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REDUCIBILITY OF DIFFERENTIAL EQUATIONS AND PSEUDO-ISOMORPHISM OF AUTOMORPHISM PSEUDO-GROUPS

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Introduction

Lie's theory of the integration for a category $\mathcal{C}$ of differential equations is composed of the following four kinds of problems:

(1) Determine all canonical forms of differential equations belonging to $\mathcal{C}$.

(2) Discriminate the canonical form of a differential equation belonging to $\mathcal{C}$.

(3) Translate a differential equation in $\mathcal{C}$ to its canonical form.

(4) Integrate all canonical forms in $\mathcal{C}$.

Problem (1) is a classification problem, and Problem (2) is an equivalence problem.

For example, let $\mathcal{C}(M)$ be the set of local vector fields with no singularity on a manifold M. Then it is well-known that each element X of $\mathcal{C}(M)$ possesses the caonical form $\partial/\partial x$ as a germ. That is, in this case, Problems (1), (2) and (4) are trivial. Problem (3) is to seek for a local transformation of X to $\partial/\partial x$.

Now we shall pose the following problem: Let $\mathcal{G}$ be a pseudo-group on a manifold Q, and let $\mathcal{C}_\Gamma$ be the set of a differential equation E such that the automorphism pseudo-group $\mathcal{Q}(E)$ of E is equal to Γ. Then the problem to consider is to classify the category $\mathcal{C}_\Gamma$.

As the associated problem to this, we consider the reducibility of a differential equation E to another differential equation E'.

This is similar to the reduction of the classification problem of pseudo-groups to the primitive case.

Let (Q, Q', π) be a fibred manifold, and let E or E' be a differential

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equation at $j_x^\alpha(f) \in J^\alpha(N, Q)$ or $j_x^{\alpha'}(f') \in J^{\alpha'}(N, Q')$, respectively. Roughly speaking, the pair (E, E') is called a reduced pair if π induces a one to one correspondence between the solution space of E and that of E'.

The purpose of this paper is to formulate such a reduced pair of differential equations and to characterize it by their automorphism pseudo-groups and their resolvent systems.

In §1, we shall introduce the notion of orbit systems of a weak Lie algebra sheaf.

In §2, we shall state the structure of a differential equation E as a family of orbit systems of the weak Lie algebra sheaf which is induced from the automorphism pseudo-group $\mathcal{Q}(E)$ of E. The family is parametrized by the solution space of a resolvent differential equation of E (Corollary 2.2.1). E. Vessiot or H. H. Johnson refers to the contents of this section in [7] or [1].

In §3, we shall reduce the equivalence problem of differential equations to that of their resolvent differential equations (Proposition 3.3.1). This will be applied to the compatibility of reduction and equivalence in §6.

In §4, we shall define the reducibility of a differential equation E and prove that under some conditions the reduction is of orbit-system-preserving (Lemma 4.1.1). As one of the applications of Lemma 4.1.1, we shall show that if E is an $\mathfrak{L}_{\mathcal{Q}(E)}$-orbit system, any reduced form E' is also an $\mathfrak{L}_{\mathcal{Q}(E')}$-orbit system. Furthermore we shall prove that if E is $\mathcal{Q}(E)$-automorphic, then E' is $\mathcal{Q}(E')$-automorphic (Proposition 4.1.1).

In §5, we shall state the notion of pseudo-isomorphism of pseudo-groups (Definition 5.1.1) which is given in [3] in a vague form, and shall characterize the reducibility of a differential equation E to a differential equation E' by using the pseudo-isomorphism of $\mathcal{Q}(E)$ to $\mathcal{Q}(E')$ (Theorems 5.2.1, 5.2.2).

In §6, we shall study the compatibility of reduction and equivalence of differential equations, namely, for two pairs (E_1, E_1') and (E_2, E_2') satisfying some conditions, if E_1 is isomorphic to E_2, then E_1' is isomorphic to E_2' in a sense (Theorem 6.2.1).

In §7, we shall give an example of a pseudo-group which is k-closed at (x, f) for some integer k where f is a local immersion (Proposition 7.1.1).

The completeness and the order of pseudo-groups are explained in §8 and §9 as appendices.

Throughout this paper, we assume the differentiability of class C^ω. By a differential equation at $p \in J^l(N, Q)$, we mean a system of functions defined on a neighborhood of p. For a pseudo-group Γ, we always assume that any element of Γ is near to the identity. As to a (weak) Lie algebra sheaf, refer to [5] or [6].

Index of notation

	Section
$I(E)$: the set of integral points of a differential equation E	4
$p^k(E)$: standard k-th prolongation of E	4
$\mathcal{S}(E)$: the set of solutions of E	2
$\mathcal{Q}(E)$: automorphism pseudo-group of E	2
$D^k(\theta^l, E) = D^k(E)$: resolvent system of E	5, 6
$\mathcal{W}^k(\theta^l, E)$: resolvent space of E	4
D^k_E: involutive distribution induced from a differential equation E	6
$\mathfrak{N}(\Gamma)$: normalizer of a pseudo-group Γ on Q in the pseudo-group of all local transformations of Q	5
$D(\theta^l, \mathfrak{N}(\Gamma)) = D(\mathfrak{N}(\Gamma))$: pseudo-group induced from $\mathfrak{N}(\Gamma)$ via the map $\theta^l = (\theta^l_1, \ldots, \theta^l_{m_l})$	5
$F(\theta^l)$: Γ-differential invariant	3
$\mathfrak{L}$: a sheaf of vector fields associated to a pseudo-group Γ	3
$\mathfrak{L}(l, x, f)$: $\mathfrak{L}$-orbit system at (l, x, f)	1

1. Orbit systems

1.1. We denote by $J^l(N, Q)$ the space of l-jets $j^l_x(f)$ of local maps f of a manifold N to another manifold Q, and if $\dim N \geqslant \dim Q \geqslant 1$, we denote by $\tilde{J}^l(N, Q)$ the open submanifold of $J^l(N, Q)$ which is the l-jet space of local submersions. If $\dim Q > \dim N \geqslant 1$, $\tilde{J}^l(N, Q)$ means the l-jet space of local immersions which is also an open submanifold of $J^l(N, Q)$. In case $l = 0$, $J^0(N, Q)$ or $\tilde{J}^0(N, Q)$ means the manifold $N \times Q$, and $j^0_x(f)$ the point $(x, f(x)) \in N \times Q$.

Let ϕ be any local diffeomorphism of Q. Then ϕ is prolonged to a local diffeomorphism $\phi^{(l)}$ of $\tilde{J}^l(N, Q)$, which is defined by $\phi^{(l)}(j^l_x(f)) = j^l_x(\phi \circ f)$, $j^l_x(f) \in \tilde{J}^l(N, Q)$.

Let X be a germ of a local vector field on Q. Then, by considering the local 1-parameter group of local transformations generated by X, we can easily see that X is prolonged to a germ $X^{(l)}$ of a local vector field on $\tilde{J}^l(N, Q)$. Therefore a weak Lie algebra sheaf $\mathfrak{L}$ on Q is prolonged to a weak Lie algebra sheaf $\mathfrak{L}^{(l)}$ on $\tilde{J}^l(N, Q)$.

Let ${}^0\mathfrak{L}^{(l)}_p$ denote the isotropy of the stalk $\mathfrak{L}^{(l)}_p$, and set $D^{(l)}_p = \mathfrak{L}^{(l)}_p / {}^0\mathfrak{L}^{(l)}_p$.

Definition 1.1.1. A weak Lie algebra sheaf $\mathfrak{L}$ on Q is said to be (l, N)-regular if $\dim D^{(l)}_p$ is constant. If $\mathfrak{L}$ is (l, N)-regular for any integer $l \geqslant 0$, then $\mathfrak{L}$ is said to be N-regular.

Definition 1.1.2. A function ϕ given on an open subset $\mathcal{O}^l$ of $\tilde{J}^l(N, Q)$ is called a differential invariant of $\mathfrak{L}$ at $j_x^l(f) \in \mathcal{O}^l$ if there is a neighborhood $\mathcal{U}^l \subset \mathcal{O}^l$ of $j_x^l(f)$ such that, for any $p \in \mathcal{U}^l$ and any germ $X^{(l)}$ of $\mathfrak{L}_p^{(l)}$, we have $X^{(l)} \cdot \phi_p = 0$ where ϕ_p is a germ of ϕ at p.

Note that if $\mathfrak{L}$ is N-regular, then a differential invariant of $\mathfrak{L}$ at $j_x^l(f)$ is a first integral of the involutive distribution $D^{(l)}$ at $j_x^l(f)$.

Definition 1.1.3. A fundamental system of first integrals of $D^{(l)}$ at $j_x^l(f)$ is called a fundamental system of differential invariants of $\mathfrak{L}$ at $j_x^l(f)$.

Suppose $\mathfrak{L}$ is an N-regular weak Lie algebra sheaf on Q.

Lemma 1.1.1. *Let $\{\theta_j^l\}_{j=1}^{m_l}$ and $\{\tilde{\theta}_j^l\}_{j=1}^{m_l}$ be two fundamental system of differential invariants of $\mathfrak{L}$ at $j_{x_0}^l(f)$, and set $\lambda_j(x) = \theta_j^l(j_x^l(f))$ and $\tilde{\lambda}_j(x) = \tilde{\theta}_j^l(j_x^l(f))$. We denote by $(*)^l$ (respectively $(\tilde{*})^l$) the system of differential equations generated by $\theta_j^l - \lambda_j$ $(1 \leqslant j \leqslant m_l)$ (respectively $\tilde{\theta}_j^l - \tilde{\lambda}_j$ $(1 \leqslant j \leqslant m_l)$). Then the set of solutions of $(*)^l$ is equal to that of $(\tilde{*})^l$.*

Proof. We have the analytic expressions $\tilde{\theta}_j^l = \xi_j(\theta_1^l, \cdots, \theta_{m_l}^l)$, $1 \leqslant j \leqslant m_l$. Then we get $\tilde{\lambda}_j = \xi_j(\lambda_1, \cdots, \lambda_{m_l})$, $1 \leqslant j \leqslant m_l$. Let s be any solution of $(*)^l$. Then $\theta_j^l(j_x^l(s)) = \lambda_j(x)$, $1 \leqslant j \leqslant m_l$. Therefore we have $\tilde{\theta}_j(j_x^l(s)) = \xi_j(\theta_1^l(j_x^l(s))), \cdots, \theta_{m_l}^l(j_x^l(s)) = \xi_j(\lambda_1(x), \cdots, \lambda_{m_l}(x)) = \tilde{\lambda}_j(x)$, $1 \leqslant j \leqslant m_l$. This implies that s is a solution of $(\tilde{*})^l$. Similarly any solution of $(\tilde{*})^l$ is also a solution of $(*)^l$. Hence the proof is completed.

Definition 1.1.4. The differential equation $(*)^l$ is called the $\mathfrak{L}$-orbit system at (l, x_0, f) and denoted by $\mathfrak{L}(l, x_0, f)$.

2. Structures of differential equations

2.1. Let E be a differential equation at $j_x^l(f) \in \tilde{J}^l(N, Q)$. We denote by $\mathfrak{S}(E)$ the set of solutions of E. For any neighborhood $\mathcal{U}^k$ of $j_x^k(f)$, we denote by $\mathfrak{S}(E)|\mathcal{U}^k$ the set of solutions s of E such that the image of $j^k(s) \subset \mathcal{U}^k$.

Definition 2.1.1. A differential equation E^1 at $j_x^l(f^1)$ is said to be isomorphic to a differential equation E^2 at $j_x^l(f^2)$ if there exist a diffeomorphism ψ of a neighborhood $\mathcal{U}^1$ of $f^1(x) \in Q$ to a neighborhood $\mathcal{U}^2$ of $f^2(x) \in Q$ and a neighborhood $\mathcal{W}$ of $x \in N$ such that $\mathfrak{S}(E^1)|\mathcal{W} \times \mathcal{U}^1 \ni s$ if and only if $\psi \circ s \in \mathfrak{S}(E^2)|\mathcal{W} \times \mathcal{U}^2$. ψ is called an isomorphism of E^1 to E^2.

Definition 2.1.2. If $E^1 = E^2 = E$ and $f^1 = f^2 = f$, we denote by $\mathcal{A}(E)$ the pseudo-group on a neighborhood of $f(x)$ generated by the set of isomorphisms of E to E. $\mathcal{A}(E)$ is called the automorphism pseudo-group of E, and each element of $\mathcal{A}(E)$ is called an automorphism of E.

2.2. Let Γ be a pseudo-group on Q such that $\mathfrak{L}_\Gamma$ is an N-regular weak Lie algebra sheaf, where $\mathfrak{L}_\Gamma$ is the sheaf of germs of local vector fields X such that the local 1-parameter group of local transormations generated by X is contained in Γ. Let $\{\theta_j^l\}_{j=1}^{m_l}$ be a fundamental system of differential invariants of $\mathfrak{L}_\Gamma$ at $j_{x_0}^l(f)$. We set $\theta^l = (\theta_1^l, \cdots, \theta_{m_l}^l)$. Then θ^l is a submersion of a neighborhood $\mathfrak{U}^l$ of $j_{x_0}^l(f)$ onto a neighborhood W of $\theta^l(j_{x_0}^l(f)) = (\theta_1^l(j_{x_0}^l(f)), \cdots, \theta_{m_l}^l(j_{x_0}^l(f))) \in \mathbf{R}^{m_l}$. Let ρ_l^{l+k} be the projection of $\tilde{J}^{l+k}(N, Q)$ onto $\tilde{J}^l(N, Q)$ and set $\mathfrak{U}^{l+k} = (\rho_l^{l+k})^{-1}(\mathfrak{U}^l)$. Similarly let α^l and β^l be the projection of $J^l(N, Q)$ onto N and Q respectively. We denote by $p^k\theta^l$ the map of $\mathfrak{U}^{l+k}$ to $J^k(\mathfrak{V}, W) \subset J^k(N, \mathbf{R}^{m_l})$, where $\mathfrak{V} = \alpha^l(\mathfrak{U}^l)$, defined by $p^k\theta^l(j_x^{l+k}(f)) = j_x^k(\theta^l(j^l(f)))$. We set $q^k = j_{x_0}^k(\theta^l(j^l(f)))$. Then $q^k \in J^k(\mathfrak{V}, W)$. For any function F locally defined at q^k, we set $F(\theta^l) = (p^k\theta^l)^*F$, which is a function locally defined at $J_{x_0}^{l+k}(f)$. Let $\{\tilde{\theta}_j^l\}_{j=1}^{m_l}$ be any other fundamental system of differential invariants of $\mathfrak{L}_\Gamma$ at $j_{x_0}^l(f)$. If y is a differential invariant of $\mathfrak{L}_\Gamma$ at $j_{x_0}^{l+k}(f)$ of the form $F(\theta^l)$, then y is also of the form $\tilde{F}(\tilde{\theta}^l)$.

Definition 2.2.1. A differential invariant of $\mathfrak{L}_\Gamma$ at $j_{x_0}^{l+k}(f)$ of the form $F(\theta^l)$ is called a Γ-differential invariant of type l at $j_{x_0}^{l+k}(f)$.

Definition 2.2.2. A family of linearly independent differential invariants $\mathfrak{H} = \{\mathfrak{h}_j\}_{j=1}^r$ of $\mathfrak{L}_\Gamma$ at $j_{x_0}^{l+k}(f)$ is called a Γ-family at $j_{x_0}^{l+k}(f)$ of type (l, r) if $\mathfrak{H}$ satisfies the following conditions:

(1) $\mathfrak{h}_j$ is a Γ-differential invariant of type l at $j_{x_0}^{l+k}(f)$, $1 \leqslant j \leqslant r$.

(2) The differential equation at $j_{x_0}^{l+k}(f)$ generated by $\mathfrak{h}_j$, $1 \leqslant j \leqslant r$, possesses a solution.

