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UPPER ESTIMATES OF THE LENGTH OF A CURVE IN A RIEMANNIAN MANIFOLD WITH BOUNDARY

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1. Introduction

1.1. We establish here upper bounds of the length of a curve in a class of Riemannian manifolds with boundaries. The upper bounds are expressed in terms of the curvature of the curve and some characteristics of the manifold. All manifolds, submanifolds and curves here are supposed to be of class C^∞ unless otherwise stated. A curve parametrized by the arc length will be said to be normal.

1.2. In Euclidean case, a similar result is represented by Rešetnyak theorem [6, p. 262]. Its simplified version is as follows.

Rešetnyak Theorem. *Let* $x\colon [0, L] \to R^n$ *be a normal piecewise regular curve and* $\delta = \max(\widehat{\dot{x}(a), \dot{x}}(b))$, *where the dot denotes the derivative with respect to the arc length of the curve,* $\wedge$ *means the angle between the two vectors, and the maximum is taken over all regular points* $a, b \in [0, L]$. *If all vectors* $\dot{x}(s)$ *(at regular points) are directed into the same half-space and* $\cos\delta > -1/(n-1)$, *then*

$$L \leqslant \frac{r\sqrt{n}}{\sqrt{1 + (n-1)\cos\delta}}, \tag{1.1}$$

where r *is the distance between* $x(0)$ *and* $x(L)$.

1.3. The length of a curve in a 2-dimensional surface was estimated by A. D. Aleksandrov and V. V. Strel′cov [1] in 1953. Their estimates and ours (when the dimension $n = 2$) do not follow from one another.

In 1969, Gromoll and Meyer [5, Lemma 6] proved that for any compact set D in a complete open manifold of positive curvature, there exists a bound λ such that the length of any geodesic in D is less than λ.

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Recently [3, Introduction], the number λ was calculated explicitly for a class of the sets D. More precisely, the following Theorem 1.5 was proved there.

1.4. Basic notation and assumptions. Let M be a compact convex n-dimensional ($n \geqslant 2$) Riemannian manifold with boundary Γ. Denote by k_s and κ some lower bounds of the sectional curvature in M and the normal curvature of the surface Γ (on the side of the interior normal) respectively. Let P^2 and P^n be the simply connected 2-and n-dimensional manifolds of constant curvature k_s (sphere, Euclidean or hyperbolic space). P^2 is supposed to be oriented.

For a normal curve c: $[0, L] \to P^2$ with the ordinary curvature $|\ddot{c}|$, we consider the *oriented* curvature, i.e., $|\ddot{c}|$ with assigned sign + (resp., –) if $\dot{c}$ rotates in the positive (resp., negative) direction. The total curvature of a curve and the total oriented curvature of a curve in P^2 are the integrals of the appropriate curvatures along the curve. These definitions are naturally generalized to a piecewise C^2-curve.

We always assume that $\kappa > 0$ and $k_s > -\kappa^2$. Then in P^2 there exists a circle M_0 whose boundary Γ_0 has curvature κ. We denote the center and the radius of M_0 by 0 and $R_0 = R_0(\kappa, k_s)$ respectively.

The distance in M is denoted by $\rho(\,.\,,\,.\,)$, and that in P^2, P^n by $\rho_0(\,.\,,\,.\,)$.

A minimal geodesic with end points a, b (in any space) is denoted sometimes by ab, its length by $\overline{ab}$, and its direction (the unit tangent vector at a point of ab) by $\overrightarrow{ab}$. We use $(\,.\widehat{\;,}\,.\,)$ and $\measuredangle \ldots$ to denote angles.

1.5. Theorem (*proved in* [3]). *Any curve in M with curvature at every point not greater than $\chi \leqslant \kappa$ is not longer than a circular arc in M_0 of curvature χ whose end points are opposite points of Γ_0. (In particular, any geodesic in M is not longer than the diameter $2R_0$ of M_0.)*

We prove the following theorem in §2.5.

1.6. Theorem. *Let a curve of length L lie in M and have the total curvature θ satisfying*

$$\theta \in \begin{cases} [0, \theta^*), & \text{if } k_s > \kappa^2, \\ [0, \pi/2], & \text{otherwise,} \end{cases} \tag{1.2}$$

where the precise value of $\theta^* = \theta^*(\kappa, k_s)$ is given in §1.7. *Denote by l_θ the supremum of the lengths of piecewise C^2-curves in M_0 such that the straight line of support of each curve rotates in the same direction as along the curve and such that the total curvature of each curve does not exceed θ. Then l_θ is finite and*

$$L \leqslant l_\theta. \tag{1.3}$$

1.7. By [3, (1.1)],

$$(1.4)\qquad \max_{X\in M}\rho(X,\Gamma)\leqslant R_0=\begin{cases}\dfrac{1}{\sqrt{k_s}}\cot^{-1}\dfrac{\kappa}{\sqrt{k_s}}, & \text{if } k_s>0,\\ \dfrac{1}{\kappa}, & \text{if } k_s=0,\\ \dfrac{1}{\sqrt{-k_s}}\coth^{-1}\dfrac{\kappa}{\sqrt{-k_s}}, & \text{if } k_s<0.\end{cases}$$

For $k_s > \kappa^2$, (1.4) implies $2R_0 > (2/\sqrt{k_s})\cot^{-1} 1 = \pi/(2\sqrt{k_s})$. Let Y_0Z_0 be a diameter of M_0, and Z_0V_0 be a chord of the length $\pi/(2\sqrt{k_s})$. Put $\delta = \sphericalangle Y_0Z_0V_0$, and denote by σ the area of the part of M_0 bounded by Y_0Z_0, Z_0V_0 and the (shortest) arc $\overset{\frown}{V_0Y_0} \subset \Gamma_0$; see Fig. 1. We set $\theta^* = \pi/2 + \delta - k_s\sigma$. It will be shown in §2.4 that $\theta^* \in (0, \pi/2)$.

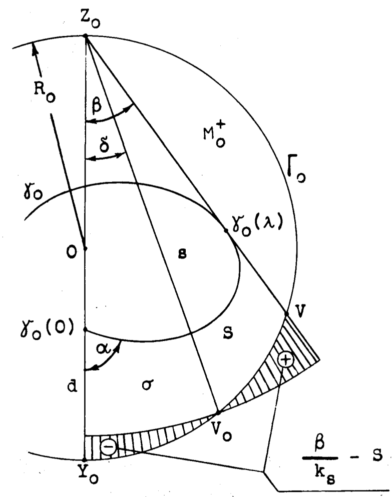

FIG. 1

1.8. In the case $-\kappa^2 < k_s \leqslant \kappa^2$, l_θ is the length of a polygonal line ACB with A and B at diametrically opposite points of Γ_0, $\overline{AC} = \overline{CB}$, and $\sphericalangle ACB = \pi - \theta$. For $k_s > \kappa^2$, the extremal line is more complicated. All of these will be considered in §4.

1.9. Corollary. *Let a curve of length L lie in M and have the total curvature $\theta \geqslant 0$. Let numbers θ_i, $i = 1, 2, \cdots, m$, satisfy* (1.2) *and $\Sigma_{i=1}^m \theta_i = \theta$. Then*

$$(1.5)\qquad L \leqslant \sum_{i=1}^{m} l_{\theta_i}.$$

(*We can prove* (1.5) *by dividing the curve into m parts with total curvatures* θ_i.)

1.10. A nondecreasing piecewise C^1-function Ξ: $[0, \infty) \to R$ with $\Xi(0) = 0$ will be called a *turn-function* if it is continuous from the left and any of its jumps is less than π.

Our main technical result is the following.

