paper consists in rederiving their estimates with careful attention as to how the bounds depend on the coefficients of the linear problem. We have found it convenient to state the theorems not in terms of a first degree quadratic form but instead in terms of the associated second degree linear operator. Note that our self-adjoint elliptic operators are self-adjoint only in the sense of symbols, i.e., the highest order terms. We have included some sections on spectral theory which will be useful in constructing universal families; this approach was introduced by Kuranishi in [6].

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PART 3. SELF-ADJOINT ELLIPTIC BOUNDARY VALUE PROBLEMS

3.1. Definition

Let X be a compact manifold with boundary ∂X, and let F be a vector bundle over X and P, Q vector bundles over ∂X, each equipped with a hermitian inner product $\langle\ ,\ \rangle$. We consider an elliptic boundary value problem

$$\mathcal{E}: \mathcal{C}^\infty(X; F) \to \mathcal{C}^\infty(X; F) \oplus \mathcal{C}^\infty(\partial X; P) \oplus \mathcal{C}^\infty(\partial X; Q),$$
$$\mathcal{E}f = (Ef, pf, qf),$$

where E: $\mathcal{C}^\infty(X; F) \to \mathcal{C}^\infty(X; F)$ is a linear partial differential operator of degree 2, and p: $\mathcal{C}^\infty(X; F) \to \mathcal{C}^\infty(\partial X; P)$ and q: $\mathcal{C}^\infty(X; F) \to \mathcal{C}^\infty(\partial X; Q)$ are linear partial differential operators at the boundary of degree 0 and 1. Then E, p, q have principal symbols $\sigma_E(\xi)$: $F \to F$, $\sigma_p(\xi)$: $F|\partial X \to P$ and $\sigma_q(\xi)$: $F|\partial X \to Q$ which are homogeneous polynomials of degrees 2, 0 and 1. Note that $\sigma_p = \sigma_p(\xi)$ is independent of ξ since p has degree 0. We write

$$D\sigma_E(\xi; \eta) = \lim_{t\to 0}\left[\sigma_E(\xi + t\eta) - \sigma_E(\xi)\right]/t$$

for the derivative of the symbol. Note we take the derivative only in the ξ-directions and not in the x-directions. Since p and q have degrees 0 and 1, we do not need their derivatives, for

$$D\sigma_p(\xi; \eta) = 0, \quad D\sigma_q(\xi; \eta) = \sigma_q(\eta).$$

We say that E is a self-adjoint elliptic boundary value problem if the principal symbols σ_E, σ_p, σ_q satisfy the following conditions with respect to the hermitian metrics on F, P, Q (here ν is the unit normal covector):

(1) $\sigma_E(\xi)$ is positive-definite, i.e.,

$$\langle \sigma_E(\xi)f, f\rangle > 0 \quad \text{if } f \neq 0 \text{ and } \xi \neq 0.$$

(2) $\sigma_E(\xi)$ is hermitian symmetric, i.e.,

$$\langle \sigma_E(\xi)f, g\rangle = \langle f, \sigma_E(\xi)g\rangle.$$

(3) $\langle \sigma_E(\nu)f, g\rangle = \langle \sigma_p f, \sigma_p g\rangle + \langle \sigma_p(\nu)f, \sigma_q(\nu)g\rangle$.

(4) If $\sigma_p f = 0$ and $\sigma_p g = 0$, then for all real η

$$\langle D\sigma_E(\nu; \eta)f, g\rangle = \langle \sigma_q(\nu)f, \sigma_q(\eta)g\rangle + \langle \sigma_q(\eta)f, \sigma_q(\nu)g\rangle.$$

(5) $\dim F = \dim P + \dim Q$.

We remark on the meaning of these conditions. The first makes E strongly elliptic, and the second makes E essentially self-adjoint, i.e., if we choose a volume element dV on X and form the inner product

$$\langle\langle f, g\rangle\rangle = \int_X \langle f, g\rangle \, dV,$$

then $\langle\langle Ef, g\rangle\rangle = \langle\langle f, E^*g\rangle\rangle$ for all f and g with compact support in the interior of X, and $E^* - E$ has degree 1. The third condition guarantees that we can integrate by parts to find a hermitian bilinear form $Q(f, g)$ such that

$$\langle\langle Ef, g\rangle\rangle = Q(f, g) \text{ when } qf = 0 \text{ and } pg = 0.$$

The fourth condition assures us that Q can be chosen to be essentially hermitian symmetric, i.e., if $Q^*(f, g) = \overline{Q(g, f)}$, then $Q - Q^*$ involves no product of first derivatives. Of course Q is still not unique. The fifth condition assures us of the right number of boundary conditions.

3.2. Integration by parts

Here we justify the previous remarks. Let $K_p = \operatorname{Ker} \sigma_p$ and $K_q = \operatorname{Ker} \sigma_q(\nu)$. If $f \in K_p \cap K_q$, then $\langle \sigma_E(\nu)f, f\rangle = 0$ by (3) so $f = 0$ by (1). Therefore

$$\sigma_p \oplus \sigma_q(\nu)\colon F|\partial X \to P \oplus Q$$

is injective, and is also surjective since $\dim F = \dim P + \dim Q$ by (5). We have a direct sum decomposition $F|\partial X = K_p \oplus K_q$. We choose coordinates

$\{f^1, \cdots, f^m\}$ for the bundle F agreeing with this decomposition. Thus

$$K_p = \{f: f^\alpha = 0 \text{ for } 1 \leqslant \alpha \leqslant l\},$$
$$K_q = \{f: f^\omega = 0 \text{ for } l+1 \leqslant \omega \leqslant m\}.$$

In general we adopt the convention that α, β, γ are restricted to $1 \leqslant \alpha \leqslant l$ while φ, ψ, ω are restricted to $l+1 \leqslant \omega \leqslant m$. Latin indices i, j, k are unrestricted, i.e., $1 \leqslant i \leqslant m$. We also choose coordinates $\{x, y^1, \cdots, y^n\}$ for X so that $X = \{x \geqslant 0\}$ and $\partial X = \{x = 0\}$. We can easily make $\nu = dx$ and $dV = dx\, dy^1 \cdots dy^n$.We let r, s, t denote indices $1 \leqslant r \leqslant n$ for y.

The boundary operators will now be of the following form, for the appropriate choice of bases in P and Q:

$$pf^\alpha = f^\alpha, \quad qf^\omega = \frac{\partial f^\omega}{\partial x} + q_\varphi^{\omega r}\frac{\partial f^\varphi}{\partial y^r} + q_\alpha^{\omega r}\frac{\partial f^\alpha}{\partial y^r} + \cdots,$$

where dots denote terms of lower degrees. The hermitian metric will have local representatives h_{ij} consisting of $h_{\alpha\beta}$, $h_{\alpha\omega}$ and $h_{\varphi\omega}$. By a proper choice of the basis in F we can make $h_{\alpha\beta} = \delta_{\alpha\beta}$ and $h_{\varphi\omega} = \delta_{\varphi\omega}$; however $h_{\alpha\omega} \neq 0$ in general since K_p and K_q may not be orthogonal.

The operator E has the form

$$Ef^i = a_j^i\frac{\partial^2 f^j}{\partial x^2} + b_j^{ir}\frac{\partial^2 f^j}{\partial x \partial y^r} + c_j^{irs}\frac{\partial^2 f^j}{\partial y^r \partial y^s} + \cdots.$$

When we integrate by parts it is clear that the c-terms will cause no problems since they involve only tangential derivatives; therefore we neglect them and relegate them to the dots. We have

$$\langle\langle Ef, g\rangle\rangle = \int\int Ef^i \bar{g}^k h_{ik}\, dV$$
$$= \int\int \left\{ a_j^i\frac{\partial^2 f^j}{\partial x^2} + b_j^{ir}\frac{\partial^2 f^j}{\partial x \partial y^r} + \cdots \right\} \bar{g}^k h_{ik}\, dV.$$

We write $a_{jk} = a_j^i h_{ik}$ and $b_{jk}^r = b_j^{ir} h_{ik}$. Then

$$\langle\langle Ef, g\rangle\rangle = \int\int a_{jk}\frac{\partial^2 f^j}{\partial x^2}\bar{g}^k + b_{jk}^r\frac{\partial^2 f^j}{\partial x \partial y^r}\bar{g}^k\, dV + \cdots.$$

The hypothesis (1) that $\sigma_E(\xi)$ is hermitian symmetric with respect to h guarantees that a_{ij} and b_{ij}^r are hermitian symmetric:

$$a_{ij} = \overline{a_{ji}}, \, b_{ij}^r = \bar{b}_{ji}^r.$$

Condition (3) says that

$$\langle \sigma_E(\nu)f, g\rangle = \langle \sigma_p f, \sigma_p g\rangle + \langle \sigma_q(\nu)f, \sigma_q(\nu)g\rangle.$$

In local coordinates $\sigma_p f = \{f^\alpha\}$, $\sigma_q(\nu)f = \{f^\omega\}$ and

$$\begin{aligned}\langle \sigma_E(\nu)f, g\rangle &= a_{ij}f^i\bar{g}^j \\ &= a_{\alpha\beta}f^\alpha\bar{g}^\beta + a_{\alpha\omega}f^\alpha\bar{g}^\omega + a_{\varphi\beta}f^\varphi\bar{g}^\beta + a_{\varphi\omega}f^\varphi\bar{g}^\omega.\end{aligned}$$

Therefore condition (3) implies that $a_{\alpha\beta}$ is the representative of the metric on P, and $a_{\varphi\omega}$ is the representative of the metric on Q, while $a_{\alpha\omega} = 0$ and $a_{\varphi\beta} = 0$.

What does condition (4) say? If $\nu = dx$ and $\eta^r = dy^r$, then

$$\begin{aligned}D\sigma_E(\nu; \eta^r)^i_j &= b^{ir}_j, \\ \langle D\sigma_E(\nu; \eta^r)f, g\rangle &= b^r_{jk}f^j\bar{g}^k, \\ \sigma_q(\eta^r)f^\omega &= q^{\omega r}_\varphi f^\varphi + q^{\omega r}_\alpha f^\alpha, \\ \sigma_q(\nu)f^\omega &= f^\omega.\end{aligned}$$

If $\sigma_p f = 0$ and $\sigma_p g = 0$, then $f^\alpha = 0$ and $g^\beta = 0$, so

$$\langle D\sigma_E(\nu, \eta^r)f, g\rangle = b^r_{\psi\omega}f^\psi\bar{g}^\omega.$$

Moreover the metric $\langle\ ,\ \rangle$ on Q is given by $a_{\varphi\omega}$, so

$$\begin{aligned}\langle \sigma_q(\eta^r)f, \sigma_q(\nu)g\rangle &= a_{\varphi\omega}q^{\varphi r}_\psi f^\psi\bar{g}^\omega, \\ \langle \sigma_q(\nu)f, \sigma_q(\eta^r)g\rangle &= a_{\psi\varphi}\bar{q}^{\varphi r}_\omega f^\psi\bar{g}^\omega.\end{aligned}$$

Therefore condition (4) says

$$b^r_{\psi\omega} = a_{\varphi\omega}q^{\varphi r}_j + a_{\psi\varphi}\bar{q}^{\varphi r}_\omega.$$

We now proceed with the integration by parts. We must integrate

$$\begin{aligned}\int\int\Big\{ & a_{\alpha\beta}\frac{\partial^2 f^\alpha}{\partial x^2}\bar{g}^\beta + a_{\alpha\omega}\frac{\partial^2 f^\alpha}{\partial x^2}\bar{g}^\omega + a_{\varphi\beta}\frac{\partial^2 f^\varphi}{\partial x^2}\bar{g}^\beta + a_{\varphi\omega}\frac{\partial^2 f^\varphi}{\partial x^2}\bar{g}^\omega \\ & + b^r_{\alpha\beta}\frac{\partial^2 f^\alpha}{\partial x\partial y^r}\bar{g}^\beta + b^r_{\alpha\omega}\frac{\partial^2 f^\alpha}{\partial x\partial y^r}\bar{g}^\omega + b^r_{\varphi\beta}\frac{\partial^2 f^\varphi}{\partial x\partial y^r}\bar{g}^\beta + b^r_{\varphi\omega}\frac{\partial^2 f^\varphi}{\partial x\partial y^r}\bar{g}^\omega\Big\}\, dV\end{aligned}$$

into an essentially hermitian symmetric bilinear form, using the boundary conditions $qf = 0$ and $pg = 0$; so we have

$$\frac{\partial f^\varphi}{\partial x} + q^{\varphi r}_\psi\frac{\partial f^\psi}{\partial y^r} + q^{\varphi r}_\alpha\frac{\partial f^\alpha}{\partial y^r} = 0 \quad \text{on } \partial X,$$

$$g^\beta = 0 \quad \text{on } \partial X.$$

First observe that $a_{\alpha\omega} = 0$ and $a_{\varphi\beta} = 0$ by condition (3), so we can eliminate these terms. Next since $g^\beta = 0$ on ∂X, we can transfer $\partial/\partial x$ or

$\partial/\partial y^r$ onto g^β with equal ease; therefore the terms

$$\iint a_{\alpha\beta}\frac{\partial^2 f^\alpha}{\partial x^2}\bar{g}^\beta\,dV,\quad \iint b^r_{\alpha\beta}\frac{\partial^2 f^\alpha}{\partial x\partial y^r}\bar{g}^\beta\,dV,$$

$$\iint b^r_{\alpha\omega}\frac{\partial^2 f^\alpha}{\partial x\partial y^r}\bar{g}^\omega + b^r_{\varphi\beta}\frac{\partial^2 f^\varphi}{\partial x\partial y^r}\bar{g}^\beta\,dv$$

all can be integrated by parts into essentially hermitian symmetric bilinear forms. We are left with only two terms with $a_{\varphi\omega}$ and $b^r_{\varphi\omega}$ as coefficients. Since we can ignore the terms of lower orders, we may treat the coefficients as constants. Using the boundary condition for f

$$\begin{aligned}\iint a_{\varphi\omega}\frac{\partial f^\varphi}{\partial x^2}\bar{g}^\omega\,dV &= \iint a_{\varphi\omega}\frac{\partial}{\partial x}\left\{\frac{\partial f^\varphi}{\partial x} + q^{\varphi r}_\psi\frac{\partial f^\psi}{\partial y^r} + q^{\varphi r}_\alpha\frac{\partial f^\alpha}{\partial y^r}\right\}\bar{g}^\omega\,dV\\ &\quad -\iint a_{\varphi\omega}\left\{q^{\varphi r}_\psi\frac{\partial^2 f^\psi}{\partial x\partial y^r} + q^{\varphi r}_\theta\frac{\partial f^\alpha}{\partial x\partial y^r}\right\}\bar{g}^\omega\,dV\\ &= -\iint a_{\varphi\omega}\left\{\frac{\partial f^\varphi}{\partial x} + q^{\varphi r}_\psi\frac{\partial f^\psi}{\partial y^r} + q^{\varphi r}_\alpha\frac{\partial f^\alpha}{\partial y^r}\right\}\frac{\partial\bar{g}^\omega}{\partial x}\,dV\\ &\quad +\iint a_{\varphi\omega}\left\{q^{\varphi r}_\psi\frac{\partial f^\psi}{\partial x} + q^{\varphi r}_\alpha\frac{\partial f^\alpha}{\partial x}\right\}\frac{\partial\bar{g}^\omega}{\partial y^r}\,dV.\end{aligned}$$

Now

$$\iint a_{\varphi\omega}\frac{\partial f^\varphi}{\partial x}\frac{\partial\bar{g}^\omega}{\partial x}\,dV$$

is hermitian symmetric, so we may ignore it. We are left with two expressions

$$\iint a_{\varphi\omega}q^{\varphi r}_\psi\left\{\frac{\partial f^\psi}{\partial x}\frac{\partial\bar{g}^\omega}{\partial y^r} - \frac{\partial f^\psi}{\partial y^r}\frac{\partial\bar{g}^\omega}{\partial x}\right\}dV$$
$$+\iint a_{\varphi\omega}q^{\varphi r}_\alpha\left\{\frac{\partial f^\alpha}{\partial x}\frac{\partial\bar{g}^\omega}{\partial y^r} - \frac{\partial f^\alpha}{\partial y^r}\frac{\partial\bar{g}^\omega}{\partial x}\right\}dV.$$

We deal with the second expression first. This differs from a hermitian symmetric expression by the conjugate expression

$$\iint \bar{a}_{\varphi\omega}\bar{q}^{\varphi r}_\alpha\left\{\frac{\partial f^\omega}{\partial y^r}\frac{\partial\bar{g}^\alpha}{\partial x} - \frac{\partial\bar{f}^\omega}{\partial x}\frac{\partial\bar{g}^\alpha}{\partial y^r}\right\}dV.$$

But $\bar{g}^\alpha = 0$ on ∂X, so this can be integrated by parts into an expression of lower degree (i.e., transfer $\partial/\partial x$ off of g and then $\partial/\partial y^r$ onto g in the first term and then it cancels with the second).

Now for the first expression (the one with $a_{\varphi\omega}$), the second term plus its conjugate will equal a hermitian symmetric expression. Therefore we can replace the first expression by the equivalent expression

$$\int\int \{ a_{\varphi\omega} q_{\psi}^{\varphi r} + a_{\psi\varphi} \bar{q}_{\omega}^{\varphi r} \} \frac{\partial f^{\psi}}{\partial x} \frac{\partial \bar{g}^{\omega}}{\partial y^{r}} \, dV.$$

But we also have the $b^{r}_{\varphi\omega}$ expression

$$\int\int b^{r}_{\varphi\omega} \frac{\partial^{2} f^{\varphi}}{\partial x \partial y^{r}} \bar{g}^{\omega} \, dV = -\int\int b^{r}_{\psi\omega} \frac{\partial f^{\psi}}{\partial x} \frac{\partial \bar{g}^{\omega}}{\partial y^{r}} \, dV.$$

As we saw before, condition (4) says

$$b^{r}_{\psi\omega} = a_{\varphi\omega} q_{\psi}^{\varphi r} + a_{\psi\varphi} \bar{q}_{\omega}^{\varphi r},$$

so these two expressions cancel. This completes the proof of the integration by parts.

3.3. Norms

We introduce the Sobolev norms $\|f\|_{n}$ for $f \in \mathcal{C}^{\infty}(X; F)$ which measure the L_{2} norm over X of f and its partial derivatives of degree n or less. We also introduce norms $|f|_{n}$ and $|f|_{n-1/2}$ for sections $f \in \mathcal{C}^{\infty}(\partial X; F|\partial X)$ defined only on the boundary. The norm $|f|_{n}$ is just the Sobolev norm on ∂X which measures the L_{2} norm over ∂X of f and its partial derivatives of degree n or less in directions tangent to the boundary only. The norms $|f|_{n-1/2}$ can be defined in local coordinates using the Fourier transform and the multiplier $(1 + |\eta|)^{n-1/2}$. There is an equivalent and more useful definition, which is that for integer $n \geqslant 1$, $|f|_{n-1/2}$ is just the norm $\|\tilde{f}\|_{n}$ of the best extension $\tilde{f}$ of f to all of X

$$|f|_{n-1/2} = \inf\{ \|\tilde{f}\|_{n} : \tilde{f} \in \mathcal{C}^{\infty}(X; F) \text{ and } \tilde{f}|\partial X = f \}.$$

Since expressions with norms invariably involve arbitrary constants, we adopt the convention of Kohn and Folland:

$$V_{1}(f) \lesssim V_{2}(f) \text{ means } \exists C, \forall f, V_{1}(f) \leqslant C V_{2}(f).$$

In case of more terms we say

$$V_{1}(f) \lesssim V_{2}(f) + V_{3}(f) \text{ means } \exists C_{2}, \exists C_{3}, \forall f, V_{1}(f) \leqslant C_{2} V_{2}(f) + C_{3} V_{3}(f).$$

Note that if V_{2}(f) becomes negative, we may need C_{3} larger than C_{2}.

We have the relation

$$|f|_{n-1} \lesssim |f|_{n-1/2} \lesssim |f|_{n}.$$

More precisely one can show by interpolation methods that

$$|f|^{2}_{n-1/2} \lesssim |f|_{n} |f|_{n-1}, \quad |f|^{2}_{n} \lesssim |f|_{n+1/2} |f|_{n-1/2}.$$

We do not need this good a result; all we use is that (for $n \geqslant 1$)

$$\forall \varepsilon > 0, \exists C, |f|_{n-1/2} \leqslant \varepsilon |f|_n + C|f|_0.$$

We can find a continuous linear extension operator

$$T\colon \mathcal{C}^\infty(\partial X; F|\partial X) \to \mathcal{C}^\infty(X; F)$$

such that $Tf|\partial X = f$ and

$$\|Tf\|_n \lesssim |f|_{n-1/2}$$

for all integers $n \geqslant 1$. More generally, let

$$\frac{\partial}{\partial n}\colon \mathcal{C}^\infty(X; F) \to \mathcal{C}^\infty(\partial X; F|\partial X)$$

be a linear partial differential operator at the boundary whose symbol is

$$\sigma_{\partial/\partial n}(\xi) = \xi(v)I,$$

where v is a vector field pointing outward at ∂X, i.e., with $\nu(v) = 1$ where ν is an outward normal cotangent vector field. We call $\partial/\partial n$ a normal derivative, since we can always choose a coordinate chart on X and a vector bundle chart on F such that $\partial/\partial n$ is exactly the normal derivative in local coodinates. Then we can choose a continuous linear extension

$$T\colon \mathcal{C}^\infty(\partial X; F|\partial X) \oplus \mathcal{C}^\infty(\partial X; F|\partial X) \to \mathcal{C}^\infty(X; F)$$

such that $T(g, h)|\partial X = g$, $\partial/\partial n T(g, h)|\partial X = h$, and

$$\|T(g, h)\|_n \lesssim |g|_{n-1/2} + |h|_{n-3/2}$$

for all integers $n \geqslant 2$.

On the other hand, we can approximate as closely as we wish in $\| \, \|_1$ without any restriction on the normal derivative.

Lemma. *Given any $f \in \mathcal{C}^\infty(X; F)$ and any $h \in \mathcal{C}^\infty(\partial X; F|\partial X)$ we can find a sequence $f_j \in \mathcal{C}^\infty(X; F)$ such that $f_j|\partial X = f$, $(\partial/\partial n)f_j = h$ and $\|f_j - f\|_1 \to 0$ as $j \to \infty$.*

Proof. It is enough to verify the lemma when $f = 0$, and h has compact support in a local coordinate chart. Choose a chart with coordinates $\{x, y^1, \cdots, y^n\}$ on X and $\{f^1, \cdots, f^m\}$ on F so that $\partial/\partial n$ becomes $\partial/\partial x$. Choose a sequence of functions $\varphi_j(x)$ such that $\varphi_j(x) = 0$ for $x \geqslant 1/j$, $0 \leqslant \varphi_j'(x) \leqslant 1$ for all x, $\varphi_j(0) = 0$ and $\varphi_j'(0) = 1$. Put $f_j(x, y) = \varphi_j(x)h(y)$. It is then clear that $f_j|\partial X = 0$, $(\partial/\partial n)f_j = h$ and $\|f_j\|_1 \to 0$ as claimed.

3.4. Coercive boundary value problems

Let $\mathcal{E}f = (Ef, pf, qf)$ be a self-adjoint elliptic bound value problem as described in §1. We say $\mathcal{E}$ is coercive if satisfies a coercive estimate or

Gårding's inequality

$$\|f\|_1^2 \lesssim \mathrm{Re}\langle\langle Ef, f\rangle\rangle + \|f\|_0^2$$

for all f with $pf = 0$ and $qf = 0$. This has several well known consequences. For every integer $n \geq 2$ we have estimates

$$\|f\|_n \lesssim \|Ef\|_{n-2} + |pf|_{n-1/2} + |qf|_{n-3/2} + \|f\|_0.$$

The map

$$\mathcal{E}: \mathcal{C}^\infty(X; F) \to \mathcal{C}^\infty(X; F) \oplus \mathcal{C}^\infty(\partial X; P) \oplus \mathcal{C}^\infty(\partial X; Q)$$

has finite dimensional kernel and closed range with finite codimension. We can solve $Ef = g$, $pf = h$, $qf = k$ if and only if g, h, k satisfy a finite number of linear relations of the form

$$\langle\langle g, \gamma\rangle\rangle + \langle h, \eta\rangle + \langle k, \kappa\rangle = 0,$$

where $\gamma \in \mathcal{C}^\infty(X; F)$, $\eta \in \mathcal{C}^\infty(\partial X; P)$, $\kappa \in \mathcal{C}^\infty(\partial X; Q)$ and

$$\langle\langle g, \gamma\rangle\rangle = \int\!\!\int_X \langle g, \gamma\rangle\, dV, \quad \langle h, \eta\rangle = \int_{\partial X} \langle h, \eta\rangle\, dS, \quad \langle k, \kappa\rangle = \int_{\partial X} \langle k, \kappa\rangle\, dS,$$

where dS is the "surface area" on ∂X with $dV = dS \wedge \nu$. Moreover the classical Fredholm alternative holds.

Lemma. *If $\mathcal{E}$ is a coercive self-adjoint elliptic boundary value problem then*

$$\dim \ker \mathcal{E} = \operatorname{codim} \operatorname{Im} \mathcal{E}.$$

Hence in particular if $\mathcal{E}$ is injective then it is also surjective.