(3) The automorphism pseudo-group of the differential equation is equal to Γ on a neighborhood of $f(x_0)$.

Proposition 2.2.1. *Suppose $\mathfrak{H} = \{\mathfrak{h}_j\}_{j=1}^r$ is a Γ-family at $j_{x_0}^{l+k}(f)$ of type (l, r). Then Γ is locally determined at $f(x_0)$ by $\mathfrak{H}$. Namely, assume that each $\mathfrak{h}_j$ is defined on a neighborhood V^{l+k} of $j_{x_0}^{l+k}(f)$. We denote by $\tilde{\Gamma}$ a pseudo-group on $\beta^{l+k}(V^{l+k})$ which is defined by the following way $\tilde{\Gamma} \ni \phi\colon \mathfrak{U} \to \mathfrak{V}$ if and only if $\phi^{(l+k)*}\mathfrak{h}_j = \mathfrak{h}_j$ $(1 \leqslant j \leqslant r)$ on $V^{l+k} \cap \tilde{J}^{l+k}(N, \mathfrak{U})$. Then we have $\tilde{\Gamma} = \Gamma$ on a neighborhood $\tilde{\mathfrak{U}} \subset \beta^{l+k}(V^{l+k})$ of $f(x_0)$.*

Proof. Since $\mathfrak{h}_j$ is a differential invariant of $\mathfrak{L}_\Gamma$ at $j_{x_0}^{l+k}(f)$, it is clear that $\tilde{\Gamma}$ contains Γ on a neighborhood of $f(x_0)$. On the other hand, $\tilde{\Gamma}$ is contained in the automorphism pseudo-group of the differential equation generated by $\mathfrak{h}_j$ $(1 \leqslant j \leqslant r)$, which is, by the assumption that $\mathfrak{H}$ is a Γ-family at $j_{x_0}^{l+k}(f)$ of type (l, r), equal to Γ on a neighborhood of $f(x_0)$. Therefore $\tilde{\Gamma}$ is equal to Γ on a neighborhood $\tilde{\mathfrak{U}} \subset \beta^{l+k}(V^{l+k})$ of $f(x_0)$.

2.3. Definition 2.3.1. A family $\Theta^l = \{\theta_j^l\}_{j=1}^m$ of such functions that θ_j^l $(1 \leqslant j \leqslant m)$ are defined on a neighborhood $\tilde{\mathfrak{U}}^l$ of $j_{x_0}^l(f) \in \tilde{J}^l(N, Q)$ is said to be regular at (x_0, f) if the following conditions are satisfied:

(1) We set $\theta^l = (\theta_1^l, \cdots, \theta_m^l)$. Then $p^k\theta^l$ is of rank constant on a neighborhood $\mathfrak{U}^{l+k} \subset \tilde{\mathfrak{U}}^{l+k}$ of $j_{x_0}^{l+k}(f)$ for each $k \geqslant 0$.

(2) We set $\mathcal{W}^k = p^k\theta^l(\mathcal{U}^{l+k})$, $k = 0, 1, 2, \cdots$. For a map λ of a neighborhood $\tilde{\mathcal{V}}$ of x_0 to $\mathbf{R}^m$ such that $j_x^k(\lambda) \in \mathcal{W}^k$ for any $x \in \tilde{\mathcal{V}}$ and any integer $k \geqslant 0$, there exists a map s of a neighborhood $\mathcal{V} \subset \tilde{\mathcal{V}}$ of x_0 to Q such that $\theta^l(j_x^l(s)) = \lambda(x)$ on $\mathcal{V}$.

Proposition 2.3.1. *Let $\Theta^l = \{\theta_j^l\}_{J=1}^m$ be a family of functions at $j_{x_0}^l(f) \in \tilde{J}^l(N, Q)$ which is regular at (x_0, f). Then we have an integer K and a submanifold $\mathcal{W}^K(\theta^l) \subset J^K(N, \mathbf{R}^m)$ satisfying the following conditions*:

(1) *$\mathcal{W}^K(\theta^l) \ni p^K\theta^l(j_{x_0}^{l+K}(f))$ and $(\mathcal{W}^K(\theta^l), \alpha^K(\mathcal{W}^K(\theta^l)), \alpha^K)$ is a fibred manifold, where α^K is the projection of $J^K(N, \mathbf{R}^m)$ onto N.*

(2) *For a local map λ of N to $\mathbf{R}^m$, the differential equation $\theta^l = \lambda$ possesses a solution if and only if λ is a local cross secton of $(\mathcal{W}^K(\theta^l), \alpha^K(\mathcal{W}^K(\theta^l))\ \alpha^k)$.*

Proof. By the regularity condition (1) of Θ^l at (x_0, f), $\mathcal{V}^k = p^k\theta^l\ (\mathcal{U}^{l+k})$ is a submanifold of $J^k(N, \mathbf{R}^m)$ and $(\mathcal{U}^{l+k}, \mathcal{W}^k, p^k\theta^l)$ is a fibred manifold. Moreover it is clear that $(\mathcal{W}^k, \alpha^k(\mathcal{W}^k), \alpha^k)$ is also a fibred manifold. Since there is a neighborhood $\mathcal{U}^{l+k} \supset {'\mathcal{U}^{l+k}}$ of $j_{x_0}^{l+k}(f)$ (resp. $\mathcal{U}^{l+k+1} \supset {'\mathcal{U}^{l+k+1}}$ of $j_{x_0}^{l+k+1}(f)$) such that $('\mathcal{U}^{l+k+1}, {'\mathcal{U}^{l+k}}, \rho_{l+k}^{l+k+1})$ is a fibred manifold and since $(\mathcal{U}^{l+k+1}, \mathcal{W}^{k+1}, p^{k+1}\theta^l)$ and $(\mathcal{U}^{l+k}, \mathcal{W}^k, p^k\theta^l)$ are fibred manifolds, there exist neighborhoods $\mathcal{W}^{k+1} \supset {'\mathcal{W}^{k+1}}$ of $p^{k+1}\theta^l(j_{x_0}^{l+k+1}(f))$ and $\mathcal{W}^k \supset {'\mathcal{W}^k}$ of $p^k\theta^l(J_{x_0}^{l+k}(f))$ such that $('\mathcal{W}^{1+k}, {'\mathcal{W}^k}, \rho_k^{k+1})$ is a fibred manifold. It is clear that we have $p\mathcal{W}^k \supset \mathcal{W}^{k+1}$ and $\theta^l(j^l(f))$ is a solution of $\mathcal{W}^k$ for any $k \geqslant 0$. Therefore by Kuranishi's prolongation theorem, there exists an integer K such that, for any $k \geqslant K$, $\mathcal{W}^k$ is involutive at $p^k\theta^l(j_{x_0}^{l+k}(f))$ and $p\mathcal{W}^k = \mathcal{W}^{k+1}$ on a neighborhood of $p^{k+1}\theta^l(j_{x_0}^{l+k+1}(f))$. We set $\mathcal{W}^K(\theta^l) = \mathcal{W}^K$. Let λ be a solution of $\mathcal{W}^K(\theta^l)$. Then λ is also a solution of $\mathcal{W}^k$ for any $k \geqslant K$. Therefore, by the regularity condition (2) of Θ^l at (x_0, f), for a fixed $\tilde{x}_0 \in N$, we have a local map s of N to Q such that $j_{\tilde{x}_0}^\alpha(\theta^l(j^l(s))) = j_{\tilde{x}_0}^\alpha(\lambda)$ for any integer $\alpha \geqslant 0$. This implies that if λ is a local cross secton of $(\mathcal{W}^K(\theta^l), \alpha^K(\mathcal{W}^K(\theta^l)), \alpha^K)$, then the differential equation $\theta^l = \lambda$ possesses a solution. Clearly if $\theta^l = \lambda$ possesses a solution, λ is a local cross section of $(\mathcal{W}^K(\theta^l), \alpha^K(\mathcal{W}^K(\theta^l)), \alpha^K)$. This completes the proof of Proposition 2.3.1.

Definition 2.3.2. The space $\mathcal{W}^K(\theta^l)$ in Proposition 4.1 is called a resolvent space of θ^l. It is clear that if $\mathcal{W}^K(\theta^l)$ is a resolvent space of θ^l, we have a resolvent space $\mathcal{W}^H(\theta^l)$ of θ^l for any integer $H \geqslant K$.

Definition 2.3.3. Let $\Theta^l = \{\theta_j^l\}_{j=1}^h$ be a family of functions at $j_{x_0}^l(f)$ which is regular at (x_0, f) and let $\mathcal{W}^K(^l) = p^K\theta^l(\mathcal{U}^{l+K})$ be any resolvent space of θ^l. We set $\mathcal{W}^K(\theta^l, E) = p^K\theta^l(I(p^{l+K-\alpha}(E)) \cap \mathcal{U}^{l+K})$, where $p^{l+K-\alpha}(E)$ is the $(l + K - \alpha)$-th prolongation of a differential equation E at $j_{x_0}^\alpha(f)$. $\mathcal{W}^K(\theta^l, E)$ is called a resolvent space of (θ^l, E). Θ^l is said to be E-regular at (x_0, f) if $\mathcal{W}^K(\theta^l, E)$ is a regular submanifold of $\mathcal{W}^K(\theta^l)$ for a neighborhood $\mathcal{U}^{l+K}$ and any integer $K \geqslant K_0$ where K_0 is the minimum integer for which θ^l possesses a resolvent space.

Proposition 2.3.2. *Let $\Theta^l = \{\theta_j^l\}_{j=1}^h$ be a family of functions at $j_{x_0}^l(f) \in \tilde{J}^l(N, Q)$ which is regular at (x_0, f). Let E be an involutive differential equation at $j_{x_0}^\alpha(f)$, $\alpha \leqslant l$. Suppose that Θ^l is E-regular at (x_0, f) and that, for any solution λ of the resolvent space $\mathcal{W}^K(\theta^l, E)$ of (θ^l, E), we have $I(\theta^l(\lambda)) \subset I(p^{l-\alpha}(E))$, where $\theta^l(\lambda)$ is the differential equation generated by $\theta_j^l - \lambda_j$ $(1 \leqslant j \leqslant h)$. Then there exists a generator $\mathcal{H} = \{\mathfrak{h}_j\}_{j=1}^r$ of $p^\beta(E)$, $\beta = l + K - \alpha$, such that $\mathfrak{h}_j$ is of the form $(p^K\theta^l)^* F_j$, where F_j is a function at $p^K\theta^l(j_{x_0}^{l+k}(f)) \in J^{l+K}(N, \mathbf{R}^h)$, $1 \leqslant j \leqslant r$.*

Proof. Let $\{F_1, \cdots, F_r\}$ be a family of linearly independent functions at $p^K\theta^l(j_{x_0}^{l+K}(f)) = j_{x_0}^K(\theta^l(f))$ by which $\mathcal{W}^K(\theta^l, E)$ is locally defined at $j_{x_0}^K(\theta^l(f))$. We set $\mathfrak{h}_j = (p^K\theta^l)^* F_j$ $(1 \leqslant j \leqslant r)$. We denote by $(*)$ the differential equation generated by $\{\mathfrak{h}_j\}_{j=1}^r$. Then it is clear that $\mathcal{S}(*) \supset \mathcal{S}(p^\beta(E))$. On the other hand by the assumption, we have $I(\theta^l(\lambda)) \subset I(p^{l-\alpha}(E))$ for any solution λ of $\mathcal{W}^K(\theta^l, E)$. This implies that $\mathcal{S}(*) \subset \mathcal{S}(p^\beta(E))$. Therefore we get $\mathcal{S}(*) = \mathcal{S}(p^\beta(E))$. Then it is easy to see that $I(*) = I(p^\beta(E))$ because E is involutive and the resolvent space $\mathcal{W}^K(\theta^l, E)$ is defined by $F_1, \cdots, F_r$. This means that $p^\beta(E)$ is generated by $\{\mathfrak{h}_j\}_{j=1}^r$. The proof is completed.

Definition 2.2.4. A differential equation E at $j_x^k(f) \in \tilde{J}^k(N, Q)$ is said to be Γ-closed if the automorphism pseudo-group $\mathcal{Q}(E)$ of E is equal to Γ on a neighborhood of $f(x)$.

Corollary 2.2.1. *Suppose $\mathfrak{L}_\Gamma$ is an N-regular weak Lie algebra sheaf and let $\Theta^l = \{\theta_j^l\}_{j=1}^{m_l}$ be a fundamental system of differential invariants of $\mathfrak{L}_\Gamma$ at $j_{x_0}^l(f)$. Let E be a Γ-closed involutive differential equation at $j_{x_0}^\alpha(f)$, where $l + K - \alpha \geqslant 0$. Suppose Θ^l is regular and E-regular at (x_0, f). Then there exists a Γ-family $\mathcal{H} = \{\mathfrak{h}_j\}_{j=1}^r$ at $j_{x_0}^{l+K}(f)$ of type (l, r) such that $\mathcal{H}$ is a generator of $p^\beta(E)$ at $j_{x_0}^{l+K}(f)$, $\beta = l + K - \alpha$.*

Proof. Since E is Γ-closed, it is easy to see that, for any solution λ of $\mathcal{W}^K(\theta^l, E)$ we have $I(\theta^l(\lambda)) \subset I(p^{l-\alpha}(E))$, where $\theta^l(\lambda)$ is the $\mathfrak{L}_\Gamma$-orbit system generated by $\theta_j^l - \lambda_j$ $(1 \leqslant j \leqslant m_l)$. Therefore, by Proposition 2.3.2, $p^\beta(E)$ possesses a generator $\mathcal{H} = \{\mathfrak{h}_j\}_{j=1}^r$ such that $\mathfrak{h}_j$ is a Γ-differential invariant. It is clear that $\mathcal{H} = \{\mathfrak{h}_j\}_{j=1}^r$ is a Γ-family at $j_{x_0}^{l+K}(f)$ of type (l, r).

3. Reduction of equivalence

3.1. Let Γ be a pseudo-group on Q, such that $\mathfrak{L}_\Gamma$ is an N-regular weak Lie algebra sheaf. For a fundamental system of differential invariants $\Theta^l =$

$\{\theta_j^l\}_{j=1}^{m_l}$ of $\mathfrak{L}_\Gamma$ at $j_{x_0}^l(f) \in \tilde{J}^l(N, Q)$, there are neighborhoods $\mathcal{U}^l$ of $j_{x_0}^l(f)$ and $\mathcal{W}$ of $\theta^l(j_{x_0}^l(f)) \in \mathbf{R}^{m_l}$ such that $(\mathcal{U}^l, \mathcal{W}, \theta^l)$ is a fibred manifold. Let $\mathcal{N}(\Gamma)$ be the normalizer of Γ in the pseudo-group of all local diffeomorphisms of Q. Then, for $\phi^{(l)} \in (\mathcal{N}(\Gamma))^{(l)}|\mathcal{U}^l$, we can induce a local diffeomorphism $D(\theta^l, \phi)$ of $\mathcal{W}$ such that $\theta^l \circ \phi^{(l)} = D(\theta^l, \phi) \circ \theta^l$. We denote by $D(\theta^l, \mathcal{N}(\Gamma))$ or simply by $D(\mathcal{N}(\Gamma))$ the pseudo-group on $\mathcal{W}$ generated by $D(\theta^l, \phi)$, $\phi \in \mathcal{N}(\Gamma)$. $D(\theta^l, \phi)$ is also simply denoted by $D(\phi)$.