Theorem. *Let* $d \in [0, R_0]$, $\alpha \in [0, \pi]$ *and* Ξ *be a turn-function. Denote by* M_0^+ *the closed semicircle separated from* M_0 *by a diameter* Y_0Z_0 *and such that the rotation of the radius* OY_0 *to the radius* OZ_0 *within* M_0^+ *is positive; see Fig.* 2. *Suppose there is a mapping* γ_0 *such that*

(i) γ_0 *is a normal piecewise* C^2-*curve* $[0, \infty) \to P^2$;

(ii) $\gamma_0(0) \in OY_0$ *and* $\overline{Y_0\gamma_0(0)} = d$;

(iii) $\dot\gamma_0(0)$ *is directed into* M_0^+, *and* $(\widehat{\dot\gamma_0(0), \overrightarrow{OY_0}}) = \alpha$;

(iv) *the total oriented curvature of* $\gamma_{0|[0,s]}$ *is equal to* $\Xi(s)$, $s > 0$;

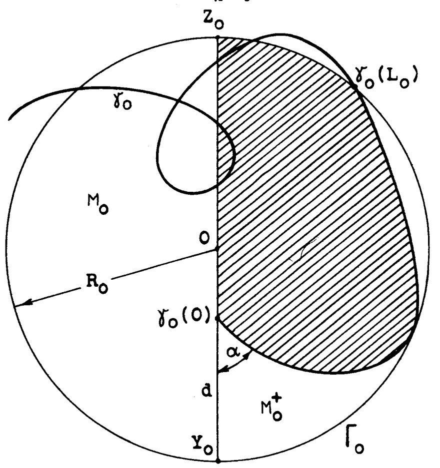

FIG. 2

(v) *there is a maximum number* $L_0 > 0$ *such that* $\gamma_0([0, L_0]) \subset M_0^+$;

(vi) $\gamma_0(L_0) \in \Gamma_0 \cap M_0^+ \setminus Z_0$ *and the curve* $\gamma_{0|[0,L_0]}$, *the segment* $\gamma_0(0)Z_0$ *and the arc* $\overset{\frown}{\gamma_0(L_0)Z_0}$ *of the semicircle* $\Gamma_0 \cap M_0^+$ *bound a nondegenerate region.* (*This implies that the region is convex and* $\alpha < \pi$.) *Then any normal curve* γ: $[0, L] \to M$ *is not longer than* $\gamma_{0|[0,L_0]}$ (*i.e.*, $L \leq L_0$) *if*

(I) $\rho(\gamma(0), \Gamma) = d$;

(II) $\dot\gamma(0)$ *forms an angle* $\phi \leq \alpha$ *with a shortest path* $\gamma(0)Y$, $Y \in \Gamma$, *of the length* $\rho(\gamma(0), \Gamma)$ (*or with the exterior normal, if* $d = 0$);

(III) $|\ddot\gamma(t)| \leq \Xi'(t)$ $(= |\ddot\gamma_0(t)|)$ *for those* $t \in [0, L]$ *where* Ξ' *exists.*

1.11. Corollary. *If* $\Xi(s) = \chi s$ *with* $\chi =$ const ≥ 0, *then* $\gamma_{0|[0,L_0]}$ *is the appropriate arc of curvature* χ *and Theorem* 1.10 *is a generalisation of* [3, *Theorem* 1(1)] (*for the case when* $\alpha > \pi/2$, $\chi \geq \kappa$). *For* $\chi = 0$, *it means that any* *geodesic in M which starts at a distance d from the boundary and forms an angle α with a shortest path realizing that distance d is not longer than the longest segment in M_0 with the same properties. Then obviously the length of any geodesic in M starting at a distance d from Γ does not exceed $\gamma_0(0)Z_0 = 2R_0 - d$.*

1.12. We prove also in §2.6 that

$$V(M) \leqslant V(M_0^n), \tag{1.6}$$

where $V(\,.\,)$ means the volume and M_0^n is a ball of radius R_0 in P^n.

1.13. The equalities hold in the estimates of §§1.6, 1.10, 1.11, and 1.12, when $M = M_0^n$ and the curve γ lies in a 2-dimensional plane P^2 passing through the center of M so that $P^2 \cap M = M_0$. In addition, in the cases of §§1.10 and 1.11, γ should coincide with $\gamma_{0|[0,L_0]}$ and the appropriate circular arc (segment) respectively.

In the case of §1.6, γ should approach the appropriate extremal line in D_0; see §1.8. (The results of §§1.6, 1.10, and 1.11 can be easily generalized to a piecewise regular γ; then γ should coincide with the above mentioned extremal line.)

1.14. Let, for example, M be a ball of radius R_0 in R^n and $\theta = \pi/2$. We can take $k_s = 0$, $\kappa = 1/R_0$. Then M_0 is a circle of radius R_0 in R^2 and we may imbed M_0 into M. According to §1.8, $l_\theta = l_{\pi/2} = 2\sqrt{2}\, R_0$. So by Theorem 1.6 any curve with total curvature $\leqslant \pi/2$ in the ball M has the length $\leqslant 2\sqrt{2}\, R_0$. This estimate is realized by a polygonal line $ABC \subset M_0 \subset M$ with $A, B, C \in \operatorname{bd} M_0 \subset \operatorname{bd} M$, $\overline{AB} = \overline{BC}$, $\measuredangle ABC = \pi/2$.

The estimate (1.1) of Rešetnyak theorem (which is also exact) applied to the line ABC yields

$$\overline{AB} + \overline{BC} \leqslant \frac{2R_0\sqrt{n}}{\sqrt{1 + (n-1)\cos \pi/2}} = 2\sqrt{n}\, R_0.$$

This is rougher than $2\sqrt{2}\, R_0$. (Of course, there are examples where (1.1) works better than Theorem 1.6.)

1.15. The restriction $k_s > -\kappa^2$ in Theorem 1.5 and in the further results can hardly be omitted. Indeed, if $k_s < -\kappa^2$, then one can construct a compact region of an arbitrary large volume where infinitely long (or closed) geodesics exist. The example is a tube of curvature -1, the curvature of whose edges grows within the interval $(0, 1)$ when the tube becomes longer.

1.16. Applying the estimate of §1.11 to all geodesics emanating, from a fixed point in D and using Rauch comparison theorem, we establish in §2.6 the inequality (1.6): $V(M) \leqslant V(M_0^n)$. The idea of such an application was given to the author by M. L. Gromov whom the author thanks very much.

2. Proof of Theorem 1.6 and inequality (1.6)

2.1. Let $d \in [0, R_0]$, $\alpha \in [0, \pi]$, and let Ξ be a turn-function. The set (d, α, Ξ) will be said to be *admissible* if there is a curve γ_0 satisfying the conditions (i)–(vi) of §1.10. (This implies $\alpha < \pi$.)

Let (d, α, Ξ) be an admissible set, $d > 0$, and γ_0, L_0 be as in §1.10. Rotating γ_0 about the point $\gamma_0(0)$, it is easy to notice that (d, α', Ξ) is an admissible set for any $\alpha' \in [0, \alpha]$, and $L_0 = L_0(\alpha')$ increases in $[0, \alpha]$.

2.2. We use the notation of §§1.10, 1.7 (assuming that $Z_0 V_0 \subset M_0^+$).

Lemma. *Let $d \in [0, R_0]$, $\alpha \in [0, \pi)$, and Ξ be a turn-function with $\Xi(\infty) = \lim_{t\to\infty} \Xi(t) \leqslant \theta$ satisfying*

$$0 \leqslant \theta \begin{cases} < \pi - \alpha + \delta - k_s\sigma, & \text{if } k_s > \kappa^2, \\ \leqslant \pi - \alpha, & \text{otherwise}. \end{cases} \tag{2.1}$$

Consider the curve γ_0 determined identically by the conditions (i)–(iv) of §1.10. Suppose $\gamma_0([0, \varepsilon]) \subset M_0^+$ for sme $\varepsilon > 0$. (This is always true when $d \neq 0$.) Then the set (d, α, Ξ) is admissible.

2.3. Proof. Suppose the contrary. Then obviously there is the minimal $\lambda > 0$ such that a chord $Z_0 V$ of the circle M_0 passes through $\gamma_0(\lambda) \in M_0^+$ and is situated in a supporting straight line of γ_0 at the point $\gamma_0(\lambda)$; see Fig. 1.

