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REAL 4-DIMENSIONAL KÄHLERIAN MANIFOLDS OF CONSTANT SCALAR CURVATURE

SHÛKICHI TANNO

1. Introduction

We denote by (M, g) a compact oriented Riemannian manifold of dimension 4 with metric g, and by (M, g, J) a compact Kählerian manifold of real dimension 4 with almost complex structure tensor J and Kählerian metric g. Let sign (M) and $\chi(M)$ be the sign ature and the Euler-Poincaré characteristic of M. It is well known that $\chi(M) \geqslant 0$ holds for an Einstein (M, g) (M. Berger [1]). With respect to the relation between sign (M) and $\chi(M)$, N. Hitchin [4] showed

$$3|\operatorname{sign}(M)| \leqslant 2\chi(M) \tag{1.1}$$

for an Einstein (M, g). Later H. Donnelly [3] showed

$$-2\chi(M) \leqslant 3\operatorname{sign}(M) \leqslant \chi(M) \tag{1.2}$$

for an Einstein (M, g, J). Here the equality of the second inequality of (1.2) holds if and only if (M, g, J) is of constant holomorphic sectional curvature. This second inequality is decomposed as

$$3\operatorname{sign}(M) \leqslant \frac{1}{96}\pi^{-2}S^2\operatorname{Vol}(M) \leqslant \chi(M), \tag{1.3}$$

where $\operatorname{Vol}(M)$ and S denote the volume and the scalar curvature of (M, g, J), respectively (cf. Remark 5).

Generalizing (1.3) for (M, g, J) of constant scalar curvature, we obtain Theorems A, B and C.

Let $S^2(K)$ and $H^2(-K)$ be a Euclidean 2-sphere of constant curvature K and a hyperbolic 2-space of constant curvature $-K$, with the natural Kählerian structures.

Theorem A. *If a compact Kählerian manifold (M, g, J) of real dimension 4 has constant scalar curvature S, then*

$$3\operatorname{sign}(M) \leqslant \frac{1}{96}\pi^{-2}S^2\operatorname{Vol}(M) \tag{1.4}$$

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holds. If the equality of (1.4) *holds, then either*

(i) (M, g, J) *is of constant holomorphic sectional curvature, or*

(ii) (M, g, J) *is locally a product* $S^2(K) \times H^2(-K)$.

For the case $S > 0$, we have a characterization of a complex projective 2-space.

Theorem B. *Let* (M, g, J) *be a real* 4*-dimensional compact Kählerian manifold with positive constant scalar curvature* S. *Then we have inequality* (1.4), *and the equality holds if and only if* (M, g, J) *is holomorphically isometric to a complex projective* 2*-space* (CP^2, g_0, J_0) *with Fubini-Study metric* g_0 *of constant holomorphic sectional curvature* $H = S/6$.

By CE^2 we denote a complex Euclidean 2-space.

Theorem C. *In a compact Kählerian manifold* (M, g, J) *of real dimension* 4, *if the scalar curvature* $S = 0$ *and* sign $(M) = 0$, *then* (M, g, J) *is one of the following spaces;*

$$CE^2/\Gamma_1, \; S^2(K) \times H^2(-K)/\Gamma_2,$$

where Γ_1 *or* Γ_2 *denotes some discrete subgroup of the automorphism group of* CE^2 *or* $S^2(K) \times H^2(-K)$.

2. Preliminaries

By $R = (R^i{}_{jkl})$, $Ric = (R_{jl} = R^i{}_{jil})$ and $S = (g^{ji}R_{ji})$ we denote the Riemannian curvature tensor, the Ricci curvature tensor and the scalar curvature of a Riemannian manifold (M, g). The first Pontrjagin form p_1 of a 4-dimensional compact oriented Riemannian manifold (M, g) is given by

$$p_1 = \frac{1}{4}\pi^{-2}(R_{rs12}R^{rs34} - R_{rs13}R^{rs24} + R_{rs14}R^{rs23})\, dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4,$$

where (x^i) denotes a local coordinate system which is positively related to the given orientation of M. The sign ature of M is given by

$$3 \operatorname{sign}(M) = \int_M p_1.$$

Now let (M, g, J) be a compact Kählerian manifold of real dimension 4. The orientation of M is defined by considering (e_1, Je_1, e_2, Je_2) as a positive frame. Then the sign ature of (M, g, J) is expressed as

$$-3 \operatorname{sign}(M) = \frac{1}{16}\pi^{-2}\int_M (|R|^2 - 2|\mathrm{Ric}|^2)\, dM \tag{2.1}$$

by a result of H. Donnelly [3], where dM denotes the volume element of (M, g, J), and

$$|R|^2 = R_{ijkl}R^{ijkl}, \; |\mathrm{Ric}|^2 = R_{ji}lR^{ji}.$$

To get (2.1) H. Donnelly applied his invariance theorem [2] to Kählerian manifolds. Namely, denoting by $*$ the Hodge star operator associated with the Kählerian structure, the function $*p_1$ is an invariant of order 4 in his sense.