Let E^1 and E^2 be Γ-closed differential equations at $j_{x_0}^{\alpha^1}(f)$ and at $j_{x_0}^{\alpha^2}(f)$ respectively. Suppose there is an integer K such that $l + K - \alpha^i = \beta^i \geqslant 0$, $i = 1, 2$, and such that $p^{\beta^i}(E^i)$ possesses a generator $F^i(\theta^l) = \{F_j^i(\theta^l)\}_{j=1}^r$ which is a Γ-family at $j_{x_0}^{l+K}(f)$ of type (l, r). We denote by $D^K(\theta^l, E^i)$ or simply by $D^K(E^i)$ the differential equation at $p^K\theta^l(j_{x_0}^{l+K}(f))$ generated by $\{F_1^i, \cdots, F_r^i\}$.

Proposition 3.1.1. *E^1 is isomorphic to E^2 if and only if there is an element $\phi \in \mathcal{N}(\Gamma)$ such that $D(\phi)$ is an isomorphism of $D^K(E^1)$ to $D^K(E^2)$.*

Proof. Let ϕ be an isomorphism of E^1 to E^2. Since $\mathcal{Q}(E^i) = \Gamma$ on a neighborhood of $f(x_0)$, it is easy to see that $\phi \in \mathcal{N}(\Gamma)$. Let λ^1 be a solution of $D^K(E^1)$ and consider the differential equation $\theta^l = \lambda^1$. We denote by $\mathfrak{S}(\lambda^1)$ the solution space of the equation. Then, since $p^{\beta^i}(E^i)$ possesses as a generator a Γ-family $F^i(\theta^l)$ at $j_{x_0}^{l+K}(f)$ of type (l, r) it is clear that $\mathfrak{S}(E^i) = \bigcup_{\lambda^i \in \mathfrak{S}(D^K(E^i))} \mathfrak{S}(\lambda^i)$, where $\mathfrak{S}(\lambda^i) \cap \mathfrak{S}(\mu^i) \neq \varnothing$ if and only if $\mathfrak{S}(\lambda^i) = \mathfrak{S}(\mu^i)$. Since $\phi \in \mathcal{N}(\Gamma)$, $\phi^{(l)*}\theta^l = \xi(\theta^l)$. Therefore $\theta^l(j_x^l(\phi \circ s)) = (\phi^{(l)*}\theta^l)(j_x^l(s)) = \xi(\lambda^1)$ for $s \in \mathfrak{S}(\lambda^1)$. This implies that ϕ maps $\mathfrak{S}(\lambda^1)$ to $\mathfrak{S}(\lambda^2)$, where $\lambda^2 = \xi(\lambda^1)$, that is, for each $\lambda^1 \in \mathfrak{S}(D^K(E^1))$, there corresponds $\lambda^2 \in \mathfrak{S}(D^K(E^2))$ such that $\lambda^2 = D(\phi) \circ \lambda^1$ and such that the isomorphism ϕ of E^1 to E^2 maps $\mathfrak{S}(\lambda^1)$ to $\mathfrak{S}(\lambda^2)$. This means that $D(\phi)$ is an isomorphism of $D^K(E^1)$ to $D^K(E^2)$. Conversely, assume that there is an element $\phi \in \mathcal{N}(\Gamma)$ such that $D(\phi)$ is an isomorphism of $D^K(E^1)$ to $D^K(E^2)$. Then, for $s \in \mathfrak{S}(E^1)$, if $s \in \mathfrak{S}(\lambda^1)$, we have $\theta^l(j_x^l(\phi \circ s)) = \phi^{(l)*}\theta^l(j_x^l(s)) = D(\phi) \circ \theta^l(j_x^l(s)) = D(\phi) \circ \lambda^1 = \lambda^2 \in \mathfrak{S}(D^K(E^2))$. Therefore $\phi \circ s \in \mathfrak{S}(E^2)$. This means that ϕ is an isomorphism of E^1 to E^2. This completes the proof of Proposition 3.1.1.

Remark 3.3.1. Let E be a differential equation at $j_x^\alpha(f) \in \tilde{J}^\alpha(N, Q)$. Then, roughly speaking, Corollary 2.2.1 implies that E is a family $\mathcal{F}$ of $\mathcal{Q}(E)$-orbit systems, where $\mathcal{F}$ is parametrized by the solution space of a differential equation $D^K(E)$. That is, the structure of E is decomposed into that of $\mathcal{Q}(E)$-orbit systems and that of $D^K(E)$. Proposition 3.1.1 implies that the equivalence problem of Γ-closed differential equations E_1 and E_2 is reduced to that of $D^K(E_1)$ and $D^K(E_2)$ under the pseudo-group $D(\mathcal{N}(\Gamma))$.

4. Reducibility of differential equations

4.1. Definition 4.1.1. Let (Q, Q', π) be a fibred manifold. A differential equation E at $j^{\alpha}_{x_0}(f) \in J^{\alpha}(N, Q)$ is said to be weakly K-reducible to a differential equation E' at $j^{\alpha'}_{x_0}(\pi \circ f) \in J^{\alpha'}(N, Q')$ by π if there exist a nonnegative integer K and, for $k \geqslant K$, neighborhoods $\mathfrak{U}^k$ and $\mathfrak{U}'^k$ of $j^k_{x_0}(f)$ and $j^k_{x_0}(\pi \circ f)$, respectively, satisfying the following conditions:

(1) $(\mathfrak{U}^k, \mathfrak{U}'^k, \pi^k)$ is a fibred manifold, where π^k is the map naturally induced from π.

(2) (i) For any $s \in \mathbb{S}(E)|\mathfrak{U}^k$, we have $\pi \circ s \in \mathbb{S}(E')|\mathfrak{U}'^k$. (ii) For any $s' \in \mathbb{S}(E')|\mathfrak{U}'^k$, we have an $s \in \mathbb{S}(E)|\mathfrak{U}^k$ with $s' = \pi \circ s$. (iii) If $s\colon U \to V$ or $\tilde{s}\colon \tilde{U} \to \tilde{V}$ belongs to $\mathbb{S}(E)|\mathfrak{U}^k$ with $\pi \circ s = \pi \circ \tilde{s}$ on $U \cap \tilde{U}$, then we have $s = \tilde{s}$ on $U \cap \tilde{U}$, where $\mathbb{S}(E)|\mathfrak{U}^k = \{g \in \mathbb{S}(E); \operatorname{Im} j^k(g) \subset \mathfrak{U}^k\}$.

Furthermore if E satisfies the following condition (3), E is said to be K-reducible to E' by π: We set $\mathcal{Q}(E)|\mathfrak{U}^k = \{g \in \mathcal{Q}(E);\ ^{\exists}\psi \in \mathcal{Q}(E)^{(k)}|\mathfrak{U}^k$ such that $g \circ \beta^k = \beta^k \circ \psi\}$. Then

(3) For any $g\colon U \to V$ which belongs to $\mathcal{Q}(\mathrm{E})|\mathfrak{U}^k$, there is a diffeomorphism g' of $\pi(U)$ to $\pi(V)$ such that $g' \circ \pi = \pi \circ g$ for $k \geqslant K$.

Then we denote by $\mathcal{Q}(E)'_k$ the pseudo-group on a neighborhood of $f(x_0)$ generated by $\{g'; g \in \mathcal{Q}(E)|\mathfrak{U}^k, \pi \circ g = g' \circ \pi\}$.

We say that E' is a (weakly) K-reduced form of E and the pair (E, E') is called a (weakly) K-reduced pair.

Let E be a differential equation at $j^l_{x_0}(f) \in \tilde{J}^l(N, Q)$. For a neighborhood $\mathfrak{U}^l$ of $j^l_{x_0}(f)$, we set $S(E, \mathfrak{U}^l) = \bigcup_{s \in \mathbb{S}(E)} \operatorname{Im} j^l(s) \cap \mathfrak{U}^l$.

Corollary 2.2.1 makes the following definition significant.

Definition 4.1.2. A differential equation E at $j^{\alpha}_x(f) \in \tilde{J}^{\alpha}(N, Q)$ is said to be l-regular at x if E satisfies the following conditions (1), (2) and (3):

(1) The sheaf $\mathfrak{L}_{\mathcal{Q}(E)}$ associated to the automorphism pseudo-group $\mathcal{Q}(E)$ is an N-regular weak Lie algebra sheaf on a neighborhood of $f(x)$.

(2) There are an integer $K^{\tilde{l}}$ and an $\mathcal{Q}(E)$-family $\mathfrak{H}^{\tilde{l}} = \{\mathfrak{h}^{\tilde{l}}_j\}^r_{j=1}$ at $j^{\tilde{l}+K^{\tilde{l}}}_x(f)$ of type $(\tilde{l}, r)$ such that $\tilde{l} + K^{\tilde{l}} - \alpha \geqslant 0$ and $p^{\tilde{l}+K^{\tilde{l}}-\alpha}(E)$ is generated by $\mathfrak{H}^{\tilde{l}}$ for $\tilde{l} \geqslant l$.

(3) There is a neighborhood $\mathfrak{U}^k$ of $j^k_x(f)$ such that $S(E, \mathfrak{U}^k)$ is a regular submanifold of $\tilde{J}^k(N, Q)$ for $k \geqslant l$.

We denote by $D^{K^{\tilde{l}}}(\theta^{\tilde{l}}, E)$ the differential equation generated by $F^{\tilde{l}}_1, \cdots, F^{\tilde{l}}_r$ where $\mathfrak{h}^{\tilde{l}}_j = F^{\tilde{l}}_j(\theta^{\tilde{l}})$. It is called a resolvent system of E. (Refer to Definition 2.3.3.)

Remark 4.1.1. If a differential equation E at $j^{\alpha}_x(f)$ is l-regular at x, then it is clear that E is k-regular at x for any $k \geqslant l$.

Let Γ^1 and Γ^2 be pseudo-groups on Q, and $j^l_x(f)$ a point of $\tilde{J}^l(N, Q)$. We say that $\Gamma^1 = \Gamma^2$ on a neighborhood $\mathfrak{U}^l$ of $j_x^l(f)$ if $\Gamma^1|\mathfrak{U}^l = \Gamma^2|\mathfrak{U}^l$.

Definition 4.1.3. Let Γ be a pseudo-group on Q such that $\mathcal{L}_\Gamma$ is an N-regular weak Lie algebra sheaf. Γ is said to be $\tilde{l}$-closed at (x, f) if there is an integer $\tilde{l}$ such that for $l \geqslant \tilde{l}$, $\mathcal{G}(\mathcal{L}_\Gamma(l, x, f)) = \Gamma$ on a neighborhood of $j_x^l(f)$, where $\mathcal{L}_\Gamma(l, x, f)$ is the $\mathcal{L}_\Gamma$-orbit system at (l, x, f).

Remark 4.1.2. When $\dim N \geqslant \dim Q$, in Lemma 5.1.2 we shall show that any pseudo-group which is complete at $(z_0, 1)$ is l-closed at (x_0, f), where $z_0 = f(x_0)$, and l is the order of the pseudo-group at (x_0, f). When $\dim N < \dim Q$, in §7 we shall give an example of a pseudo-group which is closed at (x_0, f) where f is a local immersion of N to Q.

Let f be a local map of a neighborhood of $x \in N$ to Q which is an immersion (resp. a submersion) if $\dim N < \dim Q$ (resp. $\dim N \geqslant \dim Q$).

Definition 4.1.4. A pseudo-group Γ on Q is said to be l-automorphic at (x, f) where l is a nonnegative integer, if the following conditions are satisfied:

(1) $\mathcal{L}_\Gamma$ is an N-regular weak Lie algebra sheaf.

(2) The orbit system $\mathcal{L}_\Gamma(l, x, f)$ is Γ-automorphic.

As for the definition of "Γ-automorphic system", refer to Definition 9.1.2.

Let (Q, Q', π) be a fibred manifold, and let E (resp. E') be a differential equation at $j_{x_0}^{\alpha}(f) \in \tilde{J}^{\alpha}(N, Q)$ (resp. $j_{x_0}^{\alpha'}(\pi \circ f) \in \tilde{J}^{\alpha'}(N, Q')$). Suppose E and E' are l-regular at x_0, and let $\{\theta_j^k\}_{j=1}^{m_k}$ (resp. $\{\theta_j'^k\}_{j=1}^{m_k'}$) be a fundamental system of differential invariants of $\mathcal{L}_{\mathcal{G}(E)}$ at $j_{x_0}^k(f)$ (resp. $\mathcal{L}_{\mathcal{G}(E')}$ at $j_{x_0}^k(\pi \circ f)$). We denote by D_E^k (resp. $D_{E'}^k$) the involutive distribution on a neighborhood of $j_{f(x_0)}^k(1) \in \tilde{J}^k(Q, Q)$ (resp. $j_{\pi \circ f(x_0)}^k(\pi \circ 1) \in \tilde{J}^k(Q, Q')$) induced from $\mathcal{L}_{\mathcal{G}(E)}$ (resp. $\mathcal{L}_{\mathcal{G}(E')}$).

Lemma 4.1.1. *Suppose E is K-reducible to E' by π. Then the following two assertions* (A) *and* (B) *hold.*

(A) *For any $\tilde{l} \geqslant l' = \max(l, K)$, we have a fibred manifold $(\Theta^{\tilde{l}}, \Theta'^{\tilde{l}}, D^{\tilde{l}}(\pi))$ satisfying the following properties*:

(1) *$\Theta^{\tilde{l}}$ of $\Theta'^{\tilde{l}}$ is a neighborhood of $\theta^{\tilde{l}}(j_{x_0}^{\tilde{l}}(f)) \in \mathbf{R}^{m_{\tilde{l}}}$ or $\theta'^{\tilde{l}}(j_{x_0}^{\tilde{l}}(\pi \circ f)) \in \mathbf{R}^{m_{\tilde{l}}'}$, respectively.*

(2) *$D^{\tilde{l}}(\pi)$ induces an onto map $O_E^{\tilde{l}}$ of $\mathcal{S}(D^{K^{\tilde{l}}}(\theta^{\tilde{l}}, E))|\mathfrak{W}^{K^{\tilde{l}}}$ to $\mathcal{S}(D^{K'^{\tilde{l}}}(\theta'^{\tilde{l}}, E'))|\mathfrak{W}'^{K'^{\tilde{l}}}$ for some neighborhoods $\mathfrak{W}^{K^{\tilde{l}}}$ of $j_{x_0}^{K^{\tilde{l}}}(\theta^{\tilde{l}}(j^{\tilde{l}}(f)))$ and $\mathfrak{W}'^{K'^{\tilde{l}}}$ of $j_{x_0}^{K'^{\tilde{l}}}(\theta'^{\tilde{l}}(j^{\tilde{l}}(\pi \circ f)))$.*

(3) *For any $\lambda \in \mathcal{S}(D^{K^{\tilde{l}}}(\theta^{\tilde{l}}, E))|\mathfrak{W}^{K^{\tilde{l}}}$, the space $\mathcal{S}(\theta^{\tilde{l}}(\lambda))$ is transferred by π into $\mathcal{S}(\theta'^{\tilde{l}}(\lambda'))$, where $\lambda' = D^{\tilde{l}}(\pi) \circ \lambda$, $\theta^{\tilde{l}}(\lambda)$ is an $\mathcal{L}_{\mathcal{G}(E)}$-orbit system generated by $\theta_j^{\tilde{l}} - \lambda_j$ $(1 \leqslant j \leqslant m_{\tilde{l}})$, and $\theta'^{\tilde{l}}(\lambda')$ is an $\mathcal{L}_{\mathcal{G}(E')}$-orbit system generated by $\theta_j'^{\tilde{l}} - \lambda_j'$ $(1 \leqslant j \leqslant m_{\tilde{l}}')$.*

(B) *If one of the following two conditions holds, $\theta^k(\lambda)$ is weakly k-reducible to*

$\theta'^k(\lambda')$ by π for $k \geq l'$, where $\lambda = \theta^k(j^k(f))$, and $\lambda' = D^k(\pi) \circ \lambda$:

(i) *The onto map O_E^k is one to one.*

(ii) $\dim (\pi^k)_* D_E^k = \dim D_{E'}^k$, *$\mathfrak{A}(E')$ and $\mathfrak{A}(E)'_k$ are complete at $(\pi \circ f(x_0), 1)$, and $\mathfrak{A}(E')$ is k-automorphic at $(x_0, \pi \circ f)$ for a sufficiently large integer $k \geq l'$. Moreover, if $\mathfrak{A}(E)$ is k-closed at (x_0, f), then $\theta^k(\lambda)$ with $\lambda = \theta^k(j^k(f))$ is k-reducible to $\theta'^k(\lambda')$ with $\lambda' = D^k(\pi)\lambda$ by π.*

Proof. Let $g \in \mathfrak{A}(E)|\mathfrak{U}^{l'}$. Then we have a local diffeomorphism g' of Q' such that $g' \circ \pi = \pi \circ g$. Let $s' \in \mathfrak{S}(E')|\mathfrak{U}'^{l'}$, and set $\bar{s}' = g' \circ s'$. Since E is l'-reducible to E' by π, we have an $s \in \mathfrak{S}(E)|\mathfrak{U}^{l'}$ such that $s' = \pi \circ s$. Therefore we get $\bar{s}' = g' \circ s' = g' \circ \pi \circ s = \pi \circ g \circ s$. Since $g \in \mathfrak{A}(E)|\mathfrak{U}^{l'}$ and $s \in \mathfrak{S}(E)|\mathfrak{U}^{l'}$, we have $g \circ s \in \mathfrak{S}(E)|\mathfrak{U}^{l'}$. Thus we get $\bar{s}' = \pi \circ g \circ s \in \mathfrak{S}(E')|\mathfrak{U}'^{l'}$ because of the l'-reducibility of E to E' by π. Since s' is any element of $\mathfrak{S}(E')|\mathfrak{U}^{l'}$, we get $g' \in \mathfrak{A}(E')|\mathfrak{U}'^{l'}$.