Proof. Since $\mathcal{E}$ is self-adjoint it follows from §2 that we can integrate by parts to obtain an essentially hermitian symmetric bilinear form $Q(f, g)$ with

$$\langle\langle Ef, g\rangle\rangle = Q(f, g) \text{ when } qf = 0 \text{ and } pg = 0.$$

Moreover from the construction it is clear that there are boundary linear partial differential operators

$$p': \mathcal{C}^\infty(X; F) \to \mathcal{C}^\infty(\partial X; P) \quad \text{of degree 1}$$
$$q': \mathcal{C}^\infty(X; F) \to \mathcal{C}^\infty(\partial X; Q) \quad \text{of degree 0}$$

such that

$$\langle\langle Ef, g\rangle\rangle + \langle p'f, pg\rangle + \langle qf, q'g\rangle = Q(f, g).$$

Hence also

$$\langle\langle f, Eg\rangle\rangle + \langle pf, p'g\rangle + \langle q'f, qg\rangle = \overline{Q(g, f)},$$

and $R(f, g) = Q(f, g) - \overline{Q(g, f)}$ contains no product of derivatives. Thus

$$R(f, g) = -\langle\langle Lf, g\rangle\rangle + \langle\langle f, Mg\rangle\rangle,$$

where L and M are linear partial differential operators of degree 1:

$$L, M: \mathcal{C}^\infty(X; F) \to \mathcal{C}^\infty(X; F).$$

Thus we have the formula

$$\langle\!\langle (E+L)f, g\rangle\!\rangle + \langle p'f, pg\rangle + \langle qf, q'g\rangle$$
$$= \langle\!\langle f, (E+M)g\rangle\!\rangle + \langle pf, p'g\rangle + \langle q'f, qg\rangle$$

Let

$$\mathcal{E}_L f = ((E+L)f, pf, qf), \quad \mathcal{E}_M f = ((E+M)f, pf, qf).$$

Then $\mathcal{E}_L$ and $\mathcal{E}_M$ are also coercive self-adjoint elliptic boundary value problems. Let

$$\text{index } \mathcal{E} = \dim \operatorname{Ker} \mathcal{E} - \operatorname{codim} \operatorname{Im} \mathcal{E}.$$

Since $\mathcal{E}$, $\mathcal{E}_L$, $\mathcal{E}_M$ all differ only by operators of lower degrees, it is a classical result of Fredholm theory that they all have the same index. Therefore index $\mathcal{E} = 0$ follows from the relations

$$\dim \operatorname{Ker} \mathcal{E}_L = \operatorname{codim} \operatorname{Im} \mathcal{E}_M, \quad \dim \operatorname{Ker} \mathcal{E}_M = \operatorname{codim} \operatorname{Im} \mathcal{E}_L.$$

By symmetry it suffices to prove only the second. Suppose $g \in \operatorname{Ker} \mathcal{E}_M$. Then $(E+M)g = 0$, $pg = 0$, $qg = 0$ so

$$\langle\!\langle (E+L)f, g\rangle\!\rangle - \langle pf, p'g\rangle + \langle qf, q'g\rangle = 0$$

for all f. Thus $(g, -p'g, q'g)$ defines a linear relation on Im $\mathcal{E}_L$. Conversely suppose $(g, -h, k)$ defines a relation on Im $\mathcal{E}_L$, so that

$$\langle\!\langle (E+L)f, g\rangle\!\rangle - \langle pf, h\rangle + \langle qf, k\rangle = 0$$

for all f. Then

$$\langle\!\langle f, (E+M)g\rangle\!\rangle + \langle pf, p'g - h\rangle + \langle qf, k - q'g\rangle$$
$$-\langle p'f, pg\rangle + \langle q'f, qg\rangle = 0$$

for all f. If f has support in the interior of X, then $\langle\!\langle f, (E+M)g\rangle\!\rangle = 0$. This forces $(E+M)g = 0$. Now pf, $q'f$, $p'f$, qf can all be specified arbitrarily at ∂X. Thus $pg = 0$, $qg = 0$, $p'g = h$ and $q'g = k$. Hence $g \in \operatorname{Ker} \mathcal{E}_M$ and the linear relation is of the form considered before. Thus there is a 1-1 correspondence between Ker $\mathcal{E}_M$ and linear relations on Im $\mathcal{E}_L$, so

$$\dim \operatorname{Ker} \mathcal{E}_M = \operatorname{codim} \operatorname{Im} \mathcal{E}_L$$

as we asserted before.

We still have to show that p, p', q, q' are independent as claimed. The following assertions are clearly equivalent since $\dim F = \dim P + \dim Q$:

(a) the map

$$p \oplus p' \oplus q \oplus q' : \mathcal{C}^\infty(X; F)$$
$$\to \mathcal{C}^\infty(X; P) \oplus \mathcal{C}^\infty(X; P) \oplus \mathcal{C}^\infty(X; Q) \oplus \mathcal{C}^\infty(X; Q)$$

is surjective;

(b) the maps

$$\sigma_p \oplus \sigma_{q'}\colon F|\partial X \to P \oplus Q, \quad \sigma_{p'}(\nu) \oplus \sigma_q(\nu)\colon F|\partial X \to P \oplus Q$$

are isomorphisms;

(c) if $pf = 0$, $p'f = 0$, $qf = 0$, $q'f = 0$, then $f = 0$ and $\partial f/\partial n = 0$ on ∂X.

We prove (c). If $pf = 0, p'f = 0$, $qf = 0$, $q'f = 0$, then

$$\langle\langle (E + L)f, g\rangle\rangle = \langle\langle f, (E + M)g\rangle\rangle$$

holds for all g. Moreover we can find two boundary linear partial differential operators of degree 1:

$$r, s\colon \mathcal{C}^\infty(X; F) \to \mathcal{C}^\infty(\partial X; F|\partial X),$$

so that

$$\langle\langle (E + L)f, g\rangle\rangle + \langle rf, g\rangle = \langle\langle f, (E + M)g\rangle\rangle + \langle f, sg\rangle$$

holds for all f and g. In local coordinates

$$\langle\langle (E + L)f, g\rangle\rangle = \int\int_X a_j^i \frac{\partial^2 f^j}{\partial x^2} \bar{g}^k h_{ik}\, dV + \cdots = -\int_{\partial X} a_j^i \frac{\partial f^j}{\partial x} \bar{g}^k h_{ik}\, dS + \cdots,$$

so $\sigma_r(\nu) = \sigma_E(\nu)$. By symmetry $\sigma_s(\nu) = \sigma_E(\nu)$ as well. Since $\sigma_E(\nu)$ is invertible, g and sg are completely arbitrary on ∂X. Thus, if $pf = 0$, $p'f = 0$, $qf = 0$ and $q'f = 0$, we must have $f = 0$ and $rf = 0$ on ∂X. But again since $\sigma_E(\nu)$ is invertible we must have $f = 0$ and $\partial f/\partial n = 0$ on ∂X. This proves the assertion (c).

3.5. Persuasive boundary value problems

Let $\mathcal{E}f = (Ef, pf, qf)$ be a self-adjoint elliptic boundary value problem as defined in §1. A coercive estimate or Gårding's inequality,

$$\|f\|_1^2 \lesssim \operatorname{Re}\langle\langle Ef, f\rangle\rangle + \|f\|_0^2$$

when $pf = 0$ and $qf = 0$, is very strong and fails in certain interesting cases. Nevertheless many of the important results are still valid, if we have instead a persuasive estimate or subelliptic inequality,

$$|f|_0^2 \lesssim \operatorname{Re}\langle\langle Ef, f\rangle\rangle + \|f\|_0^2,$$

when $pf = 0$ and $qf = 0$. Such an estimate occurs for example in $\bar{\partial}$-Neumann problem by a clever integration by parts. The consequences of such an estimate are discussed at length by Kohn and Nirenberg [5]; we give a brief review of their argument since we shall shortly need to rederive their main estimate with uniform bounds in terms of the coefficients.

Theorem (*Kohn-Nirenberg*). *Let* $\mathcal{E}f = (Ef, pf, qf)$ *be a self-adjoint elliptic boundary value problem as defined in* §1, *satisfying conditions* (1)–(5). *Suppose*

$\mathcal{E}$ satisfies a persuasive estimate or subelliptic inequality

$$|f|_0^2 \lesssim \operatorname{Re}\langle\langle Ef, f\rangle\rangle + \|f\|_0^2$$

when $pf = 0$ and $qf = 0$. Then

$$\mathcal{E} : \mathcal{C}^\infty(X; F) \to \mathcal{C}^\infty(X; F) \oplus \mathcal{C}^\infty(\partial X; P) \oplus \mathcal{C}^\infty(\partial X; Q)$$

has finite dimensional kernel and closed range with finite codimension; *moreover*

$$\dim \operatorname{Ker} \mathcal{E} = \operatorname{codim} \operatorname{Im} \mathcal{E}.$$

The first step is to integrate by parts as in §2 to obtain an essentially hermitian symmetric bilinear form $Q(f, g)$ with

$$\langle\langle Ef, g\rangle\rangle + \langle p'f, pg\rangle + \langle qf, q'g\rangle = Q(f, g).$$

Then the persuasive estimate says

$$|f|_0^2 \lesssim \operatorname{Re} Q(f, f) + \|f\|_0^2$$

when $pf = 0$ and $qf = 0$. Observe that now it is unnecessary to require $qf = 0$. For given any $f \in \mathcal{C}^\infty(X; F)$, we can find a sequence f_j according to the lemma of §3 such that $\|f_j - f\|_1 \to 0$, $f_j|\partial X = f$ and the normal derivatives $(\partial/\partial n)f_j = h$ for any given h. Since $\sigma_q(\nu)$ is surjective, for an appropriate choice of h we will have $qf_j = 0$ for all j. If $pf = 0$, then $pf_j = 0$ also. Apply the persuasive estimate to f_j;

$$|f_j|_0^2 \lesssim \operatorname{Re} Q(f_j, f_j) + \|f_j\|_0^2.$$

Since $\|f_j - f\|_1 \to 0$, surely $Q(f_j, f_j) \to Q(f, f)$. Therefore $|f|_0^2 \lesssim \operatorname{Re} Q(f, f) + \|f\|_0^2$ when $pf = 0$ without any restriction on qf.

Let v be a vector field on X. We can choose a linear partial differential operator ∇ of degree 1:

$$\nabla : \mathcal{C}^\infty(X; F) \to \mathcal{C}^\infty(X; F)$$

with symbol $\sigma \nabla(\xi) = \xi(v)I$. This determines ∇ up to an operator of degree 0. If v is tangent to the boundary of X, then by restriction

$$\nabla : \mathcal{C}^\infty(\partial X; F|\partial X) \to \mathcal{C}^\infty(\partial X; F|\partial X).$$

In this case, by changing ∇ by an operator of degree 0 we can make ∇ preserve $\operatorname{Ker} p$, so that $pf = 0 \Rightarrow \nabla pf = 0$. It is sufficient for this to construct ∇ in local charts where $\operatorname{Ker} p = \{f : f^\alpha = 0 \text{ for } 1 \leq \alpha \leq l\}$ and patch together with a partition of unity. We call such an operator ∇ a simple operator. Notice that there are enough simple operators so that we can find a finite number of them $\nabla_1, \cdots, \nabla_N$ with

$$|f|_n^2 \leq \sum_{k=0}^{n} \sum_{j=1}^{N} |\nabla_j^k f|_0^2.$$

Let ∇ be a simple operator. Then $pf = 0 \Rightarrow p\nabla f = 0 \Rightarrow \cdots \Rightarrow p\nabla^n f = 0.$

Hence if $pf = 0$, we can apply the persuasive estimate to $\nabla^n f$;

$$|\nabla^n f|_0^2 \lesssim \operatorname{Re} Q(\nabla^n f, \nabla^n f) + \|\nabla^n f\|_0^2.$$

At this point Kohn and Nirenberg [5] perform a careful shifting of derivatives to transform $\operatorname{Re} Q(\nabla^n f, \nabla^n f)$ into $\operatorname{Re} Q(f, \nabla^{2n} f)$ with only small error terms. In particular they prove the following result.

Lemma (*Kohn-Nirenberg*). *Let $Q(f, g)$ be an essentially hermitian symmetric bilinear form of degree* 1 *in f and g. Let ∇ be an operator of degree* 1 *with symbol $\sigma\,\nabla(\xi) = \xi(v)I$ where v is a vector field tangent to ∂X. Then for any n*

$$|\operatorname{Re} Q(\nabla^n f, \nabla^n f) - (-1)^n \operatorname{Re} Q(f, \nabla^{2n} f)| \lesssim \|f\|_n^2.$$

Assuming this result for the moment, we complete the derivation of the main a-priori estimate. Assume $pf = 0$, $qf = 0$. Then $p\nabla^{2n} f = 0$ as well, so

$$Q(f, \nabla^{2n} f) = \langle\langle Ef, \nabla^{2n} f\rangle\rangle.$$

Since ∇ is a differentiation parallel to the boundary, $\nabla(\nu) = 0$. Therefore ∇ will have an adjoint operator ∇^* of degree 1 such that

$$\langle\langle \nabla^* f, g\rangle\rangle + \langle\langle f, \nabla g\rangle\rangle = 0$$

for all f and g without restriction at the boundary. Thus

$$\langle\langle Ef, \nabla^{2n} f\rangle\rangle = (-1)^n \langle\langle \nabla^{*n} Ef, \nabla^n f\rangle\rangle.$$

Now

$$|\langle\langle \nabla^{*n} Ef, \nabla^n f\rangle\rangle| \lesssim \|Ef\|_n \|f\|_n,$$

so also

$$|Q(f, \nabla^{2n} f)| \lesssim \|Ef\|_n \|f\|_n.$$

Then by the Lemma of Kohn-Nirenberg

$$|Q(\nabla^n f, \nabla^n f)| \lesssim \|Ef\|_n \|f\|_n + \|f\|_n^2.$$

Therefore

$$|\nabla^n f|_0^2 \lesssim \|Ef\|_n \|f\|_n + \|f\|_n^2.$$

Summing over a finite number of ∇

$$|f|_n^2 \lesssim \|Ef\|_n \|f\|_n + \|f\|_n^2$$

when $pf = 0$ and $qf = 0$. Hence

$$|f|_n \lesssim \|Ef\|_n + \|f\|_n.$$

For any elliptic operator Ef the Dirichlet boundary conditions $f|\partial X$ are always coercive. Thus

$$\|f\|_n \lesssim \|Ef\|_{n-2} + |f|_{n-1/2} + \|f\|_0$$

by Gårding's inequality. But

$$\forall \varepsilon > 0, \exists C_\varepsilon, |f|_{n-1/2} \leqslant \varepsilon |f|_n + C_\varepsilon |f|_0.$$

Therefore

$$|f|_{n-1/2} \leqslant \varepsilon C(\|Ef\|_n + \|f\|_n) + C_\varepsilon |f|_0,$$
$$\|f\|_n \leqslant \varepsilon C(\|Ef\|_n + \|f\|_n) + C\|Ef\|_{n-2} + C\|f\|_0 + C_\varepsilon |f|_0.$$

If $\varepsilon C \leqslant 1/2$, we have

$$\|f\|_n \leqslant \varepsilon C\|Ef\|_n + C\|Ef\|_{n-2} + C\|f\|_0 + C_\varepsilon |f|_0.$$

But $\forall \varepsilon > 0, \exists C_\varepsilon$ with

$$\|Ef\|_{n-2} \leqslant \varepsilon \|Ef\|_n + C_\varepsilon \|Ef\|_0,$$

and $\|Ef\|_0 \leqslant C\|f\|_2$. Thus

$$\|f\|_n \leqslant \varepsilon C\|Ef\|_n + C_\varepsilon(\|f\|_2 + |f|_0).$$

But $\forall \eta > 0, \exists C_\eta$ with

$$\|f\|_2 + |f|_0 \leqslant \eta \|f\|_n + C_\eta \|f\|_0.$$

Choose η so small that $C_\varepsilon \eta \leqslant 1/2$. This makes η a function of ε, so we can write C_ε for C_η. Then $\forall \varepsilon > 0, \exists C_\varepsilon$ with

$$\|f\|_n \leqslant \varepsilon \|Ef\|_n + C_\varepsilon \|f\|_0$$

when $pf = 0$ and $qf = 0$. In particular

$$\|f\|_n \lesssim \|Ef\|_n + \|f\|_0,$$

when $pf = 0$ and $qf = 0$. (We need the better version with any $\varepsilon > 0$ at one point in discussing the spectral theory.)

It is now easy to obtain an estimate in the case of inhomogeneous boundary data. Recall that $pf, p'f, qf, q'f$ uniquely determine $f|\partial X$ and $\partial f/\partial n$. Therefore we can find two linear partial differential operators φ and ψ on ∂X,

$$\varphi: \mathcal{C}^\infty(\partial X; P) \oplus \mathcal{C}^\infty(\partial X; Q) \to \mathcal{C}^\infty(\partial X; F|\partial X),$$
$$\psi: \mathcal{C}^\infty(\partial X; P) \oplus \mathcal{C}^\infty(\partial X; P) \oplus \mathcal{C}^\infty(\partial X; Q) \oplus \mathcal{C}^\infty(\partial X; Q) \to \mathcal{C}^\infty(\partial X; F|\partial X),$$

where $\varphi(h, l)$ is of degree 0 in h and l, and $\psi(h, j, k, l)$ is of degree 1 in h and l and degree 0 in j and k such that if $f|\partial X = \varphi(h, l)$ and $\partial f/\partial n = \psi(h, j, k, l)$, then $pf = h$, $p'f = j$, $qf = k$, $q'f = l$. Now let T be a continuous linear extension

$$T: \mathcal{C}^\infty(\partial X; F|\partial X) \oplus \mathcal{C}^\infty(\partial X; F|\partial X) \to \mathcal{C}^\infty(X; F)$$

as described in §3, so that if $T(u, v) = f$, then $f|\partial X = u$ and $\partial f/\partial n = v$. Let $f = T(\varphi(h, l), \psi(h, j, k, l))$. Then $pf = h$, $p'f = j$, $qf = k$, $q'f = l$. Moreover for each $n \geqslant 2$

$$\|f\|_n \lesssim |h|_{n-1/2} + |j|_{n-3/2} + |k|_{n-3/2} + |l|_{n-1/2}.$$

If we choose for simplicity to lose half a derivative, then

$$\|f\|_n \lesssim |h|_n + |j|_{n-1} + |k|_{n-1} + |l|_n.$$

Suppose now that we are given an arbitrary f. Put

$$f' = T(\varphi(pf, 0), \psi(pf, 0, qf, 0)).$$

Then $pf' = pf$ and $qf' = qf$, while

$$\|f'\|_n \lesssim |pf|_n + |qf|_{n-1}.$$

Let $f = f' + f''$. Then $pf'' = 0$ and $qf'' = 0$. Applying the previous a-priori estimate to f'' we have

$$\|f''\|_n \lesssim \|Ef''\|_n + \|f''\|_0.$$

Then

$$\|f\|_n \lesssim \|Ef\|_n + |pf|_{n+2} + |qf|_{n+1} + \|f\|_0.$$

This proves the following.

Main a-priori estimate. *Let $\mathfrak{E}f = (Ef, pf, qf)$ be a self-adjoint elliptic boundary value problem. Suppose $\mathfrak{E}$ satisfies a persuasive estimate (or subelliptic inequality)*

$$|f|_0^2 \lesssim \langle\langle Ef, f\rangle\rangle + \|f\|_0^2,$$

when $pf = 0$ and $qf = 0$. Then for all f without restriction and all n

$$\|f\|_n \lesssim \|Ef\|_n + |pf|_{n+2} + |qf|_{n+1} + \|f\|_0.$$

Also we have the more precise estimate

$$\forall \varepsilon > 0, \exists C_\varepsilon, \forall f \text{ with } pf = 0 \text{ and } qf = 0,$$

$$\|f\|_n \leqslant \varepsilon\|Ef\|_n + C_\varepsilon\|f\|_0.$$

Remark. This is not the best possible estimate. With a little more work we could have proven

$$\|f\|_n \lesssim \|Ef\|_{n-1} + |pf|_{n+1/2} + |qf|_{n-1/2} + \|f\|_0.$$

However it is not clear that there is any advantage to justify the work involved.

3.6. The lemma of Kohn and Nirenberg

We now prove the lemma which we used in the last section. We state it again briefly for reference. Recall that ∇ is an operator of degree 1 with symbol $\sigma\,\nabla(\xi) = \xi(v)I$ for a vector field v tangent to ∂X.

Lemma (*Kohn-Nirenberg*). *If $Q(f, g)$ is an essentially hermitian symmetric bilinear form of degree* 1, *then for any n*

$$|\mathrm{Re}\, Q(\nabla^n f, \nabla^n f) - (-1)^n \,\mathrm{Re}\, Q(f, \nabla^{2n} f)| \lesssim \|f\|^2.$$

Proof. For any hermitian bilinear form $Q(f, g)$ of degree 1, there is (uniquely) defined another such form $\nabla Q(f, g)$ with

$$Q(\nabla f, g) + Q(f, \nabla g) = \nabla Q(f, g)$$

for all f and g. The coefficients of ∇Q are obtained by differentiating the coefficients of Q with the operator ∇ (in some sense). If Q is essentially hermitian symmetric, then so is ∇Q. Using the above relation we can transfer derivatives ∇ from one side to the other in $Q(\nabla^n f, \nabla^n f)$. In doing so we generate various terms

$$\nabla^i Q(\nabla^j f, \nabla^k f)$$

with $i + j + k = 2n$. We call $\max(j, k)$ the degree of the term, and i the rank.

Consider first the expression

$$Q(\nabla^{2n} f, f) - 2(-1)^n Q(\nabla^n f, \nabla^n f) + Q(f, \nabla^{2n} f).$$

We claim we can rewrite this as a sum of terms $\Sigma\, \nabla^i Q(\nabla^j f, \nabla^k f)$ of degree $\max(j, k) \leqslant n - 1$ and rank $2 \leqslant i \leqslant n + 1$. We have

$$\begin{aligned} &Q(\nabla^{2n} f, f) - 2(-1)^n Q(\nabla^n f, \nabla^n f) + Q(f, \nabla^{2n} f) \\ &\quad = \nabla Q(\nabla^{2n-1} f, f) - \nabla Q(\nabla^{2n-2} f, \nabla f) + \cdots \\ &\quad\quad - (-1)^n \nabla Q(\nabla^n f, \nabla^{n-1} f) - (-1)^n \nabla Q(\nabla^{n-1} f, \nabla^n f) + \cdots \\ &\quad\quad - \nabla Q(\nabla f, \nabla^{2n-2} f) + \nabla Q(f, \nabla^{2n-1} f) \\ &\quad = \nabla^2 Q(\nabla^{2n-2} f, f) - 2\nabla^2 Q(\nabla^{2n-3} f, \nabla f) + \cdots \\ &\quad\quad - n(-)^n \nabla^2 Q(\nabla^{n-1} f, \nabla^{n-1} f) + \cdots \\ &\quad\quad - 2\nabla^2 Q(\nabla f, \nabla^{2n-3} f) + \nabla^2 Q(\nabla^{2n-2} f, f). \end{aligned}$$

Thus we have reduced the expression to a sum of terms of rank $i \geqslant 2$. If we make a substitution.

$$\nabla^i Q(\nabla^j f, \nabla^k f) = -\nabla^i Q(\nabla^{j-1} f, \nabla^{k+1} f) + \nabla^{i+1} Q(\nabla^{j-1} f, \nabla^k f),$$

we will obtain terms of strictly lower degrees and equal or greater rank, provided $j \geqslant k + 2$. A similar remark applies if $k \geqslant j + 2$. Therefore we can reduce the expression to a sum of terms with $i \geqslant 2$ and either $\max(j, k) \leqslant n - 1$ or $|j - k| \leqslant 1$. Since $i + j + k = 2n$, these relations imply $\max(j, k) \leqslant n - 1$. Moreover we do not need to make such a reduction on a term unless either j or $k \geqslant n$. Therefore, if we only make such reductions when necessary, we only generate terms of rank $i \leqslant n+1$. This remark will be important later when we study uniform estimates in terms of the coefficients, as it guarantees that we do not differentiate the coefficient of Q more than $n+1$ times. Now any term of degree $\max(j, k) \leqslant n-1$ satisfies

$$|\nabla^i Q(\nabla^j f, \nabla^k f)| \lesssim \|f\|_n^2.$$

We would be done with the proof if Q were completely hermitian symmetric.

Since Q is essentially hermitian symmetric, the difference form

$$R(f, g) = Q(f, g) - \overline{Q(g, f)}$$

involves no product of first derivatives of f and g. Moreover we can write $R(f, g) = R_1(f, g) + R_2(f, g)$ where R_1 involves no derivatives of f, and R_2 involves no derivatives of g. If we apply the previous argument to Re Q, we have

$$|\operatorname{Re} Q(\nabla^{2n} f, f) - 2(-1)^n \operatorname{Re} Q(\nabla^n f, \nabla f) + \operatorname{Re} Q(f, \nabla^{2n} f)| \lesssim \|f\|_n^2.$$

We then have to estimate the difference

$$\operatorname{Re} Q(\nabla^{2n} f, f) - \operatorname{Re} Q(f, \nabla^{2n} f) = \tfrac{1}{2}\{R(\nabla^{2n} f, f) - R(f, \nabla^{2n} f)\}.$$

We estimate $R_1(\nabla^{2n} f, f) - R_1(f, \nabla^{2n} f)$; the estimate for R_2 is symmetrical. We can write

$$\begin{aligned} R_1(\nabla^{2n} f, f) - R_1(f, \nabla^{2n} f) = {} & \nabla R_1(\nabla^{2n-1} f, f) - \nabla R_1(\nabla^{2n-2} f, \nabla f) + \cdots \\ & + \nabla R_1(\nabla f, \nabla^{2n-2} f) - \nabla R_1(f, \nabla^{2n-1} f). \end{aligned}$$

Further rearrangements produce terms of the form $\nabla^i R_1(\nabla^j f, \nabla^k f)$. As before we can shift ∇ from one side to the other until in each term either $j \leqslant n$ and $k \leqslant n-1$ or else $j = k$ or $k+1$, and we can do so without producing a term of lower i than we start with. Since we have already reduced to $i \geqslant 1$, we continue to have $i \geqslant 1$. Then, if $j = k$ or $k+1$, we will have $j \leqslant n$ and $k \leqslant n-1$. Moreover if we only make reductions when necessary, then in the final terms we will have either $j = n$ or $k = n-1$ so $i \leqslant n$. Therefore we can rewrite the expression $R_1(\nabla^{2n} f, f) - R_1(f, \nabla^{2n} f)$ as a sum of terms $\nabla^i R_1(\nabla^j f, \nabla^k f)$ with $1 \leqslant i \leqslant n, j \leqslant n, k \leqslant n-1$. Since $R_1(f, g)$ has no derivatives on f we have

$$|R_1(\nabla^{2n} f, f) - R_1(f, \nabla^{2n} f)| \lesssim \|f\|_n^2.$$

The same holds for R_2. This proves that

$$|\operatorname{Re} Q(\nabla^n f, \nabla^n f) - (-1)^n \operatorname{Re} Q(f, \nabla^{2n} f)| \lesssim \|f\|_n^2$$

as claimed. Moreover we have shown that

$$\operatorname{Re} Q(\nabla^n f, \nabla^n f) - (-1)^n \operatorname{Re} Q(f, \nabla^{2n} f)$$

can be written as a sum of expressions $\nabla^i Q(\nabla^j f, \nabla^k f)$ and their conjugates with $2 \leqslant i \leqslant n+1$, $j \leqslant n-1$, $k \leqslant n-1$, and expressions $\nabla^i R_1(\nabla^j f, \nabla^k f)$ with $1 \leqslant i \leqslant n$, $j \leqslant n$, $k \leqslant n-1$, and expressions $\nabla^i R_2(\nabla^j f, \nabla^k f)$ with $1 \leqslant i \leqslant n$, $j \leqslant n-1$, $k \leqslant n$. We shall need the bounds on i later.