Denote by s the area of the region bounded by the segments $Z_0\gamma_0(0)$, $Z_0\gamma_0(\lambda)$ and the arc $\gamma_{0|[0,\lambda]}$. Let S be the area of the figure bounded by $Z_0 Y_0$, $Z_0 V$ and the arc $\breve{Y_0 V} \subset \Gamma_0 \cap M_0^+$. Put $\beta = \measuredangle Y_0 Z_0 V$; see Fig. 1. (In the case where $Z_0 = \gamma_0(\lambda) = V$, we assign $-\dot{\gamma}_0(\lambda - 0)$ as the direction $Z_0 V$.) Obviously, $\beta \in (0, \pi/2]$. Notice that total curvature Ξ_λ of the curve consisting of $\gamma_{0|[0,\lambda]}$ and the segment $\gamma_0(\lambda)Z_0$ satisfies $\Xi(\lambda) \leqslant \Xi_\lambda \leqslant \Xi(\lambda + 0) \leqslant \Xi(\infty)$. By Gauss-Bonnet theorem, $\alpha + \Xi_\lambda + (\pi - \beta) = 2\pi - k_s \cdot s$, so that

$$\Xi(\infty) \geqslant \Xi_\lambda = \pi - \alpha + \beta - k_s \cdot s. \tag{2.2}$$

For $k_s \leqslant 0$, (2.2) and (2.1) imply $\Xi(\infty) > \pi - \alpha \geqslant \theta$ which contradicts to the assumption $\Xi(\infty) \leqslant \theta$.

For $k_s \in (0, \kappa^2]$, (1.4) implies $2R_0 \leqslant \pi/2\sqrt{k_s}$. Note that β/k_s is the area of a circular sector of radius $\pi/2\sqrt{k_s}$ with central angle β in P^2. Then obviously $\beta - k_s s \geqslant \beta - k_s S > 0$ for $\beta \in (0, \pi/2]$. Again, (2.2) and (2.1) imply $\Xi(\infty) > \pi - \alpha \geqslant \theta$ which contradicts $\Xi(\infty) \leqslant \theta$.

For $k_s > \kappa^2$, the above mentioned interpretation of β/k_s shows that $\beta - k_s S$ as a function of $\beta \in (0, \pi/2]$ is minimal (and negative) at $\beta = \delta$; see Fig. 1. By (2.2) and (2.1), we have

$$\Xi(\infty) \geqslant \pi - \alpha + \delta - k_s\sigma > \theta, \tag{2.3}$$

which contradicts $\Xi(\infty) \leqslant \theta$.

2.4. The expression $E_\alpha = \pi - \alpha + \delta - k_s\sigma$ in (2.1) can be negative. (Then (2.1) makes no sense.) But $E_\alpha > 0$ if $\alpha \leqslant \pi/2$. In fact, $E_\alpha \geqslant E_{\pi/2} = \pi/2 + \delta - k_s\sigma = \theta^*$; see §1.7. By Gauss-Bonnet theorem, θ^* is the total curvature of the curve consisting of the arc $\smile{Y_0V_0} \subset \Gamma_0$ and the chord V_0Z_0, so that $\theta^* > 0$. As it was mentioned at the end of §2.3, $\delta - k_s\sigma < 0$, so that $\theta^* \in (0, \pi/2)$.

2.5. Proof of Theorem 1.6. Let a normal curve $\gamma\colon [0, L] \to M$ be that one mentioned in §1.6. Applying [4, Lemma 2] to the one-dimensional manifold $[0, L]$ with the immersion γ, we find that the point $\gamma(\lambda)$ most distant from Γ has the following property. There exist (possibly coincident) shortest paths $\gamma(\lambda)Y_1$ and $\gamma(\lambda)Y_2$, $Y_1 \in \Gamma$, $Y_2 \in \Gamma$, of the same length $d = \rho(\gamma(\lambda), \Gamma) > 0$ such that $\phi_1 \overset{\text{def}}{=} \widehat{(-\dot\gamma(\lambda), \overrightarrow{\gamma(\lambda)Y_1})} \leqslant \pi/2$ (if $\lambda \neq 0$), $\phi_2 \overset{\text{def}}{=} \widehat{(\dot\gamma(\lambda), \overrightarrow{\gamma(\lambda)Y_2})} \leqslant \pi/2$ (if $\lambda \neq L$).

Consider the curves $\gamma_1\colon [0, \lambda] \to M$ and $\gamma_2\colon [0, L - \lambda] \to M$ given by the formulas $\gamma_1(t) = \gamma(\lambda - t)$, $\gamma_2(t) = \gamma(\lambda + t)$. Then $\widehat{(\dot\gamma_1(0), \overrightarrow{\gamma_1(0)Y_1})} = \phi_1 \leqslant \pi/2$, $\widehat{(\dot\gamma_2(0), \overrightarrow{\gamma_2(0)Y_2})} = \phi_2 \leqslant \pi/2$. Put

$$\Xi_1(t) = \begin{cases} \int_0^t |\ddot\gamma_1(x)|\,dx, & t \in [0, \lambda], \\ \int_0^\lambda |\ddot\gamma_1(x)|\,dx, & t > \lambda, \end{cases}$$

$$\Xi_2(t) = \begin{cases} \int_0^t |\ddot\gamma_2(x)|\,dx, & t \in [0, L - \lambda], \\ \int_0^{L-\lambda} |\ddot\gamma_2(x)|\,dx, & t > L - \lambda. \end{cases}$$

Then $\theta = \theta_1 + \theta_2$ where $\theta_i = \Xi_i(\infty) \geqslant 0$, $i = 1, 2$. So θ_i satisfies (1.2) and consequently (2.1) with $\alpha = \pi/2$. By Lemma 2.2, the set $(d, \pi/2, \Xi_i)$ is admissible. Let γ_{0i} and L_{0i} satisfy the conditions (i)–(vi) of §1.10 for the set $(d, \alpha, \Xi) = (d, \pi/2, \Xi_i)$.

By Theorem 1.10 we have $\lambda \leqslant L_{01}$ and $L - \lambda \leqslant L_{02}$. It means in particular that the total curvature of $\gamma_{oi|[0, L_{oi}]}$ is equal to θ_i. Addition yields $L \leqslant L_{o1} + L_{o2}$. But $L_{o1} + L_{o2}$ is the length of a curve which consist of $\gamma_{o1|[0, L_{o1}]}$ and the specular reflection of the arc $\gamma_{o2|[0, L_{o2}]}$ with respect to the diameter Z_0Y_0. The straight line of support of such a curve rotates in the same direction as along the curve, and its total curvature is $\theta_1 + \theta_2 = \theta$. That is why $L \leqslant l_\theta$.

Finiteness of l_θ is quite obvious (see also §4.2).

2.6. Proof of inequality (1.6). Let $X \in \operatorname{int} M$, and let XY, $Y \in \Gamma$, be a shortest path of the length $\rho(X, \Gamma)$. Let $u \in T_XM$ be a unit vector, and $z(u)$ be the maximum number such that the geodesic $g_u\colon [0, z(u)] \to M$ with $g_u(t) = \exp_X(ut)$ is a shortest path.

Let us put $\Phi = 0 \cup \{a \in T_X M: |a| \neq 0, |a| < z(a/|a|)\}$. Obviously, the set $\Psi \overset{\text{def}}{=} \exp_X(\Phi) = M \setminus (C \cup \Gamma)$ where C is the cut locus for the point X. The mapping $\exp_X$ restricted to Φ is one-to-one. As in [2, Lemma 9], one can see that the set C has n-dimensional measure zero. Therefore

$$V(M) = V(\Psi), \tag{2.4}$$

where $V(\,.\,)$ denote the volume.