We set $\eta_j^{\bar{l}} = (\pi^{\bar{l}})^* \theta_j'^{\bar{l}}$ $(1 \leq j \leq m'_{\bar{l}})$. Then, by the above stated fact, $\eta_j^{\bar{l}}$ $(1 \leq j \leq m'_{\bar{l}})$ is a linearly independent differential invariant of $\mathfrak{L}_{\mathfrak{A}(E)}$ at $j_{x_0}^{\bar{l}}(f)$. Let $\{\eta_j^{\bar{l}}\}_{j=1}^{m_{\bar{l}}}$ be a fundamental system of differential invariants of $\mathfrak{L}_{\mathfrak{A}(E)}$ at $j_{x_0}^{\bar{l}}(f)$. Then we have expressions $\eta_j^{\bar{l}} = \phi_j^{\bar{l}}(\theta_1^{\bar{l}}, \cdots, \theta_{m_{\bar{l}}}^{\bar{l}})$ $(1 \leq j \leq m_{\bar{l}})$. For s_1 and $s_2 \in \mathfrak{S}(\theta^{\bar{l}}(\lambda))$, if we set $s'_i = \pi \circ s_i$, we have $\theta_j'^{\bar{l}}(j_x^{\bar{l}}(s'_i)) = \theta_j'^{\bar{l}}(j_x^{\bar{l}}(\pi \circ s_i)) = (\pi^{\bar{l}})^* \theta_j'^{\bar{l}}(j_x^{\bar{l}}(s_i)) = \eta_j^{\bar{l}}(j_x^{\bar{l}}(s_i))$. Since $\theta_j^{\bar{l}}(j_x^{\bar{l}}(s_1)) = \theta_j^{\bar{l}}(j_x^{\bar{l}}(s_2)) = \lambda_j(x)$ $(1 \leq j \leq m_{\bar{l}})$, we get $\theta_j'^{\bar{l}}(j_x^{\bar{l}}(s'_1)) = \theta_j'^{\bar{l}}(j_x^{\bar{l}}(s_2)) = \lambda'_j(x) = \phi_j^{\bar{l}}(\lambda_1(x), \cdots, \lambda_{m_{\bar{l}}}(x))$ $(1 \leq j \leq m')$. This means that $\mathfrak{S}(\theta^{\bar{l}}(\lambda))$ is transferred into $\mathfrak{S}(\theta'^{\bar{l}}(\lambda'))$ by π. Since E is K-reducible to E' by π, we have a fibred manifold $(\mathfrak{U}^k, \mathfrak{U}'^k, \pi^k)$ such that $\mathfrak{U}^k$ (resp. $\mathfrak{U}'^k$) is a neighborhood of $j_{x_0}^k(f)$ (resp. $j_{x_0}^k(\pi \circ f)$) and such that π induces a map of $\mathfrak{S}(E)|\mathfrak{U}^k$ onto $\mathfrak{S}(E')|\mathfrak{U}'^k$, $k \geq K$. Therefore, if we set $\mathfrak{W}^{K^{\bar{l}}} = p^{K^{\bar{l}}}\theta^{\bar{l}}(\mathfrak{U}^{\bar{l}+K^{\bar{l}}})$ and $\mathfrak{W}'^{K'^{\bar{l}}} = p^{K'^{\bar{l}}}\theta'^{\bar{l}}(\mathfrak{U}'^{\bar{l}+K'^{\bar{l}}})$, then $\mathfrak{S}(\theta^{\bar{l}}(\lambda))$ with $\lambda \in \mathfrak{S}(D^{K^{\bar{l}}}(\theta^{\bar{l}}, E))|\mathfrak{W}^{K^{\bar{l}}}$ is transferred into $\mathfrak{S}(\theta'^{\bar{l}}(\lambda'))$ for some $\lambda' \in \mathfrak{S}(D^{K'^{\bar{l}}}(\theta'^{\bar{l}}, E'))|\mathfrak{W}'^{K'^{\bar{l}}}$. We set $\phi^{\bar{l}} = (\phi_1^{\bar{l}}, \cdots, \phi_{m_{\bar{l}}}^{\bar{l}})$, and denote by p the projection of $\mathbf{R}^{m_{\bar{l}}}$ onto $\mathbf{R}^{m'_{\bar{l}}}$ defined by $p(x_1, \cdots, x_{m_{\bar{l}}}) = (x_1, \cdots, x_{m'_{\bar{l}}})$. We set $\Theta^{\bar{l}} = \theta^{\bar{l}}(\mathfrak{U}^{\bar{l}})$, $\Theta'^{\bar{l}} = \theta'^{\bar{l}}(\mathfrak{U}'^{\bar{l}})$ and $D^{\bar{l}}(\pi) = p \circ \phi^{\bar{l}}$. Then $(\Theta^{\bar{l}}, \Theta'^{\bar{l}}, D^{\bar{l}}(\pi))$ is a fibred manifold satisfying conditions (1), (2) and (3). This proves (A).

If the condition (i) is satisfied, it is now clear that $\theta^k(\lambda)$ is weakly k-reducible to $\theta'^k(\lambda')$ by π for $k \geq l'$.

Now we assume the condition (ii). Since E is K-reducible to E', we see that $(\pi^k)_* D_E^k = D_{E'}^k$ for a sufficiently large k. On the other hand, since $\mathfrak{A}(E')$ and $\mathfrak{A}(E)'_k$ are complete at $(\pi \circ f(x_0), 1)$, by [6, Proposition 8.1] we can easily see that $\mathfrak{A}(E)'_k|\mathfrak{U}'^k = \mathfrak{A}(E')|\mathfrak{U}'^k$ for a neighborhood $\mathfrak{U}'^k$ of $j_{x_0}^k(\pi \circ f)$. Since $\mathfrak{A}(E')$ is k-automorphic at $(x_0, \pi \circ f)$, $\theta'^k(\lambda')$ is $\mathfrak{A}(E)'_k$-automorphic.

Since π maps $\mathfrak{S}(\theta^k(\lambda))|\mathfrak{U}^k$ into $\mathfrak{S}(\theta'^k(\lambda'))|\mathfrak{U}'^k$, and since $\theta'^k(\lambda')$ is $\mathfrak{A}(E)'_k$-

automorphic, $\mathfrak{S}(\theta^k(\lambda))|\mathfrak{U}^k$ is transferred onto $\mathfrak{S}(\theta'^k(\lambda'))|\mathfrak{U}'^k$. Since E is k-reducible to E', we see that $\theta^k(\lambda)$ is weakly k-reducible to $\theta'^k(\lambda')$.

If $\mathfrak{A}(E)$ is k-closed at (x_0, f), we see that $\mathfrak{A}(\theta^k(\lambda)) = \mathfrak{A}(E)$ on a neighborhood of $j^k_{x_0}(f)$. Therefore the condition (3) in Definition 6.1 is also satisfied, namely, $\theta^k(\lambda)$ is k-reducible to $\theta'^k(\lambda')$ by π. This completes the proof of Lemma 4.1.1.

Remark 4.1.3. For a pair (E, E') which does not satisfy the reducibility, $D^k(\pi)$ is also defined if $\mathfrak{A}(E)|\mathfrak{U}^k$ is transferred by π to $\mathfrak{A}(E')|\mathfrak{U}'^k$ for a neighborhood $\mathfrak{U}^k$ or $\mathfrak{U}'^k$.

Proposition 4.1.1. *Let (Q, Q', π) be a fibred manifold, and let E be a differential equation at $j^\alpha_{x_0}(f) \in \tilde{J}^\alpha(N, Q)$ which is K-reducible to a differential equation E' at $j^{\alpha'}_{x_0}(\pi \circ f) \in \tilde{J}^{\alpha'}(N, Q')$ by π. Suppose that E and E' are l-regular ($l \geqslant \alpha, \alpha'$) at x_0. We set $l' = \max(l, K)$. Then the following two assertions hold*:

(1) *If $p^{l'-\alpha}(E)$ is an $\mathfrak{L}_{\mathfrak{A}(E)}$-orbit system at (l', x_0, f), then $p^{l'-\alpha'}(E')$ is also an $\mathfrak{L}_{\mathfrak{A}(E')}$-orbit system at $(l', x_0, \pi \circ f)$.*

(2) *If $K = 0$, and E is $\mathfrak{A}(E)$-automorphic, then E' is $\mathfrak{A}(E')$-automorphic.*

Proof. By Lemma 4.1.1. $D^{l'}(\pi)$ induces a map of $\mathfrak{S}(D^{Kl'}(\theta^{l'}, E))|\mathfrak{W}^{Kl'}$ onto $\mathfrak{S}(D^{Kl'}(\theta'^{l'}, E'))|\mathfrak{W}'^{Kl'}$. Since $p^{l'-\alpha}(E)$ is an $\mathfrak{L}_{\mathfrak{A}(E)}$-orbit system at (l', x_0, f), $\mathfrak{S}(D^{Kl'}(\theta^{l'}, E))|\mathfrak{W}^{Kl'}$ consists of a single point. Therefore $\mathfrak{S}(D^{Kl'}(\theta'^{l'}, E'))|\mathfrak{W}'^{Kl'}$ also consists of a single point. This means that $p^{l'-\alpha'}(E')$ is also an $\mathfrak{L}_{\mathfrak{A}(E')}$-orbit system at $(l', x_0, \pi \circ f)$. This proves (1).

Suppose E is $\mathfrak{A}(E)$-automorphic. Then any $s \in \mathfrak{S}(E)$ is of the form $g \circ f$, $g \in \mathfrak{A}(E)$. Let s' be any element of $\mathfrak{S}(E')|\mathfrak{U}'^0$ where $\mathfrak{U}'^0$ is a neighborhood of $(x_0, \pi \circ f(x_0)) \in N \times Q'$. Then by the 0-reducibility of E to E' by π, there is an $\tilde{s} \in \mathfrak{S}(E)|\mathfrak{U}^0$ such that $s' = \pi \circ \tilde{s}$, and that $\mathfrak{U}^0$ is a neighborhood of $(x_0, f(x_0)) \in N \times Q$. As was proved in Lemma 4.1.1, for any $g \in \mathfrak{A}(E)|\mathfrak{U}^0$, we have a unique $g' \in \mathfrak{A}(E')|\mathfrak{U}'^0$ with $g' \circ \pi = \pi \circ g$. Therefore we get $\tilde{s}' = \pi \circ \tilde{s} = \pi \circ g \circ f = g' \circ \pi \circ f$. This implies that E' is $\mathfrak{A}(E')$-automorphic. This proves (2).

5. Pseudo-isomorphism of automorphism pseudo-groups

5.1. Let (Q, Q', π) be a fibred manifold, and let Γ and Γ' be pseudo-groups on Q and Q' respectively.

Definition 5.1.1. Γ is said to be k-pseudo-isomorphic to Γ' by π at $(q, \pi(q)) \in Q \times Q'$ if there exist neighborhoods $\mathfrak{U}^k$ and $\mathfrak{U}'^k$ of $j^k_q(1) \in \tilde{J}^k(Q, Q)$ and $j^k_{\pi(q)}(1) \in \tilde{J}^k(Q', Q')$, respectively, satisfying the following conditions:

(1) For any g: $U \to V$ which belongs to $\Gamma|\mathfrak{U}^k$, there exists a map g': $\pi(U) \to \pi(V)$ belonging to $\Gamma'|\mathfrak{U}'^k$ such that $\pi \circ g = g' \circ \pi$ on U.

(2) For any $g'\colon U' \to V'$ which belongs to $\Gamma'|\mathcal{U}'^k$, there exists a map $g\colon U \to V$ belonging to $\Gamma|\mathcal{U}^k$ such that (U, U', π) is a fibred manifold and $\pi \circ g = g' \circ \pi$ on U.

(3) Let $g\colon U \to V$ and $\tilde{g}\colon \tilde{U} \to \tilde{V}$ be two maps which belong to $\Gamma|\mathcal{U}^k$ such that (U, U', π) and $(\tilde{U}, U', \pi)$ are fibred manifolds and such that $\pi \circ g = g' \circ \pi$ on U and $\pi \circ \tilde{g} = g' \circ \pi$ on $\tilde{U}$. Then we have $g = \tilde{g}$ on $U \cap \tilde{U}$.

Remark 5.1.1. If Γ is k-pseudo-isomorphic to Γ' by π at $(q, \pi(q))$, there is a neighborhood $\mathcal{U}$ of q such that Γ is k-pseudo-isomorphic to Γ' by π at $(p, \pi(p))$ for any $p \in \mathcal{U}$.

Lemma 5.1.1. *Let (Q, Q', π) be a fibred manifold, and let Γ (resp. Γ') be a pseudo-group on Q (resp. Q'), which is complete at $(z, 1)$ (resp. $(\pi(z), 1)$) for any $z \in Q$. Suppose* $\dim N \geqslant \dim Q$, *and let f be a local submersion of a neighborhood $\mathcal{V}$ of x_0 onto $\mathcal{U} \subset Q$. For a sufficiently large integer l, we assume that there are neighborhoods $\mathcal{U}^l$ of $j^l_{x_0}(f) \in \tilde{J}^l(N, Q)$ and $\mathcal{U}'^l$ of $j^l_{x_0}(\pi \circ f) \in \tilde{J}^l(N, Q')$ satisfying the following conditions*:

(1) *$(\mathcal{U}^l, \mathcal{U}'^l, \pi^l)$ is a fibred manifold.*

(2) *$\mathcal{L}_\Gamma(l, x, f)$ is weakly K-reducible to $\mathcal{L}_{\Gamma'}(l, x, \pi \circ f)$ for any $x \in \mathcal{V}$. Then Γ is l'-pseudo-isomorphic to Γ' by π at $(f(x_0), \pi \circ f(x_0))$,* $l' = \max(l, K)$.