3.7. Elliptic regularization

Let $\mathcal{E} f = (Ef, pf, qf)$ be a self-adjoint elliptic boundary value problem which satisfies a persuasive estimate

$$|f|_0^2 \lesssim \operatorname{Re}\langle\langle Ef, f\rangle\rangle + \|f\|_0^2$$

when $pf = 0$ and $qf = 0$. Then we have the main a-priori estimate of §5

$$\|f\|_n \lesssim \|Ef\|_n + |pf|_{n+2} + |qf|_{n+1} + \|f\|_0.$$

It follows that if $f \in \operatorname{Ker} \mathcal{E}$, then $\|f\|_n \lesssim \|f\|_0$ for all n. Hence the unit ball $\|f\|_0 \leqslant 1$ is compact in $\operatorname{Ker} \mathcal{E}$, so $\operatorname{Ker} \mathcal{E}$ is finite dimensional. We wish to show that $\operatorname{Im} \mathcal{E}$ is closed and has finite equal codimension. For the moment we content ourselves with the following special case.

Theorem. *If $\mathcal{E}$ is injective, then it is also surjective.*

Proof. Let $v_1, \cdots, v_n$ be vector fields on X tangent to the boundary ∂X, such that every vector field tangent to the boundary is a linear combination of the v_j (with $\mathcal{C}^\infty$ coefficients). Let ∇_j be an operator with symbol $\sigma_{\nabla_j}(\xi) = \xi(v_j)I$. The norm

$$|||f|||_1^2 = \sum \|\nabla_j f\|_0^2 + \|f\|_0^2$$

is independent of the choice of v_j up to equivalence. Let n be a vector field on X which points outward at ∂X, with $n(\nu) = 1$, and let $\partial/\partial n$ be an operator with symbol $\sigma_{\partial/\partial n}(\xi) = \xi(n)I$. Thus $\partial/\partial n$ is a normal derivative. When we integrate by parts, we get

$$\langle\langle Ef, g\rangle\rangle = Q(f, g)$$

when $qf = 0$ and $pg = 0$; and the form Q has the form

$$Q(f, f) = \int_X \int \left\langle \sigma_E(\nu) \frac{\partial}{\partial n} f, \frac{\partial}{\partial n} f \right\rangle + \cdots dV,$$

where the dots denote terms with at most one normal derivative. Since $\sigma_E(\nu)$ is hermitian symmetric and positive definite, we must have

$$\left\| \frac{\partial f}{\partial n} \right\|^2 \lesssim \operatorname{Re} Q(f, f) + \left\| \frac{\partial f}{\partial n} \right\| \cdot |||f|||_1 + |||f|||_1^2,$$

which implies

$$\|f\|_1^2 \lesssim \operatorname{Re} Q(f, f) + |||f|||_1^2,$$

since $\|f\|_1 \lesssim \|\partial f/\partial n\| + |||f|||_1$. Then

$$\|f\|_1^2 \lesssim \mathrm{Re}\langle\langle Ef, f\rangle\rangle + |||f|||_1^2,$$

when $pf = 0$ and $qf = 0$. It follows that $\mathfrak{S}$ is coercive if we have

$$|||f|||_1^2 \lesssim \mathrm{Re}\langle\langle Ef, f\rangle\rangle + \|f\|_0^2,$$

when $pf = 0$ and $qf = 0$. Let ∇_j^* be the adjoint of ∇_j, and put $C = -\sum \nabla_j^* \nabla_j + I$. Then $\langle\langle Cf, f\rangle\rangle = \sum\|\nabla_j f\|_0^2 + \|f\|_0^2 = |||f|||_1^2$. Moreover $\sigma_C(\xi) = \sum \xi(v_j)^2 I$, so $\sigma_C(\nu) = 0$ and $D\sigma_C(\nu; \eta) = 0$ for all η, because $\nu(v_j) = 0$ since the v_j are tangent to the boundary.

Therefore for any ε with $0 \leqslant \varepsilon \leqslant 1$

$$\mathfrak{S}_\varepsilon f = (Ef + \varepsilon Cf, pf, qf)$$

is a self-adjoint elliptic boundary value problem satisfying (1)–(5) of §1. If $\mathfrak{S}$ satisfies a persuasive estimate

$$|f|_0^2 \lesssim \mathrm{Re}\langle\langle Ef, f\rangle\rangle + \|f\|_0^2$$

(when $pf = 0$ and $qf = 0$), then the persuasive estimate

$$|f|_0^2 \lesssim \mathrm{Re}\langle\langle Ef + \varepsilon Cf, f\rangle\rangle + \|f\|_0^2$$

(when $pf = 0$ and $qf = 0$) holds uniformly in $0 \leqslant \varepsilon \leqslant 1$, i.e., with constants independent of ε. Consequently the main a-priori estimate

$$\|f\|_n \lesssim \|Ef + \varepsilon Cf\|_n + |pf|_{n+2} + |qf|_{n+1} + \|f\|_0$$

will also hold uniformly for $0 \leqslant \varepsilon \leqslant 1$.

However, for any $\varepsilon > 0$ we have

$$|||f|||_1^2 \lesssim \mathrm{Re}\langle\langle Ef + \varepsilon Cf, f\rangle\rangle + \|f\|_0^2,$$

when $pf = 0$ and $qf = 0$. Thus $\mathfrak{S}_\varepsilon$ is a coercive problem for $\varepsilon > 0$.

Suppose now that $\mathfrak{S}$ is injective and satisfies a persuasive estimate. We claim that $\mathfrak{S}_\varepsilon$ will also be injective for all ε sufficiently small. For suppose not. Then we can find a sequence $\varepsilon_k \to 0$ and $f_k \in \mathrm{Ker}\, \mathfrak{S}_{\varepsilon_k}$ with $\|f_k\|_0 = 1$. From the uniform main a-priori estimate we have $\|f_k\|_n \leqslant C\|f_k\|_0 = C$ with a constant independent of k. By passing to a subsequence we may assume $f_k \to f$. Then by continuity $f \in \mathrm{Ker}\, \mathfrak{S}$, and $\|f\|_0 = 1$ so $f \neq 0$, which is a contradiction. Therefore $\mathfrak{S}_\varepsilon$ is injective for ε sufficiently small.

But $\mathfrak{S}_\varepsilon$ is coercive, so by §4 it is also surjective. Let $\varepsilon_k \to 0$ be a sequence and let f_k be the unique solution of

$$\begin{aligned} Ef_k + \varepsilon_k Cf_k &= g, \quad \text{on } X,\\ pf_k &= h, \quad \text{on } \partial X,\\ qf_k &= j, \quad \text{on } \partial X, \end{aligned}$$

for any given g, h, j. From the uniform main estimate

$$\|f_k\|_n \leqslant C(\|g\|_n + |h|_{n+2} + |j|_{n+1} + \|f_k\|_0)$$

with a constant independent of k.

We claim $\|f_k\|_0$ is bounded. For if not, by passing to a subsequence we would have $\|f_k\|_0 \to \infty$. Then put $\tilde{f}_k = f_k/\|f_k\|_0$. We would have

$$\|\tilde{f}_k\|_n \leqslant C(\|g\|_n + |h|_{n+2} + |j|_{n+1})/\|f_k\|_0 + C,$$

so $\|\tilde{f}_k\|_n \leqslant C$. Then by passing to another subsequence we would have $\tilde{f}_k \to \tilde{f}$. Then $E\tilde{f} = \lim E\tilde{f}_k + \varepsilon_k C\tilde{f}_k = \lim g/\|f_k\|_0 = 0$, and likewise $p\tilde{f} = 0$ and $q\tilde{f} = 0$. But $\|\tilde{f}_k\|_0 = 1$ so $\|\tilde{f}\|_0 = 1$. This contradicts the hypothesis that $\mathcal{E}$ is injective. Therefore the $\|f_k\|_0$ are bounded.

Inserting this in the uniform main estimate, we have $\|f_k\|_n \leqslant C$ for all n, independent of k. Therefore, by passing to a subsequence, we have $f_k \to f$. Then by continuity

$$Ef = g, pf = h, qf = j.$$

This proves $\mathcal{E}$ is also surjective.

3.8. Spectral theory

Let $\mathcal{E}f = (Ef, pf, qf)$ be a self-adjoint elliptic boundary value problem. Then for every complex number $\lambda \in \mathbf{C}$ so is $\mathcal{E}_\lambda f = (f + \lambda f, pf, qf)$. We say that λ belongs to the spectrum Σ of $\mathcal{E}$ if $\mathcal{E}_\lambda$ is not invertible.

Theorem. *If $\mathcal{E}$ satisfies a persuasive estimate*

$$|f|_0^2 \lesssim \langle\langle Ef, f\rangle\rangle + \|f\|_0^2$$

when $pf = 0$ and $qf = 0$, then its spectrum Σ consists of a set of isolated points.

Proof. First note that if $\operatorname{Re}\lambda$ is sufficiently large, then when $pf = 0$ and $qf = 0$,

$$|f|_0^2 + \|f\|_0^2 \lesssim \operatorname{Re}\langle\langle Ef + \lambda f, f\rangle\rangle.$$

Thus $\mathcal{E}_\lambda$ is injective if $\operatorname{Re}\lambda$ is large. Hence the spectrum Σ lies in a half plane $\operatorname{Re}\lambda < \mu$.

Since $\mathcal{E}_\lambda$ is always invertible for some λ, it suffices to prove that Σ is a set of isolated points when $\mathcal{E}$ is invertible. Write $\mathcal{I}f = (f, 0, 0)$. Then $\mathcal{E}_\lambda = \mathcal{E} + \lambda\mathcal{I}$. Let $T_\lambda = \mathcal{E}^{-1} \circ \mathcal{E}_\lambda$, and $T = \mathcal{E}^{-1}\mathcal{I}$. Then

$$T_\lambda = \mathcal{E}^{-1} \circ (\mathcal{E} + \lambda\mathcal{I}) = \mathcal{I} + \lambda T.$$

Notice that $Tg = f$ is the unique solution of $Ef = g$, $pf = 0$, $qf = 0$. The operator T is a map T: $\mathcal{C}^\infty(X; F) \to \mathcal{C}^\infty(X; F)$ which by (the more precise version of) the main a-priori estimate satisfies

$$\forall \varepsilon > 0, \quad \exists C_\varepsilon, \quad \forall G$$
$$\| Tg \|_n \leqslant \varepsilon \| g \|_n + C_\varepsilon \| g \|_0.$$

Fix any suitably large value of n, and let $L_2^n(X; F)$ denote the completion of $\mathcal{E}^\infty(X; F)$ in the norm $\| \ \|_n$. By the previous estimate, T extends to a continuous linear map

$$\tilde{T}: L_2^n(X; F) \to L_2^n(X; F),$$

and in fact $\tilde{T}$ is compact. Therefore by the classical Riesz theory, the operator $\tilde{T}_\lambda = I + \lambda \tilde{T}$ is invertible for all but a discrete set $\tilde{\Sigma}$ of numbers $\lambda \in \mathbf{C}$. Note that $\Sigma \subseteq \tilde{\Sigma}$. For if $\mathcal{E}_\lambda$ is not invertible, then it is not injective, and if f_λ lies in the kernel of $\mathcal{E}_\lambda$, then f_λ is smooth (by definition since $\mathcal{C}^\infty(X; F)$ is the domain of $\mathcal{E}_\lambda$) and hence $f_\lambda \in L_2^n(X; F)$ all the more; and

$$\tilde{T}_\lambda f_\lambda = T_\lambda f_\lambda = \mathcal{E}^{-1} \mathcal{E}_\lambda f_\lambda = 0,$$

so f_λ also lies in the kernel of $\tilde{T}_\lambda$ and $\tilde{T}_\lambda$ is not invertible either. This proves that the spectrum Σ of $\mathcal{E}$ is discrete.

The harmonic space H_λ is defined as $H_\lambda = \ker \mathcal{E}_\lambda$, or

$$H_\lambda = \{ h: Eh + \lambda h = 0, ph = 0, qh = 0 \}.$$

We define the eigenspace $\hat{H}_\lambda$ as the smallest subspace such that

$$Eh + \lambda h \in \hat{H}_\lambda, pf = 0, qf = 0 \Rightarrow h \in \hat{H}_\lambda.$$

Theorem. *If $\mathcal{E}$ satisfies a persuasive estimate, then each $\hat{H}_\lambda$ is finite dimensional.*

Proof. Again by a translation we may assume $\mathcal{E}$ is invertible, and set $T_\lambda = I + \lambda T = \mathcal{E}^{-1} \mathcal{E}_\lambda$ with $T = \mathcal{E}^{-1} \mathcal{G}$. Recall that T extends to a compact linear map $\tilde{T}: L_2^n(X; F) \to L_2^n(X; F)$. By the Riesz theory, each eigenspace

$$\tilde{K}_\lambda = \{ f \in L_2^n(X; F): \tilde{T}_\lambda^k f = 0 \quad \text{for some } k \}$$

is finite dimensional. Let

$$\bar{K}_\lambda = \{ f \in \mathcal{C}^\infty(X; F): T_\lambda^k f = 0 \text{ for some } k \}.$$

Then clearly $K_\lambda \subseteq \tilde{K}_\lambda$ and hence is also finite dimensional. We claim $\hat{H}_\lambda \subseteq \hat{K}_\lambda$. For if $Ef + \lambda f \in \hat{K}_\lambda, pf = 0, qf = 0$, then $\mathcal{E}_\lambda f = (Ef + \lambda f, pf, qf) \in \mathcal{G} \hat{K}_\lambda$, so $\mathcal{E}^{-1} \mathcal{E}_\lambda f \in \mathcal{E}^{-1} \mathcal{G} \hat{K}_\lambda$ or $T_\lambda f \in T \hat{K}_\lambda$. But $T(\hat{K}_\lambda) \subseteq \hat{K}_\lambda$ so $T_\lambda f \in \hat{K}_\lambda$, which implies $f \in \hat{K}_\lambda$. Thus $\hat{K}_\lambda$ has the property for which H_λ is minimal, so $\hat{H}_\lambda \subseteq \hat{K}_\lambda$. Thus $\hat{H}_\lambda$ is also finite dimensional.

The complement of the spectrum Σ is called the resolvent set $\Sigma^c = \mathbf{C} - \Sigma$. It is open and omits a discrete set of points. $\mathcal{E}_\lambda$ is invertible for all $\lambda \in \Sigma^c$.

Lemma. *If $\mathfrak{E}$ is invertible, then for all n sufficiently large*

$$\|f\|_n \lesssim \|Ef\|_n + |pf|_{n+2} + |qf|_{n+1}.$$

Proof. We already know

$$\|f\|_n \lesssim \|Ef\|_n + |pf|_{n+2} + |qf|_{n+1} + \|f\|_0.$$

Therefore it suffices to show that

$$\|f\|_0 \lesssim \|Ef\|_k + |pf|_{k+2} + |qf|_{k+1}$$

for some k. If not, we can find a sequence f_j with $\|f_j\|_0 = 1$, but $Ef_j \to 0$, $pf_j \to 0$ and $qf_j \to 0$. Then $\|f_j\|_n \leqslant C$ for all n, so by passing to a subsequence we have $f_j \to f$ for some f. Then $Ef = 0$, $pf = 0$, $qf = 0$, so $f \in \ker \mathfrak{E}$. But $\|f\|_0 = 1$, so $f \neq 0$. This contradicts the hypothesis that $\mathfrak{E}$ is invertible.

Theorem. *The resolvent*

$$R\colon \Sigma^c \times \{\mathcal{C}^\infty(X; F) \oplus \mathcal{C}^\infty(\partial X; P) \oplus \mathcal{C}^\infty(\partial X; Q)\} \to \mathcal{C}^\infty(X; F)$$

defined by

$$R(\lambda)(g, h, k) = \mathfrak{E}_\lambda^{-1}(g, h, k)$$

is complex-analytic in λ and linear in (g, h, k).

Proof. First we claim that R is continuous. It is sufficient to prove that if $0 \in \Sigma^c$, then R is continuous in a neighborhood of 0. By the previous Lemma

$$\|f\|_n \leqslant C(\|Ef\|_n + |pf|_{n+2} + |qf|_{n+1}).$$

Therefore

$$\|f\|_n \leqslant C(\|Ef + \lambda f\|_n + |pf|_{n+2} + |qf|_{n+1}) + C\lambda\|f\|_n.$$

When $|\lambda| < \varepsilon$ with $\varepsilon = \frac{1}{2}C$,

$$\|f\|_n \leqslant C(\|Ef + \lambda f\|_n + |pf|_{n+2} + |qf|_{n+1})$$

with a constant independent of λ. Then

$$\|R(\lambda)(g, h, k)\|_n \leqslant C(\|g\|_n + |h|_{n+2} + |k|_{n+1})$$

with a constant independent of λ for all n sufficiently large. Now

$$\mathfrak{E}_\lambda\{R(\lambda) - R(\mu)\}\mathfrak{E}_\mu = \mathfrak{E}_\mu - \mathfrak{E}_\lambda = (\mu - \lambda)\mathfrak{I},$$

where $\mathfrak{I}(f) = (f, 0, 0)$. Thus

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)\mathfrak{I}R(\mu),$$

which together with the previous estimate proves R is continuous. Moreover

$$R(\lambda + \theta) - R(\lambda) = -\theta R(\lambda + \theta)\mathfrak{I}R(\lambda).$$

Therefore

$$DR(\lambda) = \lim_{\theta\to 0}[R(\lambda + \theta) - R(\lambda)]/\theta, \quad DR(\lambda) = -R(\lambda)\mathfrak{I}R(\lambda).$$

Hence R is continuously differentiable in the complex sense, so surely R is analytic. (All these formulas are to be understood as being applied to a fixed (g, h, k) which we omit from the notation for simplicity.)

Now assume $\mathfrak{G}$ is not invertible, and let γ be any path in the resolvent set Σ^c enclosing the origin. Let

$$N = \int_\gamma R(\lambda)\frac{d\lambda}{\lambda}, \quad \sigma = \int_\gamma R(\lambda)\, d\lambda.$$

Lemma. *We have the following relations*:

$$\begin{aligned} N\mathfrak{G} + \sigma\mathfrak{J} &= I, & N\mathfrak{G}N &= N,\\ \mathfrak{G}N + \mathfrak{J}\sigma &= I, & \sigma\mathfrak{J}\sigma &= \sigma,\\ \sigma\mathfrak{J}N &= 0, & \sigma\mathfrak{G}N &= 0,\\ N\mathfrak{J}\sigma &= 0, & N\mathfrak{G}\sigma &= 0. \end{aligned}$$

As a consequence if $\pi = \sigma\mathfrak{J}$ *and* $\rho = \mathfrak{J}\sigma$, *then* π *and* ρ *are projections*

$$\pi^2 = \pi, \quad \rho^2 = \rho,$$

such that $\operatorname{Ker} \mathfrak{G} \subseteq \operatorname{Im} \pi$ *and* $\operatorname{Ker} \rho \subseteq \operatorname{Im} \mathfrak{G}$.

Proof. The first two are trivial. For example

$$\begin{aligned} N\mathfrak{G} + \sigma\mathfrak{J} &= \int_\gamma R(\lambda)\mathfrak{G}\frac{d\lambda}{\lambda} + \int_\gamma R(\lambda)\mathfrak{J}\, d\lambda\\ &= \int_\gamma R(\lambda)[\mathfrak{G} + \lambda\mathfrak{J}]\frac{d\lambda}{\lambda} = \int_\gamma \frac{d\lambda}{\lambda} = I. \end{aligned}$$

The others follow from the identity

$$R(\lambda)\mathfrak{J}R(\mu) = \frac{1}{\lambda - \mu}\{R(\mu) - R(\lambda)\}.$$

Let γ' be a curve close to γ but inside, let $\lambda \in \gamma$ and $\mu \in \gamma'$. Then for example

$$\begin{aligned} \sigma\mathfrak{J}\sigma &= \int_\gamma\int_{\gamma'} R(\lambda)\mathfrak{J}R(\mu)\, d\mu\, d\lambda\\ &= \int_\gamma\int_{\gamma'} \frac{1}{\lambda - \mu}\{R(\mu) - R(\lambda)\}\, d\mu\, d\lambda = \int_{\gamma'} R(\mu)\, d\mu = \sigma, \end{aligned}$$

since $\int_\gamma d\lambda/(\lambda - \mu) = 1$ and $\int_{\gamma'} d\mu/(\lambda - \mu) = 0$. The others are proved in the same way.

Lemma. *If* γ *is a path in the resolvent set enclosing the origin, then*

$$pN(g, h, k) = h, \quad qN(g, h, k) = k.$$

Consequently

$$pN\mathfrak{J} = 0, \quad qN\mathfrak{J} = 0.$$

Also

$$p\sigma = 0, \quad q\sigma = 0.$$

Proof. Observe that from the definition of $R(\lambda)$

$$pR(\lambda)(g, h, k) = h, \quad qR(\lambda)(g, h, k) = k.$$

Now $N = \int_\gamma R(\lambda)d\lambda/\lambda$ and $\sigma = \int_\gamma R(\lambda)\, d\lambda$. The result is immediate.

Lemma. $\hat{H}_0 \subseteq \operatorname{Im} \pi$.

Proof. Recall that $\hat{H}_0$ is the smallest subspace such that

$$Ef \in \hat{H}_0, \quad pf = 0, \quad qf = 0 \Rightarrow f \in \hat{H}_0.$$

Suppose f is such that

$$Ef \in \operatorname{Im} \pi, \quad pf = 0, \quad qf = 0.$$

Then $\mathcal{E}f = \mathcal{G}g$ where $g = Ef$, so $\pi g = g$. Then using the previous relations

$$N\mathcal{E}f = N\mathcal{G}g = N\mathcal{G}\pi g = N\mathcal{G}\sigma\mathcal{G}g = 0, \quad f = N\mathcal{E}f + \pi f = \pi f,$$

so $f \in \operatorname{Im} \pi$ also. Hence $\operatorname{Im} \pi$ has the property for which $\hat{H}_0$ is minimal, so $\check{H}_0 \subseteq \operatorname{Im} \pi$.

Theorem. *If γ contains no eigenvalue other than* 0, *then* $\hat{H}_0 = \operatorname{Im} \pi$.

Proof. Choose a subspace S complementary to $\hat{H}_0$; this is surely possible since $\dim \hat{H}_0 < \infty$. Then $\mathcal{C}^\infty(X; F) = S \oplus \hat{H}_0$. We write $f \in \mathcal{C}^\infty(X; F)$ as $f = s + h$ with $s \in S$ and $h \in \hat{H}_0$. The quotient semi-norm $\|h/S\|_k = \inf\{\|s + h\|_k: s \in S\}$ is in fact a norm, and all norms on a finite dimensional space are equivalent, so $\|h\|_k \lesssim \|s + h\|_k$. Then also $\|s\|_k \lesssim \|s + h\|_k$.

Lemma. *For some m and all λ sufficiently small, if $h \in \hat{H}_0$, then*

$$\|h\|_0 \lesssim \lambda^{-m}\|Eh + \lambda h\|_0.$$

Proof. E takes the finite dimensional space $\hat{H}_0$ into itself. Therefore the above estimate holds for all λ sufficiently small, i.e., if $m = \dim \hat{H}_0$ since $\det(E + \lambda I) = \lambda^m$ on H_0.

From the main a-priori inequality

$$\|f\|_n \lesssim \|Ef\|_n + |pf|_{n+2} + |qf|_{n+1} + \|f\|_0.$$

This implies that for all λ sufficiently small

$$\|f\|_n \lesssim \|Ef + \lambda f\|_n + |pf|_{n+2} + |qf|_{n+1} + \|f\|_0$$

with a constant independent of λ.

Lemma. *For all λ sufficiently small and n sufficiently large,*

$$\|s\|_n \lesssim \|Es + \lambda s\|_n + |ps|_{n+2} + |qs|_{n+1},$$

when $s \in S$, with a constant independent of λ.

Proof. By the previous estimate it is only necessary to show that for some k

$$\|s\|_0 \lesssim \|Es + \lambda s\|_k + |ps|_{k+2} + |qs|_{k+1},$$

then the Lemma will hold for $n \geqslant k$. If this were false, we could find sequences $s_j \in S$ and $\lambda_j \to 0$ with $\|s_j\|_0 = 1$, $Es_j + \lambda_j s_j \to 0$, $ps_j \to 0$ and $qs_j \to 0$. By the previous estimate we would have $\|s_j\|_n \leqslant C$ for all n. Hence passing to a subsequence we would have $s_j \to s \in S$. Then by continuity $Es = 0$, $ps = 0$, $qs = 0$ so $s \in \hat{H}_0$ also. But $S \cap \hat{H}_0 = \{0\}$ so $s = 0$. But $\|s_j\|_0 = 1$ so $\|s\|_0 = 1$. This is a contradiction. Hence the lemma must be true.

Lemma. *For all λ sufficiently small and for some k,*

$$\|s\|_0 + \|h\|_0 \lesssim \|Es + \lambda s + h\|_k + |ps|_{k+2} + |qs|_{k+1}$$

for all $s \in S$ and $h \in \hat{H}_0$, with a constant independent of λ.

