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# KAEHLER MANIFOLDS ISOMETRICALLY IMMERSED IN EUCLIDEAN SPACE

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## 0. Introduction

In this article we investigate the properties of Kaehler manifolds which can be isometrically immersed in Euclidean space with low codimension. For such manifolds, the curvature tensor together with the complex structure gives certain restrictions on the topology. Using Morse theory we obtain the following theorem (compare [2, Theorem 1] or [4, Theorem 7.2]).

**Theorem 1.** *Suppose that a complete Kaehler manifold $M$ of real dimension $2n$ is properly isometrically immersed in a Euclidean space $E^{2n+d}$ $(n > d)$. Then $M$ has the homotopy type of a* CW*-complex with cells of dimension at most $n + d$, and hence $H_k(M; Z) = 0$ for $k > n + d$.*

A compact Kaehler manifold is orientable and its top dimensional homology group is nonzero, and every immersion of a compact manifold is proper. Thus we have an immediate consequence:

**Corollary.** *A compact Kaehler manifold of real dimension $2n$ cannot be isometrically immersed in $E^{2n+d}$ $(n > d)$.*

Furthermore, if we suppose that $M$ is of nonnegative sectional curvature, then we can say more about the geometric structure of $M$ (compare [1]).

**Theorem 2.** *Let $M$ be a complete Kaehler manifold of real dimension $2n$ with nonnegative sectional curvature. If $\varphi$: $M \to E^{2n+d}$ $(n > d)$ is an isometric immersion, then $M = \mathbf{C}^m \times K$, the Riemannian product of $\mathbf{C}^m$ $(m \geqslant n - d)$ and a Kaehler manifold $K$ of real dimension $2n - 2m$. Moreover, $\varphi = 1 \times \psi$, the product of the identity map of $\mathbf{C}^m$ onto $E^{2m}$ and an isometric immersion of $K$ into $E^{2n-2m+d}$.*

Throughout this article, we suppose all the manifolds and maps are smooth. For general notation and definitions we refer to [3].

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## 1. Preliminaries

Let $M$ be a Kaehler manifold. For each point $p \in M$, $T_pM$ denotes the tangent space at $p$. An almost complex structure is a linear map $J$: $T_pM \to$

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$T_pM$ with $J^2 = -$ identity. Let $\langle\ ,\ \rangle$ be the Riemannian metric on $M$, and $\nabla$ the Levi-Civita connection. For tangent vector fields $X$, $Y$ we have $\langle JX, JY\rangle = \langle X, Y\rangle$ and $\nabla_X JY = J\nabla_X Y$. The Riemann-Christoffel curvature tensor $R$ satisfies $JR(x,y)z = R(x,y)Jz$, and as a consequence,

$$\langle R(x,y)Jw, Jz\rangle = \langle R(x,y)w, z\rangle. \tag{1}$$

Suppose $\varphi\colon M \to E^{2n+d}$ is an isometric immersion with second fundamental form $\alpha\colon T_pM \times T_pM \to N_pM$, where $N_pM$ is the normal space at $p$ induced by $\varphi$. Without danger of confusion, we may denote the metric on $N_pM$ by $\langle\ ,\ \rangle$ also. By the Gauss equation we have

$$\langle R(x,y)w, z\rangle = \langle \alpha(x,z), \alpha(y,w)\rangle - \langle \alpha(x,w), \alpha(y,z)\rangle, \tag{2}$$

$$\langle R(x,y)Jw, Jz\rangle = \langle \alpha(x,Jz), \alpha(y,Jw)\rangle - \langle \alpha(x,Jw), \alpha(y,Jz)\rangle, \tag{3}$$

for all $x, y, z$ and $w$ in $T_pM$. (1), (2), and (3) lead to the following definition: Let $W_p = N_pM \oplus N_pM$ be the direct sum of two copies of $N_pM$. An indefinite inner product on $W_p$ is defined by $\langle\langle \xi \oplus \eta, \zeta \oplus \lambda\rangle\rangle = \langle \xi, \zeta\rangle - \langle \eta, \lambda\rangle$. We define a bilinear map $\beta\colon T_pM \times T_pM \to W_p$ as follows: $\beta(x,y) = \alpha(x,y) \oplus \alpha(x, Jy)$. In view of (1), (2) and (3) we have

$$\langle\langle \beta(x,z), \beta(y,w)\rangle\rangle - \langle\langle \beta(x,w), \beta(y,z)\rangle\rangle = 0. \tag{4}$$