Proof. We set $f' = \pi \circ f$. For a map $g\colon U \to V$ belonging to $\Gamma|\mathcal{U}^{l'}$, $g \circ f$ is also in $\mathcal{S}(\mathcal{L}_\Gamma(l, x_0, f), \mathcal{U}^{l'})$. If we set $\tilde{f} = g \circ f$ and $\tilde{f}' = \pi \circ \tilde{f}$, $\tilde{f}'\colon f^{-1}(U) \to \pi(V)$ is a solution of $\mathcal{L}_{\Gamma'}(l, x_0, f')$ and is also a solution of $\mathcal{L}_{\Gamma'}(l, x, f')$ for $x \in f^{-1}(U)$. Since $\mathcal{L}_{\Gamma'}(l, x, f')$ is, by [6, Theorem 6.1], Γ'-automorphic for a sufficiently large integer l, there is a $g'_x \in \Gamma'|\mathcal{U}'^{l'}$ such that $\tilde{f}' = g'_x \circ f'$ on a neighborhood of x. Since f' is a submersion, we have clearly $g'_x \circ \pi = \pi \circ g$ on a neighborhood of $U_x \subset U$ of $f(x)$. It is obvious that if $U_x \cap U_y \neq \varnothing$, we have $g'_x = g'_y$ on $\pi(U_x) \cap \pi(U_y)$. This implies that there exists a map $g'\colon \pi(U) \to \pi(V)$ belonging to $\Gamma'|\mathcal{U}'^{l'}$ such that $\pi \circ g = g' \circ \pi$ on U.

Conversely let $g'\colon U' \to V'$ belong to $\Gamma'|\mathcal{U}'^{l'}$ and set $\tilde{f}' = g' \circ f'$. Then $\tilde{f}'$ is a solution of $\mathcal{L}_{\Gamma'}(l, x, f')$ for any $x \in (f')^{-1}(U')$. Since $\mathcal{L}_\Gamma(l, x, f)$ is weakly K-reducible to $\mathcal{L}_{\Gamma'}(l, x, f')$, we have an element $s_x \in \mathcal{S}(\mathcal{L}_\Gamma(l, x, f))|\mathcal{U}^{l'}$ such that $\tilde{f}' = \pi \circ s_x$. Then since $\mathcal{L}_\Gamma(l, x, f)$ is Γ-automorphic, s_x is of the form $s_x = g_x \circ f$ on a neighborhood of x, where $g_x \in \Gamma|\mathcal{U}^{l'}$ and g_x is defined on U_x which satisfies $\pi(U_x) \subset U'$. We set $U = \bigcup_{x \in (f')^{-1}(U')} U_x$. Then we can easily see that (U, U', π) is a fibred manifold. Moreover if $U_x \cap U_y \neq \varnothing$, then we have $\pi \circ s_x = \pi \circ s_y$ on $f^{-1}(U_x \cap U_y)$. For $\tilde{x} \in f^{-1}(U_x \cap U_y)$, both s_x and s_y are elements of $\mathcal{S}(\mathcal{L}_\Gamma(l, \tilde{x}, f))|\mathcal{U}^{l'}$. Therefore, by the reducibility, $s_x = s_y$ on a neighborhood of $\tilde{x}$. Thus we get $s_x = s_y$ on $f^{-1}(U_x \cap U_y)$, that is, $g_x \circ f = g_y \circ f$ on $f^{-1}(U_x \cap U_y)$. Since f is a submersion, we get $g_x = g_y$ on $U_x \cap U_y$. This implies that we have an element $g \in \Gamma|\mathcal{U}^{l'}$ defined on U such that $\pi \circ g = g' \circ \pi$.

Let $\tilde{g}: \tilde{U} \to \tilde{V}$ be another element of $\Gamma|\mathfrak{U}^{l'}$ such that $g' \circ \pi = \pi \circ \tilde{g}$ and $(\tilde{U}, U', \pi)$ is a fibred manifold. Then for any $x \in f^{-1}(U \cap \tilde{U})$, $g \circ f$ and $\tilde{g} \circ f$ are elements of $\mathfrak{S}(\mathfrak{L}_\Gamma(l, x, f))|\mathfrak{U}^{l'}$. Since $\pi \circ g \circ f = \pi \circ \tilde{g} \circ f \in \mathfrak{S}(\mathfrak{L}_{\Gamma'}(l, x, f'))|\mathfrak{U}'^{l'}$, we get $g \circ f = \tilde{g} \circ f$ on a neighborhood of x. Since x is an arbitrary point of $f^{-1}(U \cap \tilde{U})$, we have $g \circ f = \tilde{g} \circ f$ on $f^{-1}(U \cap \tilde{U})$. Since f is a submersion, we obtain $= \tilde{g}$ on $U \cap \tilde{U}$. This proves that Γ is l'-pseudo-isomorphic to Γ' at $(f(x_0), \pi \circ f(x_0))$.

Lemma 5.1.2. *Suppose* $\dim N \geqslant \dim Q$. *Let* Γ *be a pseudo-group on* Q *which is complete at* $(z_0, 1)$, *and let* f *be a submersion of a neighborhood of* $x_0 \in N$ *to* Q *with* $f(x_0) = z_0$. *We denote by* $\tilde{l}$ *the order of* $\mathfrak{L}_\Gamma$ *at* (x_0, f). *Then, for* $l \geqslant \tilde{l}$, *there is a neighborhood* $\mathfrak{U}^l$ *of* $j^l_{x_0}(f)$ *such that* $\mathfrak{A}(\mathfrak{L}_\Gamma(\tilde{l}, x_0, f))|\mathfrak{U}^l = \Gamma|\mathfrak{U}^l$.

Proof. Let $\{\theta_j^{\tilde{l}}\}_{j=1}^{m_{\tilde{l}}}$ be a fundamental system of differential invariants of $\mathfrak{L}_\Gamma$ at $j^{\tilde{l}}_{x_0}(f)$. We assume that each $\theta_j^{\tilde{l}}$ is defined on a neighborhood $\mathfrak{V}^{\tilde{l}}$ of $j^{\tilde{l}}_{x_0}(f)$, and $\beta^{\tilde{l}}(\mathfrak{V}^{\tilde{l}}) \supset \operatorname{Im} f$. We set $\mathfrak{U} = \operatorname{Im} f$. Then $\mathfrak{L}_\Gamma(\tilde{l}, x, f)$ is defined for any $x \in \mathfrak{U}$. Let $\phi: U(\ni z_0) \to V(\ni z_0)$ belong to $\mathfrak{A}(\mathfrak{L}_\Gamma(\tilde{l}, x_0, f))|\mathfrak{U}$. Then $\tilde{f} = \phi \circ f$ is a solution of $\mathfrak{L}_\Gamma(\tilde{l}, x, f)$ for any $x \in f^{-1}(U)$. Since by [6, Theorem 6.1], $\mathfrak{L}_\Gamma(\tilde{l}, x, f)$ is Γ-automorphic, we have a $g_x \in \Gamma$ such that $\tilde{f} = g_x \circ f$ on a neighborhood of x. Since f is a submersion, we see that $\phi = g_x$ on a neighborhood $U_x \subset U$ of $f(x)$. Moreover we have clearly $g_x = g_y$ on $U_x \cap U_y$. This implies that $\phi \in \Gamma|\mathfrak{U}$ and any restriction of ϕ to an open subset of U belongs to $\Gamma|\mathfrak{U}$. This means that $\mathfrak{A}(\mathfrak{L}_\Gamma(\tilde{l}, x_0, f))|\mathfrak{U} \subset \Gamma|\mathfrak{U}$. Let $\mathfrak{U}^l$, $l \geqslant \tilde{l}$, be a neighborhood of $j^l_{x_0}(f)$ such that $(\mathfrak{U}^l, \mathfrak{U}, \pi^l)$ is a fibred manifold. Then we have $\mathfrak{A}(\mathfrak{L}_\Gamma(\tilde{l}, x_0, f))|\mathfrak{U}^l \subset \Gamma|\mathfrak{U}^l$. Conversely we have clearly $\Gamma|\mathfrak{U}^l \subset \mathfrak{A}(\mathfrak{L}_\Gamma(\tilde{l}, x_0, f))|\mathfrak{U}^l$. This completes the proof of Lemma 7.2.

5.2. Theorem 5.2.1. *Let* (Q, Q', π) *be a fibred manifold, and let* $\mathfrak{L}$ (*resp.* $\mathfrak{L}'$) *be a regular weak Lie algebra sheaf on* Q (*resp.* Q'). *Assume* $\dim N \geqslant \dim Q$. *If* $\mathfrak{L}(l, x_0, f)$ *is weakly* K*-reducible to* $\mathfrak{L}'(l, x_0, \pi \circ f)$ *by* π *for a sufficiently large integer* l, *then* $\mathfrak{A}(\mathfrak{L}(l, x_0, f))$ *is* k*-pseudo-isomorphic to* $\mathfrak{A}(\mathfrak{L}'(l, x_0, \pi \circ f))$ *by* π *at* $(f(x_0), \pi \circ f(x_0))$ *for* $k \geqslant l' = \max(l, K)$.

Conversely, if $\mathfrak{A}(\mathfrak{L}(l, x_0, f))$ *is* k*-pseudo-isomorphic to* $\mathfrak{A}(\mathfrak{L}'(l, x_0, \pi \circ f))$ *for a sufficiently large* l *and an integer* $k \geqslant l$, *then* $\mathfrak{L}(l, x_0, f)$ *is* k*-reducible to* $\mathfrak{L}'(l, x_0, \pi \circ f)$ *by* π.

Proof. Let Γ or Γ' be a pseudo-group on Q or Q' such that $\mathfrak{L}_\Gamma = \mathfrak{L}$ or $\mathfrak{L}_{\Gamma'} = \mathfrak{L}'$ and such that Γ or Γ' is complete at $(z, 1)$ or $(\pi(z), 1)$ for any $z \in Q$. By Lemma 5.1.2 we have $\mathfrak{A}(\mathfrak{L}_\Gamma(l, x_0, f))|\mathfrak{U}^{l'} = \Gamma|\mathfrak{U}^{l'}$ and $\mathfrak{A}(\mathfrak{L}_{\Gamma'}(l, x_0, \pi \circ f))|\mathfrak{U}'^{l'} = \Gamma'|\mathfrak{U}'^{l'}$ for some neighborhoods $\mathfrak{U}^{l'}$ and $\mathfrak{U}'^{l'}$ of $j^{l'}_{x_0}(f)$ and $j^{l'}_{x_0}(\pi \circ f)$, respectively.

On the other hand, it is easy to see that there is a neighborhood $\mathfrak{V}$ of x_0 such that, for any $x \in \mathfrak{V}$, $\mathfrak{L}(l, x, f)$ is weakly K-reducible to $\mathfrak{L}'(l, x, \pi \circ f)$ by

π. Therefore by Lemma 5.1.1, the assertion of the former half is obtained. Since $\mathcal{L}(l, x_0, f)$ or $\mathcal{L}'(l, x_0, \pi \circ f)$ is $\mathcal{A}(\mathcal{L}(l, x_0, f))$-automorphic or $\mathcal{A}(\mathcal{L}'(l, x_0, \pi \circ f))$-autormorphic, the latter half is now clear, and the proof is completed.

Theorem 5.2.2. *Let (Q, Q', π) be a fibred manifold, and let E (resp. E') be a differential equation at $j^{\alpha}_{x_0}(f) \in \tilde{J}^{\alpha}(N, Q)$ (resp. $J^{\alpha'}_{x_0}(\pi \circ f) \in \tilde{J}^{\alpha'}(N, Q')$). Assume that* $\dim N \geqslant \dim Q$, *that E and E' are l-regular at x_0 and that $\mathcal{A}(E)$ or $\mathcal{A}(E')$ is complete at $(f(x_0), 1)$ or $(\pi \circ f(x_0), 1)$. If the following conditions* (i) *and* (ii) *are satisfied, then $\mathcal{A}(E)$ is k-pseudo-isomorphic to $\mathcal{A}(E')$ by π at $(f(x_0), \pi \circ f(x_0))$, and $D^{\tilde{K}}(\theta^k, E)$ is weakly $\tilde{K}$-reducible to $D^{\tilde{K}}(\theta'^k, E')$ by $D^k(\pi)$ for an integer $\tilde{K}$*:

(i) *E is K-reducible to E' by π.*

(ii) *For a sufficiently large integer k, $\dim(\pi^k)_* D^k_E = \dim D^k_{E'}$ and $\mathcal{A}(E)'_k$ is complete at $(\pi \circ f(x_0), 1)$.*

Conversely, for a sufficiently large integer k, if $\mathcal{A}(E)$ is k-pseudo-isomorphic to $\mathcal{A}(E')$ by π at $(f(x_0), \pi \circ f(x_0))$, and $D^K(\theta^k, E)$ is weakly K-reducible to $D^K(\theta'^k, E')$ by $D^k(\pi)$, then E is $(k + K)$-reducible to E' by π, and $\dim(\pi^k)_* D^k_E = \dim D^k_{E'}$.

Proof. By Lemma 4.1.1 and Remark 4.1.1, for a sufficiently large $k >$ $\max(l, K)$, $\mathcal{O}^k_E$ is one-to-one, and $\mathcal{L}_{\mathcal{A}(E)}(k, x_0, f)$ is weakly k-reducible to $\mathcal{L}_{\mathcal{A}(E')}(k, x_0, \pi \circ f)$. Therefore by Theorem 5.2.1, the assertion of the former half follows. By Remark 5.1.1 and the fact that $\mathcal{L}_{\mathcal{A}(E)}(k, x, f)$ or $\mathcal{L}_{\mathcal{A}(E')}(k, x, \pi \circ f)$ is $\mathcal{A}(E)$- or $\mathcal{A}(E')$-automorphic, the latter half easily follows, and the proof is completed.

Now we shall modify the condition (i) in Theorem 5.2.2.

Let E be a differential equation at $j^{\alpha}_x(f)$ which is l-regular at x, and let $\{\theta^l_j\}^{m_l}_{j=1}$ be a fundamental system of differential invariants of $\mathcal{L}_{\mathcal{A}(E)}$ at $j^l_x(f)$. For a subset $\mathcal{F} \subset \mathcal{S}(E)$, we set $\mathcal{O}^{\mathcal{F}}_E = \{\lambda\colon {}^{\exists}s \in \mathcal{F}$ such that $\theta^l(j^l(s)) = \lambda\}$.

Definition 5.2.1. A differential equation E is said to be $\mathcal{F}$-trivial if, for any $\lambda, \mu \in \mathcal{O}^{\mathcal{F}}_E$, there exists an element $\Delta \in D(\theta^l, \mathcal{N}(\mathcal{A}(E)))$ such that $\mu = \Delta \circ \lambda$.

Let (Q, Q', π) be a fibred manifold, and let E and E' be differential equations at $j^{\alpha}_x(f)$ and $j^{\alpha'}_x(\pi \circ f)$ respectively. Suppose E is weakly K-reducible to E' by π and $f \in \mathcal{S}(E)$. We set $\tilde{\mathcal{F}} = \{\pi \circ s;\ s$ is of the form $\phi \circ f, \phi \in \mathcal{A}(E)\}$. Then for a neighborhood $\mathcal{U}'^K$ of $j^K_x(\pi \circ f)$, $\mathcal{F} = \tilde{\mathcal{F}}|\mathcal{U}^K$ is a subset of $\mathcal{S}(E')$.