Proof. Suppose not. Then we could find sequences $\lambda_j \to 0$, $s_j \in S$, $h_j \in \check{H}_0$ with $\|s_j\|_0 + \|h_j\|_0 = 1$ and

$$Es_j + \lambda_j s_j + h_j \to 0, \quad ps_j \to 0, \quad qs_j \to 0.$$

Since $\hat{H}_0$ is finite dimensional and $\|h_j\|_0 \leqslant 1$, by passing to a subsequence we may assume $h_j \to h \in \hat{H}_0$. Then $Es_j + \lambda_j s_j \to h$ also, so $Es_j + \lambda s_j$, ps_j, qs_j are all bounded. By the previous Lemma we have $\|s_j\|_n \leqslant C$ for all n. Then by passing to another subsequence we may assume $s_j \to s \in S$. By continuity $Es = h$, $ps = 0$, $qs = 0$. Since $h \in \hat{H}_0$, this implies $s \in \hat{H}_0$ from its definition. But $S \cap \hat{H}_0 = \{0\}$, so $s = 0$. Then $h = 0$ also. But $\|s_j\|_0 + \|h_j\|_0 = 1$, so $\|s\|_0 + \|h\|_0 = 1$. This is a contradiction. Hence the Lemma must be true.

We apply this Lemma not with s and h but with s and $Eh + \lambda h$, for if $h \in \hat{H}_0$, then $Eh + \lambda h \in \hat{H}_0$ also. Thus for all λ sufficiently small and for some k,

$$\|s\|_0 + \|Eh + \lambda h\|_0 \lesssim \|Es + Eh + \lambda h\|_k + |ps|_{k+2} + |qs|_{k+1}.$$

Now given $f \in \mathcal{C}^\infty(X; F)$ write $f = s + h$ with $s \in S$ and $h \in \hat{H}_0$. We saw before that $\|s\|_k \lesssim \|f\|_k$. Since $h \in \hat{H}_0$, we have $ph = 0$ and $qh = 0$, so $pf = ps$ and $qf = qs$. Also $Es + Eh + \lambda h = Ef + \lambda f - \lambda s$. Therefore for all λ sufficiently small and for some k,

$$\|s\|_0 + \|Eh + \lambda h\|_0 \lesssim \|Ef + \lambda f\|_k + \lambda\|f\|_k + |pf|_{k+2} + |qf|_{k+1}.$$

Also $\|h\|_0 \lesssim \lambda^{-m}\|Eh + \lambda h\|_0$,

$$\|f\|_n \lesssim \|Ef + \lambda f\|_n + |pf|_{n+2} + |qf|_{n+1} + \|f\|_0,$$

and $\|f\|_0 \leqslant \|s\|_0 + \|h\|_0$. Combining these, when λ is sufficiently small (so we can ignore $C\lambda\|f\|_k$ by subtraction when $C\lambda \leqslant 1/2$) and n is sufficiently large (i.e., $n \geqslant k$) we have

$$\|f\|_n \lesssim \lambda^{-m}\{\|Ef + \lambda f\|_n + |pf|_{n+2} + |qf|_{n+1}\}.$$

Let us write

$$\|(g, h, j)\|_n = \|g\|_n + |h|_{n+2} + |j|_{n+1}.$$

Then we have shown the following.

Lemma. *For all λ sufficiently small, all n sufficiently large and some m we have*

$$\|f\|_n \leqslant C\lambda^{-m}\|\mathcal{E}_\lambda f\|_n.$$

Recall that $R(\lambda) = \mathcal{E}_\lambda^{-1}$. The above estimate shows that

$$\|R(\lambda)(g, h, j)\|_n \leqslant C\lambda^{-m}\|(g, h, j)\|_n$$

for all λ sufficiently small. Therefore when m is large enough, $\lambda^m R(\lambda)$ is continuous at $\lambda = 0$. Hence by the removable singularities theorem it is analytic at 0. This proves the following result.

Theorem. $R(\lambda)$ *has only poles for singularities. If m is large enough, then* $\lambda^m R(\lambda)$ *is analytic at* 0, *and if γ is a curve enclosing no point in the spectrum except* 0, *then*

$$\int_\gamma \lambda^m R(\lambda)\, d\lambda = 0.$$

This accomplished, we return to the proof that $\hat{H}_0 = \operatorname{Im} \pi$. Let

$$\sigma^m = \int_\gamma \lambda^m R(\lambda) d\lambda.$$

By the previous theorem, $\sigma^m = 0$ when m is large enough. Also $\sigma^0 = \sigma$ defined before. Let $\pi^m = \sigma^m \mathcal{J}$. Then $\pi^m = 0$ when m is large enough and $\pi^0 = \pi$ defined before.

Lemma. $E\pi^m + \pi^{m+1} = 0$ *for all m.*

Proof.

$$\begin{aligned} E\pi^m + \pi^{m+1} &= \int_\gamma E\lambda^m R(\lambda)\mathcal{J} d\lambda + \int_\gamma \lambda^{m+1} R(\lambda)\mathcal{J}\, d\lambda \\ &= \int_\gamma \lambda^m (E + \lambda I) R(\lambda)\mathcal{J}\, d\lambda = \int_\gamma \lambda^m\, d\lambda = 0, \end{aligned}$$

since $(E + \lambda I)R(\lambda)\mathcal{J} = I$.

Corollary. $E^m\pi = (-1)^m\pi^m$ *for all m.*

Lemma. *For all m*

$$pE^m\pi = 0, \qquad qE^m = 0.$$

Proof. We have

$$pR(\lambda)\mathcal{G} = 0, \qquad qR(\lambda)\mathcal{G} = 0$$

from the definition of $R(\lambda)$ and $\mathcal{G}$. Therefore

$$p\pi^m = \int_\gamma \lambda^m pR(\lambda)\mathcal{G}\, d\lambda = 0,$$

$$q\pi^m = \int_\gamma \lambda^m qR(\lambda)\mathcal{G}\, d\lambda = 0.$$

But $E^m\pi = (-1)^m\pi^m$, so the result follows.

From the results we see that if $h \in \operatorname{Im} \pi$, then $pE^m h = 0$ and $qE^m h = 0$ for all m, and $E^m h = 0$ for some m sufficiently large. Therefore $h \in \hat{H}_0$. This proves $\hat{H}_0 = \operatorname{Im} \pi$.

Now we characterize $\operatorname{Im} \rho$.

Lemma. *The map $\mathcal{G}$ defines by restriction an isomorphism*

$$\mathcal{G} : \operatorname{Im} \pi \to \operatorname{Im} \rho.$$

Thus $\operatorname{Im} \rho = \{(h, 0, 0)\colon h \in \hat{H}_0\}$.

Proof. Since $\mathcal{G}\pi = \rho\mathcal{G}$, we have $\mathcal{G}(\operatorname{Im} \pi) \subseteq \operatorname{Im} \rho$. Since $\mathcal{G}$ is one-to-one, so is its restriction. Since $\rho = \mathcal{G}\sigma$, if $(g, h, k) \in \operatorname{Im} \rho$, then $(g, h, k) \in \operatorname{Im} \mathcal{G}$ so $h = 0$ and $k = 0$. Moreover $(g, 0, 0) = \rho(g, 0, 0) = \rho\mathcal{G}g = \mathcal{G}\pi g$ so $\operatorname{Im} \rho \subseteq \mathcal{G}(\operatorname{Im} \pi)$. Thus $\mathcal{G} : \operatorname{Im} \pi \to \operatorname{Im} \rho$ is an isomorphism as claimed. It follows that $\operatorname{Im} \rho$ is finite-dimensional, and in fact has the same dimension as $\operatorname{Im} \pi = \hat{H}_0$.

Corollary. *If $\mathcal{E}$ is a self-adjoint elliptic boundary value problem which satisfies a persuasive estimate, then $\mathcal{E}$ has closed range with finite codimension.*

We stated this result back in §7. Since $\mathcal{E}N + \rho = I$, we can solve $\mathcal{E}f = (g, h, k)$ by $f = N(g, h, k)$ if $\rho(g, h, k) = 0$. Since $\operatorname{Im} \rho$ has finite dimension, $\operatorname{Ker} \rho$ is closed with finite codimension, and $\operatorname{Im} \mathcal{E} \supseteq \operatorname{Ker} \rho$ (it may be larger).

Moreover $E(\hat{H}_0) \subseteq \hat{H}_0$ so $E(\operatorname{Im} \pi) \subseteq \operatorname{Im} \pi$. Therefore $E\pi = \pi E\pi$ and

$$\mathcal{E}\pi = \mathcal{G}E\pi = \mathcal{G}\pi E\pi = \rho\mathcal{G}E\pi = \rho\mathcal{E}\pi,$$

which shows $\mathcal{E}$ maps $\operatorname{Im} \pi$ into $\operatorname{Im} \rho$:

$$\mathcal{E} : \operatorname{Im} \pi \to \operatorname{Im} \rho.$$

Moreover $\operatorname{Ker} \mathcal{E} \subseteq \operatorname{Im} \pi$ and $\operatorname{Im} \mathcal{E} \supseteq \operatorname{Ker} \rho$. Therefore

$$\operatorname{index} \mathcal{E} : \mathcal{C}^\infty(X; F) \to \mathcal{C}^\infty(X; F) \oplus \mathcal{C}^\infty(\partial X; P) \oplus \mathcal{C}^\infty(\partial X; Q)$$

must be the same as the index of

$$\mathcal{E} : \operatorname{Im} \pi \to \operatorname{Im} \rho.$$

But this index is 0 since $\operatorname{Im} \pi$ and $\operatorname{Im} \rho$ have the same dimension, being isomorphic under $\mathcal{G}$. Therefore we have shown the last part of the theorem in §7.

Corollary. *$\mathcal{E}$ has index* 0, *i.e.*,

$$\dim \operatorname{Ker} \mathcal{E} = \operatorname{codim} \operatorname{Im} \mathcal{E}.$$

We have a decomposition of $\mathcal{C}^\infty(X; F)$ as a direct sum

$$\mathcal{C}^\infty(X; F) = \operatorname{Im} \pi \oplus \operatorname{Ker} \pi.$$

We have already seen that $\operatorname{Im} \pi = \hat{H}_0$ is the smallest subspace such that

$$Ef \in \hat{H}_0, pf = 0, qf = 0 \Rightarrow f \in \hat{H}_0.$$

We define $\hat{K}_0$ to be the largest subspace such that

$$g \in \hat{K}_0 \Rightarrow \exists f \in \hat{K}_0 \quad \text{with } Ef = g, pf = 0, qf = 0.$$

Theorem. $\operatorname{Ker} \pi = \hat{K}_0$.

Proof. $\operatorname{Ker} \pi = \operatorname{Im} N\mathcal{E}$ which is the complementary projection. Moreover

$$\mathcal{E} R(\lambda) \mathcal{G} = \mathcal{G} - \lambda \mathcal{G} R(\lambda) \mathcal{G} = \mathcal{G} R(\lambda) \mathcal{E},$$

so

$$\mathcal{E} N \mathcal{G} = \int_\gamma \mathcal{E} R(\lambda) \mathcal{G} \frac{d\lambda}{\lambda} = \int_\gamma \mathcal{G} R(\lambda) \mathcal{E} \frac{d\lambda}{\lambda} = \mathcal{G} N \mathcal{E}.$$

Let $g \in \operatorname{Ker} \pi$. Then $g = N\mathcal{E}g$, so $\mathcal{G}g = \mathcal{G}N\mathcal{E}g = \mathcal{E}N\mathcal{G}g$. Put $f = N\mathcal{G}g$. Then $\pi f = 0$ and $\mathcal{E}f = \mathcal{G}g$. Hence $g \in \operatorname{Ker} \pi \Rightarrow \exists f \in \operatorname{Ker} \pi$ with $Ef = g$, $pf = 0$, $qf = 0$. This shows $\operatorname{Ker} \pi$ has the property for which $\hat{K}_0$ is maximal, so $\operatorname{Ker} \pi \subseteq \hat{K}_0$.

Unless $\operatorname{Ker} \pi = \hat{K}_0$ we would have a nonzero $f^0 \in \hat{H}_0 \cap \hat{K}_0$. Then we can solve for an $f^1 \in \hat{K}_0$ with $Ef^1 = f^0$, $pf^1 = 0$, $qf^1 = 0$, and then $f^1 \in \hat{H}_0$ as well, by the definition of $\hat{H}_0$, so $f^1 \in \hat{H}_0 \cap \hat{K}_0$. Continuing in this way we find a sequence $f^m \in \hat{H}_0 \cap \hat{K}_0$ with $Ef^m = f^{m-1}$. Then $E^m f = f^0$. But E takes $\hat{H}_0$ into itself, and the restriction $E: \hat{H}_0 \to \hat{H}_0$ is nilpotent, i.e., $E^m | \hat{H}_0 = 0$ for large m. Thus $f^0 = 0$.

Since there is nothing special about the eigenvalue 0, we have the following general result.

Corollary. *Let $\hat{H}_\lambda$ be the smallest subspace such that*

$$Ef + \lambda f \in \hat{H}_\lambda, pf = 0, qf = 0 \Rightarrow f \in \hat{H}_\lambda,$$

and let $\hat{K}_\lambda$ be the largest subspace such that

$$g \in \hat{K}_\lambda \Rightarrow \exists f \in \hat{K}_\lambda \quad \textit{with} \quad Ef + \lambda f = g, pf = 0, qf = 0.$$

Then $\mathcal{C}^\infty(X; F) = \hat{H}_\lambda \oplus \hat{K}_\lambda$ and $\dim \hat{H}_\lambda < \infty$.

PART 4. FAMILIES OF ELLIPTIC BOUNDARY VALUE PROBLEMS

4.1. Definition

Let M be another vector bundle over X, and $m \in \mathcal{C}^\infty(X; M)$. We consider families of linear partial differential operators whose coefficients depend smoothly but nonlinearly on m, and if we wish also on its derivatives of degree up to some number r. Let

$$E\colon U \subseteq \mathcal{C}^\infty(X; M) \times \mathcal{C}^\infty(X; F) \to \mathcal{C}^\infty(X; F) \oplus \mathcal{C}^\infty(\partial X; P) \oplus \mathcal{C}^\infty(\partial X; Q)$$

be a family of elliptic self-adjoint boundary value problems; thus

$$\mathcal{E}(m)f = (E(m)f, p(m)f, q(m)f),$$

where $E(m)f$ is a linear partial differential operator of degree 2 in f on X, and $p(m)f$ and $q(m)f$ are linear partial differential operators of degrees 0 and 1 in f on ∂X, whose coefficients depend smoothly on m and its derivatives up to some degree r. We assume that $E(m)$, $p(m)$, $q(m)$ satisfy the conditions (1)–(5) of §2.1. Thus we can find a family of hermitian metrics $\langle\ ,\ \rangle_m$ on F, P and Q whose coefficients depend smoothly on m and its derivatives up to degree r, such that for all real cotangent vectors ξ and η and any positive normal cotangent vector ν,

(1) $\langle \sigma_{E(m)}(\xi)f, f\rangle_m > 0$ if $f \neq 0$ and $\xi \neq 0$,

(2) $\langle \sigma_{E(m)}(\xi)f, g\rangle_m = \langle f, \sigma_{E(m)}(\xi)g\rangle_m$,

(3) $\langle \sigma_{E(m)}(\nu)f, g\rangle_m = \langle \sigma_{p(m)}f, \sigma_{p(m)}g\rangle_m + \langle \sigma_{q(m)}(\nu)f, \sigma_{q(m)}(\nu)g\rangle_m$,

(4) If $\sigma_{p(m)}f = 0$ and $\sigma_{p(m)}g = 0$, then

$$\langle D\sigma_{E(m)}(\nu; \eta)f, g\rangle_m = \langle \sigma_{q(m)}(\eta)f, \sigma_{q(m)}(\nu)g\rangle_m + \langle \sigma_{q(m)}(\nu)f, \sigma_{q(m)}(\eta)g\rangle_m,$$

(5) $\dim F = \dim P + \dim Q$.

Suppose that dV_m is a volume element which may also depend smoothly on m and its derivatives up to degree r. Form the inner product,

$$\langle\langle f, g\rangle\rangle_m = \int\int_X \langle f, g\rangle_m \, dV_m.$$

We say $\mathcal{E}$ satisfies a uniform persuasive estimate if for all $m \in U$,

$$|f|_0^2 \lesssim \operatorname{Re}\langle\langle Ef, f\rangle\rangle_m + \|f\|_0^2,$$

when $p(m)f = 0$ and $q(m)f = 0$, with constants independent of m and f.

We suppose $0 \in U$ and think of U as a small neighborhood of 0. We write

$$H(m) = \{h\colon E(m)h = 0, p(m)h = 0, q(m)h = 0\}.$$

Since $E(m)f$, $p(m)f$, $q(m)f$ are nonlinear partial differential operators, the map $\mathcal{E}$ is a smooth tame map in the sense of [2].

Theorem. *Suppose $\mathcal{E}$ satisfies a uniform persuasive estimate, and $H(0) = 0$. Then $H(m) = 0$ for all m in a (possibly smaller) neighborhood U of 0, and hence each linear operator $\mathcal{E}(m)$ is invertible; moreover if we define*

$$\mathcal{E}^{-1}: (U \subseteq \mathcal{C}^\infty(X; M)) \times (\mathcal{C}^\infty(X; F) \oplus \mathcal{C}^\infty(\partial X; P) \oplus \mathcal{C}^\infty(\partial X; Q)) \to \mathcal{C}^\infty(X; F)$$

by letting $\mathcal{E}^{-1}(m)(g, h, k) = f$ be the solution of $\mathcal{E}(m)f = (g, h, k)$, then the family of inverses $\mathcal{E}^{-1}$ is a smooth tame map.

4.2. Moser estimates

We derive estimates for nonlinear partial differential operators. These were first proved by Moser and are crucial for the Nash-Moser inverse function theorem. They are proved using interpolation inequalities and a close examination of the chain rule. These inequalities motivate the abstract definition of a tame map in [2].

We let $\|f\|_n$ denote the L_2 norm of f and its derivatives up to degree n. We let $[[m]]_n$ denote the supremum of f and its derivatives up to degree n. Thus $\| \ \|_n$ is the norm on the Sobolev space $L_2^n(X)$, and $[[\ ]]_n$ is the norm on the classical space $\mathcal{C}^n(X)$. However our results are all for $\mathcal{C}^\infty$ functions. The following interpolation inequalities are standard.

Interpolation theorem. If $k \leqslant i \leqslant n$, then

$$\|f\|_i \lesssim \|f\|_n^{(i-k)/(n-k)} \|f\|_k^{(n-i)/(n-k)},$$

$$[[m]]_i \lesssim [[m]]_n^{(i-k)/(n-k)} [[m]]_k^{(n-i)/(n-k)}.$$

For simplicity we discuss estimates for m in a neighborhood of 0, of the form $[[m]]_r < \varepsilon$. Of course the same will be true in a neighborhood of any m_0.

Moser estimate 1. *Let $P(m)$ be a nonlinear partial differential operator of degree r in m. Then for all m in a neighborhood $[[m]]_r < \varepsilon$ of zero we have estimates*

$$[[P(m)]]_n \lesssim [[m]]_{n+r} + 1.$$

Proof. For simplicity we take m and $P(m)$ to be real-valued; the same argument works in a vector bundle. We have

$$P(m) = \varphi(m, \cdots, D^\alpha m, \cdots), |\alpha| \leqslant r,$$

where φ is a smooth function

$$\varphi(y, \cdots, y^\alpha, \cdots)$$

defined in a neighborhood of some compact set $K = \{|y^\alpha| \leqslant \varepsilon_\alpha\}$. On K every derivative of φ is uniformly bounded. Moreover we can find an $\varepsilon > 0$ such that if $[[m]]_r < \varepsilon$, then $|D^\alpha m| < \varepsilon_\alpha$ for $|\alpha| \leqslant r$.

We are required to estimate the supremum of derivatives of $P(m)$ of degree up to n. By the chain rule each such derivative is a product of a derivative of φ (with respect to the y^α) times derivatives D^β of the arguments $D^\alpha m$. As remarked before, the derivatives of φ are uniformly bounded for $[[m]]_r < \varepsilon$. Moreover any product of derivatives of m which occurs is of the form

$$D^{\beta_1+\alpha_1}m \cdot D^{\beta_2+\alpha_2}m \cdots D^{\beta_k+\alpha_k}m$$

with $|\alpha_j| \leqslant r$ and $|\beta_1| + |\beta_2| + \cdots + |\beta_k| \leqslant n$. The supremum of the product is the product of the suprema. Therefore we must estimate the products

$$[[m]]_{a_1+b_1}[[m]]_{a_2+b_2} \cdots [[m]]_{a_k+b_k}$$

with $\max a_j \leqslant r$ and $\Sigma\, b_j \leqslant n$. By interpolation if $a + b \geqslant r$, then

$$[[m]]_{a+b} \lesssim [[m]]_{n+r}^{(a+b-r)/n}[[m]]_r^{(n-a-b+r)/n},$$

and always $[[m]]_r < \varepsilon$ is bounded. Therefore each product is bounded by $[[m]]_{n+r} + 1$, since $\Sigma(a_j + b_j - r)/n \leqslant 1$.

Moser estimate 2. *Let $L(m)f$ be a partial differential operator, nonlinear of degree r in m and linear of degree s in f. Then for all m in a neighborhood $[[m]]_r < \varepsilon$ of zero we have estimates*

$$\|L(m)f\|_n \lesssim \|f\|_{n+s} + [[m]]_{n+r}\|f\|_s.$$

Proof. We can write $L(m)f$ as $L(m) * j^s f$, where $j^s f$ is the sth jet extension of f, $L(m)$ is a nonlinear partial differential operator of degree r in m with values in a bundle of linear maps of the jet bundle into another bundle, and $*$ denotes a bilinear bundle product. (This is to say in local coordinates

$$L(m)f = \sum_{|\alpha| \leqslant s} L_\alpha(m)D^\alpha f,$$

where $L_\alpha(m)$ is a nonlinear operator of degree r in m which is the coefficient of a derivative of f.) Then by the product rule

$$\|L(m)f\|_n \lesssim \sum_{i+j=n} [[L(m)]]_i \|f\|_{j+s}.$$

By Moser estimate 1,

$$[[L(m)]]_i \lesssim [[m]]_{i+r} + 1,$$

and by interpolation

$$[[m]]_{i+r} \lesssim [[m]]_{n+r}^{i/n}[[m]]_r^{(n-i)/n},$$
$$\|f\|_{j+s} \lesssim \|f\|_{n+s}^{j/n}\|f\|_s^{(n-j)/n},$$

and $[[m]]_r < \varepsilon$ is bounded, so

$$\sum_{i+j=n} [[m]]_{i+r}\|f\|_{j+s} \lesssim \|f\|_{n+s} + [[m]]_{n+r}\|f\|_s,$$

$$\|L(m)f\|_n \lesssim \|f\|_{n+s} + [[m]]_{n+r}\|f\|_s.$$

Moser estimate 3. *Let $P(m)$ be a nonlinear partial differential operator of degree r in m. Suppose $P(0) = 0$. Then for all m in a neighborhood $[[m]]_r < \varepsilon$ of zero, we have estimates*

$$[[P(m)]]_n \lesssim [[m]]_{n+r}.$$

Proof. The derivative $DP(m)f$ is again a partial differential operator, nonlinear in m and linear in f, of degree r in each. By the fundamental theorem of calculus

$$P(m) = P(0) + \int_0^1 DP(tm)m\, dt.$$

By Moser estimate 2 we have

$$\|DP(tm)m\|_n \lesssim [[m]]_{n+r},$$

(that is, we have $[[m]]_{n+r} + [[tm]]_{n+r}[[m]]_r$ but $[[m]]_r < \varepsilon$ is bounded and $0 \leqslant t \leqslant 1$). Integrating this gives

$$\|P(m)\|_n \lesssim \int_0^1 [[m]]_{n+r}\, dt \lesssim [[m]]_{n+r}.$$

Moser estimate 4. *Let $L(m)f$ be a partial differential operator, nonlinear of degree r in m and linear of degree s in f. Suppose $L(0)f = 0$ for all f. Then for all m in a neighborhood $[[m]]_r < \varepsilon$ of zero we have estimates*

$$\|L(m)f\|_n \lesssim [[m]]_r\|f\|_{n+s} + [[m]]_{n+r}\|f\|_s.$$

Proof. Again write $L(m)f = L(m) * j^s f$. Then $L(0) = 0$. Using Moser estimate 3 and the product rule, we obtain

$$\|L(m)f\|_n \lesssim \sum_{i+j=n} [[L(m)]]_i\|f\|_{j+s} \lesssim \sum_{i+j=n} [[m]]_{i+r}\|f\|_{j+s}.$$

Using the previous interpolation inequalities and since $n - i = j$ and $n - j = i$, this is

$$\lesssim \sum_{i+j=n} [[m]]_{n+r}^{i/n}[[m]]_r^{j/n}\|f\|_{n+s}^{j/n}\|f\|_s^{i/n}$$

$$\lesssim [[m]]_r\|f\|_{n+s} + [[m]]_{n+r}\|f\|_s.$$

4.3. Coercive families

The theorem of §1 is particularly easy to prove for coercive families. Moreover we need the uniform Gårding inequality even for the persuasive case (when we argue from the Dirichlet problem). Therefore we argue first for the coercive case. We consider equations of degree 2 to preserve continuity in the notation, but these results are completely general.

Let $\mathfrak{S}(m)f = (E(m)f, p(m)f, q(m)f)$ be a family of boundary value problems:

$$\mathfrak{S} : (U \subseteq \mathcal{C}^\infty(X; M)) \times \mathcal{C}^\infty(X; F) \\ \to \mathcal{C}^\infty(X; F) \oplus \mathcal{C}^\infty(\partial X; P) \oplus \mathcal{C}^\infty(\partial X; Q),$$

where $E(m)f$ is a linear partial differential operator of degree 2 in f, and $p(m)f$ and $q(m)f$ are linear partial differential boundary operators of degrees 0 and 1, all of which have coefficients which are smooth functions of m and its derivatives up to degree r. Suppose that $\mathfrak{S}(0)f = (E(0)f, p(0)f, q(0)f)$ is a coercive elliptic boundary value problem (in the sense of Agmon, Douglis and Nirenberg). Then so is $\mathfrak{S}(m)f = (E(m)f, p(m)f, q(m)f)$ for all m in a neighborhood U of 0, since coercivity is equivalent to a certain matrix formed from the coefficients being invertible. We prove a generalization of Gårding's estimate which gives the dependence of the constants on m.