Let $\tilde{0}$ be the center of the ball M_0^n, $\tilde{0}\tilde{Y}$ be its radius, and the point $\tilde{X} \in \tilde{0}\tilde{Y}$ be such that $\tilde{X}\tilde{Y} = XY$ ($\leqslant R_0$; see (1.4)). Denote by $i\colon T_X M \to T_{\tilde{X}} M_0^n$ an arbitrary isometric mapping which transforms the direction of the shortest path XY into the direction of the segment $\tilde{X}\tilde{Y}$. Let $f \overset{\text{def}}{=} \exp_{\tilde{X}} \circ i \circ \exp_X^{-1}$: $\Psi \to P^n$. We want to show that $\tilde{\Psi} \overset{\text{def}}{=} f(\Psi) \subset M_0^n$ and consequently that

$$V(M_0^n) \geqslant V(\tilde{\Psi}). \tag{2.5}$$

Let a point $B \in \Psi$, $B \neq X$, and let XB be the (unique) shortest path with the end points X, B. Consider P^2 passing through the radius $\tilde{0}\tilde{Y}$ and the geodesic $f(XB)$. Obviously, $P^2 \cap M_0^n = M_0$, and the geodesic $f(XB) \subset P^2$ has length XB and forms the same angle with $\tilde{X}\tilde{Y}$ as the shortest path XB with XY. Now by §1.11 we have $f(XB) \subset M_0$, so that $f(B) \in M_0^n$.

On the strength of (2.4) and (2.5), the proof will be completed if we show that $V(\Psi) \leqslant V(\tilde{\Psi})$. For this, it is enough to establish that $f_*\colon T\Psi \to T\tilde{\Psi}$ does not decrease the length of tangent vectors; then the mapping f does not decrease volume since for any point $B \in \Psi$ the mapping $f_{*B}\colon T_B\Psi \to T_{\tilde{B}}\tilde{\Psi}$ ($\tilde{B} = f(B)$) can be reduced to expansions into n pair-wise orthogonal directions and an orthogonal transformation.

Let $B \in \Psi$, $B \neq X$ and $w \in T_B\Psi$. We need to show that $|f_*(w)| \geqslant |w|$. Let vectors w^- and $w^\perp \in T_B\Psi$ be such that $w = w^- + w^\perp$, $w^- \perp w^\perp$, and w^- is directed along the shortest path XB. It follows easily from the definition of f that $f_*(w^-)$ is directed along $\tilde{X}\tilde{B}$, $|f_*(w^-)| = |w^-|$, and $f_*(w^\perp) \perp f_*(w^-)$. Since $f_*(w) = f_*(w^-) + f_*(w^\perp)$, it will be enough to prove that $|f_*(w^\perp)| \geqslant |w^\perp|$.

Let $B(t)$, $t \in [0, \varepsilon]$, be a curve such that $B(0) = B$, $\dot{B}(0) = w^\perp$, and $XB(t) = XB$. Put $\tilde{B}(t) = f(B(t))$. Obviously, $\tilde{X}\tilde{B}(t) = XB(t) = XB$ and $\sphericalangle \tilde{B}\tilde{X}\tilde{B}(t) = \sphericalangle BXB(t)$. By Rauch comparison theorem, $|\dot{\tilde{B}}(0)| \geqslant |\dot{B}(0)|$, i.e., $|f_*(w^\perp)| \geqslant |w^\perp|$.

3. Proof of Theorem 1.10

3.1. By a simple limit reasoning, we may assume that $\Xi \in C^1$, $\Xi' > 0$, $d > 0$ and $\gamma([0, L]) \subset \operatorname{int} M$. It follows from §2.1 that we may also assume that $\phi = \alpha$.

3.2. Denote by F the cut locus of M from its boundary Γ. By [2, Lemma 8], F is closed and has n-dimensional measure zero. Without loss of generality, one may assume that the (closed) set $\Phi \overset{\text{def}}{=} \{t \in [0, L]\colon \gamma(t) \in F\}$ has linear measure zero. Indeed, γ can be included in a suitable (n-1)-parametric family of curves. Then in any neighborhood of γ there should be a curve whose set Φ has measure zero; otherwise, by Fubini theorem, the measure of F would be positive. Obviously, one may assume that $0 \notin \Phi$.

3.3. Put $r(t) = \rho(\gamma(t), \Gamma)$ (> 0 since $\gamma([0, L]) \subset \operatorname{int} M$). Obviously, $|r(t_2) - r(t_1)| \leqslant |t_2 - t_1|$, $r_{|[0,L]\setminus\Phi} \in C^\infty$, and $|r'| \leqslant 1$ in $[0, L] \setminus \Phi$. Varying the curve γ within the intervals composing the set $[0, L] \setminus \Phi$, one can easily see that we may also assume the set $\Psi \overset{\text{def}}{=} \{t \in [0, L] \setminus \Phi\colon |r'(t)| = 1\}$ countable, not containing 0 and having no points of condensation in $[0, L] \setminus \Phi$. (One can succeed even more in getting rid of the points with $|r'| = 1$; but in 2-dimensional case, a finite number of such points is inevitable.)

3.4. Put $\Omega = [0, L) \setminus (\Phi \cup \Psi)$. Obviously, $\Omega \setminus 0$ is an open set of complete measure, Ω contains 0 with a (positive) half-neighborhood, and $r_{|\Omega} \in C^\infty$.

3.5. We may also assume that $r(t) < R_0$. Indeed, suppose Theorem 1.10 has been proved under this assumption. Take $\tilde{\kappa} \in (0, \kappa)$. By (1.4), $r(t) \leqslant R_0(\kappa, k_s) < R_0(\tilde{\kappa}, k_s)$. Now by Theorem 1.10, $L \leqslant L_0$ ($= L_0(\tilde{\kappa})$). Passing here to the limit as $\tilde{\kappa} \to \kappa$, we prove Theorem 1.10 in the general case.

3.6. Denote by p_X, ω_X (or just p, ω) the polar coordinates of a point $X \in P^2$ with the pole at the center 0 of M_0 and the angle ω counted in positive direction from the radius $0Y_0$.

3.7. Consider the curve $\gamma_1\colon [0, L] \to M_0$ with the equations

$$p = R_0 - r(t); \tag{3.1}$$

$$\omega = \begin{cases} \sin R_0\sqrt{k_s} \displaystyle\int_0^t \frac{\sqrt{1 - r'^2(x)}}{\sin(R_0 - r(x))\sqrt{k_s}}\,dx, & \text{if } k_s > 0,\\[2ex] R_0 \displaystyle\int_0^t \frac{\sqrt{1 - r'^2(x)}}{R_0 - r(x)}\,dx, & \text{if } k_s = 0,\\[2ex] \sinh R_0\sqrt{-k_s} \displaystyle\int_0^t \frac{\sqrt{1 - r'^2(x)}}{\sinh(R_0 - r(x))\sqrt{-k_s}}\,dx, & \text{if } k_s < 0. \end{cases} \tag{3.2}$$

Obviously, $\gamma_{1|_\Omega} \in C^\infty$. One can check that $|\dot{\gamma}_1(t)| = 1$ for $t \in \Omega$ and that the mapping $\gamma_1\colon [0, L] \to M_0$ is Lipschitz. ($|p(t + \Delta t) - p(t)| \leqslant |r(t + \Delta t) - r(t)| \leqslant \Delta t$ and, say for $k_s = 0$, $|\omega(t + \Delta t) - \omega(t)| \leqslant R_0/[R_0 - \max_{x \in [0,L]} r(x)] \int_t^{t+\Delta t} 1 \cdot dx$.) Therefore γ_1 is a normal curve. Moreover, $\widehat{(\dot{\gamma}_1, \overline{0Y_0})} = \cos^{-1} - r'(0) = \phi$.

3.8. Let us now construct a function $f: \Omega \to R$ as follows. For $t \in \Omega$, let us move the curve γ_0 in P^2 in such a way that the new position of the vector $\dot{\gamma}_0(t)$ coincides with $\dot{\gamma}_1(t)$. Denote by $W(t)$ the new position of the point $\gamma_0(L_0)$. We put $f(t) = p_{W(t)}$. (For $t = a$, the construction is shown in Fig. 3.)