Theorem 5.2.3. *The condition* (i) *in Theorem* 5.2.2 *is equivalent to the following condition* (i′):

(i′) *E is weakly K-reducible to E' by π, and E' is $\mathcal{F}$-trivial.*

Proof. Let $\phi \in \mathcal{A}(E)|\mathcal{U}^K$ and set $g = \phi \circ f$. Then $g \in \mathcal{S}(E)|\mathcal{U}^K$. Since E' is $\mathcal{F}$-trivial, $\mathcal{L}_{\mathcal{A}(E')}(l, x, \pi \circ f)$ is isomorphic to $\mathcal{L}_{\mathcal{A}(E')}(l, x, \pi \circ g)$ by Proposition 3.1.1. Let φ be such an isomorphism. We set $f' = \pi \circ f$ and $\varphi \circ f' = g''$. Then $g'' \in \mathcal{S}(\mathfrak{L}_{\mathfrak{A}(E')}(l, x, \pi \circ g))$. Since $\mathfrak{L}_{\mathfrak{A}(E')}(l, x, \pi \circ g)$ is $\mathfrak{A}(E')$-automorphic, we have an $\xi \in \mathfrak{A}(E')$ such that $\xi \circ g'' = \pi \circ g$. We set $\phi' = \xi \circ \varphi$. Then we have $\pi \circ \phi \circ f = \phi' \circ \pi \circ f$. Since f is a submersion, we get $\pi \circ \phi = \phi' \circ \pi$. Sinc (E, E') is a weakly K-reduced pair, we see that $\phi' \in \mathfrak{A}(E')$. That is, E is K-reducible to E' by π. Conversely, if we assume (i), it is easy to see that the condition (i′) is satisfied. Hence the proof is completed.

5.3. Example 1. We set $\mathbf{R}^2_* = \{(z_1, z_2) \in \mathbf{R}^2 | z_2 \neq 0\}$, and let E be the differential equaton at $j^1_{x_0}(f) \in \tilde{J}^1(\mathbf{R}^2, \mathbf{R}^2_*)$ generated by $z_2 \cdot \partial z_1/\partial x_1 - \alpha(x_1, x_2)$ and $z_2 \cdot \partial z_1/\partial x_2 - \beta(x_1, x_2)$, where $\{x_1, x_2\}$ is the coordinate system on $N = \mathbf{R}^2$, and $\alpha(x) = [z_2 \cdot \partial z_1/\partial x_1](j^1_x(f))$, $\beta(x) = [z_2 \cdot \partial z_1/\partial x_2](j^1_x(f))$.

On the other hand, we assume $\alpha(x) \neq 0$ and denote by E' the differential equation at $j^1_{x_0}(f') \in \tilde{J}^1(\mathbf{R}^2, \mathbf{R})$ generated by $(\partial z_1/\partial x_2)/(\partial z_1/\partial x_1) - \beta/\alpha$, where $f' = \pi \circ f$, and π is the projection of $\mathbf{R}^2_*$ onto $\mathbf{R}$ defined by $z_1 = \pi(z_1, z_2)$.

We shall show the following two assertions:

(1) E is 1-reducible to E' by π.

(2) $\mathfrak{A}(E)$ is 1-pseudo-isomorphic to $\mathfrak{A}(E')$ by π at $(f(x_0), f'(x_0))$.

Let $\mathfrak{L}$ be the sheaf of germs of local vector fields on $\mathbf{R}_{*2}$ of the form $\xi(z_1) \cdot \partial/\partial z_1 - \xi'(z_1) \cdot z_2 \cdot \partial/\partial z_2$, where ξ is any local function on $\mathbf{R}$. ($\mathfrak{L}$ is given in [4] and [7]). Then $\mathfrak{L}$ is a regular Lie algebra sheaf on $\mathbf{R}^2_*$, and we can easily see that the family $\{x_1, x_2, z_2 \cdot \partial z_1/\partial x_1, z_2 \cdot \partial z_1/\partial x_2, D(z_1, z_2)/D(x_1, x_2)\}$ is a fundamental system of differential invariants of $\mathfrak{L}$ at $j^1_x(f)$, and the order of $\mathfrak{L}$ at (x_0, f) is 1. Therefore, if we denote by $\tilde{E}$ the differential equation at $j^1_x(f)$ generated by $z_2 \cdot \partial z_1/\partial x_1 - \alpha$, $z_2 \cdot \partial z_1/\partial x_2 - \beta$ and $D(z_1, z_2)/D(x_1, x_2) - (\partial\alpha/\partial x_2 - \partial\beta/\partial x_1)$, then $\tilde{E}$ possesses a solution, and $\mathfrak{A}(\tilde{E}) = \Gamma$ on a neighborhood $\mathfrak{U}^1$ of $j^1_{x_0}(f)$, where Γ is a pseudo-group given on a neighborhood of $f(x_0)$ such that Γ is complete at $(f(x_0), 1)$ and $\mathfrak{L}_\Gamma = \mathfrak{L}$.

On the other hand, it is easy to see that we have $\mathcal{S}(\tilde{E}) = \mathcal{S}(E)$. Therefore we get $\mathfrak{A}(E) = \mathfrak{A}(\tilde{E})$.

Now let $\mathfrak{L}'$ be the sheaf of germs of all local vector fields on $\mathbf{R}$. Then clearly $\mathfrak{L}'$ is a regular Lie algebra sheaf on $\mathbf{R}$, and $\{x_1, x_2, (\partial z_1/\partial x_2)/(\partial z_1/\partial x_1)\}$ is a fundamental system of differential invariants of $\mathfrak{L}'$ at $j^1_x(f') \in \tilde{J}^1(\mathbf{R}^2, \mathbf{R})$. Therefore $\mathfrak{A}(E')$ is the pseudo-group on a neighborhood $\mathfrak{U}'$ of $f'(x_0)$ consisting of all local diffeomorphism of $\mathfrak{U}'$. We set $\mathfrak{U} = \pi^{-1}(\mathfrak{U}')$, $(\beta^k)^{-1}(\mathfrak{U}) = \mathfrak{U}^k$, $(\beta^k)^{-1}(\mathfrak{U}') = \mathfrak{U}'^k$ for $k \geqslant 1$. Then $(\mathfrak{U}^k, \mathfrak{U}'^k, \pi^k)$ is a fibred manifold $(k \geqslant 1)$. Let $s \in \mathcal{S}(E)|\mathfrak{U}^k$. Then it is clear that $s' = \pi \circ s \in \mathcal{S}(E')|\mathfrak{U}'^k$. Conversely, let $s' \in \mathcal{S}(E')|\mathfrak{U}'^k$. Then $(\partial s'/\partial x_2)/(\partial s'/\partial x_1) = \beta/\alpha$. Therefore $\alpha/(\partial s'/\partial x_1) = \beta/(\partial s'/\partial x_2)$. We shall define a local map s of $\mathbf{R}^2$ to $\mathbf{R}^2_*$ by $z_1(s) = s'$ and $z_2(s) = \alpha/(\partial s'/\partial x_1)$. Then, since $(D(z_1(f), z_2(f))/D(x_1, x_2))_{x=x_0} \neq 0$, there exists a neighborhood $\mathcal{V}'^k \subset \mathcal{U}'^k$ of $j^k_x(f')$ such that, for any $s' \in \mathcal{S}(E')|\mathcal{V}'^k$, if we construct s from s' by the above stated way, we have $(D(z_1(s), z_2(s))/D(x_1, x_2))_{x=x_0} \neq 0$. We set $(\pi^k)^{-1}(\mathcal{V}'^k) = \mathcal{V}^k$. Then $(\mathcal{V}^k, \mathcal{V}'^k, \pi^k)$ is a fibred manifold, and $\mathcal{S}(E)|\mathcal{V}^k$ and $\mathcal{S}(E')|\mathcal{V}'^k$ satisfy the condition (2) of Definition 4.1.1 for $k \geqslant 1$. This proves that $\tilde{E}$ and therefore E are weakly 1-reducible to E' by π. By Theorem 5.2.1, $\mathcal{Q}(\tilde{E})$ and therefore $\mathcal{Q}(E)$ are 1-pseudo-isomorphic to $\mathcal{Q}(E')$ by π at $(f(x_0), f'(x_0))$.

Example 2. Let E denote the system of differential equation at $j^1_{x_0}(f) \in \tilde{J}^1(\mathbf{R}^2, \mathbf{R}^2_*)$ given by

$$\frac{\partial z_1}{\partial x_1} = z_2, \qquad \frac{\partial z_1}{\partial x_2} = 0,$$

and let E' be the differential equation at $j^2_{x_0}(f') \in \tilde{J}^2(\mathbf{R}^2, \mathbf{R})$ given by

$$\frac{\partial^2 z_1}{\partial x_1 \partial x_2} = 0,$$

where $f' = \pi \circ f$, and π is the projection of $\mathbf{R}^2_*$ to $\mathbf{R}$ defined by $z_1 = \pi(z_1, z_2)$. We shall prove that E is 1-reducible to E' by π, and that $\mathcal{Q}(E)$ is 0-pseudo-isomorphic to $\mathcal{Q}(E')$ by π at $(f(x_0), f'(x_0))$.

It is clear that E is weakly 1-reducible to E' by π.

Now we shall calculate the automorphism pseudo-groups $\mathcal{Q}(E)$ and $\mathcal{Q}(E')$. Let $\phi \in \mathcal{Q}(E')$, and set $\overline{s'} = \phi \circ s'$. Then, if $\partial^2 s'/\partial x_1 \partial x_2 = 0$, we have $\partial^2 \overline{s'}/\partial x_1 \partial x_2 = 0$. Since

$$\frac{\partial^2 \overline{s'}}{\partial x_1 \partial x_2} = \frac{\partial^2 \phi}{\partial z_1^2} \cdot \frac{\partial s'}{\partial x_1} \cdot \frac{\partial s'}{\partial x_2} + \frac{\partial \phi}{\partial z_1} \frac{\partial^2 s'}{\partial x_1 \partial x_2}$$

and $\partial^2 s'/\partial x_1 \partial x_2 = 0$, we get $\partial^2 \phi/\partial z_1^2 \cdot \partial s'/\partial x_1 \cdot \partial s'/\partial x_2 = 0$ for any $s' \in \mathcal{S}(E')$. For any $(z_0, p_0, q_0) \in \mathbf{R}^3$, we have an $s' \in \mathcal{S}(E')$ such that $s(x_0) = z_0$, $(\partial s'/\partial x_1)(x_0) = p_0$ and $(\partial s'/\partial x_2)(x_0) = q_0$. Therefore we get $\partial^2 \phi/\partial z_1^2 = 0$. This implies that $\phi(z_1) = a \cdot z_1 + b$, where a and b are constants such that $a \neq 0$. Conversely, let ϕ be a local diffeomorphism of $\mathbf{R}$ such that $\phi(z_1) = a \cdot z_1 + b$, where a and b are constants such that $a \neq 0$. Then it is clear that $\phi \in \mathcal{Q}(E')$.

Next let $\psi \in \mathcal{Q}(E)$ and $s \in \mathcal{S}(E)$, and set $\overline{z_1} = z_1 \circ \psi$ and $\overline{z_2} = z_2 \circ \psi$. Then we have

$$\frac{\partial z_1(s)}{\partial x_1} = z_2(s), \qquad \frac{\partial z_2(s)}{\partial x_2} = 0;$$

$$\frac{\partial z_1(\psi \circ s)}{\partial x_1} = z_2(\psi \circ s), \qquad \frac{\partial z_2(\psi \circ s)}{\partial x_2} = 0.$$

Since

$$\begin{aligned}\frac{\partial z_2(\psi \circ s)}{\partial x_2} &= \frac{\partial}{\partial x_2}\left(\frac{\partial z_1(\Psi \circ s)}{\partial x_1}\right)\\ &= \frac{\partial^2 \bar{z}_1}{\partial z_1^2}\cdot\frac{\partial z_1(s)}{\partial x_1}\cdot\frac{\partial z_1(s)}{\partial x_2} + \frac{\partial^2 \bar{z}_1}{\partial z_1 \partial z_2}\cdot\frac{\partial z_1(s)}{\partial x_1}\cdot\frac{\partial z_2(s)}{\partial x_2}\\ &\quad + \frac{\partial \bar{z}_1}{\partial z_1}\cdot\frac{\partial^2 z_1(s)}{\partial x_1 \partial x_2} + \frac{\partial^2 \bar{z}_1}{\partial z_1 \partial z_2}\cdot\frac{\partial z_2(s)}{\partial x_1}\cdot\frac{\partial z_1(s)}{\partial x_2}\\ &\quad + \frac{\partial^2 \bar{z}_1}{\partial z_2^2}\cdot\frac{\partial z_2(s)}{\partial x_1}\cdot\frac{\partial z_2(s)}{\partial x_2} + \frac{\partial \bar{z}_1}{\partial z_2}\cdot\frac{\partial^2 z_2(s)}{\partial x_1 \partial x_2},\end{aligned}$$

we get

$$\frac{\partial^2 \bar{z}_1}{\partial z_1^2}\cdot\frac{\partial z_1(s)}{\partial x_1}\cdot\frac{\partial z_1(s)}{\partial x_2} + \frac{\partial^2 \bar{z}_1}{\partial z_1 \partial z_2}\cdot\frac{\partial z_2(s)}{\partial x_1}\cdot\frac{\partial z_1(s)}{\partial x_2} = 0.$$

Since for any $(z_1^0, z_2^0, p_1^0, p_2^0, q^0) \in \mathbf{R}^5$ with $z_2^0 = p_1^0 \neq 0$ there is an $s \in \mathcal{S}(E)$ such that

$$s(x_0) = (z_1^0, z_2^0), \quad \frac{\partial z_1(s)}{\partial x_1}(x_0) = p_1^0, \quad \frac{\partial z_1(s)}{\partial x_2}(x_0) = p_2^0$$

and $(\partial z_2(s)/\partial x_1)(x_0) = q^0$, we get $\partial^2 \bar{z}_1/\partial z_1^2 = 0$ and $\partial^2 \bar{z}_1/\partial z_1 \partial z_2 = 0$, which implies that $\bar{z}_1(z_1, z_2) = h(z_2)z_1 + k(z_2)$ and $h(z_2) =$ constant $a \neq 0$.

On the other hand, we have

$$\frac{\partial \bar{z}_2(s)}{\partial x_2} = \frac{\partial \bar{z}_2}{\partial z_1}\cdot\frac{\partial \bar{z}_1(s)}{\partial x_2} + \frac{\partial \bar{z}_2}{\partial z_2}\cdot\frac{\partial z_2(s)}{\partial x_2},$$

so that $(\partial \bar{z}_2/\partial z_1)\partial z_1(s)/\partial x_2 = 0$. Since for any $(z_1^0, z_2^0, p_2^0) \in \mathbf{R}^3$ there is an $s \in \mathcal{S}(E)$ such that $s(x_0) = (z_1^0, z_2^0)$ and $(\partial z_1(s)/\partial x_2)(x_0) = p_2^0$, we have $\partial \bar{z}_2/\partial z_1 = 0$ which implies that $\bar{z}_2(z_1, z_2)$ is of the form $\eta(z_2)$. Since $\bar{z}_1(z_1, z_2) = a \cdot z_1 + k(z_2)$, we get

$$\frac{\partial \bar{z}_1}{\partial x_1} = a \cdot \frac{\partial z_1}{\partial x_1} + k'(z_2)\frac{\partial z_2}{\partial x_1}.$$

Since $z_2(s) = (\partial z_1/\partial x_1)(s)$ and $\bar{z}_2(s) = (\partial \bar{z}_1/\partial x_1)(s)$ for any $s \in \mathcal{S}(E)$, we have $\eta(z_2(s)) = a \cdot z_2(s) + k'(z_2(s)) \cdot \partial z_2(s)/\partial x_1$ for $s \in \mathcal{S}(E)$. Now for any $(q^0, z_2^0) \in \mathbf{R}^2$ there is an $s \in \mathcal{S}(E)$ such that $z_2(s)(x_0) = z_2^0$ and $(\partial z_2(s)/\partial x_1)(x_0) = q^0$. Therefore $k'(z_2) = 0$, that is, k is also constant $(=b)$. Thus $\bar{z}_1(z_1, z_z) = a \cdot z_1 + b$. Since $\bar{z}_2(s) = (\partial \bar{z}_1/\partial x_1)(s) = a \cdot z_2(s)$ for $s \in \mathcal{S}(E)$ and for any $z^0 \in \mathbf{R}$ there is an $s \in \mathcal{S}(E)$ with $s(x_0) = z^0$, we have $\bar{z}_2 = a \cdot z_2$. This proves that for any $\psi \in \mathcal{Q}(E)$ we have constants a and b

such that $a \neq 0$, $\bar{z}_1(z_1, z_2) = a \cdot z_1 + b$ and $\bar{z}_2(z_1, z_2) = a \cdot z_2$. Conversely, let ψ be a local diffeomorphism of $\mathbf{R}^2$ such that $z_1 \circ \psi = a \cdot z_1 + b$ and $z_2 \circ \psi = a \cdot z_2$, where a and b are constants and $a \neq 0$. Then it is obvious that $\psi \in \mathcal{Q}(E)$.