More generally, a bilinear map $\beta\colon V \times V \to W$ from a vector space $V$ into another $W$ with an inner product $\langle\langle\ ,\ \rangle\rangle$ is called a flat bilinear form if it satisfies (4). This definition was first given by J. D. Moore in [6], where flat bilinear forms were used to determine the topology of conformally flat submanifolds.

For each fixed vector $x \in T_pM$ the expression $\beta(x)(y) = \beta(x,y)$ defines a linear map $\beta(x)\colon T_pM \to W_p$. A vector $x_0 \in T_pM$ is said to be left regular if rank $\beta(x_0) = \max\{\text{rank } \beta(x)\colon\ x \in T_pM\} = q$. Define $N(\beta; x) = \{n \in T_pM\colon \beta(x,n) = 0$ for all $x \in T_pM\}$ to be the null space of $\beta(x)$. A vector $e \in W_p$ is null if $\langle\langle e, e\rangle\rangle = 0$. Part (i) of the following lemma can be found in [6].

**Lemma.** *Suppose $x_0$ is a left regular vector and $N = N(\beta, x_0)$. Then clearly $N$ is invariant under $J$ and* dim $N \geq 2n - 2d$. *For all $(x, n) \in T_pM \times N$, we have*

(i) $\beta(x,n) \in \beta(x_0)(T_pM)$,

(ii) *there is an orthogonal transformation $\tilde{J}\colon N_pM \to N_pM$ such that $\alpha(x, Jn) = \tilde{J}\alpha(x,n)$,*

(iii) $\tilde{J}^2\alpha(x,n) = -\alpha(x,n)$.

*Proof.* (i) There exist $\{v_i\colon 1 \leq i \leq q\}$ in $T_pM$ such that $\{\beta(x_0, v_i)\colon 1 \leq i \leq q\}$ is a basis of $\beta(x_0)(T_pM)$. For any $(x,n) \in T_pM \times N$ and small $t \neq 0$, continuity of $\beta$ implies that $\{\beta(x_0 + tx, v_i)\colon 1 \leq i \leq q\}$ is a family of $q$

linearly independent vectors of $\beta(x_0 + tx)(T_pM)$, and hence a basis of $\beta(x_0 + tx)(T_pM)$, since $\dim \beta(y)(T_pM) \leqslant q$ for any $y \in T_pM$. Now $\beta(x_0 + tx, n) = t\beta(x, n)$ is in $\beta(x_0 + tx)(T_pM)$, so for all small $t \neq 0$

$$\beta(x, n) \wedge \beta(x_0 + tx, v_1) \wedge \cdots \wedge \beta(x_0 + tx, v_q) = 0.$$

By continuity of $\beta$ again we have

$$\beta(x, n) \wedge \beta(x_0, v_1) \wedge \cdots \wedge \beta(x_0, v_q) = 0,$$

which implies $\beta(x, n) \in \beta(x_0)(T_pM)$.

(ii) Let $W_1$ be the subspace generated by $\beta(T_pM, N)$ in $W_p$. Suppose that $(y_i, n_i) \in T_pM \times N$ for $i = 1, 2$. By (i) there is $v_1$ in $T_pM$ such that $\beta(x_0, v_1) = \beta(y_1, n_1)$. Since $\beta$ is flat,

$$\begin{aligned}\langle\langle \beta(y_1, n_1), \beta(y_2, n_2)\rangle\rangle &= \langle\langle \beta(x_0, v_1), \beta(y_2, n_2)\rangle\rangle \\ &= \langle\langle \beta(x_0, n_2), \beta(y_2, v_1)\rangle\rangle \\ &= 0.\end{aligned}$$