3.9. If the function f is nondecreasing, then the proof is complete. Indeed, suppose Theorem 1.10 is false i.e., suppose $L > L_0$. Take $u \in \Omega \cap [L_0, L)$ such that $0 \leqslant u - L_0 < \tilde{r} = \min_{t \in [0,L]} r(t)$. ($\tilde{r}$ is positive as γ is interior to M; see §3.1.) By (3.1), $\rho_0(\gamma_1(u), \Gamma_0) \geqslant \tilde{r}$. Obviously, $\rho_0(\gamma_1(u), W(u)) = \rho_0(\gamma_0(u), \gamma_0(L_0)) \leqslant u - L_0 < \tilde{r}$. Therefore $W(u) \in \operatorname{int} M$ and $f(u) < R_0$. This is impossible since $f(0) = R_0$.

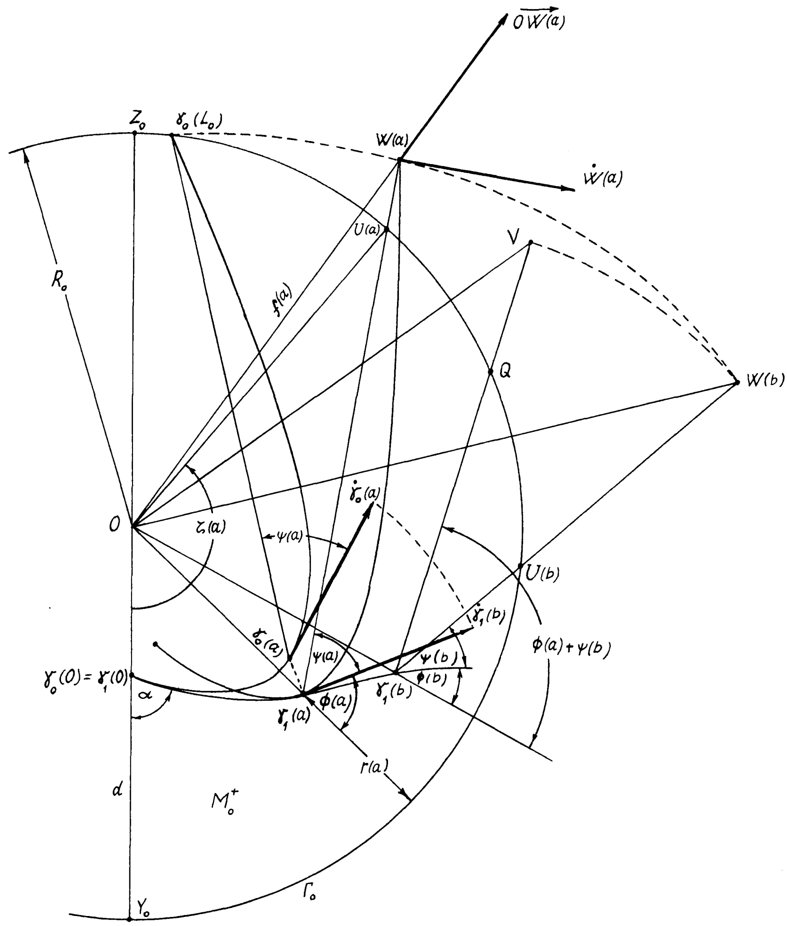

FIG. 3

3.10. A triangle $\Delta(t) \overset{\text{def}}{=} 0\gamma_1(t)W(t)$ is said to be *regular* if its vertices are pairwise different, each side is shorter than the sum of the others and if the triangle is positively oriented (in the order 0, $\gamma_1(t)$, $W(t)$ of the vertices). It is easy to check that, under the assumptions §§3.1–3.5, $\Delta(0)$ is regular. Since Ω contains a half-neighborhood of zero and by continuity, $\Delta(t)$ is regular for small t. Therefore $s \overset{\text{def}}{=} \sup\{t \in \Omega: \Delta(t) \text{ is regular}\} > 0$.

We show first that f does not decrease in $\Omega \cap [0, s)$ and then, in §§3.14–3.16, that $s = L$. The consideration is based essentially on [3, Theorem 2] which establishes some properties of the function $r(t)$.

3.11. Let an interval I be a component of $\Omega \cap [0, s)$. The curve $W(t)$, $t \in I$, is just the trajectory of the point $\gamma_0(L_0)$ when γ_0 rolls along $\gamma_{1|_I}$. A simple calculation based on (3.1), (3.2) shows that the oriented curvature $\xi_1(t)$ of $\gamma_{1|_\Omega}$ satisfies

$$r'' = \sqrt{1 - r'^2} \cdot \xi_1 - (1 - r'^2)\kappa_r, \tag{3.3}$$

where κ_r (> 0) is curvature of the circle of radius $R_0 - r$ centered at 0. By [3, (3.9)] and the consequent inequality ($K \geqslant \kappa_r$), we have

$$r'' \leqslant \sqrt{1 - r'^2} \cdot |\ddot{\gamma}| - (1 - r'^2)\kappa_r. \tag{3.4}$$

As $|r'| < 1$ in Ω, (3.3), (3.4) and the condition (III) of Theorem 1.10 imply $\xi_1 \leqslant |\ddot{\gamma}| \leqslant \Xi'$. So, at the moment $t \in I$, γ_0 rotates (instantaneously) about the point $\gamma_1(t)$ in negative direction. Regularity of $\Delta(t)$ implies $\overline{OW(t)} < \pi/\sqrt{k_s}$ in the case where $k_s > 0$ and $f'(t) = \langle \dot{W}(t), \overrightarrow{OW(t)} \rangle \geqslant 0$, $t \in \Omega \cap [0, s)$; see Fig. 3 for $t = a$. ($f' \geqslant 0$ is not sufficient for f to increase.)

3.12. For $t \in \Omega$, denote by h_t: $[0, r(t)] \to M$ the (unique) normal shortest path with $h_t(0) = \gamma(t)$, $h_t(r(t)) \in \Gamma$. Since $|\dot{\gamma}_1| = 1$ in Ω (see §3.7.) and by (3.1), $(\widehat{\overrightarrow{0\gamma_1(t)}, \dot{\gamma}_1(t)}) = (\widehat{\dot{h}_t(0), \dot{\gamma}(t)}) = \cos^{-1} - r'(t) \overset{\text{def}}{=} \phi(t)$. Let a, $b \in \Omega \cap [0, s)$, $a < b$, and g: $[0, \lambda] \to M$ be a normal shortest path with $g(0) = \gamma(a)$, $g(\lambda) = \gamma(b)$. By the remark at the end of [3, §3], the function $\bar{r}(u) \overset{\text{def}}{=} \rho(g(u), \Gamma)$, $u \in [0, \lambda]$, is convex. It is differentiable at $u = 0$ and $u = \lambda$ since the ends of g lie outside the cut locus F. Then $\bar{r}'(0) \geqslant \bar{r}'(\lambda)$ and

$$\big(\widehat{\dot{h}_a(0), \dot{g}(0)}\big) = \cos^{-1} - \bar{r}'(0) \geqslant \cos^{-1} - \bar{r}'(\lambda) = \big(\widehat{\dot{h}_b(0), \dot{g}(\lambda)}\big). \tag{3.5}$$

We denote by C (different) positive constants depending only on M, γ, s, and Ξ. By a compactness reasoning, $(\widehat{\dot{g}(0), \dot{\gamma}(a)}) \leqslant C(b - a)$, $(\widehat{\dot{g}(\lambda), \dot{\gamma}(b)}) \leqslant C(b - a)$. Therefore by (3.5) we have

$$\phi(b) - \phi(a) \leqslant C(b - a). \tag{3.6}$$

3.13. Put $\psi(t) = (\widehat{\dot{\gamma}_0(t), \overrightarrow{\gamma_0(t)\gamma_0(L_0)}})$, $t \in [0, L_0)$. Notice that $s \leqslant L_0$ because otherwise the triangle $\Delta(t)$ is not regular for $t \in \Omega \cap [L_0, s)$ close to L_0.