It is now clear that $\mathcal{Q}(E)$ is 0-pseudo-isomorphic to $\mathcal{Q}(E')$ by π at any $(p, p') \in \mathbf{R}^2 \times \mathbf{R}$ with $p' = \pi(p)$. Therefore E is 1-reducible to E' by π.

6. Compatibility of reduction and equivalence

6.1. In this section we shall prove the compatibility of reducibility and equivalence of differential equations.

Let E be a differential equation at $j_x^\alpha(f) \in \tilde{J}^\alpha(N, Q)$, and let $\mathcal{S}(E)$ denote the space of solutions of E. Let $\mathcal{U}^k$ be a neighborhood of $j_x^k(f)$, and set $\mathcal{S}(E)|\mathcal{U}^k = \{s \in \mathcal{S}(E); \operatorname{Im} j^k(s) \subset \mathcal{U}^k\}$.

Definition 6.1.1. A differential equation E_1 at $j_x^{\alpha_1}(f^1) \in \tilde{J}^{\alpha_1}(N, Q)$ is said to be k-isomorphic to a differential equation E_2 at $j_x^{\alpha_2}(f^2) \in \tilde{J}^{\alpha_2}(N, Q)$, $k \geqslant 1$, if there are neighborhoods $\mathcal{U}_1^k$ and $\mathcal{U}_2^k$ of $j_x^k(f^1)$ and $j_x^k(f^2)$, respectively, and a diffeomorphism ϕ of $\mathcal{U}_1 = \beta^k(\mathcal{U}_1^k)$ onto $\mathcal{U}_2 = \beta^k(\mathcal{U}_2^k)$ such that ϕ induces a one-to-one correspondence ϕ^k of $\mathcal{S}(E_1)|\mathcal{U}_1^k$ to $\mathcal{S}(E_2)|\mathcal{U}_2^k$ by $\phi^k(s) = \phi \circ s$. Then ϕ is called a k-isomorphism of E_1 to E_2. An isomorphism is also called a 0-isomorphism, and "0-isomorphic" means "isomorphic".

Proposition 6.1.1. *If E_1 is k-isomorphic to E_2, then E_1 is l-isomorphic to E_2 for $l \geqslant k$.*

Proof. Let ϕ be a k-isomorphism of E_1 to E_2 such that ϕ^k is a one-to-one map of $\mathcal{S}(E_1)|\mathcal{U}_1^k$ to $\mathcal{S}(E_2)|\mathcal{U}_2^k$. We set, for $l \geqslant k$, $\mathcal{U}_i^l = (\rho_k^l)^{-1}(\mathcal{U}_i^k)$. Then it is clear that ϕ induces a one-to-one map ϕ^l of $\mathcal{S}(E_1)|\mathcal{U}_1^l$ to $\mathcal{S}(E_2)|\mathcal{U}_2^l$. This proves Proposition 6.1.1.

Let E_i and E_i' be differential equations at $j_x^{\alpha_i}(f^i) \in \tilde{J}^{\alpha_i}(N, Q)$ and $j_x^{\alpha_i'}(f'^i) \in J^{\alpha_i'}(N, Q')$, respectively, for $i = 1, 2$, and let (Q, Q', π) be a fibred manifold. We suppose $f'^i = \pi \circ f^i$.

Lemma 6.1.1. *Assume that E_i is weakly K-reducible to E_i' by π, and that there is a k-isomorphism ϕ of E_1 to E_2. If there is a local diffeomorphism ϕ' of Q' such that $\phi' \circ \pi = \pi \circ \phi$, then ϕ' is a k'-isomorphism of E_1' to E_2', $k' =$ $\max(k, K)$.*

Proof. Since E_i is weakly K-reducible to E_i', for any $\tilde{k} \geqslant K$ there exist neighborhoods $\mathcal{U}_i^{\tilde{k}}$ of $j_x^{\tilde{k}}(f^i)$ and $\mathcal{U}_i'^{\tilde{k}}$ of $j_x^{\tilde{k}}(f')$, respectively, such that $(\mathcal{U}_i^{\tilde{k}}, \mathcal{U}_i'^{\tilde{k}}, \pi^{\tilde{k}})$ is a fibred manifold and such that π induces a map of $\mathcal{S}(E_i)|\mathcal{U}_i^{\tilde{k}}$ onto $\mathcal{S}(E_i')|\mathcal{U}_i'^{\tilde{k}}$ for $i = 1, 2$. On the other hand, since ϕ is a k-isomorphism, by Proposition 6.1.1, ϕ is a k'-isomorphsim, and $k' =$ $\max(k, K)$. We may assume that $\phi^{k'}$ is a one-to-one map of $\mathcal{S}(E_1)|\mathcal{U}_1^{k'}$ to $\mathcal{S}(E_2)|\mathcal{U}_2^{k'}$. Let $s' \in \mathcal{S}(E_1')|\mathcal{U}_1'^{k'}$. Then we have an $s \in \mathcal{S}(E_1)|\mathcal{U}_1^{k'}$ such that

$s' = \pi \circ s$. Thus we get $\phi' \circ s' = \phi' \circ \pi \circ s = \pi \circ \phi \circ s$. Since $\phi \circ s \in \mathfrak{S}(E_2)|\mathfrak{U}_2^{k'}$, we have $\pi \circ \phi \circ s \in \mathfrak{S}(E_2')|\mathfrak{U}_2'^{k'}$. This proves that $\phi' \circ s' \in \mathfrak{S}(E_2')|\mathfrak{U}_2'^{k'}$. Therefore ϕ' is a k'-isomorphism of E_1' to E_2', and the proof is completed.

6.2. Let (Q, Q', π) be a fibred manifold and let Γ (resp. Γ') be a pseudo-group on Q (resp. Q') which is complete at $(z, 1)$ for any $z \in Q$ (resp. at $(z', 1)$ for any $z' \in Q'$).

Let E_i (resp. E_i') be a differential equation at $j_x^{\alpha_i}(f) \in \tilde{J}^{\alpha_i}(N, Q)$ (resp. $j_x^{\alpha_i'}(f') \in \tilde{J}^{\alpha_i'}(N, Q')$) such that $\mathfrak{A}(E_i) = \Gamma$ on a neighborhood of $f(x)$ (resp. $\mathfrak{A}(E_i') = \Gamma'$ on a neighborhood of $f'(x)$ with $f' = \pi \circ f$), $i = 1, 2$.

Let $\{\theta_j'^l\}_{j=1}^{m_l'}$ be a fundamental system of differential invariants of $\mathfrak{L}_{\Gamma'}$ at $j_x^l(f')$, and set $\theta'^l = (\theta_1'^l, \cdots, \theta_{m_l'}'^l)$, where $l \geqslant$ the order of $\mathfrak{L}_{\Gamma'}$ at $(f'(x), 1)$.

Let $j_x^l(g')$ be a point of $\tilde{J}^l(N, Q')$ near to $j_x^l(f')$ such that we can choose sufficiently small neighborhoods $\mathfrak{V}'^l$ and $\mathfrak{U}'^l$ of $j_x^l(g')$ and $j_x^l(f')$, respectively, which satisfy $\mathfrak{U}'^l \ni j_x^l(g')$ and $\mathfrak{V}'^l \supset \mathfrak{U}'^l$.

Theorem 6.2.1. *Set $\theta'^l(j^l(f')) = \lambda'$ and $\theta'^l(j^l(g')) = \mu'$, and assume the following conditions*:

(1) $\dim N \geqslant \dim Q$.

(2) $f \in \mathfrak{S}(E_1)$.

(3) *(E_i, E_i') is a weakly K-reduced pair, and E_i' is l-regular at x.*

(4) *There is an m-isomorphism ϕ of E_1 to E_2 such that $g' = \pi \circ \phi \circ f$.*

(5) *There is an element $\Delta \in D(\theta'^l, \mathfrak{N}(\Gamma'))$ such that $\Delta \circ \lambda' = \mu'$.*

Then we have an m'-isomorphism ϕ' of E_1' to E_2' such that $\phi' \circ \pi = \pi \circ \phi$ where $m' = \max(m, K)$.

Proof. Since $\dim N \geqslant \dim Q'$, by Lemma 5.1.2, $\mathfrak{L}_{\Gamma'}(l, x, f')$ and $\mathfrak{L}_{\Gamma'}(l, x, g')$ which are defined on $\mathfrak{U}'^l$ are Γ''-closed, where $\Gamma'' = \Gamma'|\mathfrak{U}'^l$. Since $\Delta \circ \lambda' = \mu'$, by Proposition 3.1.1 there is a 0-isomorphism φ' of $\mathfrak{L}_{\Gamma'}(l, x, f')$ to $\mathfrak{L}_{\Gamma'}(l, x, g')$ such that $D(\theta'^l, \varphi') = \Delta$. Set $g'' = \varphi' \circ f'$. Since $\mathfrak{L}_{\Gamma'}(l, x, g')$ is Γ'-automorphic, there exists an element $\gamma' \in \Gamma'$ such that $g' = \gamma' \circ g''$. Set $\phi' = \gamma' \circ \varphi'$. Then we have $g' = \phi' \circ f'$ and therefore $\pi \circ \phi \circ f = \phi' \circ \pi \circ f$. Since f is a submersion, we get $\pi \circ \phi = \phi' \circ \pi$ on a neighborhood of $f(x)$. Therefore ϕ' is an m'-isomorphism of E_1' to E_2' by Lemma 6.1.1, and the proof is completed.

6.3. Example 3. Let $\mathfrak{L}$ and Γ be the same as in Example 1. Let $\phi \in \mathfrak{N}(\Gamma)$, and $X = \xi(z_1)\partial/\partial z_1 - \xi'(z_1)z_2\partial/\partial z_2$ be a local cross section of $\mathfrak{L}$. Then $\phi_* X$ is also a local cross section of $\mathfrak{L}$, and therefore $\phi_* X$ is of the form $\eta(\tilde{z}_1)\partial/\partial\tilde{z}_1 - \eta'(\tilde{Z}_1)\tilde{z}_2\partial/\partial\tilde{z}_2$, where $\tilde{z}_i = z_i \circ \phi = \phi^i$ $(i = 1, 2)$. Thus we have

$$\frac{D(\phi^1, \phi^2)}{D(x_1, x_2)}\begin{pmatrix} \xi(z_1) \\ -\xi'(z_1)z_2 \end{pmatrix} = \begin{pmatrix} \eta(\tilde{z}_1) \\ -\eta'(\tilde{z}_1)\tilde{z}_2 \end{pmatrix},$$

so that

$$\eta(\tilde{z}_1) = \eta(\phi^1(z_1, z_2)) = \frac{\partial\phi^1}{\partial z_1}\cdot \xi(z_1) - \frac{\partial\phi^1}{\partial z_2}\cdot \xi'(z_1)\cdot z_2,$$

$$-\eta'(\tilde{z}_1)\tilde{z}_2 = -\eta'(\phi^1(z_1, z_2))\phi^2 = \frac{\partial\phi^2}{\partial z_1}\cdot \xi(z_1) - \frac{\partial\phi^2}{\partial z_2}\cdot \xi'(z_1)z_2.$$

On the other hand,

$$\begin{aligned}-\eta'(\tilde{z}_1)\tilde{z}_2 &= -\left(\frac{\partial\eta(\phi^1(z_1, z_2))}{\partial z_1}\cdot\frac{\partial z_1}{\partial\tilde{z}_1} + \frac{\partial\eta(\phi^1(z_1, z_2))}{\partial z_2}\cdot\frac{\partial z_2}{\partial\tilde{z}_1}\right)\phi^2\\ &= -\left\{\left[\frac{\partial^2\phi^1}{\partial z_1^2}\cdot\xi(z_1) + \frac{\partial\phi^1}{\partial z_1}\cdot\xi'(z_1) - \frac{\partial^2\phi^1}{\partial z_1\partial z_2}\cdot\xi'(z_1)\cdot z_2\right.\right.\\ &\qquad \left. -\frac{\partial\phi^1}{\partial z_2}\cdot\xi''(z_1)\cdot z_2\right]\frac{\partial z_1}{\partial\tilde{z}_1} + \left[\frac{\partial^2\phi^1}{\partial z_1\partial z_2}\cdot\xi(z_1)\right.\\ &\qquad \left.\left. -\frac{\partial^2\phi^1}{\partial z_2^2}\cdot\xi'(z_1)\cdot z_2 - \frac{\partial\phi^1}{\partial z_2}\cdot\xi'(z_1)\right]\frac{\partial z_2}{\partial\tilde{z}_1}\right\}\cdot\phi^2.\end{aligned}$$

Since ξ is arbitrary, we get the following equations:

$$\frac{\partial\phi^2}{\partial z_1} + \left(\frac{\partial^2\phi^1}{\partial z_1^2}\cdot\frac{\partial z_1}{\partial\tilde{z}_1} + \frac{\partial^2\phi^1}{\partial z_1\partial z_2}\cdot\frac{\partial z_2}{\partial\tilde{z}_1}\right)\cdot\phi^2 = 0,$$

$$\frac{\partial\phi^2}{\partial z_2}\cdot z_2 - \left\{\left(\frac{\partial\phi^1}{\partial z_1} - \frac{\partial^2\phi^1}{\partial z_1\partial z_2}\cdot z_2\right)\cdot\frac{\partial z_1}{\partial\tilde{z}_1} + \left(\frac{\partial^2\phi^1}{\partial z_2^2}\cdot z_2 + \frac{\partial\phi^1}{\partial z_2}\right)\cdot\frac{\partial z_2}{\partial\tilde{z}_1}\right\}\cdot\phi^2 = 0,$$

$$\frac{\partial\phi^1}{\partial z_2} = 0.$$

Therefore ϕ^1 is a function of the form $\alpha(z_1)$, and from the first and the second of the above equations we get

$$\frac{\partial\phi^2}{\partial z_1} + \frac{\partial^2\phi^1}{\partial z_1^2}\cdot\frac{\partial z_1}{\partial\tilde{z}_1}\cdot\phi^2 = 0, \quad \frac{\partial\phi^2}{\partial z_2}\cdot z_2 - \frac{\partial\phi^1}{\partial z_1}\cdot\frac{\partial z_1}{\partial\tilde{z}_1}\cdot\phi^2 = 0.$$

which are equivalent to

$$\frac{\partial\phi^1}{\partial z_1}\cdot\frac{\partial\phi^2}{\partial z_1} + \frac{\partial^2\phi^1}{\partial z_1^2}\cdot\phi^2 = 0, \qquad \frac{\partial\phi^2}{\partial z_2}\cdot z_2 - \phi^2 = 0,$$

since $\partial z_1/\partial\tilde{z}_1 = (\partial\phi^1/\partial z_1)^{-1}$. By the above second equation we see that $\phi^2(z_1, z_2)$ is of the form $\beta(z_1)\cdot z_2$. Replacing $\phi^1(z_1)$ and $\phi^2(z_1, z_2)$ by $\alpha(z_1)$ and $\beta(z_1)\cdot z_2$ respectively, from the above first equation we get the following ordinary differential equation with respect to one unknown function β:

$$\beta'(z_1) = -\frac{\alpha''(z_1)}{\alpha'(z_1)} \cdot \beta(z_1),$$

the general solution of which is $c/\alpha'(z_1)$. This proves that if $\phi \in \mathcal{N}(\Gamma)$ then ϕ must be of the form $\phi^1(z_1, z_2) = \alpha(z_1)$ and $\phi^2(z_1, z_2) = c \cdot z_2/\alpha'(z_1)$.

