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CERTAIN ISOPARAMETRIC FAMILIES OF HYPERSURFACES IN SYMMETRIC SPACES

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The aim of this paper is to explore a generalization of the theory of isoparametric families of hypersurfaces in a space of constant curvature as initiated by É. Cartan and developed by H. Münzer, K. Nomizu, H. Ozeki, R. Takagi, T. Takahashi, and M. Takeuchi. Since many examples occur as orbits of a group of isometries of the ambient space, we will try to extend the theory to more general ambient spaces by restricting consideration to such orbits. In this paper, we will mostly study the case when the ambient space is Riemannian symmetric and will get our most complete results when the symmetric space is of rank one.

In the original theory, one considers a hypersurface N in a space M of constant curvature c such that N has constant principal curvatures. One has the "isoparametric family" $\{N_t\}$ where N_t is obtained by moving N a distance t along the field of normal geodesics. N_t is again a hypersurface except when it lies in the focal set. The results which we will generalize include: (a) the fact that the focal set is the union of minimal submanifolds when $c > 0$ and N is compact (our Theorem 1.8), (b) the formula giving the distances to the focal points in terms of the principal curvatures of N (our Theorem 3.3), (c) the formula giving the principal curvatures and curvature directions of N_t (when N_t is not in the focal set) in terms of those of N (our Theorem 3.7), and (d) Cartan's formula relating the principal curvatures of N (our Theorem 3.9 and comments which follow). In the classical theory, one also considers the distributions of vector fields on N which at each point are eigenvectors of the second fundamental form belonging to the same eigenvalue and shows that these are integrable. This is in general false for our spaces.

In §1 we set out general results about orbits of a group of isometries which hold for any Riemannian space M. In §2 we restrict to the case when M is Riemannian symmetric, and define the notion of an amenable hypersurface orbit N. In §3 we prove our main results for amenable hypersurfaces. In §4

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we observe that results of R. Takagi show that N is always amenable when M is a complex projective space, and that these techniques can be extended to show the same when M is a quaternionic projective space. In the process, we get a complete classification of these hypersurfaces (related to work of Wolf [21]) and information about their geometry.

Our proof that all hypersurface orbits in a complex and quaternionic projective space are amenable is based on a classification of the possible Lie algebras. Since amenability here is equivalent to a simple geometric property (see (2.6.2)), one would like to find a more geometric proof. Even more interesting would be a proof of the analogs of (a)–(d) using only the fact that N was a hypersurface with constant principal curvatures (and not necessarily the orbit of an isometry group). We do not know if there are hypersurfaces with constant principal curvatures in a complex or quaternionic projective space which are not orbits, but the results of Ozeki and Takeuchi [16], which came to our attention after the body of this paper was written, show that there are many such hypersurfaces in the sphere.

Finally, we mention that much of this work can be extended to the case where N is a principal orbit of a group of isometries with codimension greater than one.

1. Let K be a connected Lie subgroup of the isometry group of a connected Riemannian manifold M, and let N be the orbit through a point 0. For $X \in \mathfrak{k}$, the Lie algebra of K, one has the Killing vector field X^* on M whose one-parameter group is $\exp tX$. The mapping $X \to X^*$ is an injective Lie algebra antihomomorphism because K acts on M on the left. Obviously, at each point $p \in M$, $\{X_p^*: X \in \mathfrak{k}\}$ is the tangent space to the orbit through p. Fix a unit normal vector field ε to N in a neighborhood of 0, and let γ be the arc length parametrized geodesic with $\gamma(0) = 0$, $\gamma'(0) = \varepsilon_0$. Let σ be the corresponding shape operator on N, and let Π be the field of curvature transformations along γ given by $\Pi(V) = R(\gamma', V)\gamma'$. The following lemmas are known or obvious.