Now $(\overrightarrow{0\gamma_1(t)}, \overrightarrow{\gamma_1(t)W(t)})\widehat{\ } = \phi(t) + \psi(t)$ for $t \in \Omega \cap [0, s)$. Let the segment $\gamma_1(b)V$ satisfy $\overline{\gamma_1(b)V} = \overline{\gamma_1(b)W(b)}$, $(\overrightarrow{0\gamma_1(b)}, \overrightarrow{\gamma_1(b)V})\widehat{\ } = \phi(a) + \psi(b)$; see Fig. 3. By §3.1, $\gamma_0 \in C^2$. Then

$$|\psi(b) - \psi(a)| \leqslant C(b - a). \tag{3.7}$$

Along with the relations $p_{\gamma_1(t)} = R_0 - r(t) \geqslant C > 0$ for $t \in [0, L]$ (see §3.5), $\rho_0(\gamma_1(a), \gamma_1(b)) \leqslant b - a$ (see §3.7) and $|\overline{\gamma_1(b)V} - \overline{\gamma_1(a)W(a)}| = |\overline{\gamma_0(b)\gamma_0(L_0)} - \overline{\gamma_0(a)\gamma_0(L_0)}| \leqslant b - a$, (3.7) implies

$$\rho_0(V, W(a)) \leqslant C(b - a). \tag{3.8}$$

If $\phi(b) - \phi(a)$ in (3.6) is nonnegative, then $\measuredangle W(b)\gamma_1(b)V = |\phi(b) + \psi(b) - (\phi(a) + \psi(b))| = |\phi(b) - \phi(a)| \leqslant C(b - a)$ and therefore $\rho_0(V, W(b)) \leqslant C(b - a)$, which together with (3.8) implies $\rho_0(W(b), W(a)) \leqslant C(b - a)$ and

$$f(b) - f(a) \geqslant -C(b - a). \tag{3.9}$$

If $\phi(b) - \phi(a) < 0$, then comparison of the triangles $0\gamma_1(b)V$ and $\Delta(b)$ yields $f(b) \geqslant \overline{0V}$; see Fig. 3. A combination of this with (3.8) results in (3.9) again. The inequalities $f' \geqslant 0$ and (3.9) show that $f_{|\Omega \cap [0,s)}$ is nondecreasing.

3.14. Let $t \in \Omega \cap [0, s)$. Since $f(t) \geqslant f(0) = R_0$ and γ_1 lies inside M_0, there is the unique point $U(t) = \gamma_1(t)W(t) \cap \Gamma_0$. We show here that $\zeta(t) \stackrel{\text{def}}{=} \omega_{U(t)}$ is a nonincreasing function in $\Omega \cap [0, s)$.

The "rolling" described in §3.11 shows that $\zeta'(t) \leqslant 0$.

Let $a, b \in \Omega \cap [0, s)$, $a < b$. Obviously, any straight line intersecting γ_1 forms with Γ_0 angles $\geqslant C > 0$. Now following the way (3.8) was obtained, one can see that

$$\rho_0(Q, U(a)) \leqslant C(b - a). \tag{3.10}$$

where Q is the intersection of Γ_0 with $\gamma_1(b)V$ (or its extension beyond V); see Fig. 3. If $\phi(b) - \phi(a) \geqslant 0$ then, analogously to §3.13, one obtains $\rho_0(U(b), U(a)) \leqslant C(b - a)$ and

$$\zeta(b) - \zeta(a) \leqslant C(b - a). \tag{3.11}$$

If $\phi(b) - \phi(a) < 0$ then obviously $\zeta(b) \leqslant \omega_Q$; see Fig. 3. Combining it with (3.10), one obtains (3.11) again.

Along with $\zeta' \geqslant 0$, (3.11) shows that $\zeta_{|\omega \cap [0,s)}$ is nonincreasing. Thus

$$\zeta(t) \leqslant \zeta(0) = \omega_{\gamma_0(L_0)} < \pi, \; t \in \Omega \cap [0, s). \tag{3.12}$$

3.15. Put for short $\lim_{x \in \Omega, t > x \to t} y(x) = y(t - 0)$, $\lim_{x \in \Omega, t < x \to t} y(x) = y(t + 0)$. By [5, Theorem 2(3), (4)], there are left and right derivatives $r'_-(t)$, $r'_+(t)$ of the function $r(t)$ satisfying $r'_-(t) \geqslant r'_+(t)$, $t \in (0, L)$. Moreover, $r'(t - 0) = r'_-(t)$, $r'(t + 0) = r'_+(t)$. It follows now from (3.1), (3.2) and §3.5 that there are left and right tangent vectors $\dot{\gamma}_{1-}$ and $\dot{\gamma}_{1+}$ satisfying

(3.13) $$\phi_-(t) \overset{\text{def}}{=} (\widehat{\dot\gamma_{1-}(t), \overrightarrow{0\gamma_1(t)}}) = \cos^{-1} - r'_-(t) \geqslant \cos^{-1} - r'_+(t) = (\widehat{\dot\gamma_{1+}(t), \overrightarrow{0\gamma_1(t)}}) \overset{\text{def}}{=} \phi_+(t).$$

Moreover, $\phi(t-0) = \phi_-(t)$, $\phi(t+0) = \phi_+(t)$. Denote by $\Delta_-(t)$, $\Delta_+(t)$ the triangles $0\gamma_1(t)W_-(t)$, $0\gamma_1(t)W_+(t)$ constructed as $\Delta(t)$ (see §§3.10, 3.8) with replacement of $\dot\gamma_1(t)$ by $\dot\gamma_{1-}(t)$ and $\dot\gamma_{1+}$ respectively. Obviously

(3.14) $$\Delta_-(t) = \Delta(t-0), \Delta_+(t) = \Delta(t+0).$$

3.16. Notice that $\overline{\gamma_1(s)W_-(s)} = \overline{\gamma_1(s)W_+(s)} = \overline{\gamma_0(s)\gamma_0(L_0)} > 0$ since $\gamma_1(s) \in \operatorname{int} M_0$ and $\overline{0W_-(s)} = f(s-0) \geqslant R_0$. (According to §3.13, $s \leqslant L_0$. We see now that $s < L_0$.) It follows easily from (3.12) that

(3.15) $$\phi_-(s) + \psi(s) = \phi(s-0) + \psi(s) < \pi.$$

Since $s < L_0$ and $\Xi' > 0$ (see §3.1), $\phi_-(s) + \psi(s) \geqslant \psi(s) > 0$. Thus $0 < \phi_-(s) + \psi(s) < \pi$. Since $0\gamma_1(s)$ and $\gamma_1(s)W_-(s)$ are shorter than meridian of P^2 (when $k_s > 0$), $\Delta_-(s)$ is regular.

By (3.13), $0 < \psi(s) \leqslant \phi_+(s) + \psi(s) \leqslant \phi_-(s) + \psi(s) < \pi$. So $\Delta_+(s)$ is regular as well. If $s < L$ then, for values of $t \in \Omega \cap (s, L)$ sufficiently close to s, the triangles $\Delta(t)$ are also regular, see (3.14). This contradicts Definition 3.10 of s. Thus $s = L$.

4. The longest curve of a given total curvature in M_0

4.1. Notation. The angle between an arc of Γ_0 with the central angle $\pi/2$ and the chord joining its end points will be denoted throughout the section by $\omega = \omega(\kappa, k_s)$. (Say, $\omega(\kappa, 0) \equiv \pi/4$.)

Let u: $[0, 2a] \to \Gamma_0$ be a normal curve, and v: $[0, 2a] \to \Gamma_0$ be a mapping such that the total curvature of the curve composed of $u_{|[0,x]}$ and the chord $\overline{u(x)v(x)}$ does not depend on $x \in (0, 2a)$. Suppose the length $l(x)$ $(= x + \overline{u(x)v(x)})$ of the latter curve satisfies $l'(a) = 0$. We always denote by $\psi = \psi(\kappa, k_s)$ the angle between $\overrightarrow{u(a)v(a)}$ and $\dot u(a)$. (One can check that $\psi(\kappa, 0) \equiv \pi/3$, $\psi = \cos^{-1}(-e + \sqrt{e^2 + e + 1})$ for $k_s > 0$, $\psi = \cos^{-1}(e - \sqrt{e^2 - e + 1})$ for $k_s < 0$ where $e = \kappa^2/|k_s|$.) A calculation shows that $\omega < \psi$. Moreover, $l'(a) > 0$ (< 0) if $(\widehat{\overrightarrow{u(a)v(a)}, \dot u(a)}) > \psi$ $(< \psi)$.