Gårding-Moser estimate. *Let $\mathfrak{S}(m)f = (E(m)f, p(m)f, q(m)f)$ be a family of coercive elliptic boundary value problems of degree 2, whose coefficients depend smoothly on m of degree r. Then for all m in a neighborhood $[[m]]_r < \varepsilon$ of zero we have estimates*

$$\|f\|_n \lesssim \|E(m)f\|_{n-2} + |p(m)f|_{n-1/2} \\ + |q(m)f|_{n-3/2} + ([[m]]_{n+r} + 1)\|f\|_0.$$

Proof. By Gårding's inequality for $m = 0$

$$\|f\|_2 \lesssim \|E(0)f\|_0 + |p(0)f|_{3/2} + |q(0)f|_{1/2} + \|f\|_0.$$

By Moser estimate 4

$$\|E(m)f - E(0)f\|_0 \lesssim [[m]]_r \|f\|_2.$$

We can find an operator $P(m)$ on X with $p(m)f = P(m)f|\partial X$. Then

$$|p(m)f - p(0)f|_{3/2} \lesssim \|P(m)f - P(0)f\|_2 \\ \lesssim [[m]]_r \|f\|_2 + [[m]]_{r+2} \|f\|_0.$$

Likewise if $q(m)f = Q(m)f|\partial X$ then

$$|q(m)f - q(0)f|_{1/2} \lesssim \|Q(m)f - Q(0)f\|_1 \\ \lesssim [[m]]_r \|f\|_2 + [[m]]_{r+1} \|f\|_1.$$

By interpolation if $0 \leqslant j \leqslant n$

$$[[m]]_{r+j}\|f\|_{n-j} \lesssim [[m]]_r\|f\|_n + [[m]]_{r+n}\|f\|_0.$$

Therefore we have the estimate

$$\|f\|_2 \lesssim \|E(m)f\|_0 + |p(m)f|_{3/2} + |q(m)f|_{1/2} + [[m]]_r\|f\|_2 + ([[m]]_{r+2} + 1)\|f\|_0.$$

If $[[m]]_r < \varepsilon$ with ε sufficiently small, we can subtract the term $[[m]]_r\|f\|_2$ from the other side. This proves the estimate for $n = 2$.

We proceed by induction on n. Suppose the estimate holds for n as written. Choose operators ∇_j as in §3.7 on the bundles F, P and Q. For simplicity we include $\nabla_0 = I$. Then by the induction hypothesis

$$\|\nabla_j f\|_n \lesssim \|E(m)\nabla_j f\|_{n-2} + |p(m)\nabla_j f|_{n-1/2} + |q(m)\nabla_j f|_{n-3/2} + ([[m]]_{n+r} + 1)\|\nabla_j f\|_0.$$

The commutator $[E(m), \nabla_j]f$ involves only second derivatives of f with coefficients depending non-linearly on r derivatives of m and linearly on the $(r + 1)$st derivatives of m. Hence we can estimate it by

$$\|[E(m), \nabla_j]f\|_{n-2} \lesssim ([[m]]_{r+1} + 1)\|f\|_n + [[m]]_{n+r-1}\|f\|_2.$$

Since ∇_j involves only derivatives parallel to the boundary, $[p(m), \nabla_j]f = [P(m), \nabla_j]f|\partial$ and $[q(m), \nabla_j]f = [Q(m), \nabla_j]f|\partial$. Therefore

$$|[p(m), \nabla_j]f|_{n-1/2} \lesssim ([[m]]_{r+1} + 1)\|f\|_n + [[m]]_{n+r+1}\|f\|_0,$$
$$|[q(m), \nabla_j]f|_{n-3/2} \lesssim ([[m]]_{r+1} + 1)\|f\|_n + [[m]]_{n+r}\|f\|_1.$$

Thus we have the estimate

$$\|\nabla_j f\|_n \lesssim \|E(m)f\|_{n-1} + |p(m)f|_{n+1/2} + |q(m)f|_{n-1/2} + ([[m]]_{r+1} + 1)\|f\|_n + [[m]]_{n+r+1}\|f\|_0.$$

We can solve the equation $E(m)f$ for the second normal derivative $\partial^2 f/\partial n^2$. Thus we can write

$$\frac{\partial^2 f}{\partial n^2} = \sigma_{E(m)}(\nu)^{-1}E(m)f + \sum A_j(m)\nabla_j f,$$

where the $A_j(m)$ are linear operators of degree 1 with coefficients depending nonlinearly on m and its derivatives of degree r or less. Since $\|E(m)f\|_0 \lesssim \|f\|_2$ we have

$$\left\|\frac{\partial^2 f}{\partial n^2}\right\|_{n-1} \lesssim \|E(m)f\|_{n-1} + \sum \|\nabla_j f\|_n + [[m]]_{n+r-1}\|f\|_2.$$

Then $\|f\|_{n+1} \lesssim \Sigma\|\nabla_j f\|_n + \|\partial^2 f/\partial n^2\|_{n-1}$ so we have

$$\|f\|_{n+1} \lesssim \|E(m)f\|_{n-1} + |p(m)f|_{n+1/2} + |q(m)f|_{n-1/2} + ([[m]]_{r+1} + 1)\|f\|_n + [[m]]_{n+r+1}\|f\|_0.$$

By interpolation, for every $\delta > 0$ we can find a constant C_δ with

$$([[m]]_{r+1} + 1)\|f\|_n \leqslant \delta[[m]]_r\|f\|_{n+1} + C_\delta([[m]]_{n+r+1} + 1)\|f\|_0.$$

Since $[[m]]_r < \varepsilon$, if we take δ sufficiently small we may subtract the term involving $C\delta([[m]]_r + 1)\|f\|_{n+1}$ from the left hand side. Then

$$\|f\|_{n+1} \lesssim \|E(m)f\|_{n-1} + |p(m)f|_{n+1/2} + |q(m)f|_{n-1/2} + ([[m]]_{n+r+1} + 1)\|f\|_0.$$

This completes the induction.

Lemma. *Suppose $\mathfrak{S}(0)$ is invertible. Then so is $\mathfrak{S}(m)$ for all m in a neighborhood $[[m]]_{r+2} < \varepsilon$. Moreover*

$$\|f\|_2 \lesssim \|E(m)f\|_0 + |p(m)f|_{3/2} + |q(m)f|_{1/2}.$$

Proof. From the standard coercive theory, if $\mathfrak{S}(0)$ is invertible, then

$$\|f\|_2 \lesssim \|E(0)f\|_0 + |p(0)f|_{3/2} + |q(0)f|_{1/2}.$$

Again we have

$$\|E(m)f - E(0)f\|_0 \lesssim [[m]]_r\|f\|_2,$$

$$|p(m)f - p(0)f|_{3/2} \lesssim \|P(m)f - P(0)f\|_2 \lesssim [[m]]_r\|f\|_2 + [[m]]_{r+2}\|f\|_0,$$

$$|q(m)f - q(0)f|_{1/2} \lesssim \|Q(m)f - Q(0)f\|_1 \lesssim [[m]]_r\|f\|_2 + [[m]]_{r+1}\|f\|_1.$$

The sum of all these is $\lesssim [[m]]_{r+2}\|f\|_2$. Hence if $[[m]]_{r+2} < \varepsilon$ with ε sufficiently small, we have

$$\|f\|_2 \lesssim \|E(m)f\|_0 + |p(m)f|_{3/2} + |q(m)f|_{1/2}.$$

This shows $\mathfrak{S}(m)$ is injective if $[[m]]_{r+2} < \varepsilon$. But $\mathfrak{S}(0)$ is an isomorphism, so its index is zero. By Fredholm theory $\mathfrak{S}(m)$ also has index 0, so if it is injective then it is surjective as well.

Write $\|(g, h, k)\|_n = \|g\|_{n-2} + |h|_{n-1/2} + |k|_{n-3/2}$. Combining the lemma with the Gårding-Moser estimate, we see that if $\mathfrak{S}(0)$ is invertible, then for all m in a neighborhood U of 0 we have the estimate

$$\|f\|_n \lesssim \|\mathfrak{S}f\|_n + [[m]]_{n+r}\|\mathfrak{S}f\|_0.$$

This proves the following.

Corollary. *Let* $\mathcal{E}$ *be a family of coercive elliptic boundary value problems. If* $\mathcal{E}(0)$ *is invertible, so is* $\mathcal{E}(m)$ *for all* m *in a neighborhood of the form* $[[m]]_{r+2} < \varepsilon$ U *of* 0, *and the family of inverses*

$$\mathcal{E}^{-1}: (R \subseteq \mathcal{C}^\infty(X; M)) \times (\mathcal{C}^\infty(X; F) \oplus \mathcal{C}^\infty(\partial X; P) \oplus \mathcal{C}^\infty(\partial X; Q)) \to \mathcal{C}^\infty(X; F)$$

is a smooth tame map, where $\mathcal{E}^{-1}(m)(g, h, k) = f$ *is the unique solution of* $E(m)f = g, p(m)f = h, g(m)f = k$.

Proof. That $\mathcal{E}^{-1}$ is tame is guaranteed by the estimate. That $\mathcal{E}^{-1}$ is smooth and all its derivatives are tame follows automatically; see [2, Theorems 5 and 6].

This done, we turn our attention to the non-coercive case.

4.4. Normalizing the first boundary condition

The first step in the proof of the theorem in §1 is to simplify the problem to the case where the boundary condition $p(m)$ is independent of m. We do this as follows.

Fix a smooth normal cotangent vector field ν. The map

$$\sigma_{p(m)} \oplus \sigma_{q(m)}(\nu): F|\partial X \to P \oplus Q$$

is an isomorphism by condition (3). Therefore we can choose a family of vector bundle maps $\varphi(m): F|\partial X \to F|\partial X$ such that

$$\{\sigma_{p(m)} \oplus \sigma_{q(m)}(\nu)\} = \{\sigma_{p(0)} \oplus \sigma_{q(0)}(\nu)\} \circ \varphi(m).$$

Moreover $\varphi(m)$ will be an isomorphism and will depend smoothly on the values of m and its derivatives up to order r at each point. We can extend $\varphi(m)$ to such a family of isomorphisms over all of X. Thus

$$\varphi(m): F \to F$$

induces an operator of degree zero:

$$\varphi(m): \mathcal{C}^\infty(X; F) \to \mathcal{C}^\infty(X; F).$$

Then we can regard φ as a nonlinear partial differential operator

$$\varphi: (U \subseteq \mathcal{C}^\infty(X; M)) \times \mathcal{C}^\infty(X; F) \to \mathcal{C}^\infty(X; F)$$

of degree r in m, and linear of degree zero in f. Moreover for each m the map $\varphi(m)$ is invertible; and writing $\varphi^{-1}(m)g = \varphi(m)^{-1}g$, we can regard φ^{-1} as another nonlinear partial differential operator

$$\varphi^{-1}: (U \subseteq \mathcal{C}^\infty(X; M)) \times \mathcal{C}^\infty(X; F) \to \mathcal{C}^\infty(X; F)$$

of degree r in m, and linear of degree zero in g. Clearly both φ and φ^{-1} are smooth tame maps. Moreover by the choice of φ, $p(m) \circ \varphi(m) = p(0)$ since $p(m) = \sigma_{p(m)}$ (because p has degree zero in f).

We now define a new family of elliptic boundary value problems:

$$\tilde{\mathcal{E}}(m)\tilde{f} = \big(\tilde{E}(m)\tilde{f}, \tilde{p}(m)\tilde{f}, \tilde{q}(m)\tilde{f}\big),$$

where

$$\tilde{E}(m)\tilde{f} = \varphi(m)^{-1}E(m)\varphi(m)\tilde{f},$$

$$\tilde{p}(m)\tilde{f} = p(m)\varphi(m)\tilde{f},$$

$$\tilde{q}(m)\tilde{f} = q(m)\varphi(m)\tilde{f}.$$

Then the new system $\tilde{\mathcal{E}}$ will still satisfy condition (1)–(5) in the pull-back hermitian metric

$$\langle \tilde{f}, \tilde{g}\rangle_m^{\sim} = \langle \varphi(m)\tilde{f}, \varphi(m)\tilde{g}\rangle_m.$$

Also if the old system satisfies a uniform persuasive estimate, then so will the new system, at least on a neighborhood U of 0, since $|\varphi(m)\tilde{f}|_0 \leqslant C|\tilde{f}|_0$ and $\|\varphi(m)\tilde{f}\|_0 \leqslant C\|\tilde{f}\|_0$ uniformly in m, once m and its first r derivatives are uniformly bounded; for if we write $f = \varphi(m)\tilde{f}$, then $\tilde{p}(m)\tilde{f} = p(m)f$, so if $\tilde{p}(m)\tilde{f} = 0$ then $p(m)f = 0$ and

$$\begin{aligned} |\tilde{f}|_0^2 \lesssim |f|_0^2 &\lesssim \mathrm{Re}\langle\!\langle E(m)f, f\rangle\!\rangle_m + \|f\|_0^2 \\ &\lesssim \mathrm{Re}\langle\!\langle E(m)\varphi(m)\tilde{f}, \varphi(m)\tilde{f}\rangle\!\rangle_m + \|\tilde{f}\|_0^2 \\ &= \mathrm{Re}\langle\!\langle \varphi(m)\tilde{E}(m)\tilde{f}, \varphi(m)\tilde{f}\rangle\!\rangle_m + \|\tilde{f}\|_0^2 \\ &= \mathrm{Re}\langle\!\langle \tilde{E}(m)\tilde{f}, \tilde{f}\rangle\!\rangle_m^{\sim} + \|\tilde{f}\|_0^2. \end{aligned}$$

Suppose that the new system $\tilde{\mathcal{E}}$ has a smooth tame inverse. Then so does $\mathcal{E}$, namely

$$\mathcal{E}^{-1}(m)(g, h, k) = \varphi(m)\tilde{\mathcal{E}}^{-1}(m)\big(\varphi(m)^{-1}g, h, k\big),$$

since a composition of smooth tame maps is a smooth tame map.

Now in the system $\tilde{\mathcal{E}}$ the boundary condition $\tilde{p}(m) = p(m)\varphi(m) = p(0)$ is independent of m. Therefore we now drop the $\sim$ and assume that $p(m) = p$ is independent of m.

Next we repeat the derivation of the estimates in the previous section, paying attention to the influence of m on the coefficients in the estimates. Integrating by parts as before (using conditions (3)–(5) to transfer derivatives) we can find a smooth essentially hermitian symmetric bilinear form $Q(m)(f, g)$ whose coefficients are smooth nonlinear functions of m and its first $r + 1$ derivatives (since we may transfer a derivative onto a term involving $m!$) such that

$$Q(m)(f, g) = \langle\!\langle E(m)f, g \rangle\!\rangle_m$$

when $q(m)f = 0$ and $pg = 0$. Then we will have a uniform persuasive estimate

$$|f|_0^2 \lesssim \operatorname{Re} Q(m)(f, f) + C\|f\|_0^2$$

when $pf = 0$. At first we seem to need $q(m)f = 0$ also; but arguing as before, if $pf = 0$ we can find a sequence f_k with $pf_k = 0$ and $q(m)f_k = 0$ such that $\|f_k - f\|_1 \to 0$ as $k \to \infty$, and applying the estimate to f_k and taking the limit, we see that the estimate holds for f also, with the same constants.

Now we let ∇ be a simple operator as before. Since p is independent of m, we can choose ∇ so that $pf = 0 \Rightarrow p\nabla f = 0$, independently of m. Applying the uniform Morrey inequality to $\nabla^n f$, we have

$$|\nabla^n f|_0^2 \lesssim \operatorname{Re} Q(m)(\nabla^n f, \nabla^n f) + \|\nabla^n f\|_0^2$$

when $pf = 0$. Now we must consider how to estimate the error term

$$\operatorname{Re} Q(m)(\nabla^n f, \nabla^n f) - (-1)^n \operatorname{Re} Q(m)(f, \nabla^{2n} f).$$

4.5. The uniform Kohn-Nirenberg lemma

In this section we derive an estimate on the error in replacing

$$\operatorname{Re} Q(m)(\nabla^n f, \nabla^n f) \text{ by } \operatorname{Re} Q(m)(f, \nabla^{2n} f)$$

paying careful attention to the growth in terms of the coefficient m. This is done by applying the methods used in the Moser estimates to the Kohn-Nirenberg lemma.

Consider the bilinear form

$$\nabla Q(m)(f, g) = Q(m)(\nabla f, g) + Q(m)(f, \nabla g).$$

The form $\nabla Q(m)$ is again of degree 1 in f and g and its coefficients are found by differentiating the coefficients of $Q(m)$ by ∇. If the coefficients of $Q(m)$ depend smoothly on m and its derivatives up to some degree $r + 1$, then the coefficients of $\nabla^n Q(m)$ will depend smoothly on m and its derivatives up to degree $n + r + 1$. Therefore we can estimate

$$|\nabla^n Q(m)(f, g)| \lesssim ([[m]]_{n+r+1} + 1)\|f\|_1\|g\|_1.$$

Recall that we can write

$$\operatorname{Re} Q(m)(\nabla^n f, \nabla^n f) - (-1)^m \operatorname{Re} Q(m)(f, \nabla^{2n} f)$$

as a sum of terms $\nabla^i Q(m)(\nabla^j f, \nabla^k f)$ with $2 \leqslant i \leqslant n + 1$ and $j, k \leqslant n - 1$ and $i + j + k = 2n$, and terms $\nabla^i R_1(m)(\nabla^j f, \nabla^k f)$ with $1 \leqslant i \leqslant n$ and $j \leqslant n, k \leqslant n - 1$ and $i + j + k = 2n$, and terms $\nabla^i R_2(m)(\nabla^j f, \nabla^k f)$ with $1 \leqslant i \leqslant n$ and

$j \leqslant n - 1$, $k \leqslant n$ and $i + j + k = 2n$, by the argument of part 2 §6. Here $R(m)(f, g) = Q(m)(f, g) - \overline{Q(m)(g, f)}$ involves no product of first derivatives, and $R(m)(f, g) = R_1(m)(f, g) + R_2(m)(f, g)$ where R_1 involves R_1 involves no f-derivatives and R_2 involves no g derivatives. Hence the coefficients of R_1 and R_2 will also be smooth functions of m and its derivatives of degree up to $r + 1$.

We have

$$|\nabla^i Q(m)(\nabla^j f, \nabla^k f)| \lesssim ([[m]]_{i+r+1} + 1)\|f\|_{j+1}\|f\|_{k+1}.$$

By interpolation

$$[[m]]_{i+r+1} \lesssim [[m]]_{n+r+2}^{(i-2)/(n-1)}[[m]]_{r+3}^{(n-i+1)/(n-1)},$$
$$\|f\|_{j+1} \lesssim \|f\|_n^{j/(n-1)}\|f\|_1^{(n-j-1)/(n-1)},$$
$$\|f\|_{k+1} \lesssim \|f\|_n^{k/(n-1)}\|f\|_1^{(n-k-1)/(n-1)}.$$

Choose U small enough that $[[m]]_{r+3} \leqslant C$. Since $j + k = 2n - i$, $2 \leqslant i \leqslant n + 1$, $j \leqslant n - 1$, $k \leqslant n - 1$,

$$[[m]]_{i+r+1}\|f\|_{j+1}\|f\|_{k+1} \lesssim [[m]]_{n+r+2}^{(i-2)/(n-1)}\|f\|_n^{(2n-i)/(n-1)}\|f\|_1^{(i-2)/(n-1)}$$
$$\lesssim ([[m]]_{n+r+2}\|f\|_1 + \|f\|_n)^2.$$

Therefore each term

$$|\nabla^i Q(m)(\nabla^j f, \nabla^k f)| \lesssim ([[m]]_{n+r+2}\|f\|_1 + \|f\|_n)^2.$$

Similarly

$$|\nabla^i R_1(m)(\nabla^j f, \nabla^k f)| \lesssim ([[m]]_{i+r+1} + 1)\|f\|_j\|f\|_{k+1}.$$

By interpolation, since $j + k = 2n - i$, $1 \leqslant i \leqslant n$, $j \leqslant n$, $k \leqslant n - 1$,

$$[[m]]_{i+r+1} \lesssim [[m]]_{n+r+1}^{(i-1)/(n-1)}[[m]]_{r+2}^{(n-i)/(n-1)},$$
$$\|f\|_j \lesssim \|f\|_n^{(j-1)/(n-1)}\|f\|_1^{(n-j)/(n-1)},$$
$$\|f\|_{k+1} \lesssim \|f\|_n^{k/(n-1)}\|f\|_1^{(n-k-1)/(n-1)}.$$

If U is small enough that $[[m]]_{r+2} \leqslant C$,

$$[[m]]_{i+r+1}\|f\|_j\|f\|_{k+1} \lesssim [[m]]_{n+r+1}^{(i-1)/(n-1)}\|f\|_n^{(j+k-1)/(n-1)}\|f\|_1^{(i-1)/(n-1)}$$
$$\lesssim ([[m]]_{n+r+1}\|f\|_1 + \|f\|_n)^2.$$

Therefore for each term

$$|\nabla^i R_1(\nabla^j f, \nabla^k f)| \lesssim ([[m]]_{n+r+1}\|f\|_1 + \|f\|_n)^2.$$

The same estimate holds for R_2. This proves the following result.

Uniform Kohn-Nirenberg lemma. *Let $Q(m)(f, g)$ be a family of essentially hermitian symmetric bilinear forms whose coefficients depend smoothly on m and its derivatives of degree up to $r + 1$. Let ∇ be a linear partial differential operator of degree 1 with symbol $\sigma \nabla(\xi) = \xi(v)I$ where v is a vector field tangent to the boundary. Then there is a neighborhood U of 0 of the form $[[m]]_{r+3} < \varepsilon$ such that for all m in U and all n*

$$|\mathrm{Re}\, Q(m)(\nabla^n f, \nabla^n f) - (-1)^n \mathrm{Re}\, Q(m)(f, \nabla^{2n} f)| \lesssim ([[m]]_{n+r+2}\|f\|_1 + \|f\|_n)^2$$

with a constant independent of m and f.

4.6. The uniform a-priori estimate

We can now derive the a-priori estimate uniformly in m. Arguing as before (in 3.6) from where we left off in §4.2,

$$|\nabla^n f|_0^2 \lesssim \mathrm{Re}\, Q(m)(\nabla^n f, \nabla^n f) + \|f\|_n^2,$$

$$|\nabla^n f|_0^2 \lesssim \mathrm{Re}\, Q(m)(f, \nabla^{2n} f) + ([[m]]_{n+r+2}\|f\|_1 + \|f\|_n)^2.$$

If $pf = 0$, $q(m)f = 0$, and ∇ is adapted to the boundary condition p, which has been made independent of m, then $p\nabla^{2n} f = 0$. Thus by the choice of $Q(m)$ we have

$$Q(m)(f, \nabla^{2n} f) = \langle\langle E(m)f, \nabla^{2n} f\rangle\rangle_m.$$

The operator ∇ will have an adjoint ∇_m^* such that for all f and g

$$\langle\langle f, g\rangle\rangle_m + \langle\langle \nabla_m^* f, g\rangle\rangle_m = 0.$$

In general ∇_m^* will depend on the metric $\langle\ ,\ \rangle_m$ on F and the volume dV_m on X, both of which may depend on m. However we will have $\nabla_m^* = \nabla + a(m)$ where $a(m)$ is a linear partial differential operator of degree 0 (i.e., a multiplication operator) whose coefficients depend on m and its derivatives up to degree $r + 1$, (since in performing the integration by parts the differential operator ∇ may land on a coefficient involving m).

Lemma. *For all m in a sufficiently small neighborhood of 0, the estimate*

$$\|\nabla_m^{*k} f\|_n \lesssim \|f\|_{n+k} + [[m]]_{n+r+k}\|f\|_0$$

holds for all n and k.

Proof. We have

$$\|a(m)f\|_n \lesssim [[m]]_{n+r+1}\|f\|_0 + \|f\|_n$$

from a previous lemma. The formula is clearly true for $k = 0$; we proceed by induction. Suppose the formula holds for some value k. Then

$$\|\nabla_m^{*(k+1)}f\|_n \lesssim \|\nabla_m^* f\|_{n+k} + [[m]]_{n+r+k}\|\nabla_m^* f\|_0.$$

Now $\|\nabla_m^* f\|_0 \lesssim \|f\|_0$ and

$$\|\nabla_m^* f\|_{n+k} \lesssim \|\nabla^* f\|_{n+k} + \|a(m)f\|_{n+k}$$
$$\lesssim \|f\|_{n+k+1} + [[m]]_{n+r+k+1}\|f\|_0.$$

Therefore

$$\|\nabla_m^{*(k+1)}f\|_n \lesssim [[m]]_{n+r+k+1}\|f\|_0 + \|f\|_{n+k+1},$$

which proves the formula for $k + 1$; and hence the formula holds for all k by induction.

Returning to the main argument, we have

$$\langle\langle E(m)f, \nabla^{2n}f\rangle\rangle_m = \langle\langle \nabla_m^{*n}E(m)f, \nabla^n f\rangle\rangle_m,$$
$$|\langle\langle \nabla_m^{*n}E(m)f, \nabla^n f\rangle\rangle_m| \lesssim \|\nabla_m^{*n}E(m)f\|_0\|f\|_n.$$

By the previous lemma

$$\|\nabla_m^{*n}E(m)f\|_0 \lesssim \|E(m)f\|_n + [[m]]_{n+r}\|f\|_0.$$

Combining these facts

$$|\nabla^n f|_0^2 \lesssim (\|E(m)f\|_n + [[m]]_{n+r}\|f\|_0)\|f\|_n + ([[m]]_{n+r+2}\|f\|_1 + \|f\|_n)^2.$$

For a suitable choice of $\nabla_1, \cdots, \nabla_N$,

$$|f|_n^2 \leqslant \sum_{i=1}^{N}\sum_{j=1}^{n} |\nabla_i^j f|_0^2.$$

Therefore

$$|f|_n \lesssim \|E(m)f\|_n + \|f\|_n + [[m]]_{n+r+2}\|f\|_1$$

for all m in some fixed neighborhood of 0 (independent of n) and all f with $pf = 0$ and $q(m)f = 0$.