Lemma 1.1. *If N is a hypersurface, then ε and σ are locally invariant under the action of K (i.e., invariant under the action of a neighborhood of the identity).*

Lemma 1.2. *If ε is locally invariant under the action of K, then the principal curvatures of N (with respect to the shape operator σ determined by ε) are constant.*

Lemma 1.3. *For $X \in \mathfrak{k}$, $\sigma(X_0^*) = -(\nabla_{\varepsilon_0}X^*)_N$ where the subscript indicates projection on the tangent space T_0N,* [19].

Lemma 1.4. *For $X \in \mathfrak{k}$, X^* is an N-transverse Jacobi field along γ, everywhere normal to γ.*

Proof. That X^* is Jacobi is in [12, Vol. II, p. 66]. Clearly $X_0^* \in T_0N$, so to show X^* is N-transverse [8, p. 8] we observe that $\langle(\nabla_e X^* + \sigma X^*)_0, T_0N\rangle$ vanishes by Lemma 1.3. Finally

$$\gamma' \cdot \langle X^*, \gamma'\rangle = \langle \nabla_{\gamma'} X^*, \gamma'\rangle + \langle X^*, \nabla_{\gamma'}\gamma'\rangle = 0,$$

where we use that X^* is Killing. Since X^* is normal to γ at 0, this shows X^* is everywhere normal to γ.

Definition 1.5. Let $\mathcal{V} = \mathcal{V}(N, \gamma)$ denote the set of N-transverse Jacobi fields along γ everywhere normal to γ. It is well known [8] that $\mathcal{V}$ is a vector space of dimension $m - 1$ where m is the dimension of M. A point $p = \gamma(s)$ is called a focal point for N if there exists a nonzero vector field in $\mathcal{V}$ which vanishes at s and the order of the focal point is the dimension of the subspace of such vector fields.

Lemma 1.6. *Suppose N is a hypersurface. Choose $X_1, \cdots, X_n \in \mathfrak{k}$ such that $\{X_1^*, \cdots, X_n^*\}$ form at 0 a basis of T_0N. Then $\{X_i^*|\gamma\}$ is a basis of $\mathcal{V}$ which at every nonfocal point $\gamma(s)$ gives linearly independent tangent vectors.*

Assume from now on that N is a hypersurface. Suppose $\gamma(t)$ is a focal point. Each $f \in K$ is an isometry of M leaving N invariant, so $f \circ \gamma$ is a geodesic normal to N at $f(0)$ and f_* maps $\mathcal{V}(N, \gamma)$ to $\mathcal{V}(N, f \circ \gamma)$. Thus $f(\gamma(t))$ is a focal point of same order as $\gamma(t)$. Combining with (1.6), we have the following.

Lemma 1.7. *If N is a hypersurface, then the set of focal points is the union of the K-orbits of codimension greater than* 1. *In fact, the order of a focal point x is codim $K \cdot x - 1$.*

Assume now in addition that K is compact and that focal points exist. Since the principal orbit type has codimension 1, we see that the space of orbits $M^* = M/K$ is a ray if M is noncompact and a closed finite interval if M is compact; further, the subset $U^* \subset M^*$ of principal orbits is the interior of M^*, [2, pp. 205–206]. For $x \in M$, let $[x]$ denote the K-orbit of x, as a point in M^*. Then an endpoint $[x]$ of M^* comes from an orbit of focal points iff $\operatorname{codim}[x] > 1$, i.e., iff $[x]$ is a singular orbit and not an exceptional orbit [2, p. 181]. Actually, in this case, any exceptional orbit is special exceptional [2, p. 185, (3.10)], and if $H_1(M; Z_2) = 0$, there are no special exceptional orbits. Finally, it is clear that the orbits of focal points are isolated, so by results in [10] we have

Theorem 1.8. *If N is a hypersurface and K is compact, then the focal set is the union of orbits, each of which is a closed connected minimal submanifold. If the focal set $\mathcal{F}$ is nonempty and M is noncompact, then $\mathcal{F}$ consists of one orbit, while if M is compact, $\mathcal{F}$ consists of no more than two orbits* (*exactly two if* $H_1(M; Z_2) = 0$).