Denote by J_θ the class of piecewise C^2-curves in M_0, mentioned in §1.6, such that the straight line of support of each curve rotates in the same direction as along the curve, and the total curvature of each curve does not exceed θ.

4.2. Proposition. *Let θ satisfy* (1.2), *i.e., let*

$$(4.1)\qquad \begin{aligned} &\theta \in \left[0, \frac{\pi}{2} + \delta - k_s\sigma\right), && \text{if } k_s > \kappa^2, \\ &\theta \in \left[0, \frac{\pi}{2}\right], && \text{otherwise.} \end{aligned}$$

Then in J_θ there is the longest curve γ_θ, which has the total curvature θ. Moreover, each of the following holds.

(1) *If $\theta \in [0, 2\omega]$, then γ_θ is a polygonal line ACB with $\overline{AC} = \overline{CB}$ whose end points A, B and diametrically opposite points in Γ_0.*

(2) *If $\theta \in [2\omega, 2\psi]$, then γ_θ is a polygonal line ACB with $\overline{AC} = \overline{CB}$ and A, C, $B \in \Gamma_0$.*

(3) *If $\theta \in (2\psi, \infty)$, then γ_θ is a line $A\overset{\smile}{A_* B_*}B$ consisting of an arc $\overset{\smile}{A_* B_*} \subset \Gamma_0$ of total curvature $\theta - 2\psi$ and equal chords AA_* and BB_*. The line does not intersect itself.*

The family γ_θ is shown in Fig. 4.

4.3. For $k_s \leqslant 0$, only the condition (1) makes sense as $\theta \leqslant \pi/2 \leqslant 2\omega$. For $k_s > 0$, a calculation shows that the right end points of the intervals in (4.1) both are greater than 2ω, so that the conditions (1) and (2) always make sense. As for the condition (3), it makes sense for some κ, k_s but does not for others.

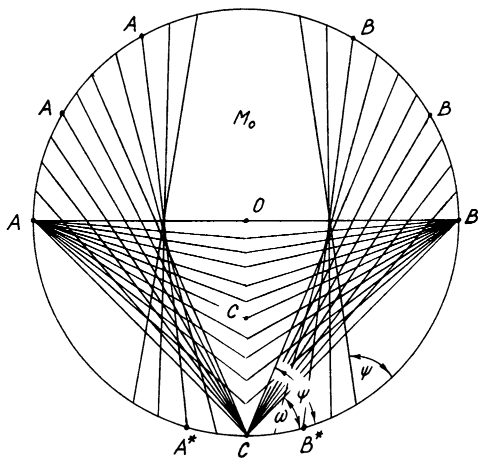

FIG. 4

4.4. The remainder of the paper is the proof of Proposition 4.2.

Let a curve $j^1 \in J_\theta$. We need to show that its length l^1 does not exceed the length l_θ of $\gamma_\theta \in J_\theta$. Without loss of generality we may assume that its end points $E, F \in \Gamma_0$. (If, say, $E \notin \Gamma_0$, one can extend j^1 beyond E as the tangent geodesic up to Γ_0 and deal with the new curve which is longer and has the same total curvature θ^1.) We assume also $\theta^1 > 0$ since the case $\theta^1 = 0$ is trivial.

Let us show that j^1 does not intersect itself. (That is why $E \neq F$ and $j^1 \cup EF$ bound a convex nondegenerate region. It means also that the curve γ_θ described in §4.2.(3) does not intersect itself.) Suppose otherwise. Then one can easily find a convex nondegenerate loop on j^1. (The important point here is that $E, F \in \Gamma_0$ and $\theta^1 \leqslant \theta \leqslant \pi/2 < \pi$.) Denote by W the point where j^1 crosses itself forming the loop. Let Y_0Z_0 be a diameter of M_0, and the semicircle M_0^+ be as in §1.10. Move the loop within M_0 until the center 0 of M_0 is inside the loop (if it was not). Let now X be the point on the loop closest to 0. Rotate the loop about 0 until $X \in 0Y_0$. If now $W \in M_0^+$, then take the specular reflection of the loop with respect to the diameter Y_0Z_0. Finally, the loop intersects $0Y_0$ at a point $X \neq 0$ at the right angle, and $W \in (M_0 \setminus M_0^+) \cup Y_0Z_0$.

Denote by a the length of that arc $\smile{XW}$ of the loop which intersects int M_0^+. Let $f(t)$, $t \in (0, a]$, be the total curvature of the arc $\smile{XF} \subset \smile{XW}$ of the length t. Put now

$$\Xi(t) = \begin{cases} f(t), & t \in (0, a], \\ f(a), & t \in [a, \infty). \end{cases}$$

Obviously, the set $(\overline{Y_0X}, \pi/2, \Xi)$ is not admissible. But $\Xi(\infty) = f(a) \leqslant \theta^1 \leqslant \theta$ where θ satisfies (4.1). By Lemma 2.2, the set above should be admissible.

4.5. In §§4.5.–4.9 we prove the inequality $l^1 \leqslant l_\theta$ for the case where $k_s > 0$.

Lemma. *Let a circle in P^2 be less than semisphere. Denote by $\beta(\xi)$ the angle between an arc of its circumference with the central angle ξ and the chord joining the end points of the arc. Denote by $b(\xi)$ the length of that chord. Let variables $\xi_1 > 0$ and $\xi_2 > 0$ satisfy $\xi_1 + \xi_2 = const. < 2\pi$. Then the function $\beta(\xi_1) + \beta(\xi_2)$ has the unique minimum, and the function $b(\xi_1) + b(\xi_2)$ has the unique maximum when $\xi_1 = \xi_2$.*

The proof is a simple discussion of the conditional extrema.

4.6. Let EE', FF' be the chords tangent to j^1 at its end point E, F. (Possibly, $E' = E$, $F' = F$.) We consider in §§4.6 and 4.7 the case when $EE' \cap FF' = \varnothing$. Denote by j^2 the curve $E\smile{E'F'}F$ consisting of the chords EE', FF' and the arc $\smile{E'F'} \subset \Gamma_0$ such that $j^2 \cup EF$ bounds a convex region containing j^1. Later on, the length and the total curvature of j^i will be denoted by l^i, θ^i, $i = 1, 2, 3$.

Obviously, $l^2 \geqslant l^1$. Gauss-Bonnet theorem applied to the region between j^1 and j^2 implies $\theta^2 \leqslant \theta^1$.

Let j^3 be the curve $\overset{\smile}{EE''F''F}$ which lies on the same side of EF as j^2 does and consists of equal chords EE'', FF'' and an arc $\overset{\smile}{E''F''} \subset \Gamma_0$ equal to $\overset{\smile}{E'F'}$. It follows easily from Lemma 4.5 that $l^3 \geqslant l^2$, $\theta^3 \leqslant \theta^2$.

4.7. We realize that the family γ_θ and j^3 have the same axis of symmetry and that their convexities face the same direction of the axis. Suppose $\theta^3 > 2\psi$. It follows easily from the properties of the angle ψ (see §4.1) that j^3 is not longer than γ_{θ^3}. If $\theta^3 \leqslant 2\psi$, then §4.1 implies that j^3 is not longer than the polygonal line ACB with total curvature θ^3 composed of equal chords AC and CB and having the same axis of symmetry. For $\theta^3 \in [2\omega, 2\psi]$, the line ACB coincides with γ_{θ^3}. For $\theta^3 \in [0, 2\omega]$, ACB is not longer than γ_{θ^3}. Thus $l^3 \leqslant l_{\theta^3}$. Since l_θ increases by θ and $l^1 \leqslant l^3$, one obtains $l^1 \leqslant l_\theta$.