Remark 1. In the case of a compact oriented Riemannian manifold (M, g) of dimension 4, denote by $*$ the Hodge star operator associated with the Riemannian structure. Then $*p_1$ is of $SO(4)$-invariant type and not of $O(4)$-invariant type, because $*p_1$ contains the so-called determinant part.

Remark 2. (2.1) is also directly proved. Let x be an arbitrary point of (M, g, J) and (e_1, Je_1, e_2, Je_2) an orthonormal frame at x. This frame may be chosen so that the Ricci tensor is diagonal. Then (2.1) is verified from the local expression of p_1.

Remark 3. Denote the Bochner curvature tensor of (M, g, J) by $B = (B^i{}_{jkl})$, define a (0, 2)-tensor G by

$$G_{ji} = R_{ji} - \tfrac{1}{4} S g_{ji},$$

and put $|B|^2 = B_{ijkl}B^{ijkl}$ and $|G|^2 = G_{ji}G^{ji}$. Then $G = 0$ holds on M if and only if (M, g) is an Einstein manifold. Furthermore, it is known that as far as curvature tensor norms are concerned the condition $|B| = |G| = 0$ is most effective to the conclusion of constancy of holomorphic sectional curvature (cf. S. Tanno [6, Theorem 4.3]).$|B|$ is given by (S. Tanno [6], [7])

$$|B|^2 = |R|^2 - 2|\mathrm{Ric}|^{12} + \frac{1}{6} S^2. \tag{2.2}$$

3. Proofs of Theorems A, B, and C

By (2.1) and (2.2) we obtain

$$48\pi^2 \operatorname{sign}(M) = -\int_M |B|^2\, dM + \frac{1}{6}\int_M S^2\, dM. \tag{3.1}$$

Hence we obtain

Proposition 3.1. *Let (M, g, J) be a compact Kählerian manifold of real dimension 4. Then*

$$288\,\pi^2 \operatorname{sign}(M) \leqslant \int_M S^2\, dM, \tag{3.2}$$

where the equality holds if and only if $B = 0$.

Proof of Theorem A. If we assume that the scalar curvature S is constant, then (1.4) follows from (3.2). The equality holds if and only if $B = 0$. Therefore to complete the proof of Theorem A it suffices to apply the following.

Proposition 3.2 (M. Matsumoto & S. Tanno [5]). *If a Kählerian manifold (M, g, J) has vanishing Bochner curvature tensor B and constant scalar curvature S, then either*

(i) *(M, g, J) is of constant holomorphic sectional curvature, or*

(ii) *(M, g, J) is locally a product of two spaces of constant holomorphic sectional curvature $K > 0$ and $-K$.*

Remark 4. For $(M, g, J) = (CP^2, g_0, J_0)$ with constant holomorphic sectional curvature $H = 4$, we have Vol $(M) = \frac{1}{2}\pi^2$, sign $(M) = 1$, and $S = 6H = 24$.

Let $N^2(-K)$ be a compact oriented Riemann surface of genus $\geqslant 2$ with constant curvature $-K < 0$. Then $(M, g, J) = S^2(K) \times N^2(-K)$ has the scalar curvature $S = 0$ and sign $(M) = 0$.

Remark 5. If (M, g, J) is a compact Einstein Kählerian manifold of real dimension 4, then

$$\frac{1}{96}\pi^{-2}S^2 \operatorname{Vol}(M) \leqslant \chi(M), \tag{3.3}$$

where the equality holds if and only if (M, g, J) is of constant holomorphic sectional curvature (S. Tanno [6]). (1.4) and (3.3) imply the second inequality of (1.2), i.e., (1.3).

Proof of Theorem B. Since $S^2(K) \times H^2(-K)$ has vanishing scalar curvature, (M, g, J) is of constant holomorphic sectional curvature $H > 0$ by Theorem A and $S > 0$. Generally we see that a complete Kählerian manifold (M^{2n}, g, J) of positive constant holomorphic sectional curvature is simply connected and hence holomorphically isometric to (CP^n, g_0, J_0). Consequently, our (M, g, J) is holomorphically isometric to (CP^2, g_0, J_0) of constant holomorphic sectional curvature $H = S/6$.

Proof of Theorem C is easy.

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Tôhoku University