So for each $\xi \oplus \eta, \zeta \oplus \lambda \in W_1$, we have $\langle\langle \xi \oplus \eta, \zeta \oplus \lambda\rangle\rangle = 0$. That is, $W_1$ consists entirely of null vectors. We choose a basis $\{\xi_\sigma \oplus \eta_\sigma: 1 \leqslant \sigma \leqslant s\}$ ($s = \dim W_1$) of $W_1$. Here $\{\xi_\sigma: 1 \leqslant \sigma \leqslant s\}$ must be a family of linearly independent vectors in $N_pM$. Otherwise, $W_1$ would contain a nontrivial linear combination $\sum a_\sigma \xi_\sigma \oplus \eta_\sigma = 0 \oplus \eta$ ($\eta \neq 0$), which is not a null vector. By the Gram-Schmidt process, we can suppose $\{\xi_\sigma: 1 \leqslant \sigma \leqslant s\}$ is a family of orthonormal vectors. Then it follows that

$$\langle \xi_\sigma, \xi_\tau\rangle - \langle \eta_\sigma, \eta_\tau\rangle = \langle\langle \xi_\sigma \oplus \eta_\sigma, \xi_\tau \oplus \eta_\tau\rangle\rangle = 0,$$

and hence $\{\eta_\sigma: 1 \leqslant \sigma \leqslant s\}$ is a family of orthonormal vectors. There is an orthogonal transformation $\tilde{J}: N_pM \to N_pM$ such that $\tilde{J}\xi_\sigma = \eta_\sigma$. For $(x, n) \in T_pM \times N$, $\alpha(x, n) \oplus \alpha(x, Jn) \in W_1$ so $\tilde{J}\alpha(x, n) = \alpha(x, Jn)$.

(iii) It suffices to prove $\langle \tilde{J}^2\alpha(x, n), \xi\rangle = -\langle \alpha(x, n), \xi\rangle$ for all $\xi \in N_pM$. Let $Q$ be the transpose of $\tilde{J}$. Define $B = \{\xi \in N_pM: (Q^2 + I)\xi = 0\}$, and let $B^\perp$ be the orthogonal complement of $B$ in $N_pM$. Since $Q$ is orthogonal, $B$ and $B^\perp$ are invariant under $Q$, and clearly $Q^2 + I: B^\perp \to B^\perp$ is nonsingular. If $\xi \in B^\perp$, then there is an $\eta \in B^\perp$ such that $\xi = (Q^2 + I)\eta$, and thus $\langle \alpha(x, n), \eta\rangle = \langle \alpha(x, -J^2n), \eta\rangle = \langle -\alpha(x, n), Q^2\eta\rangle$, so $\langle \alpha(x, n), \xi\rangle = \langle \alpha(x, n), (Q^2 + I)\eta\rangle = 0$. Similarly $\langle \tilde{J}^2\alpha(x, n), \xi\rangle = \langle \alpha(x, J^2n), \xi\rangle = 0$. On the other hand, if $\xi \in B$, then $\langle \alpha(x, n), (Q^2 + I)\xi\rangle = 0$. So $\langle \tilde{J}^2\alpha(x, n), \xi\rangle + \langle \alpha(x, n), \xi\rangle = 0$. This completes the proof.

## 2. Proof of Theorem 1

Let $\varphi: M \to E^{2n+d}$ be an isometric immersion. By Sard's Theorem, there is a point $q \in E^{2n+d}$ such that $L_q(p) = \frac{1}{2}\|\varphi(p) - q\|^2$ is a smooth function with nondegenerate critical points, where $\| \ \|$ is the standard distance in $E^{2n+d}$. Suppose $\varphi$ is a proper map; then $M^a = \{p \in M: L_q(p) \leq a\} = \varphi^{-1}\{\varphi(p): \frac{1}{2}\|\varphi(p) - q\|^2 \leq a\}$ is compact for each $a \in R$, and Morse theory can be applied. If $p$ is a critical point of $L_q$, then for each $x \in T_pM$,

$$dL_q(x) = \langle \varphi(p) - q, x \rangle = 0.$$

Hence $\varphi(p) - q = \xi$ is normal to $T_pM$ at $p$. The Hessian of $L_q$ at the critical point $p$ is given by

$$d^2L_q(x, x) = \langle \xi, \alpha(x, x) \rangle + \langle x, x \rangle,$$

where $\alpha$ is the second fundamental form of the immersion. By the lemma, there is a $J$-invariant subspace $N \subset T_pM$ with $\dim N \geq 2n - 2d$, such that if $n \in N$, then

$$\langle \xi, \alpha(Jn, Jn) \rangle = \langle \xi, \tilde{J}^2\alpha(n, n) \rangle = -\langle \xi, \alpha(n, n) \rangle.$$