4.8. Let now $EE' \cap FF' \neq \varnothing$. Since $\theta^1 > 0$ (see §4.4), the intersection is a point I distinct from E and F. Denote by j^2 the polygonal line EIF. As above $l^2 \geqslant l^1$, $\theta^2 \leqslant \theta^1$. Denote by j^3 the polygonal line $EI'F \subset M_0$ such that $\overline{EI'} = \overline{I'F}$ and I' lies on the circumference passing through E, I, F. By Lemma 4.5, $l^3 \geqslant l^2$, $\theta^3 \leqslant \theta^2$.

4.9. We realize that j^3 is located with respect to γ_θ as in §4.7. If $\theta^3 \leqslant 2\psi$, then j^3 can be matched with a part of γ_{θ^3} by a movement preserving the symmetry with respect to the axis. Suppose $\theta^3 > 2\psi$. Let us move j^3 preserving the symmetry until $I' \in \gamma_{\theta^3}$. Since $\theta^3 > 2\psi > 2\omega$, the end points E, F are in M_0 after the movement. Let the chords $I'E'$, $I'F'$ contain $I'E$, $I'F$ respectively. By the properties of the angle ψ (see §4.1), the line $E'I'F'$ is not longer than γ_{θ^3}. So again j^3 is not longer than γ_{θ^3}.

Thus $l^3 \leqslant l_{\theta^3}$. Since l_θ increases by θ and $l^1 \leqslant l^3$, one obtains $l^1 \leqslant l_\theta$.

4.10. In the remainder of the paper, we prove the inequality $l^1 \leqslant l_\theta$ for the case where $k_s \leqslant 0$.

Since j^1 can be approached with inscribed polygonal lines, we may consider only the case where j^1 is a polygonal line $EC_1C_2 \cdots C_mF$, $m \geqslant 1$, bounding (along with the chord EF) a nondegenerate convex region (see §4.4) and having pairwise distinct vertices with angles different from 0 and π at each vertex C_i.

4.11. Let $m > 1$ and $i \in \{0, 1, 2, \cdots, m-2\}$. Suppose $\sphericalangle C_iC_{i+1}C_{i+2} \leqslant \sphericalangle C_{i+1}C_{i+2}C_{i+3}$ with $C_0 = E$, $C_{m+1} = F$. We consider here the following variation j_x^1 of the polygonal line j^l.

Take $B(x) \in C_{i+2}C_{i+3}$ satisfying $\overline{B(x)C_{i+2}} = x$. For sufficiently small $x \geqslant 0$, in the extension of the segment C_iC_{i+1} there is a point $A(x)$ such that the polygons $EC_1 \cdots C_iA(x)B(x)C_{i+3} \cdots C_mFE$ and $EC_1 \cdots C_mFE$ bound equal areas (see Fig. 5). Denote the line $EC_1 \cdots C_iA(x)B(x)C_{i+3} \cdots C_mF$ by j_x^1, and put $\alpha = \sphericalangle C_iA(x)B(x)$, $\beta = \sphericalangle A(x)B(x)C_{i+3}$.

By Gauss-Bonnet theorem, the total curvature of j_x^1 is less than or equal to that of j^1, which is θ^1; it is less than if and only if $C_{i+3} = F$, $x = \overline{C_{i+2}C_{i+3}}$. Obviously α decreases while β and $\overline{C_iA(x)}$ increase by x. It is clear now that $A(x)$ exists for any x satisfying $0 \leqslant x \leqslant d \overset{\text{def}}{=} \overline{C_{i+2}C_{i+3}}$; otherwise $\overline{C_iA(x)} \to \infty$ and $\alpha \to 0$ as x grows, which is impossible since $\pi - \alpha \leqslant \theta^1 \leqslant \theta \leqslant \frac{1}{2}\pi$.

4.12. We show now that the length $l(x)$ of j_x^1 satisfies $dl/dx > 0$ for $x \in (0, d)$. Denote by $\dot{A}_=$, $\dot{B}_=$ the projections of the vectors $\dot{A}(x)$, $\dot{B}(x)$ on the direction $\overrightarrow{A(x)B(x)}$, and by $\dot{A}_\perp$, $\dot{B}_\perp$ their projections on the normal to $A(x)B(x)$ directed into the half-plane containing EF. Consider the "elementary triangles" $A(x)IA(x + \Delta x)$ and $B(x)IB(x + \Delta x)$ where $I = A(x)B(x) \cap A(x + \Delta x)B(x + \Delta x)$; see Fig. 5. Since the triangles have equal areas, it follows easily that $\dot{A}_\perp = -\dot{B}_\perp$. Thus we have

$$\dot{B}_\perp = \sin\beta,\ \dot{B}_= = -\cos\beta,\ |\dot{B}| = 1;$$

$$\dot{A}_\perp = -\sin\beta, \dot{A}_= = -\frac{\sin\beta}{\tan\alpha},\ |\dot{A}| = \frac{\sin\beta}{\sin\alpha};$$

$$\frac{dl}{dx} = |\dot{A}| - |\dot{B}| + \dot{B}_= - \dot{A}_= = \frac{\sin\beta}{\sin\alpha} - 1 - \cos\beta + \frac{\sin\beta}{\tan\alpha}$$

$$= \frac{1}{\sin\alpha}\left[\sin\beta(1 + \cos\alpha) - \sin\alpha(1 + \cos\beta)\right]$$

$$= \frac{4}{\sin\alpha}\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\left(\frac{\beta}{2} - \frac{\alpha}{2}\right).$$

Now $dl/dx > 0$ since $\alpha < \sphericalangle C_iC_{i+1}C_{i+2} \leqslant \sphericalangle C_{i+1}C_{i+2}C_{i+3} < \beta$.

4.13. Thus j_d^1, i.e., the polygonal line $EC_1 \cdots C_iA(d)C_{i+3} \cdots F$ (see Fig. 5) has the length $l(d) > l(0) = l^1$, and its curvature does not exceed θ^1 by §4.11. It also bounds along with EF a nondegenerate convex region and has pairwise different vertices with angles different from 0 and π. But j_d^1 has one vertex less than j^1. Repeating the described process $m - 1$ times, one gets a polygonal line EIF (call it j^2) satisfying $\theta^2 \leqslant \theta^1$, $l^2 \geqslant l^1$.

If $m = 1$, we put $j^2 = j^1$.

Denote by j^3 the polygonal line $EI'F$ with total curvature $\theta^3 = \theta^2$ and such that $\overline{EI'} = \overline{I'F}$. A simple discussion of the conditional maximum shows that $2\overline{EI'} = l^3 \geqslant l^2$.

4.14. As in §4.7, we realize that the family γ_θ and j^3 have the same axis of symmetry and that their convexities face the same direction of the axis. Since $\theta^3 \leqslant \theta^1 \leqslant \pi/2 \leqslant 2\omega$, the curve j^3 can be matched with a part o γ_{θ^3} by a movement preserving the symmetry with respect to the axis. Since l_θ increases by θ, $l^1 \leqslant l^3 \leqslant l_{\theta^3} \leqslant l_\theta$.

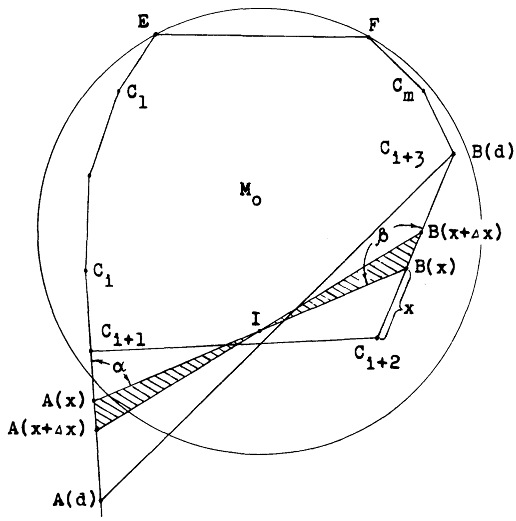

FIG. 5

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