We remark that now it is no longer necessary to assume the boundary condition $p(m)f$ is independent of m. For given any general system $\mathcal{E} = (E(m)f, p(m)f, q(m)f)$ we showed in §2 how to construct another system $\tilde{\mathcal{E}} = (\tilde{E}(m)\tilde{f}, \tilde{p}(m)\tilde{f}, \tilde{q}(m)\tilde{f})$ such that $\tilde{p}(m)\tilde{f}$ is independent of m, and if $f = \varphi(m)\tilde{f}$ then $E(m)f = \varphi(m)\tilde{E}(m)\tilde{f}$, $p(m)f = \tilde{p}(m)\tilde{f}$, $q(m)f = \tilde{q}(m)\tilde{f}$, where $\varphi(m)$ is a multiplication operator involving m and its derivatives up to degree r, and so is $\varphi^{-1}(m)$. Then

$$\|f\|_n \lesssim \|\tilde{f}\|_n + [[m]]_{n+r}\|\tilde{f}\|_0, \quad \|\tilde{f}\|_n \lesssim \|f\|_n + [[m]]_{n+r}\|f\|_0.$$

The previous estimate will hold for $\tilde{\mathcal{E}}$:

$$|\tilde{f}|_n \lesssim \|\tilde{E}(m)\tilde{f}\|_n + \|\tilde{f}\|_n + [[m]]_{n+r+2}\|\tilde{f}\|_1.$$

We have (using the argument on ∂X)

$$|f|_n \lesssim |\tilde{f}|_n + [[m]]_{n+r}|\tilde{f}|_0,$$

$$\|\tilde{E}(m)\tilde{f}\|_n \lesssim \|E(m)f\|_n + [[m]]_{n+r}\|E(m)f\|_0,$$

$$\|E(m)f\|_0 \lesssim \|f\|_2,\ \|f\|_0 \lesssim \|f\|_0, \text{ and } |f|_0 \lesssim |f|_0$$

independently of m. Therefore if $p(m)f = 0$, then

$$|f|_n \lesssim \|E(m)f\|_n + \|f\|_n + [[m]]_{n+r+2}\|f\|_1.$$

Hence the estimate is completely general.

The operator $E(m)$ is elliptic, and the Dirichlet boundary conditions $f|\partial X$ are coercive for every elliptic operator. Therefore we can appeal to the Gårding-Moser estimate of §3, and write that for all m in a neighborhood U of 0 (of the form $[[m]]_{r+2} < \varepsilon$) we have

$$\|f\|_n \lesssim \|E(m)f\|_{n-2} + |f|_{n-1/2} + ([[m]]_{n+r} + 1)\|f\|_0.$$

Again we have $\forall \varepsilon > 0, \exists C_\varepsilon$

$$|f|_{n-1/2} \leqslant \varepsilon|f|_n + C_\varepsilon|f|_0.$$

Combining this with the previous estimate

$$\begin{aligned}\|f\|_n \leqslant\ & C(\|E(m)f\|_{n-2} + [[m]]_{n+r}\|f\|_0)\\ & + \varepsilon C(\|E(m)f\|_n + \|f\|_n + [[m]]_{n+r+2}\|f\|_1)\\ & + C_\varepsilon|f|_0 + C\|f\|_0.\end{aligned}$$

We take $\varepsilon C \leqslant 1/2$; this eliminates $\|f\|_n$ from the right. Also $\forall \varepsilon > 0, \exists C_\varepsilon$ with

$$\|E(m)f\|_{n-2} \leqslant \varepsilon\|E(m)f\|_n + C_\varepsilon\|E(m)f\|_0.$$

Note C_ε does not involve m. Now for all m in a neighborhood of zero

$$\|E(m)f\|_0 \leqslant C\|f\|_2$$

with a constant independent of m. Thus $\forall \varepsilon > 0, \exists C_\varepsilon$ with

$$\|f\|_n \leqslant \varepsilon\|E(m)f\|_n + [[m]]_{n+r+2}\|f\|_1 + C_\varepsilon(\|f\|_2 + |f|_0).$$

But $\forall \eta > 0, \exists C_\eta$ with

$$\|f\|_2 + |f|_0 \leqslant \eta\|f\|_n + C_\eta\|f\|_0.$$

Take η with $\eta C_\varepsilon \leqslant 1/2$. Then η depends on ε, so we can write C_η as C_ε, which is still independent of m. This proves the following.

Theorem. *Let $\mathcal{E}(m)f = (E(m)f, p(m)f, q(m)f)$ be a family of self-adjoint* *elliptic boundary value problems. Suppose* $\mathcal{E}$ *satisfies a uniform persuasive estimate*

$$|f|_0^2 \lesssim \operatorname{Re} Q(m)(f, f) + \|f\|_0^2.$$

For all m in a neighborhood of zero we have the uniform a-priori estimate

$$\|f\|_n \lesssim \|E(m)f\|_n + \big([[m]]_{n+r+2} + 1\big)\|f\|_1,$$

when $p(m)f = 0$ *and* $q(m)f = 0$. *More precisely we can write* $\forall \varepsilon > 0, \exists C_\varepsilon$ *with*

$$\|f\|_n \leqslant \varepsilon\|E(m)f\|_n + \big([[m]]_{n+r+2} + C_\varepsilon\big)\|f\|_1,$$

when $p(m)f = 0$ *and* $q(m)f = 0$.

As before we can deduce an estimate for inhomogeneous boundary conditions. We mimic the argument at the end of §2.5. Let $p'(m)$ and $q'(m)$ be the complementary families of boundary partial differential operators for $p(m)$ and $q(m)$. We can find two families of partial differential operators $\varphi(m)(h, l)$ and $\psi(m)(h, j, k, l)$ operating on ∂X such that if $f|\partial X = \varphi(m)(h, l)$ and $\partial f/\partial n = \psi(m)(h, j, k, l)$ then $p(m)f = h$, $p'(m)f = j$, $q(m)f = k$, $q'(m)f = l$. Moreover the coefficients of φ and ψ are smooth functions of m and its derivatives up to degree $r + 1$, as in clear from the construction. Then if we apply the continuous linear extension T so that

$$f' = T(\varphi(m)(p(m)f, 0), \psi(m)(p(m)f, 0, q(m)f, 0)),$$

we have $p(m)f' = p(m)f$ and $q(m)f' = q(m)f$. Write $h = p(m)f$ and $k = q(m)f$. By the choice of T (and giving up half a derivative at that, to simplify the argument) we have (when $n \geqslant 2$)

$$\|T(u, v)\|_n \leqslant |u|_n + |v|_{n-1}.$$

Thus

$$\|f'\|_n \lesssim |\varphi(m)(h, 0)|_n + |\psi(m)(h, 0, k, 0)|_{n-1}.$$

Now φ and ψ are families of linear partial differential operators on ∂X, so we can apply Moser estimate 2 on ∂X to obtain

$$|\varphi(m)(h, 0)|_n \lesssim |h|_n + [[m]]_{n+r+1}|h|_0,$$

$$|\psi(m)(h, 0, k, 0)|_{n-1} \lesssim |h|_n + |k|_{n-1} + [[m]]_{n+r}(|h|_1 + |k|_0),$$

recalling that $\varphi(h, l)$ has degree 0 in h and l, while $\psi(h, j, k, l)$ has degree 1 in h and l and degree 0 in j and k. Now

$$|p(m)f|_1 + |q(m)f|_0 \lesssim |f|_1 \leqslant \|f\|_2.$$

Thus

$$\|f'\|_n \lesssim |p(m)f|_n + |q(m)f|_{n-1} + [[m]]_{n+r+1}\|f\|_2,$$

and so

$$\|f'\|_{n+2} \lesssim |p(m)f|_{n+2} + |q(m)f|_{n+1} + [[m]]_{n+r+3}\|f\|_2.$$

Now by Moser estimate 2

$$\|E(m)f'\|_n \lesssim \|f'\|_{n+2} + ([[m]]_{n+r+2} + 1)\|f'\|_2,$$
$$\|f'\|_2 \lesssim |p(m)f|_2 + |q(m)f|_1 + \|f\|_2 \leqslant \|f\|_3,$$

when $[[m]]_{r+3}$ is bounded. Thus

$$\|E(m)f'\|_n \lesssim |p(m)f|_{n+2} + |q(m)f|_{n+1} + [[m]]_{n+r+3}\|f\|_3.$$

Let $f = f' + f''$. Then $p(m)f'' = 0$ and $q(m)f'' = 0$. Thus we have a uniform a-priori estimate for f'':

$$\|f''\|_n \lesssim \|E(m)f''\|_n + ([[m]]_{n+r+2} + 1)\|f''\|_1.$$

Now $\|f\|_n \leqslant \|f'\|_n + \|f''\|_n$ and

$$\|E(m)f''\|_n \leqslant \|E(m)f\|_n + \|E(m)f'\|_n.$$

Combining these we have

$$\|f\|_n \lesssim \|E(m)f\|_n + |p(m)f|_{n+2} + |q(m)f|_{n+1} + ([[m]]_{n+r+3} + 1)\|f\|_3,$$

which holds without restriction on f.Now by interpolation

$$[[m]]_{n+r+3} \lesssim [[m]]_{n+r+6}^{(n-3)/n}[[m]]_{r+6}^{3/n},$$
$$\|f\|_3 \lesssim \|f\|_n^{3/n}\|f\|_0^{(n-3)/n},$$
$$[[m]]_{n+r+3}\|f\|_3 \lesssim ([[m]]_{r+6}\|f\|_n)^{3/n}([[m]]_{n+r+6}\|f\|_0)^{(n-3)/n}.$$

On a neighborhood of zero, $[[m]]_{r+6} < \varepsilon$ is bounded. Therefore $\forall\varepsilon > 0$, $\exists C_\varepsilon$ with

$$[[m]]_{n+r+3}\|f\|_3 \leqslant \varepsilon\|f\|_n + C_\varepsilon[[m]]_{n+r+6}\|f\|_0.$$

Taking $\varepsilon > 0$ small enough we have

$$\|f\|_n \lesssim \|E(m)f\|_n + |p(m)f|_{n+2} + |q(m)f|_{n+1} + ([[m]]_{n+r+6} + 1)\|f\|_0.$$

(Of course 6 is not the best possible.)

Uniform main a-priori estimate. Let $\mathcal{E}(m)f = (E(m)f, p(m)f, q(m)f)$ be a family of self-adjoint elliptic boundary value problems which satisfy a uniform persuasive estimate

$$|f|_0^2 \lesssim \operatorname{Re}\langle\!\langle E(m)f, f\rangle\!\rangle_m + \|f\|_0^2,$$

when $p(m)f = 0$ and $q(m)f = 0$. Then for all m in a neighborhood U of 0 we have

$$\|f\|_n \lesssim \|E(m)f\|_n + |p(m)f|_{n+2} + |q(m)f|_{n+1} + ([[m]]_{n+r+6} + 1)\|f\|_0.$$

4.7. Tame families of solutions

Theorem. *Let* $\mathfrak{S}(m)f = (E(m)f, p(m)f, q(m)f)$ *be a family of self-adjoint elliptic boundary value problems. Suppose* $\mathfrak{S}$ *satisfies a uniform persuasive estimate*

$$|f|_0^2 \lesssim \mathrm{Re}\langle E(m)f, f\rangle\!\rangle_m + \|f\|_0^2.$$

If $\mathfrak{S}(0)$ *is invertible, then* $\mathfrak{S}(m)$ *is invertible for all* m *in a neighborhood of* 0, *and the family of solutions*

$$\mathfrak{S}^{-1}\colon (U \subseteq \mathcal{C}^\infty(X; M)) \times (\mathcal{C}^\infty(X; F) \oplus \mathcal{C}^\infty(\partial X; P) \oplus \mathcal{C}^\infty(\partial X; Q)) \to \mathcal{C}^\infty(X, F)$$

is a smooth tame map, where $\mathfrak{S}^{-1}(m)(g, h, k) = f$ *is the unique solution of* $E(m)f = g, p(m)f = h, q(m)f = k$.

Proof. All we need is the following.

Lemma. *Under the above hypotheses we can find a neighborhood* U *of* 0 *and a number* l *such that for all* $m \in U$

$$\|f\|_0 \lesssim \|E(m)f\|_l + |p(m)f|_{l+2} + |q(m)f|_{l+1}.$$

Proof. Suppose not. Then we could find sequences $m_j \to 0$ and f_j with $\|f_j\|_0 = 1$, $E(m_j)f_j \to 0$, $p(m_j)f_j \to 0$ and $q(m_j)f_j \to 0$. By the Uniform main a-priori estimate we must prove that we would have $\|f_j\|_n \leqslant C$ for all n. Then by passing to a subsequence we could assume $f_j \to f$. In this case we would have $\|f\|_0 = 1$ and $E(0)f = 0$, $p(0)f = 0$, $q(0)f = 0$. This would make f a nonzero element in Ker $\mathfrak{S}(0)$. But $\mathfrak{S}(0)$ was assumed invertible.This would give a contradiction. Therefore the Lemma holds.

From the lemma we see that $\mathfrak{S}(m)$ is injective for ll $m \in U$. But since all the $\mathfrak{S}(m)$ satisfy persuasive estimates, if they are injective then they are surjective as well. Thus $\mathfrak{S}(m)$ is invertible for all $m \in U$.

Let us write

$$\|(g, h, k)\|_n = \|g\|_n + |h|_{n+2} + |k|_{n+1}.$$

Then combining the Uniform main a-priori estimate with the lemma we have

$$\|f\|_n \lesssim \|\mathfrak{S}(m)f\|_n + [[m]]_{n+r+6}\|\mathfrak{S}(m)f\|_l.$$

Thus with $s = r + 6$ and l fixed, for all n

$$\|\mathfrak{S}^{-1}(m)(g, h, k)\|_n \lesssim (g, h, k)\|_n + [[m]]_{n+s}\|(g, h, k)\|_l.$$

This proves that the family of solutions $\mathfrak{S}^{-1}$ is tame. That $\mathfrak{S}^{-1}$ is smooth and all its derivatives are tame follows from general considerations; see [2, Theorems 5 and 6].

4.8. Spectral families

Let $\mathcal{E}(m)f = (E(m)f, p(m)f, q(m)f)$ be a family of self-adjoint elliptic boundary value problems which satisfies a uniform persuasive estimate. We consider the family $\mathcal{E}(\lambda, m)f = (E(m)f + \lambda f, p(m)f, q(m)f)$ of self-adjoint elliptic boundary value problems for $\lambda \in \mathbf{C}$ and $m \in U$.

Lemma. *We can find a neighborhood U of 0 and $\varepsilon > 0$ such that if $m \in U$ and $|\lambda| < \varepsilon$ then for all n*

$$\|f\|_n \lesssim \|E(m)f + \lambda f\|_n + |p(m)f|_{n+2} + |q(m)f|_{n+1} + \big([[m]]_{n+r+6} + 1\big)\|f\|_0.$$

Proof. From the uniform a-priori estimate we have

$$\|f\|_n \lesssim \|E(m)f\|_n + |p(m)f|_{n+2} + |q(m)f|_{n+1} + \big([[m]]_{n+r+6} + 1\big)\|f\|_0$$

for all m in a neighborhood U of 0. If $|\lambda| < \varepsilon$ with ε sufficiently small, the lemma follows.

Lemma. *Suppose $\mathcal{E}(0, 0)$ is invertible. Then we can find a neighborhood U of 0, $\varepsilon > 0$ and a number l such that if $|\lambda| < \varepsilon$ and $m \in U$, then $\mathcal{E}(\lambda, m)$ is invertible and*

$$\|f\|_0 \lesssim \|E(m)f + \lambda f\|_l + |p(m)f|_{l+2} + |q(m)f|_{l+1}.$$

Proof. If not, we could find sequences $\lambda_j \to 0$, $m_j \to 0$, and f_j with $\|f_j\|_0 = 1$ and $E(m_j)f_j + \lambda_j f_j \to 0$, $p(m_j)f_j \to 0$ and $q(m_j)f_j \to 0$. From the previous lemma we would have $\|f_j\|_n \leqslant C$ for all n. Then by passing to a subsequence, $f_j \to f$. Now $\|f\|_0 = 1$, $E(0)f = 0$, $p(0)f = 0$ and $q(0)f = 0$. This contradicts the assumption that $\mathcal{E}(0, 0)$ is invertible. Therefore the estimate holds. This shows that $\mathcal{E}(\lambda, m)$ is injective, which implies that it is also surjective.

Let $\|(g, h, k)\|_n = \|g\|_n + |h|_{n+2} + |k|_{n+1}$. Then we have shown that if $|\lambda| < \varepsilon$ and $m \in U$, then for all n and some k and $s = r + 6$,

$$\|f\|_n \lesssim \|\mathcal{E}(\lambda, m)f\|_n + [[m]]_{n+s}\|\mathcal{E}(\lambda, m)f\|_l.$$

Let $\Sigma \subseteq \mathbf{C} \times (U \subseteq \mathcal{C}^\infty(X; M))$ denote the set of all (λ, m) for which $\mathcal{E}(\lambda, m)$ is not invertible; Σ is called the spectrum of the family. Its complement Σ^c is called the resolvent set; by the previous lemma it is open. Let $R(\lambda, m) = \mathcal{E}(\lambda, m)^{-1}$ be the family of resolvents.

Theorem. *The family of resolvents $R(\lambda, m)$ is a smooth tame map*

$$R\colon \Sigma^c \times (\mathcal{C}^\infty(X; F) \oplus \mathcal{C}^\infty(\partial X; P) \oplus \mathcal{C}^\infty(\partial X; Q)) \to \mathcal{C}^\infty(X; F).$$

Proof. Recall that $\mathbf{C}$ is graded Fréchet space in the trivial way ($\|\lambda\|_n = |\lambda|$ for all n). The previous estimate shows R is tame; that it is smooth and all its derivatives are tame follows from general considerations [2, Theorems 5 and 6].

Let γ be a closed curve in $\mathbf{C}$. The set Σ_γ^c of all m such that $(\lambda, m) \in \Sigma^c$ for all $\lambda \in \gamma$ is an open set in $U \subseteq \mathbf{C}^\infty(X; M)$. We define families of operators

$$N, \sigma\colon \Sigma_j^c \times (\mathcal{C}^\infty(X; F) \oplus \mathcal{C}^\infty(\partial X, P) \oplus \mathcal{C}^\infty(\partial X; Q)) \to \mathcal{C}^\infty(X; F),$$

$$N(m) = \int_\gamma R(\lambda, m)\frac{d\lambda}{\lambda}, \sigma(m) = \int_\gamma R(\lambda, m)\, d\lambda.$$

Theorem. *N and σ are smooth tame maps.*

Proof. We have tame estimates for $R(\lambda, m)$ or any of its derivatives in a neighborhood of each point (λ_0, m_0). Since γ is compact, for any $m_0 \in \Sigma_\gamma^c$ we can find a neighborhood U such that the same estimate holds on $\gamma \times U$. Integrating over γ produces tame estimates for N and σ or any of their derivatives.

We let $\pi(m) = \sigma(m)\mathcal{I}$ and $\rho(m) = \mathcal{I}\sigma(m)$. Then π and ρ are smooth tame families of projections. We write $\hat{H}_\gamma(m) = \operatorname{Im} \pi(m)$. By the theory for a single operator we see that

$$\hat{H}_\gamma(m) = \Sigma\, \hat{H}_\lambda(m)$$

with the sum ranging over all λ inside γ with $(\lambda, m) \in \Sigma$. However even if γ contains only one point in the spectrum of $\mathcal{E}(0)$, it may contain many points in the spectrum of $\mathcal{E}(m)$, and the nilpotency rank may be higher. Nevertheless we have the following result.

Theorem. *The spaces $\hat{H}_\gamma(m)$ have constant dimension independent of m.*

Proof. One direction is easy.

Lemma. *For all m in a neighborhood of* 0

$$\dim \hat{H}_\gamma(m) \geqslant \dim \hat{H}_\gamma(0).$$

Proof. Let $f_1, \cdots, f_n$ be a basis of $\hat{H}_\gamma(0)$. Then $\pi(0)f_j = f_j$, so by continuity the $\pi(m)f_j$ will be linearly independent for all m in a neighborhood of 0. Thus $\dim \hat{H}_\gamma(m) \geqslant \dim \hat{H}_\gamma(0)$.

The other way is a little harder. Let $\hat{H}_\lambda^j(m)$ denote the set of all solutions f of the equations

$$\begin{aligned}
&[E(m) + \lambda I]^j f = 0,\\
&p(m)[E(m) + \lambda I]^k f = 0, \quad \text{for } 0 \leqslant k \leqslant j - 1,\\
&q(m)[E(m) + \lambda I]^k f = 0, \quad \text{for } 0 \leqslant k \leqslant j - 1.
\end{aligned}$$

Then $\hat{H}_\lambda^1(m) = \operatorname{Ker} \mathcal{E}(m) + \lambda\mathcal{I}$ and $\hat{H}_\lambda(m) = \bigcup_{j=1}^\infty H_\lambda^j(m)$. Moreover we have $\hat{H}_\lambda^j(m) \subseteq \hat{H}_\lambda^{j+1}(m)$ and $\hat{H}_\lambda^j(m) = \hat{H}_\lambda(m)$ for j large enough. Also $\hat{H}_\lambda^j(m) = \hat{H}_\lambda^{j+1}(m) \Rightarrow \hat{H}_\lambda^j(m) = \hat{H}_\lambda(m)$. Thus the sequences of spaces is stable as soon as it ceases to increase. Let

$$\hat{H}^j_\gamma(m) = \Sigma\, \hat{H}^j_\lambda(m)$$

with the sum over all λ inside γ with $(\lambda, m) \in \Sigma$. Then the same properties hold for the spaces $\hat{H}^j_\gamma(m)$. In particular we have the following essential fact.

Lemma. *If* $\dim \hat{H}^j_\gamma(m) < j$, *then* $\hat{H}^j_\gamma(m) = \hat{H}_\gamma(m)$.

Proof. Unless the dimension increases by at least one each time j increases by 1, the spaces become stable.

Now to prove the theorem it is enough to show that $\dim \hat{H}_\gamma(m)$ is constant in a neighborhood of $m = 0$. Moreover we can also assume, without loss of generality, that γ contains only one eigenvalue for $m = 0$ and that that eigenvalue is $\lambda = 0$. In this case, if $(\lambda_j, m_j) \in \Sigma$ with λ_j inside γ and $m_j \to 0$, then we must have $\lambda_j \to 0$ also, since Σ is closed. In particular for any $\varepsilon > 0$ we can find a neighborhood U of 0 such that if $m \in U$, $(\lambda, m) \in \Sigma$, and λ is inside γ, then $|\lambda| < \varepsilon$.

By the first lemma in this section, we can choose $\varepsilon > 0$ and U as above so that we also have

$$\|f\|_n \lesssim \|E(m)f + \lambda f\|_n + |p(m)f|_{n+2} + |q(m)f|_{n+1} + ([[m]]_{n+r+6} + 1)\|f\|_0,$$

when $|\lambda| < \varepsilon$ and $m \in U$. Iterating this estimate j times we get

$$\|f\|_n \lesssim \|[E(m) + \lambda I]^j f\|_n + \sum_{k=0}^{j-1} |p(m)[E(m) + \lambda I]^k f|_{n+2} + \sum_{k=0}^{j-1} |q(m)[E(m) + \lambda I]^k f|_{n+1} + ([[m]]_{n+r+6} + 1) \sum_{k=0}^{j-1} \|[E(m) + \lambda I]^k f\|_0.$$

Note that for all m in a neighborhood U of zero depending on j we have

$$\sum_{k=0}^{j-1} \|[E(m) + \lambda I]^k f\|_0 \lesssim \|f\|_{2j-2}.$$

Let S be a closed subspace of $\mathcal{C}^\infty(X; F)$ complementary to $\hat{H}^j_\gamma(0)$, which surely exists by the Hahn-Banach theorem since $\hat{H}^j_\gamma(0)$ is finite dimensional.

Lemma. *We can find a neighborhood* U_j *of zero such that for all* $m \in U_j$

$$\hat{H}^j_\gamma(m) \cap S = \{0\}.$$

Proof. Suppose not. Then we can find sequences $m_i \to 0$ and $f_i \in \hat{H}^j_\gamma(m_i) \cap S$ with $f_i \neq 0$. Thus $f_i \in \hat{H}^j_{\lambda_i}(m_i) \cap S$ for some λ_i inside γ. As remarked earlier we must have $\lambda_i \to 0$. Also

$$[E(m_i) + \lambda_i I]^j f_i = 0,$$
$$p(m_i)[E(m_i) + \lambda_i I]^k f_i = 0, \quad \text{for } 0 \leqslant k \leqslant j - 1,$$
$$q(m_i)[E(m_i) + \lambda_i I]^k f_i = 0, \quad \text{for } 0 \leqslant k \leqslant j - 1.$$

Therefore by the previous estimate

$$\|f_i\|_n \lesssim ([[m]]_{n+r+6} + 1)\|f_i\|_{2j-2}.$$

Since $f_i \neq 0$ we can normalize with $\|f_i\|_{2j-2} = 1$. Then $\|f_i\|_n \leqslant C$ so by passing to a subsequence we may assume $f_i \to f$ for some $f \in \mathcal{C}^\infty(X; F)$. Then by continuity

$$E(0)^j f = 0,$$
$$p(0)E(0)^k f = 0, \quad \text{for } 0 \leqslant k \leqslant j - 1,$$
$$q(0)E(0)^k f = 0, \quad \text{for } 0 \leqslant k \leqslant j - 1,$$

so $f \in \hat{H}^j_\gamma(0)$. Also $f_i \in S$, and S is closed so $f \in S$, and $\|f_i\|_{2j-2} = 1$ so $\|f\|_{2j-2} = 1$. Thus $f \in \hat{H}^j_\gamma(0) \cap S$ and $f \neq 0$, which contradicts the choice of S as a closed complement.

Corollary. *On the neighborhood U_j of zero we have*

$$\dim \hat{H}^j_\gamma(m) \leqslant \dim \hat{H}^j_\gamma(0).$$

Now since U_j depends on j we must argue carefully. We choose j with $j > \dim \hat{H}_\gamma(0)$. Then for all $m \in U_j$

$$\dim \hat{H}^j_\gamma(m) \leqslant \dim \hat{H}^j_0(0) \leqslant \dim \hat{H}_0(0) < j.$$

As we argued before, this implies $\hat{H}^j_\gamma(m) = \hat{H}_\gamma(m)$ for all $m \in U_j$. Thus we have found a neighborhood of zero on which $\dim \hat{H}_\gamma(m) = \dim \hat{H}_\gamma(0)$. This proves the theorem.