Conversely, if a local diffeomorphism ϕ of $\mathbf{R}^2_*$ is of the form

$$\phi^1(z_1, z_2) = \alpha(z_1), \quad \phi^2(z_1, z_2) = \frac{c \cdot z_2}{\alpha'(z_1)},$$

it is easy to see that $\phi^{(1)*}(z_2 \cdot \partial z_1/\partial x_i) = c \cdot (z_2 \cdot \partial z_1/\partial x_i)$, $i = 1, 2$. Therefore we get $\phi \in \mathcal{N}(\Gamma)$.

Now we can see such an example that the compatibility of reduction and equivalence holds. Let E_i be the differential equation at $j^1_{x_0}(f_i) \in \tilde{J}^1(\mathbf{R}^2, \mathbf{R}^2_*)$ generated by

$$z_2 \cdot \frac{\partial z_1}{\partial x_1} - \alpha_i(x_1, x_2), \quad z_2 \cdot \frac{\partial z_1}{\partial x_2} - \beta_i(x_1, x_2),$$

where

$$\alpha_i(x) = \left[z_2 \cdot \frac{\partial z_1}{\partial x_1} \right] (j^1_x(f_i)), \quad \beta_i(x) = \left[z_2 \cdot \frac{\partial z_1}{\partial x_2} \right] (j^1_x(f_i)).$$

Assume that $\alpha_i(x) \neq 0$, and denote by E'_i the differential equation at $j^1_{x_0}(f'_i) \in \tilde{J}^1(\mathbf{R}^2, \mathbf{R})$ generated by

$$\frac{\partial z_1}{\partial x_2} \Big/ \frac{\partial z_1}{\partial x_1} - \beta_i/\alpha_i,$$

where $f'_i = \pi \circ f_i$ and π is the projection of $\mathbf{R}^2_*$ onto $\mathbf{R}$ defined by $z_1 = \pi(z_1, z_2)$.

Suppose E_1 is 0-isomorphic to E_2, and ϕ is such a 0-isomorphism. Since $\mathcal{Q}(E_i) = \Gamma$ on a neighborhood of $f_i(x_0)$, we get $\phi \in \mathcal{N}(\Gamma)$. By the above calculation of $\mathcal{N}(\Gamma)$, we have $\phi^{(1)*}(z_2 \cdot \partial z_1/\partial x_j) = a \cdot (z_2 \cdot \partial z_1/\partial x_j)$, where a is constant $\neq 0$. Therefore, if E_1 if 0-isomorphic to E_2, there is a constant $b \neq 0$ such that $\alpha_2 = b \cdot \alpha_1$ and $\beta_2 = b \cdot \beta_1$. Consequently we get $\beta_2/\alpha_2 = \beta_1/\alpha_1$ and therefore $E'_1 = E'_2$, in particular, E'_1 is 0-isomorphic to E'_2.

7. Closedness of the pseudo-group of isometries

7.1. Let Θ^α be the sheaf of germs of local Killing vector fields of $\mathbf{R}^q$, and let $\Gamma(\Theta^q)$ be the pseudo-group of local isometries of $\mathbf{R}^q$. Denote by $\{x_1, \cdots, x_n\}$ (resp. $\{z_1, \cdots, z_q\}$) the coordinate system on $\mathbf{R}^n$ (resp. $\mathbf{R}^q$), and set $\tilde{\rho}_j = \sum_{i=1}^q (\partial^2 z_i/\partial x_j^2)^2$. Then $\tilde{\rho}_j$ is a differential invariant of Θ^q at any

$j_x^2(f) \in \tilde{J}^2(\mathbf{R}^n, \mathbf{R}^q)$. In the following lemma, as for the property $P(2, x, f)$, refer to §8.

Lemma 7.1.1. $\Gamma(\Theta^q)$ *satisfies the property* $P(2, x, f)$.

Proof. Let ϕ be a local diffeomorphism of $\mathbf{R}^q$ such that $\phi^{(2)*}\tilde{\rho}_2 = \tilde{\rho}_1$. Then we get $\partial^2\phi_i/\partial z_j\partial z_k = 0$ and $\Sigma_{h=1}^q \partial\phi_i/\partial z_h \cdot \partial\phi_j/\partial z_h = \delta_{ij}$ $(1 \leq i, j, k \leq q)$, where $\phi_i = z_i \circ \phi$. This implies that $\phi_i(z) = \Sigma_{j=1}^q a_{ij}z_j + b_i$, $(a_{ij}) \in O(q)$, on the domain of ϕ, so that $\phi \in \Gamma(\Theta^q)$, that is, $\Gamma(\Theta^q) = \{\phi;$ local diffeo. of $\mathbf{R}^q$, $\phi^{(2)*}\tilde{\rho}_1 = \tilde{\rho}_1\}$. Thus Θ^q satisfies the property $P(2, x, f)$ for any (x, f).

Set $\lambda_j(x) = \tilde{\rho}_j(j_x^2(f))$, $1 \leq j \leq n$.

Proposition 7.1.1. *If* λ_j *is a submersion on a neighborhood of* x_0 *for some* j, *then* $\Gamma(\Theta^q)$ *is* 2-*closed at* (x_0, f).

Proof. We shall show that $\Gamma(\Theta^q)$ is 2-closed at (x_0, f) if λ_1 is a submersion. Let V^2 be a neighborhood of $j_{x_0}^2(f)$, and set

$$S(V^2) = \bigcup_{s \in \mathbb{S}(\Theta^q(l, x_0, f))} \operatorname{graph}(j^2(s)) \cap V^2.$$

If $\phi \in \mathcal{Q}(\theta^q(l, x_0, f))$, we have $\phi^{(2)*}\tilde{\rho}_1 = \tilde{\rho}_1$ on $S(V^2)$, because $\phi^{(2)}|S(V^2)$ is a local transformation of $S(V^2)$ and $\tilde{\rho}_1(j_x^2(s)) = \tilde{\rho}_1(j_x^2(t)) = \lambda_1(x)$ for any two s and $t \in \mathbb{S}(\Theta^q(l, x_0, f))$. Let $\{x_1, \cdots, x_n, z_1, \cdots, z_q, \cdots, p_j^i, \cdots, p_{jk}^i, \cdots\}$ be the natural coordinate system on $\tilde{J}^2(\mathbf{R}^n, \mathbf{R}^q)$. Then we may assume that V^2 possesses a product structure $\mathcal{H} \times \mathcal{Z} \times \mathcal{P}$, where $\mathcal{H}$, $\mathcal{Z}$ or $\mathcal{P}$ is a cubic neighborhood of a point of $\mathbf{R}^n$, $\mathbf{R}^q$ or $\mathbf{R}^m$ such that $\{x_1, \cdots, x_n\}$, $\{z_1, \cdots, z_1\}$ or $\{p_1^1, \cdots, p_j^i, \cdots, p_{jk}^i, \cdots\}$ is the coordinate system on $\mathcal{H}$, $\mathcal{Z}$ or $\mathcal{P}$. Moreover we may assume that $\mathcal{P}$ possesses a product structure $\mathcal{P}^1 \times \mathcal{P}^2$, where $\mathcal{P}^1$ is a cubic neighborhood of a point of $\mathbf{R}^q$ such that $\{p_{11}^1, \cdots, p_{11}^q\}$ is the coordinate system on $\mathcal{P}^1$ and $\mathcal{P}^2$ is a cubic neighborhood of a point of $\mathbf{R}^{m-q}$ such that $\{\cdots, p_j^i, \cdots, p_{hk}^i, \cdots\}$ $(h \neq 1$ or $k \neq 1)$ is the coordinate system on $\mathcal{P}^2$. Let π be the natural projection of $V^2 \to \mathcal{P}^1$. We shall prove that $\pi(S(V^2))$ is open in $\mathcal{P}^1$. Let $\phi \in \Gamma(\Theta^q)$. Then we have $\phi_i(z) = \Sigma_{j=1}^q a_{ij}(\phi)z_j + b_i$, $(a_{ij}(\phi)) \in O(q)$ on the domain of ϕ. For $s \in \mathbb{S}(\Theta^q(l, x_0, f))$, we set $s' = \phi \circ s$. Then

$$\frac{\partial^2 s_i'}{\partial x_1^2} = \sum_{j=1}^q a_{ij}(\phi) \cdot \frac{\partial^2 s_j}{\partial x_1^2}.$$

Since λ_1 is a submersion on a neighborhood $\mathcal{U}$ of x_0, $\lambda_1(\mathcal{U})$ is open in $\mathbf{R}$. Since

$$\tilde{\rho}_1(j_x^2(s)) = \sum_{i=1}^q \left(\frac{\partial^2 s_i}{\partial x_1^2}\right)^2 = \lambda_1(x), \quad \frac{\partial^2 s}{\partial x_1^2}(x) = \left(\frac{\partial^2 s_1}{\partial x_1^2}(x), \cdots, \frac{\partial^2 s_q}{\partial x_1^2}(x)\right)$$

is a point of $(q-1)$-sphere $S^{q-1}(\lambda_1(x))$ in $\mathbf{R}^q$ with the radius $\sqrt{\lambda_1(x)}$. On the other hand, $O(q)$ acts transitively on $S^{q-1}(\lambda_1(x))$. Since $\Gamma(\theta^q) \ni \phi \to (a_{ij}(\phi)) \in O(q)$ is onto and since $\theta^q(l, x_0, f)$ admits $\Gamma(\Theta^q)$, $\{(\partial^2(\phi \circ s)/\partial x_1^2)(x); \phi \in \Gamma(\Theta^q), x \in \mathfrak{W}\}$ is open in $\mathbf{R}^q$. Therefore $\pi(S(V^2)) = \{(\partial^2(\phi \circ s)/\partial x_1^2)(x); \phi \in \Gamma(\Theta^q), x \in \mathfrak{W}\} \cap \mathfrak{P}^1$ is open in $\mathfrak{P}^1$. Now $\tilde{\rho}_1$ is considered as a function on $\mathfrak{P}^1$. Therefore the equality $\phi^{(2)*}\tilde{\rho}_1 = \tilde{\rho}_1$ on $S(V^2)$ means the equality $\phi^{(2)*}\tilde{\rho}_1 = \tilde{\rho}_1$ on $\pi(S(V^2))$. Since $\pi(S(V^2))$ is open in $\mathfrak{P}^1$, we get $\phi^{(2)*}\tilde{\rho}_1 = \tilde{\rho}_1$ as a function. As was proved in Lemma 7.1.1, the equation $\phi^{(2)*}\tilde{\rho}_1 = \tilde{\rho}_1$ implies $\phi \in \Gamma(\Theta^q)$. Therefore we have proved that if $\phi \in \mathcal{Q}(\Theta^q(l, x_0, f))$, then $\phi \in \Gamma(\Theta^q)$. This implies that $\Gamma(\Theta^q)$ is l-closed at (x_0, f) for $l \geqslant 2$, and hence the proof is completed.

8. Appendix 1 (Completeness of pseudo-groups)

8.1. Let Γ be a pseudo-group on Q such that $\mathfrak{L}_\Gamma$ is an N-regular weak Lie algebra sheaf, and let $\{\theta_j^l\}_{j=1}^{m_l}$ be a fundamental system of differential invariants of $\mathfrak{L}_\Gamma$ at $j_x^l(f) \in \tilde{J}^l(N, Q)$.

Definition 8.1.1. Γ is said to satisfy the property $P(l, x, f)$ (resp. $P(\infty, x, f)$) if the following statement holds: Let V^l be a sufficiently small neighborhood of $j_x^l(f)$ on which θ_j^l $(1 \leqslant j \leqslant m_l)$ is defined, and let ϕ be a local diffeomorphism of Q such that $\phi^{(l)}$ maps an open subset $W^l(\ni j_{x'}^l(f'))$ of V^l into V^l. Then $'\phi$, a restriction of ϕ to a neighborhood $'W \subset \beta^l(W^l)$ of $f'(x')$, is in Γ if and only if $\phi^{(l)*}\theta_j^l = \theta_j^l$ $(1 \leqslant j \leqslant m_l)$ on a neighborhood $'W^l = (\beta^l)^{-1}('W) \cap W^l(\ni j_{x'}^l(f'))$ for an integer $l \geqslant 0$ (resp. for any integer $l \geqslant 0$).

Let Γ be a pseudo-group on Q, and let f be a diffeomorphism of a neighborhood of $x \in Q$ to a neighborhood of $f(x) \in Q$.

Definition 8.1.2. Γ is said to be complete at (x, f) if the following conditions are satisfied:

(1) $\mathfrak{L}_\Gamma$ is a regular Lie algebra sheaf around $f(x)$.

(2) Γ satisfies the property $P(\infty, x, f)$.

Proposition 8.1.1. *Suppose a pseudo-group Γ on Q satisfies the following conditions*:

(i) $\mathfrak{L}_\Gamma$ *is a regular weak Lie algebra sheaf.*

(ii) Γ *satisfies the property* $P(\infty, x, f)$.

Then Γ *is complete at* (x, f).

Proof. Let X be a vector field on a neighborhood $U \subset Q$ of $f(x)$, and assume that $\mathbf{F}_z(X(z)) \in L(z)$ for any $z \in U$. As for the definition of $\mathbf{F}_z$, refer to [6, p. 462]. Let φ_t be the local 1-parameter group of local transformation of U generated by X. Since the condition that $\mathbf{F}_z(X(z)) \in L(z)$ for any $z \in U$ implies that $X^{(l)}$ is a local cross section of $D_\Gamma^{(l)}$ defined on a neighborhood of

$j_x^l(f)$ for any $l \geqslant 0$. Since Γ satisfies the property $P(\infty, x, f)$, we can easily see that $\varphi_t \in \Gamma$. Therefore X is a local cross section of $\mathfrak{L}_\Gamma$. This means that $\mathfrak{L}_\Gamma$ is a Lie algebra sheaf. Therefore Γ is complete at (x, f), and the proof is completed.

Now let Γ be a pseudo-group on Q such that $\mathfrak{L}_\Gamma$ is a regular weak Lie algebra sheaf.

Proposition 8.1.2. *Γ is complete at $(z_0, 1)$ if and only if Γ is locally defined at z_0 by a system of differential equations $(A)^l$ at $j_{z_0}^l(1) \in \tilde{J}^l(Q, Q)$ for an integer l.*

The proof is given in [6, Propositions 8.1, 8.2].

9. Appendix 2 (Order of pseudo-groups)

9.1. Let $\mathfrak{L}$ be an N-regular weak Lie algebra sheaf on Q.

Proposition 9.1.1. *There is an integer K such that, for any $k > K$, $\mathfrak{L}(k + 1, x, f) = p\mathfrak{L}(k, x, f)$ on a neighborhood of $j_x^{k+1}(f) \in \tilde{J}^{k+1}(N, Q)$, where $p\mathfrak{L}(k, x, f)$ is the standard prolongation of $\mathfrak{L}(k, x, f)$.*

The proof is given in [6, Lemma 4.1].

Definition 9.1.1. We denote by K_0 the minimum integer K satisfying Proposition 9.1.1. The integer K_0 is called the order of $\mathfrak{L}$ at (x, f).

Let Γ be a pseudo-group on Q, and let E be a differential equation at $j_x^\alpha(f) \in \tilde{J}^\alpha(N, Q)$ where f is a solution of E.

Definition 9.1.2. E is said to be Γ-automorphic if the following conditions are satisfied:

(1) For any $\phi \in \Gamma$, $\phi \circ f$ is a solution of E if the composite is defined.

(2) Any solution s of E near to f is of the form $\phi \circ f$ for some $\phi \in \Gamma$.

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