This shows there is a subspace $N_1 \subset N$ of dimension $\frac{1}{2} \dim N \geq n - d$, on which the bilinear form $\langle \xi, \alpha( \ , \ ) \rangle$ is positive semidefinite. Thus $d^2L_q$ is positive definite on $N_1$, so the index at the critical point $p$ will be at most $2n - (n - d) = n + d$. By Morse theory [4, Theorem 3.5] $M$ has the homotopy type of a CW-complex with cells of dimensions at most $n + d$. The Morse inequalities imply that for $k > n + d$, $H_k(m; Z) = 0$.

## 3. Proof of Theorem 2

Let $\varphi$: $M \to E^{2n+d}$ be an isometric immersion with second fundamental form $\alpha$. The relative nullity space $RN_p$ at a point $p \in M$ is defined by $RN_p = \{n \in T_pM: \ \alpha(x, n) = 0 \text{ for all } x \in T_pM\}$. The dimension $\nu_p = \dim RN_p$ is called the index of relative nullity. By the lemma, there is a $J$-invariant subspace $N_p \subset T_pM$ at each point $p \in M$ with $\dim N_p \geq 2n - 2d$ such that for a unit vector $n \in N_p$,

$$K(n, Jn) = \langle \alpha(n, n), \alpha(Jn, Jn) \rangle - \langle \alpha(n, Jn), \alpha(n, Jn) \rangle = -2\|\alpha(n, n)\|^2,$$

where $K(x, y)$ is the sectional curvature of the 2-plane spanned by $x, y$. Since the sectional curvature is assumed to be nonnegative, $\alpha(n, n) = 0$. For any unit vector $x \in T_pM$ such that $\langle x, n \rangle = 0$, we have

$$K(x, n) = \langle \alpha(x, x), \alpha(n, n) \rangle - \langle \alpha(x, n), \alpha(x, n) \rangle = -\|\alpha(x, n)\|^2 \geq 0,$$

hence $\alpha(x, n) = 0$. We have proved that $N_p \subset RN_p$ and so $\nu_p \geq \dim N_p \geq 2n - 2d$. Define $\nu_0 = \min\{\nu_q: q \in M\}$, and $G = \{q \in M: \nu_q = \nu_0\}$ is an open subset of $M$. The relative nullity space $RN_p(p \in G)$ defines an involutive distribution on $G$ whose leaves are complete $\nu_0$-planes. Let $p \in G$ be contained in a leaf $L$. Since $RN_p$ contains a $J$-invariant subspace of dimension

$2m \geqslant 2n - 2d$, $L$ contains $\mathbf{C}^m$ which is immersed by $\varphi$ identically onto $E^{2m}$ in $E^{2n+d}$. $M$ is of nonnegative sectional curvature, so the Toponogov splitting theorem [7] implies that $M$ is a Riemannian product of $\mathbf{C}^m$ and $K$, where $K$ is a Reimannian manifold of dimension $2n - 2m$. Since $\nabla J = 0$ on $M$, the almost complex structure $J$ leaves the direct sum decomposition $T_pM = T_p\mathbf{C}^m \oplus T_pK$ invariant. Hence $K$ actually is a Kaehler submanifold with the induced complex structure. Furthermore, since the second fundamental form of the immersion $\varphi$ is of the form: $\alpha(x, y) = 0$ for $x$ tangent to $\mathbf{C}^m$ and $y$ tangent to $K$, [5] implies that $\varphi$ splits into a product of the identity map of $\mathbf{C}^m$ onto $E^{2m}$ and an isometric immersion of $K$ into $E^{2n-2m+d}$.

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