4.9. Tame Fredholm theory

We give a brief sketch of how to generalize classical Fredholm theory to the category of graded Fréchet spaces and smooth tame families of linear maps. Let $E, F, G, \cdots$ denote graded Fréchet spaces. A family of linear maps is a map

$$L: U \subseteq E \times F \to G$$

such that $L(m)f$ is linear in $f \in F$ for each $m \in U \subseteq E$. We take derivatives of L only with respect to m (since L is linear in f, derivatives with respect to f

produce no new information and mess up the notation). Thus

$$DL(m; n)f = \lim_{t \to 0} \left[L(m + tn)f - L(m)f \right] / t.$$

We say L is smooth if all the derivatives

$$D^k L: (U \subseteq E) \times E \times \cdots \times E \times F \to G$$

exist and are continuous. Note that $D^k L$ is again a family of linear maps.

The family of linear maps $L(m)f$ is tame if and only if for each $m_0 \in U$ we can find a neighborhood U_0 and a number s such that for all n

$$\|L(m)f\|_n \lesssim \|f\|_{n+s} + \|m\|_{n+s}\|f\|_s.$$

The family L is a smooth tame family of linear maps if each $D^n L$ exists and is tame. If $L(m)f$ is any family of linear partial differential operators in f whose coefficients are nonlinear partial differential operators in m, then L is a smooth tame family of linear maps.

A linear map L: $F \to G$ is said to be compact if there is an open set in F whose image under L is compact in G. This is equivalent to the following condition. We say the norm $|\ |$ is compact if there exists some open set which has compact closure in the $|\ |$-topology. On a Montel space like $\mathcal{C}^\infty(X; F)$ the topology can be defined by a basis of compact norms. A linear map L: $F \to G$ is compact if and only if there is a compact norm $|\ |$ on G such that for every norm $\|\ \|_n$ on F we have $\|Lf\|_n \lesssim |f|$. Now we are ready for a new definition.

Definition. The family of linear maps $K(m)f$ is tamely compact if for every m_0 we can find a neighborhood U_0, a number s and a compact norm $|\ |$ such that for every n

$$\|K(m)f\|_n \lesssim (\|m\|_{n+s} + 1)|f|.$$

We say K is a smooth tamely compact family if K is smooth and all its derivatives $D^n K$ are tamely compact.

Example. Let X be a compact manifold with a smooth volume dx, and let k: $X \times R \times X \to R$ be a smooth kernel. Define the family of linear integral operators

$$K: \mathcal{C}^\infty(X) \times \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X),$$

$$K(m)f(x) = \int_X k(x, m(x), y) f(y)\, dy.$$

Then K is a smooth tamely compact family of linear maps.

Example. If G is finite dimensional, then every smooth family of linear maps L: $U \subseteq E \times F \to G$ is a smooth tamely compact family.

Theorem. *The composition (either way) of a smooth tame family and a smooth tamely compact family is a smooth tamely compact family.*

Proof. This follows immediately from the definition.

Theorem. *Let π: $U \subseteq E \times F \to F$ be a smooth tame family of linear projections, so that $\pi(m)\pi(m)f = \pi(m)f$. Then π is a smooth tamely compact family of linear maps if and only if each space* Im $\pi(m)$ *is finite dimensional and the dimension is constant.*

Proof. First suppose that π is a smooth tamely compact family of linear projections. Since

$$\|\pi(m)f\|_n \lesssim (\|m\|_{n+s} + 1)|f|,$$

we see that if $f \in \operatorname{Im} \pi(m)$ then

$$\|f\|_n \lesssim (\|m\|_{n+s} + 1)|f|.$$

Therefore the unit ball $|f| \leqslant 1$ is compact in Im $\pi(m)$. This proves that each space Im $\pi(m)$ is finite dimensional. If $f_1, \cdots, f_N$ are a basis for Im $\pi(m_0)$, then $\pi(m_0)f_j = f_j$, so by continuity the $\pi(m)f_j$ are linearly independent for all m in a neighborhood of m_0. Thus dim Im $\pi(m) \geqslant$ dim Im $\pi(m_0)$ for all m in a neighborhood of m_0. In the other direction, let S be a closed complement for Im $\pi(m_0)$, which exists by the Hahn-Banach theorem. We claim that $\pi(m) \cap S = \{0\}$ for all m in a neighborhood of m_0. This will show that dim Im $\pi(m)$ $\leqslant$ dim Im $\pi(m_0)$ for all m in a neighborhood of m_0, and hence the dimension of Im $\pi(m)$ is constant. If the above fails, we can find sequences $m_j \to m_0$ and nonzero $f_j \in \operatorname{Im} \pi(m_j) \cap S$. We normalize with $|f_j| = 1$. Since $\pi(m_j)f_j = f_j$ we have

$$\|f_j\|_n \lesssim (\|m_j\|_{n+s} + 1)|f_j|,$$

so $\|f_j\|_n \leqslant C$ for all n. By passing to a subsequence we can assume $f_j \to f$. Then by continuity $\pi(m_0)f = f$, so $f \in \operatorname{Im} \pi(m_0) \cap S$; and $|f| = 1$ so $f \neq 0$. This gives a contradiction.

In the other direction, suppose that π is a smooth tame family of projections whose images are finite dimensional and of constant dimension. For simplicity take $m_0 = 0$. Write $H(m) = \operatorname{Im} \pi(m)$. Then if $A(m) = \pi(0)\pi(m)$,

$$A\colon U \subseteq E \times H(0) \to H(0)$$

is a smooth family of linear automorphisms of a finite dimensional vector space, and $A(0)$ is the identity. Therefore for all m in a neighborhood of 0, $A(m)$ is invertible, and the family of inverses

$$A^{-1}\colon U \subseteq E \times H(0) \to H(0)$$

defined by $A^{-1}(m)g = f$, if $A(m)f = g$, is also a smooth family of linear maps. We can see this immediately by choosing a basis for $H(0)$ and observing that the entries are smooth functions of m, and hence the determinant is a smooth nonzero function of m whose inverse is therefore smooth. Moreover A^{-1} maps into a finite dimensional space, so A^{-1} is a smooth tamely compact family. Now observe that if dim $H(m)$ is constant, then when $A(m) = \pi(0)\pi(m)$ is invertible we must have both $\pi(0)$: $H(m) \to H(0)$ and $\pi(m)$: $H(m) \to H(0)$ invertible; and therefore

$$\pi(m) = \pi(m)A^{-1}(m)\pi(0)\pi(m).$$

Hence $\pi(m)$ is a composition of smooth tame families with a smooth tamely compact family, so $\pi(m)$ is a smooth tamely compact family, as claimed.

Question. Are there any smooth tame families of projections whose image spaces are finite dimensional but not of constant dimension?

Corollary. *The projections $\pi(m)$ of* §8 *are smooth tamely compact families of linear maps. So are the projections $\rho(m)$.*

Definition. Let $L(m)f$ be a smooth tame family of linear maps. We say that L is a smooth tame Fredholm family if there exists another smooth tame family $M(m)g$ such that

$$K_1(m)g = L(m)M(m)g - g, \quad K_2(m)f = M(m)L(m)f - f$$

are smooth tamely compact families.

Example. From §8 we have

$$N(m)\mathfrak{S}(m)f + \pi(m)f = f,$$
$$\mathfrak{S}(m)N(m)(g, h, k) + \rho(m)(g, h, k) = (g, h, k).$$

Therefore $\mathfrak{S}(m)f$ is a smooth tame Fredholm family.

Theorem. *The composition of two smooth tame Fredholm families is a smooth tame Fredholm family.*

Proof. This follows directly from the definition.

Lemma. *Let K: $U \subseteq E \times F \to F$ be a smooth tamely compact family of linear maps of F into itself, and suppose that for some m_0 we have $K(m_0)f = 0$ for all f. Then for every k sufficiently large and for every $\varepsilon > 0$ we can find a neighborhood U_0 of m_0 such that if $m \in U_0$ and $f \in F$, then*

$$\|K(m)f\|_k \leqslant \varepsilon\|f\|_k.$$

Proof. Choose k so large that the unit ball $B = \{\|f\|_k \leqslant 1\}$ has compact closure in the topology of $|\ |$. For simplicity we can assume $|f| \leqslant \|f\|_k$ without loss of generality. Then for any $\eta > 0$ we can cover B by a finite number of open sets $\{|f - f_i| < \eta\}$ with $f_1, \cdots, f_N \in B$. Since K is continuous and $K(m_0) = 0$, the sets $U_i = \{\|K(m)f_i\|_k < \eta\}$ are all open in $U \subseteq E$; hence so is their intersection $U_0 = U_1 \cap \cdots \cap U_N$. For any f with $\|f\|_k \leqslant 1$ we have $|f - f_i| < \eta$ for some i, so for all $m \in U_0$

$$\begin{aligned}\|K(m)f\|_k &\leqslant \|K(m)f_i\|_k + \|K(m)(f - f_i)\|_k \\ &\leqslant \eta + C(\|m\|_{k+s} + 1)\eta,\end{aligned}$$

using the fact that since K is tamely compact

$$\|K(m)f\|_k \leqslant C(\|m\|_{k+s} + 1)|f|.$$

We further restrict U_0 so that $\|m\|_{k+s} \leqslant 1$. Then $\|K(m)f\|_k \leqslant C\eta$ with a constant C independent of η, when $\|f\|_k \leqslant 1$. By linearity, $\|K(m)f\|_k \leqslant C\eta\|f\|_k$ for all k. Choose η so small that $C\eta < \varepsilon$. Then $\|K(m)f\|_k \leqslant \varepsilon\|f\|_k$. This proves the lemma.

Theorem. *Let $K\colon U \subseteq E \times F \to F$ be a smooth tamely compact family of linear maps of F into itself with $K(m_0)f = 0$. Let $P(m)f = f + K(m)f$, so that $P\colon U \subseteq E \times F \to F$ is a smooth tame family of linear maps. Then on some neighborhood U_0 of m_0 the maps $P(m)$ are all invertible; and the family of inverses $P^{-1}\colon U_0 \subseteq E \times F \to F$ defined by $P^{-1}(m)g = f$, if $P(m)f = g$, is a smooth tame family.*

Proof. By the previous lemma, if k is large and U_0 small enough we have

$$\|K(m)f\|_k \leqslant \varepsilon\|f\|_k, \quad \|P(m)f\|_k = \|f + K(m)f\|_k \geqslant (1 - \varepsilon)\|f\|.$$

When $\varepsilon < 1$ we see that $P(m)$ is invertible for all $m \in U_0$. For by ordinary Fredholm theory $P(m)$ always has finite dimensional kernel and cokernel of equal dimension; the estimate shows that $P(m)$ is injective, and therefore it is surjective also. Next

$$\|f\|_n \leqslant \|P(m)f\|_n + \|K(m)f\|_n, \quad \|K(m)f\|_n \lesssim (\|m\|_{n+s} + 1)|f|.$$

If k is large enough, $|f| \leqslant \|f\|_k$. We saw before that $\|f\|_k \leqslant C\|P(m)f\|_k$ with $C = 1/(1 - \varepsilon)$. Thus

$$\|f\|_n \lesssim \|P(m)f\|_n + \|m\|_{n+s}\|P(m)f\|_k.$$

Let $g = P(m)f$ so $f = P^{-1}(m)g$. Then

$$\|P^{-1}(m)g\|_n \lesssim \|g\|_n + \|m\|_{n+s}\|g\|_k.$$

This proves P^{-1} is tame.

We claim P^{-1} is also continuous. It is enough to show that it is sequentially continuous. Let $m_j \to m'$ and $g_j \to g'$ be two converging sequences, and let $f_j = P^{-1}(m_j)g_j$. By the tame estimate we know at least that $\|f_j\|_n \leqslant C$ for all n. Let $f' = P^{-1}(m')g'$. Unless $f_j \to f'$ we can find a subsequence f_j which avoids a neighborhood of f'. Then yet another subsequence f_j will have the property of being Cauchy in the compact norm $|\ |$. For all n

$$\|f_i - f_j\|_n \leqslant \|g_i - g_j\|_n + \|K(m_i)f_i - K(m_j)f_j\|_n.$$

Define

$$\tilde{K}(m)f = K(m)f - K(m')f.$$

Then $\tilde{K}(m')f = 0$ for all f. By a previous lemma if n is large enough then for any $\varepsilon > 0$ we will have

$$\|\tilde{K}(m)f\|_n \leqslant \varepsilon\|f\|_n$$

for all m in a neighborhood of m'. Thus

$$\|\tilde{K}(m_j)f_j\|_n \leqslant \varepsilon\|f_j\|_n$$

for all large j, since $m_j \to m'$. Write

$$K(m_i)f_i - K(m_j)f_j = K(m')(f_i - f_j) + \tilde{K}(m_i)f_i - \tilde{K}(m_j)f_j.$$

Then for all large i and j

$$\|f_i - f_j\|_n \leqslant \|g_i - g_j\|_n + C|f_i - f_j| + \varepsilon(\|f_i\|_n + \|f_j\|_n).$$

Here C depends only on m', and $\varepsilon > 0$ is arbitrarily small if i, j are large enough. Since $g_j \to g'$, we have $\|g_i - g_j\|_n \to 0$ for all n. Then $\|f_i - f_j\|_n \to 0$ for all n also (recall f_j is Cauchy in $|\ |$). Then $f_j \to f''$ for some f'', and $P(m')f'' = g'$ by continuity. Thus $f'' = P^{-1}(m')g' = f'$ which shows P^{-1} is continuous.

Now from general considerations [2, Theorem 5] it follows that P^{-1} is smooth and all its derivatives are tame also.

Corollary. *Let $K\colon U \subseteq E \times F \to F$ be a smooth tamely compact family of linear maps, and let $P(m)f = f + K(m)f$. Suppose $P(m_0)$ is invertible. Then $P(m)$ is invertible for all m in a neighborhood U_0 of m_0, and the family of inverses $P^{-1}\colon U_0 \subseteq E \times F \to F$ is a smooth tame family. Moreover $P^{-1}(m)f = f + L(m)f$ where $L\colon U_0 \subseteq E \times F \to F$ is a smooth tamely compact family.*

Proof. Since $P(m_0)$ is invertible, write $P(m_0)^{-1} = I + L(m_0)$. Then

$$[I + K(m_0)][I + L(m_0)] = I, \quad L(m_0) = -K(m_0) - K(m_0)L(m_0).$$

Since $K(m_0)$ is compact, so is $L(m_0)$. Now let $\tilde{P}(m) = P(m_0)^{-1}P(m)$. Then $\tilde{P}(m) = I + \tilde{K}(m)$ where

$$\tilde{K}(m) = L(m_0) + K(m) + L(m_0)K(m).$$

Therefore $\tilde{K}$ is a smooth tamely compact family of linear maps with $\tilde{K}(m_0) = 0$. By the previous theorem we can find a neighborhood U_0 of m_0 on which the $\tilde{P}(m)$ are all invertible, and the family of inverses $\tilde{P}^{-1}(m)f$ is a smooth tame family of linear maps. Let

$$P^{-1}(m)f = \tilde{P}^{-1}(m)P(m_0)^{-1}f.$$

Since $P(m) = P(m_0)\tilde{P}(m)$, we see that $P(m)$ is invertible for all m in U_0 and $P(m)^{-1} = P^{-1}(m)$. Finally write

$$P^{-1}(m) = I + L(m).$$

Then as before

$$L(m) = -K(m) - K(m)L(m),$$

so $L(m)f$ is a smooth tamely compact family of linear maps.

Corollary. *Let L: $U \subseteq E \times F \to G$ be a smooth tame Fredholm family of linear maps. If $L(m_0)$ is invertible, then $L(m)$ is invertible for all m in a neighborhood U_0 of m_0, and the family of inverses L^{-1}: $U_0 \subseteq E \times G \to F$ is a smooth tame family of linear maps.*

Proof. We can find a smooth tame family $M(m)$ such that $L(m)M(m) = I + K(m)$ where $K(m)$ is a smooth tamely compact family. Let $\tilde{M}(m) = M(m) - L^{-1}(m_0)K(m_0)$. Then $\tilde{M}$ is also a smooth tame family, and $L(m)\tilde{M}(m) = I + \tilde{K}(m)$ where $\tilde{K}(m) = K(m) - L(m)L^{-1}(m_0)K(m_0)$ is also a smooth tamely compact family with $\tilde{K}(m_0) = 0$. By the previous theorem, $P(m) = I + \tilde{K}(m)$ is invertible for all m in a neighborhood U_0 of m_0, and the family of inverses P^{-1} is a smooth tame family. Then so is $\tilde{M}(m)P^{-1}(m)f$, and $L(m)\tilde{M}(m)P^{-1}(m)f = f$ so $L(m)$ is invertible and $L^{-1}(m)f = \tilde{M}(m)P^{-1}(m)f$ is a smooth tame family.

Theorem. *Let L: $U \subseteq E \times F \to G$ be a smooth tame Fredholm family of linear maps. Then the index*

$$i(m) = \dim \operatorname{Ker} L(m) - \operatorname{codim} \operatorname{Im} L(m)$$

is constant.

Proof. Fix $m_0 \in U$. Write $E = E' \oplus E''$ and $F = F' \oplus F''$ such that E' and F' are finite dimensional and the composition

$$E'' \xrightarrow{i''_E} E' \oplus E'' \xrightarrow{L(m_0)} F' \oplus F'' \xrightarrow{\pi''_F} F''$$

$\pi''_F \circ L(m_0) \circ i''_E$ is an isomorphism. Let $\tilde{L}(m) = \pi''_F \circ L(m) \circ i''_E$. Then $\tilde{L}(m)$ is also a smooth tame Fredholm family. For suppose that $M(m)f$ is another smooth tame family such that $L(m)M(m)f - f = K_1(m)f$ and $M(m)L(m)f - f = K_2(f)$ are smooth tamely compact families. Let $\tilde{M}(m) = \pi''_E \circ M(m) \circ i''_F$. Then $i''_F\pi''_F = I - \rho'_F$ and $i''_E\pi''_E = I - \rho'_E$ where ρ'_E and ρ'_F are the projections on the finite dimensional spaces E' and F'. We have

$$\tilde{L}(m)\tilde{M}(m) - I = K_1(m) + \pi''_F \circ L(m) \circ \rho'_E \circ M(m) \circ i''_F,$$

$$\tilde{M}(m)\tilde{L}(m) - I = K_2(m) + \pi''_E \circ M(m) \circ \rho' \circ L(m) \circ i''_E.$$

Thus $\tilde{L}$ is also a smooth tame Fredholm family. Moreover $\tilde{L}(m_0)$ is invertible, so $\tilde{L}(m)$ is invertible for all m in a neighborhood U_0 of m_0. It follows easily that $i(m) = \dim E' - \dim F'$ is constant on U_0.

PART 5. ELLIPTIC COMPLEXES

5.1. Definition

Let X be a compact manifold with smooth boundary ∂X, and let E, F, G be vector bundles over X. An elliptic complex of degree one consists of two linear partial differential operators A and B of degree 1 with $BA = 0$:

$$\mathcal{C}^\infty(X; E) \xrightarrow{A} \mathcal{C}^\infty(X; F) \xrightarrow{B} \mathcal{C}^\infty(X; G)$$

such that for all nonzero real cotangent vectors ξ the sequence of symbols

$$E \xrightarrow{\sigma_A(\xi)} F \xrightarrow{\sigma_B(\xi)} G$$

is exact, i.e., $\operatorname{Im} \sigma_A(\xi) = \operatorname{Ker} \sigma_B(\xi)$.

We wish to construct a splitting of the complex; this means to find two continuous linear maps K and L

$$\mathcal{C}^\infty(X; E) \underset{K}{\overset{A}{\rightleftarrows}} \mathcal{C}^\infty(X; F) \underset{L}{\overset{B}{\rightleftarrows}} \mathcal{C}^\infty(X; G)$$

such that $AK + LB = I$. This is equivalent to asserting (1) $\operatorname{Im} A = \operatorname{Ker} B$, (2) $\operatorname{Im} B$ is closed (3) $\operatorname{Ker} A$, $\operatorname{Im} A = \operatorname{Ker} B$, $\operatorname{Im} B$ are all split, i.e., are direct summands. More generally we may have only that $AK + LB + H = I$, where H is a projection on a finite dimensional subspace representing the cohomology group $\operatorname{Ker} B/\operatorname{Im} A$.

Choose hermitian metrics $\langle\ ,\ \rangle$ on E, F, G and a volume element dV on X. Form the inner product

$$\langle\langle f, g \rangle\rangle = \int_X\int \langle f, g\rangle \, dV.$$

Let ν be any nonzero normal cotangent vector field. There exists an adjoint linear partial differential operator A^* such that

$$\langle\langle Ae, f\rangle\rangle + \langle\langle e, A^* f\rangle\rangle = 0,$$

if $\sigma_A(\nu)e = 0$ or if $\sigma_{A^*}(\nu) f = 0$ on ∂X.

Since $\operatorname{Im} \sigma_A(\xi) = \operatorname{Ker} \sigma_B(\xi)$ when $\xi \neq 0$, and since the dimensions of images and kernels are semi-continuous, one above and one below, it follows that $\dim \operatorname{Im} \sigma_A(\xi) = \dim \operatorname{Ker} \sigma_B(\xi)$ is constant, and hence $\dim \operatorname{Ker} \sigma_A(\xi)$ and $\dim \operatorname{Im} \sigma_B(\xi)$ are constant also; the same holds for the adjoints. Therefore $\operatorname{Ker} \sigma_A(\nu)$ is a subbundle of $E|\partial X$. Let P be isomorphic to the quotient bundle, and let $a\colon E|\partial X \to P$ be surjective with $\operatorname{Ker} a = \operatorname{Ker} \sigma_A(\nu)$. We give P the quotient hermitian metric; thus a: $\operatorname{Im} \sigma_A(\nu)^* \to P$ is an isometry since $\operatorname{Im} \sigma_A(\nu)^* = \operatorname{Ker} \sigma_A(\nu)^\perp$. Define $a^*\colon F|\partial X \to P$ by $a^* = a\sigma_A(\nu)^*$. Then a^* is surjective and $\operatorname{Ker} a^* = \operatorname{Ker} \sigma_A(\nu)^*$; also

$$\langle a^* f, a^* g\rangle = \langle \sigma_A(\nu)^* f, \sigma_A(\nu)^* g\rangle, \quad \langle ae, a^* f\rangle = \langle e, \sigma_A(\nu)^* f\rangle.$$

Let dS be the volume on ∂X with $dV = dS \wedge \nu$, and write

$$\langle f, g\rangle = \int_{\partial X} \langle f, g\rangle \, dS.$$

By Green's formula

$$\langle\!\langle Ae, f\rangle\!\rangle + \langle\!\langle e, A^*f\rangle\!\rangle = \langle ae, a^*f\rangle$$

for all e and f.

Similarly we have an adjoint operator B such that

$$\langle\!\langle Bf, g\rangle\!\rangle + \langle\!\langle f, B^*g\rangle\!\rangle = 0$$

if $\sigma_B(\nu)f = 0$ or if $\sigma_{B^*}(\nu)g = 0$. Ker $\sigma_{B^*}(\nu)$ is a subbundle of $G|\partial X$. Let Q be isomorphic to the quotient bundle, and let $b^*\colon G|\partial X \to Q$ be surjective with Ker $b^* =$ Ker $\sigma_{B^*}(\nu)$. We give Q the quotient hermitian metric; thus b^*: Im $\sigma_B(\nu) \to Q$ is an isometry. Define $b\colon F|\partial X \to Q$ by $b = b^*\sigma_B(\nu)$. Then b is surjective and Ker $b =$ Ker $\sigma_B(\nu)$; also

$$\langle bf, bg\rangle = \langle \sigma_B(\nu)f, \sigma_B(\nu)g\rangle, \quad \langle bf, b^*g\rangle = \langle \sigma_B(\nu)f, g\rangle.$$

By Green's formula

$$\langle\!\langle Bf, g\rangle\!\rangle + \langle\!\langle f, B^*g\rangle\!\rangle = \langle bf, b^*g\rangle.$$

5.2. The associated elliptic boundary value problem

To each elliptic complex there is a natural way to associate a self-adjoint elliptic boundary value problem (in the sense of 2.1). We choose metrics and adjoints as before, and let

$$\begin{aligned} Ef &= AA^*f + B^*Bf, && \text{on } X,\\ pf &= a^*f, && \text{on } \partial X,\\ qf &= b^*Bf, && \text{on } \partial X. \end{aligned}$$

Lemma. *$\mathfrak{E} = (E, p, q)$ is a self-adjoint elliptic boundary value problem.*

Proof. First note that

$$\sigma_E(\xi) = \sigma_A(\xi)\sigma_{A^*}(\xi) + \sigma_{B^*}(\xi)\sigma_B(\xi),$$

$\sigma_{A^*}(\xi) = \sigma_A(\xi)^*$, and $\sigma_{B^*}(\xi) = \sigma_B(\xi)^*$. Then

$$\langle \sigma_E(\xi)f, f\rangle = \langle \sigma_A(\xi)^*f, \sigma_A(\xi)^*f\rangle + \langle \sigma_B(\xi)f, \sigma_B(\xi)f\rangle.$$

Since the complex AB is elliptic, Ker $\sigma_A(\xi)^* \cap$ Ker $\sigma_B(\xi) = \{0\}$ for all non-zero real cotangent vectors ξ. Thus

$$\langle \sigma_E(\xi)f, f\rangle > 0 \text{ if } f \neq 0 \quad \text{and } \xi \neq 0.$$

Also $\sigma_E(\xi)^* = \sigma_E(\xi)$, so

$$\langle \sigma_E(\xi)f, g\rangle = \langle f, \sigma_E(\xi)g\rangle.$$

Next we have

$$\begin{aligned}\langle\sigma_E(\nu)f, g\rangle &= \langle\sigma_A(\nu)^*f, \sigma_A(\nu)^*g\rangle + \langle\sigma_B(\nu)f, \sigma_B(\nu)g\rangle \\ &= \langle a^*f, a^*g\rangle + \langle bf, bg\rangle \\ &= \langle\sigma_p f, \sigma_p g\rangle + \langle\sigma_q(\nu)f, \sigma_q(\nu)g\rangle,\end{aligned}$$

since $\sigma_p = a^*$ and $\sigma_q(\nu) = b^*\sigma_B(\nu) = b$. Now $\sigma_E(\xi) = \sigma_A(\xi)\sigma_A(\xi)^* + \sigma_B(\xi)^*\sigma_B(\xi)$, so

$$D\sigma_E(\nu; \eta) = \sigma_A(\eta)\sigma_A(\nu)^* + \sigma_A(\nu)\sigma_A(\eta)^* + \sigma_B(\eta)^*\sigma_B(\nu) + \sigma_B(\nu)^*\sigma_B(\eta).$$

Suppose $\sigma_p f = 0$ and $\sigma_p g = 0$. Then $a^*f = 0$ and $a^*g = 0$, so $\sigma_A(\nu)^*f = 0$ and $\sigma_A(\nu)^*g = 0$. Therefore

$$\langle D\sigma_E(\nu; \eta)f, g\rangle = \langle\sigma_B(\nu)f, \sigma_B(\eta)g\rangle + \langle\sigma_B(\eta)f, \sigma_B(\nu)g\rangle.$$

Now b^* is an isometry on $\operatorname{Im}\sigma_B(\nu)$ and is zero on its orthogonal complement. Thus

$$\langle f, g\rangle = \langle b^*f, b^*g\rangle,$$

if $f \in \operatorname{Im}\sigma_B(\nu)$ or $g \in \operatorname{Im}\sigma_B(\nu)$. Therefore

$$\begin{aligned}\langle\sigma_B(\nu)f, \sigma_B(\eta)g\rangle &= \langle b^*\sigma_B(\nu)f, b^*\sigma_B(\eta)g\rangle, \\ \langle\sigma_B(\eta)f, \sigma_B(\nu)g\rangle &= \langle b^*\sigma_B(\eta)f, b^*\sigma_B(\nu)g\rangle.\end{aligned}$$

But $\sigma_q(\xi) = b^*\sigma_B(\xi)$. Thus

$$\langle D\sigma_E(\nu; \eta)f, g\rangle = \langle\sigma_q(\nu)f, \sigma_q(\eta)g\rangle + \langle\sigma_q(\eta)f, \sigma_q(\nu)g\rangle.$$

Therefore $\mathfrak{S} = (E, p, q)$ satisfies all the conditions (1)–(5) of §2.1. This is not surprising, for conditions (1)–(5) are just those required to integrate by parts to transform $\langle\langle Ef, g\rangle\rangle$ into an essentially hermitian symmetric bilinear form. But if $qf = b^*Bf = 0$ and $pg = a^*g = 0$, then

$$\langle\langle Ef, g\rangle\rangle = \langle\langle AA^*f, g\rangle\rangle + \langle\langle B^*Bg, g\rangle\rangle = \langle\langle A^*f, A^*g\rangle\rangle + \langle\langle Bf, Bg\rangle\rangle,$$

which is (truly) hermitian symmetric. For general f and g the boundary integrals are

$$\langle\sigma_A(\nu)A^*f, g\rangle = \langle aA^*f, a^*g\rangle, \quad \langle\sigma_B(\nu)^*Bf, g\rangle = \langle b^*Bf, bg\rangle.$$

Thus the adjoint boundary conditions are

$$p'f = aA^*f, \quad q'g = bg.$$

Recall that $H_0 = \operatorname{Ker}\mathfrak{S}$ is the set

$$H_0 = \{h\colon AA^*h + B^*Bh = 0, a^*h = 0, b^*Bh = 0\}.$$

Lemma.

$$H_0 = \{h\colon A^*h = 0, a^*h = 0, Bh = 0\}.$$

Proof. If $h \in H_0$, then

$$\|A^*h\|^2 + \|Bh\|^2 = \langle\!\langle A^*h, A^*h\rangle\!\rangle + \langle\!\langle Bh, Bh\rangle\!\rangle = \langle\!\langle AA^*h + B^*Bh, h\rangle\!\rangle = 0,$$

so $A^*h = 0$ and $Bh = 0$.

5.3. Splitting the complex

The importance of the associated self-adjoint elliptic boundary value problem is revealed by the next result.

Splitting theorem. *Suppose that the associated self-adjoint elliptic boundary value problem*

$$\mathfrak{S}f = (AA^*f + B^*Bf, a^*f, b^*Bf)$$

is an isomorphism. Then the complex AB splits. The splitting is given by two maps K and L:

$$Kf = A^*\mathfrak{S}^{-1}(f, 0, 0), \quad Lg = \mathfrak{S}^{-1}(B^*g, 0, b^*g), \quad AK + LB = I.$$

Proof. We begin with the following observation.

Lemma. *If $b^*g = 0$, then $a^*B^*g = 0$.*

Proof. If $b^*g = 0$, then for all f

$$\langle\!\langle B^*g, Af\rangle\!\rangle = \langle\!\langle g, BAf\rangle\!\rangle = 0 = \langle\!\langle A^*B^*g, f\rangle\!\rangle,$$

so we must have $a^*B^*g = 0$.

Now let $h = \mathfrak{S}^{-1}(f, 0, 0)$. Then $Kf = A^*h$. From the definition of $\mathfrak{S}$

$$\begin{aligned} AA^*h + B^*Bh &= f, && \text{on } X,\\ a^*h &= 0, && \text{on } \partial X,\\ b^*Bh &= 0, && \text{on } \partial X. \end{aligned}$$

Therefore $BB^*Bh = Bf$. Also by the previous Lemma we have $a^*B^*Bh = 0$. Therefore

$$\begin{aligned} AA^*(B^*Bh) + B^*B(B^*Bh) &= B^*Bf, && \text{on } X,\\ a^*(B^*Bh) &= 0, && \text{on } \partial X,\\ b^*B(B^*Bh) &= b^*Bf, && \text{on } \partial X. \end{aligned}$$

This shows that

$$B^*Bh = \mathfrak{S}^{-1}(B^*Bf, 0, b^*Bf) = LBf.$$

Therefore $AKf + LBf = f$ or $AK + LB = I$.

5.4. Nonzero cohomology

We say that the complex AB satisfies a persuasive estimate (or subelliptic estimate) if

$$|f|_0^2 \lesssim \|A^*f\|_0^2 + \|Bf\|_0^2 + \|f\|_0^2 \quad \text{when } a^*f = 0.$$

Since $\langle\!\langle Ef, f\rangle\!\rangle = \|A^*f\|_0^2 + \|Bf\|_0^2$ when $a^*f = 0$ and $b^*Bf = 0$, this is clearly equivalent to saying that the associated self-adjoint elliptic boundary value problem $\mathcal{E}f = (AA^*f + B^*Bf, a^*f, b^*Bf)$ satisfies a persuasive estimate:

$$|f|_0^2 \lesssim \operatorname{Re}\langle\!\langle Ef, f\rangle\!\rangle + \|f\|_0^2 \quad \text{when } a^*f = 0 \text{ and } b^*Bf = 0.$$

Let γ be a curve in the complex plane avoiding the eigenvalues of $\mathcal{E}$. Then we have the operators N and σ of §3.8 and the projections π and ρ. Recall that we defined $\mathcal{E}_\lambda = \mathcal{E} + \lambda\mathcal{I}$ and

$$H_\lambda = \operatorname{Ker} \mathcal{E}_\lambda = \{h: Eh + \lambda h = 0, ph = 0, qh = 0\},$$

$$\hat{H}_\lambda = \text{the smallest subspace such that}$$

$$Eh + \lambda h \in \hat{H}_\lambda, ph = 0, qh = 0 \Rightarrow h \in \hat{H}_\lambda.$$

We say that $\mathcal{E}$ is totally self-adjoint (and not just on the symbol level) if

$$\langle\!\langle Ef, g\rangle\!\rangle = \langle\!\langle f, Eg\rangle\!\rangle$$

when $pf = 0, qf = 0, pg = 0, qg = 0$.

Lemma. *Suppose $\mathcal{E}$ is totally self-adjoint and satisfies a persuasive estimate. Then all the eigenvalues are real. Moreover $H_\lambda = \hat{H}_\lambda$ for each λ, the projection π is an orthogonal projection and* $\operatorname{Im} \sigma = \operatorname{Im} \pi$.

Proof. First we claim that

$$\mathcal{I}f \in \operatorname{Im} \mathcal{E}_\lambda \Leftrightarrow f \in H_\lambda^\perp.$$

For if $\mathcal{E}_\lambda g = \mathcal{I}f$, then $Eg + \lambda g = f$, $pg = 0$ and $qg = 0$. If $h \in H_\lambda$, then $Eh + \lambda h = 0, ph = 0, qh = 0$. Since E is totally self-adjoint,

$$\langle\!\langle f, h\rangle\!\rangle = \langle\!\langle Eg + \lambda g, h\rangle\!\rangle = \langle\!\langle g, Eh + \lambda h\rangle\!\rangle = 0.$$

Therefore $f \in H_\lambda^\perp$. But we know that $\dim \operatorname{Ker} \mathcal{E}_\lambda = \operatorname{codim} \operatorname{Im} \mathcal{E}_\lambda$. Therefore there can be no more relations, so the reverse implication holds also. Next we claim that

$$Eh + \lambda h \in H_\lambda, \quad Ph = 0, \quad qh = 0 \Rightarrow h \in H_\lambda.$$

For given any $f \in H^\perp$, we can find g with $Eg + \lambda g = f, pg = 0, qg = 0$ by the preceding argument. Moreover we can modify g by an arbitrary element in $\operatorname{Ker} \mathcal{E}_\lambda = H_\lambda$, so we may assume $g \in H_\lambda^\perp$. Then, if $Eh + \lambda h \in H_\lambda$, $ph = 0, qh = 0$, we have

$$\langle\!\langle h, f\rangle\!\rangle = \langle\!\langle h, Eg + \lambda g\rangle\!\rangle = \langle\!\langle Eh + \lambda h, g\rangle\!\rangle = 0.$$

Thus $h \perp H_\lambda^\perp$, so $h \in H_\lambda$. But this shows H_λ has the property for which $\hat{H}_\lambda$ is minimal; hence $\hat{H}_\lambda = H_\lambda$.

Now we must show that π is an orthogonal projection and $\operatorname{Im} \pi = \operatorname{Im} \sigma$. Any closed curve γ contains only finitely many eigenvalues. Therefore it is sufficient to prove the result when γ contains only one eigenvalue λ (since the general case is just a sum of such special cases), and we may also assume $\lambda = 0$ (by translation). Then $\operatorname{Im} \pi = H_0$. Let $f \in \operatorname{Ker} \pi$ so that $\pi f = 0$.

Since $N\mathcal{E} + \pi = I$ we have $N\mathcal{E}f = f$. Then $\mathcal{G}f = \mathcal{G}N\mathcal{E}f = \mathcal{E}N\mathcal{G}f$, so $\mathcal{G}f \in \operatorname{Im} \mathcal{E}_0$. By the above $\mathcal{G}f \in \operatorname{Im} \mathcal{E}_0 \Rightarrow f \in H_0^{\perp}$. Therefore $\operatorname{Ker} \pi \perp \operatorname{Im} \pi$. This shows π is an orthogonal projection.

We saw in §3.8 that $p\sigma = 0$ and $q\sigma = 0$, so $a^*gs = 0$ and $b^*B\sigma = 0$. We claim that also $E\sigma = 0$. For $N\mathcal{E}\sigma = 0$, so

$$\operatorname{Im} \mathcal{E}\sigma \subseteq \operatorname{Ker} N = \operatorname{Im} \rho = \mathcal{G}(\operatorname{Im} \pi).$$

But

$$\mathcal{E}\sigma = (E\sigma, p\sigma, q\sigma) = (E\sigma, 0, 0) = \mathcal{G}E\sigma,$$

so $\operatorname{Im} E\sigma \subseteq \operatorname{Im} \pi$. Hence for any f we have $E\sigma f \in H_0, p\sigma f = 0, q\sigma f = 0$. Thus $\sigma f \in \hat{H}_0 = H_0$, so $E\sigma f = 0$. We also see that $\operatorname{Im} \sigma \subseteq \operatorname{Im} \pi$, so $\operatorname{Im} \sigma = \operatorname{Im} \pi$ because $\pi = \sigma\mathcal{G}$. This proves the lemma.

Lemma.

$$a^*\sigma = 0, \quad b^*B\sigma = 0, \quad a^*N(g, h, k) = h, \quad b^*BN(g, h, k) = k.$$

Proof. We saw in §3.8 that

$$p\sigma = 0, \quad q\sigma = 0, \quad pN(g, h, k) = h, \quad qN(g, h, k) = k.$$

But now $p = a^*$ and $q = b^*B$.

Corollary.

$$a^*N\mathcal{G}f = 0, \quad b^*BN\mathcal{G}f = 0.$$

The operator $\mathcal{E}$ associated to an elliptic complex AB is always totally self-adjoint, so the preceding applies. Moreover we have the following result.

Lemma. *If $f \in \operatorname{Im} \pi$, then $AA^*f \in \operatorname{Im} \pi$ and $B^*Bf \in \operatorname{Im} \pi$.*

Proof. Recall that if $b^*g = 0$, then $a^*B^*g = 0$. It is sufficient to prove the lemma in the case where γ contains only one eigenvalue λ, in which case π is orthogonal projection on

$$H_\lambda = \{h: AA^*h + B^*Bh + \lambda h = 0, a^*h = 0, b^*Bh = 0\}.$$

Suppose $h \in H_\lambda$. Then $a^*AA^*h = a^*(\lambda h - B^*Bh) = \lambda a^*h - a^*B^*Bh$, but $a^*h = 0$ and $b^*Bh = 0$ which by the previous argument implies $a^*B^*Bh = 0$. Thus $a^*AA^*h = 0$. Also $b^*BAA^*h = 0$. Moreover

$$(AA^* + B^*B + \lambda I)AA^*h = AA^*(AA^* + B^*B + \lambda I)h = 0.$$

This proves $h \in H_\lambda \Rightarrow AA^*h \in H_\lambda$. Thus $(I - \pi)AA^*\pi = 0$. Likewise $a^*B^*Bh = 0$ (as shown already) and $b^*BB^*Bh = b^*B(\lambda h - AA^*h) = \lambda b^*Bh = 0$. $(AA^* + B^*B + \lambda I)B^*Bh = B^*B(AA^* + B^*B + \lambda I)h = 0$. Thus $h \in H_\lambda \Rightarrow B^*Bh \in H_\lambda$, so $(1 - \pi)B^*B\pi = 0$ as well. This proves the lemma.

Lemma. *If $\pi f = 0$ and $a^*f = 0$, then $\pi AA^*f = 0$.*

Proof. Consider any $h \in \operatorname{Im} \pi$. Since $a^*\pi = 0$ we have $a^*h = a^*\pi h = 0$.

Therefore

$$\langle\langle AA^*f, h\rangle\rangle = \langle\langle A^*f, A^*h\rangle\rangle = \langle\langle f, AA^*h\rangle\rangle.$$

Since $h \in \operatorname{Im} \pi$, we have $AA^*h \in \operatorname{Im} \pi$ by the previous lemma. If $\pi f = 0$, then $f \perp \operatorname{Im} \pi$, so $\langle\langle f, AA^*h\rangle\rangle = 0$. Thus $\langle\langle AA^*f, h\rangle\rangle = 0$ for all $h \in \operatorname{Im} \pi$. This shows that $AA^*f \perp Im\ \pi$, so $\pi AA^*f = 0$.

Corollary. $\pi AA^*N\mathcal{G}f = 0$ *for all* f.

Proof. $\pi N = 0$ so $\pi N\mathcal{G}f = 0$. Also $pN\mathcal{G}f = 0$. Thus $a^*N\mathcal{G}f = 0$. Then $\pi AA^*N\mathcal{G}f = 0$ by the lemma.

Lemma. *Suppose* 0 *lies inside* γ. *Then* $Ef \in \operatorname{Im} \pi$, $a^*f = 0$, $b^*Bf = 0 \Rightarrow f \in \operatorname{Im} \pi$.

Proof. Let $0, \lambda_1, \cdots, \lambda_N$ be the eigenvalues lying inside γ. Then

$$\operatorname{Im} \pi = H_0 \oplus H_{\lambda_1} \oplus \cdots H_{\lambda_N}.$$

Therefore we can write

$$EF = h_0 + h_1 + \cdots + h_N,$$

where $h_0 \in H_0$, $h_1 \in H_{\lambda_1}, \cdots, h_N \in H_{\lambda_N}$. Then $a^*h_ih_j = 0$ and $b^*Bh_j = 0$ for $0 \leqslant j \leqslant N$. Also $Eh_0 = 0$ and $Eh_j = \lambda_j h_j$ for $1 \leqslant j \leqslant N$. Let $\tilde{f} = f - h_1/\lambda_1 - \cdots - h_N/\lambda_N$. Then $E\tilde{f} = Ef - h_1 - \cdots - h_N = h_0$ and $a^*\tilde{f} = 0$, $b^*B\tilde{f} = 0$. This implies that $\mathcal{G}h_0 \in \operatorname{Im} \mathcal{E}$ which by a previous argument implies $h_0 \in H_0^\perp$. But $h_0 \in H_0$ so $h_0 = 0$. Then $E\tilde{f} = 0$, $a^*\tilde{f} = 0$, $b^*B\tilde{f} = 0$, so $\tilde{f} \in H_0$. Thus

$$f = \tilde{f} + h_1/\lambda_1 + \cdots + h_N/\lambda_N \in \operatorname{Im} \pi,$$

since $\tilde{f} \in H_0$ and $h_j/\lambda_j \in H_{\lambda_j}$.

Theorem. *Let* AB *be an elliptic complex which satisfies a persuasive estimate*

$$|f|_0^2 \lesssim \|A^*f\|_0^2 + \|Bf\|_0^2 + \|f\|_0^2 \quad \text{when } a^*f = 0.$$

Form the associated self-adjoint elliptic boundary value problem

$$\mathcal{E}f = (AA^*f + B^*Bf, a^*f, b^*Bf).$$

Let γ *be any closed curve in the complex plane containing* 0 *and avoiding the eigenvalues of* $\mathcal{E}$. *Form the associated operators*

$$R(\lambda) = [\mathcal{E} + \lambda\mathcal{G}]^{-1},$$

$$N = \int_\gamma R(\lambda)\frac{d\lambda}{\lambda}, \quad \sigma = \int_\gamma R(\lambda)\, d\lambda, \quad \pi = \sigma\mathcal{G}, \quad \rho = \mathcal{G}\sigma.$$

Then π *is the orthogonal projection onto the finite dimensional subspace spanned by the eigenvectors of* $\mathcal{E}$ *with eigenvalues insides* γ.

Define operators K and L by

$$Kf = A^*N(f, 0, 0), \quad Lg = N(B^*g, 0, b^*g).$$

Then $AK + LB + \pi = I$. Also $K\pi = 0$ and $\pi L = 0$.

Proof. Since $N\mathcal{J}\pi = 0$ we have $K\pi = 0$, and since $\pi N = 0$ we have $\pi L = 0$. Let

$$h = N(f, 0, 0) = N\mathcal{J}f.$$

Then $Kf = A^*h$ and $AKf = AA^*h$. We know that $\mathfrak{S}N + \mathcal{J}\sigma = I$. Therefore

$$\mathfrak{S}h + \mathcal{J}\sigma\mathcal{J}f = (f, 0, 0).$$

Now $\mathcal{J}\sigma\mathcal{J}f = \mathcal{J}\pi f = (\pi f, 0, 0)$. Thus

$$\begin{aligned} AA^*h + B^*Bh + \pi f &= f, &&\text{on } X,\\ a^*h &= 0, &&\text{on } \partial X,\\ b^*Bh &= 0, &&\text{on } \partial X. \end{aligned}$$

Recall that $b^*g = 0 \Rightarrow a^*B^*g = 0$. Thus $a^*B^*Bh = 0$. By recent lemmas

$$(1 - \pi)B^*B\pi = 0, \quad b^*B\pi = 0,$$

so $B^*B\pi f = \pi B^*B\pi f$ and $b^*B\pi f = 0$. Then

$$\begin{aligned} AA^*(B^*Bh) + B^*B(B^*Bh) &= B^*Bf - \pi B^*B\pi f, &&\text{on } X.\\ a^*(B^*Bh) &= 0, &&\text{on } \partial X,\\ b^*B(B^*Bh) &= b^*Bf, &&\text{on } \partial X. \end{aligned}$$

Next let

$$k = LBf = N(B^*Bf, 0, b^*Bf).$$

Again since $\mathfrak{S}N + \mathcal{J}\sigma = I$ we have

$$\mathfrak{S}k + \mathcal{J}\sigma l = (B^*Bf, 0, b^*Bf)$$

with $l = (B^*Bf, 0, b^*Bf)$. Thus

$$\begin{aligned} AA^*k + B^*Bk + \sigma l &= B^*Bf, &&\text{on } X,\\ a^*k &= 0, &&\text{on } \partial X,\\ b^*Bk &= b^*Bf, &&\text{on } \partial X. \end{aligned}$$

Recall that $\operatorname{Im}\sigma = \operatorname{Im}\pi$. Put

$$m = B^*Bh - k.$$

Then from these equations and the previous set we see that

$$Em \in \operatorname{Im}\pi, \quad a^*m = 0, \quad b^*Bm = 0.$$

By a recent lemma we conclude that $m \in \operatorname{Im}\pi$. Then we have $m = \pi m$ and $B^*Bh = k + \pi m$. From $AA^*h + B^*Bh + \pi f = f$ we conclude

$$AKf + LBf + \pi m + \pi f = f.$$

Now we claim that $\pi AKf = 0$. For $h = N\mathcal{G}f$, and a recent lemma implies that $\pi AA^*h = 0$. Thus $\pi AKf = 0$. Now we see that $\pi m + \pi f = \pi f$ so $\pi m = 0$. Then we have $AKf + LBf = f$, which proves the theorem.

5.5. Families of elliptic complexes

A family of elliptic complexes consists of two partial differential operators $A(m)f$ and $B(m)g$

$$A\colon U \subseteq \mathcal{C}^\infty(X; M) \times \mathcal{C}^\infty(X; E) \to \mathcal{C}^\infty(X; F),$$
$$B\colon U \subseteq \mathcal{C}^\infty(X; M) \times \mathcal{C}^\infty(X; F) \to \mathcal{C}^\infty(X; G),$$

nonlinear of some degree r in m and linear of degree 1 in f and g, such that $B(m)A(m)f = 0$ and the sequence of symbols

$$E \xrightarrow{\sigma_{A(m)}(\xi)} F \xrightarrow{\sigma_{B(m)}(\xi)} G$$

is exact for each $m \in U$ and each nonzero real cotangent vector ξ.

We choose hermitian metrics $\langle\,,\,\rangle_m$ on the bundles E, F, G and a volume element dV_m on X, all of which may depend smoothly on m and its derivatives up to degree r, and form the adjoint operators $A^*(m)$ and $B^*(m)$. We also choose bundles P and Q with hermitian metrics $\langle\,,\,\rangle_m$ and surjective bundles maps $a^*(m)\colon F|\partial X \to P$ and $b^*(m)\colon G|\partial X \to Q$ representing the boundary conditions as before, all of which depend smoothly on m and its derivatives up to degree r. We then form the associated family of self-adjoint elliptic boundary value problems

$$\mathcal{E}(m)f = (A(m)A^*(m)f + B^*(m)B(m)f, a^*(m)f, b^*(m)B(m)f).$$

We say that the complex AB satisfies a uniform persuasive estimate if

$$|f|_0^2 \lesssim \|A^*(m)f\|_0^2 + \|B(m)f\|_0^2 + \|f\|_0^2$$

when $a^*(m)f = 0$ with a constant independent of $m \in U$. In this case the eigenspaces

$$H_\lambda(m) = \{f\colon E(m)f + \lambda f = 0, p(m)f = 0, q(m)f = 0\}$$

are all finite dimensional. Note that

$$H_0(m) = \{h\colon A^*(m)h = 0, a^*(m)h = 0, B(m)h = 0\}.$$

Theorem. *Suppose that the family of elliptic complexes AB satisfies a uniform persuasive estimate, and that $H_0(0) = 0$. Then we can find a smaller neighborhood U of 0 and smooth tame families of linear maps $K(m)f$ and $L(m)f$ which split the complex, i.e.,*

$$A(m)K(m)f + L(m)B(m)f = f.$$

Proof. By the hypotheses, $\mathcal{E}(m)f$ is invertible for all m in a neighborhood

U of 0, and the family of inverses $\mathcal{E}^{-1}(m)f$ is a smooth tame family of linear maps. We apply the previous construction uniformly in m. Thus

$$K(m)f = A^*(m)\mathcal{E}^{-1}(m)(f, 0, 0),$$
$$L(m)g = \mathcal{E}^{-1}(m)(B^*(m)g, 0, b^*(m)g).$$

Since a composition of smooth tame maps is a smooth tame map, it follows that K and L are smooth tame maps. That $AK + LB = I$ follows from the previous argument.

In case $H_0(0) \neq 0$ we still have an approximate splitting.

Theorem. *Let AB be a family of elliptic complexes which satisfies a uniform persuasive estimate. Choose a path γ which contains* 0 *but no other eigenvalue of $\mathcal{E}(0)$. Let $N(m)$ and $\pi(m)$ be the families of operators obtained by integrating around γ. Then* $\operatorname{Im} \pi(0) = H_0(0)$ *while* $\operatorname{Im} \pi(m) = \Sigma H_\lambda(m)$ *for λ inside γ. Define*

$$K(m)f = A^*(m)N(m)(f, 0, 0),$$
$$L(m)g = N(m)(B^*(m)g, 0, b^*(m)g).$$

Then K and L are smooth tame families of linear maps and

$$A(m)K(m)f + L(m)B(m)f + \pi(m)f = f.$$

Proof. K and L are smooth tame maps since N and π are. The above identity follows as before.

References

[1] G. B. Folland & J. J. Kohn, *The Neumann problem for the Cauchy-Riemann complex*, Annals of Math. Studies, No. 75, Princeton University Press, Princeton, 1972.

[2] R. S. Hamilton, *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc.

[3] ———, *Deformation of complex structures on pseudoconvex domains*, preprint, Cornell University.

[4] J. L. Kelley, *General topology*, Van Nostrand, New York, 1955.

[5] J. J. Kohn & L. Nirenberg, *Non-coercive boundary value problems*, Comm. Pure Appl. Math. **18** (1965) 443–492.

[6] M. Kuranishi, *Deformations of isolated singularities and* $\bar{\partial}_b$, to appear.

[7] J. Morrow & K. Kodaira, *Complex manifolds*, Holt, Rinehart & Winston, New York, 1971.

[8] H. Rossi, *Attaching analytic spaces to an analytic space along a pseudo-concave boundary*, Proc. Conf. Complex Analysis (Minneapolis), Springer, Berlin, 1964.

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