Sometimes it is useful to assume N is orientable and so has a global K-invariant normal field. If M is orientable, then every principal orbit is orientable, while every special exceptional orbit is nonorientable [2, p. 185]. However, if $H_1(M; Z_2) = 0$, then N will always be orientable [2, p. 188].

Definition 1.9. Suppose N orientable with global K-invariant unit normal field ε. For each $p \in N$, let γ_p be the geodesic with $\gamma_p(0) = p$, $\gamma_p'(0) = \varepsilon_p$. For each real number t, let $N_t = \{\gamma_p(t) : p \in N\}$. Since each $f \in K$ is an isometry of M, we have $f \circ \gamma_p = \gamma_{f(p)}$. Thus each N_t is an orbit, and $\{N_t : N_t \not\subset \mathfrak{F}\}$ is the isoparametric family determined by N.

2. We keep the notation of §1 except that we do not require K compact nor N orientable.

Suppose $M = G/H$ is a noneuclidean irreducible Riemannian symmetric space, where G is the connected component of the isometry group and H is the isotropy subgroup at 0. We have $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ where $\mathfrak{g}$ (respectively $\mathfrak{h}$) is the Lie algebra of G (respectively H), and $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{h}$ with respect to the Killing form B. The mapping $X \to X_0^*$ allows us to identify $\mathfrak{p}$ with T_0M, and we may assume the Riemannian metric at 0 is given by $\mp B$, depending on whether M is of compact or noncompact type. We remark that

$$X_0^* = 0 \quad \text{for } X \in \mathfrak{h},$$
$$(\nabla_{X^*} Y^*)_0 = 0 \quad \text{for } X \in \mathfrak{g},\ Y \in \mathfrak{p}.$$

Choose $A \in \mathfrak{p}$ so that $A_0^* = \varepsilon_0$. For $X \in \mathfrak{k}$ we compute

$$(\nabla_{A^*} X^*)_0 = ([A^*, X^*] + \nabla_{X^*} A^*)_0 = -[A, X]_{\mathfrak{p}} = -[A, X_{\mathfrak{h}}],$$

where the subscripts indicate projection on $\mathfrak{p}$ and $\mathfrak{h}$. Further, we have $B(A, [A, X_{\mathfrak{h}}]) = B([A, A], X_{\mathfrak{h}}) = 0$. Thus we apply Lemma 1.3 to conclude

$$\sigma(X_0^*) = [A, X_{\mathfrak{h}}] \quad \text{for } X \in \mathfrak{k}. \tag{2.1}$$

Now the geodesic γ is given by $\gamma(t) = (\exp tA) \cdot 0$. We will be interested in the vanishing of X^* at points on γ. Since for each t, $d \exp(-tA)$ is an isometry of the tangent space at $\gamma(t)$ onto the tangent space at 0 (which we have identified with $\mathfrak{p}$), it suffices to compute

$$\begin{aligned} &d \exp(-tA)(X^*_{\gamma(t)}) \\ &= (\mathrm{Ad} \exp(-tA)X)_0^* \quad = (e^{-\mathrm{ad}\, tA}X)_{\mathfrak{p}} \\ &= -\sum_{n>0} \frac{1}{(2n+1)!} t^{2n+1}(\mathrm{ad}\, A)^{2n+1} X_{\mathfrak{h}} + \sum_{n>0} \frac{1}{(2n)!} t^{2n} (\mathrm{ad}\, A)^{2n} X_{\mathfrak{p}}. \end{aligned} \tag{2.2}$$

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ containing A. For each linear form λ on $\mathfrak{a}$, let

$$\mathfrak{p}_\lambda = \{X \in \mathfrak{p}: (\operatorname{ad} H)^2 X = \pm\lambda(H)^2 X \text{ for all } H \in \mathfrak{a}\},$$
$$\mathfrak{h}_\lambda = \{X \in \mathfrak{h}: (\operatorname{ad} H)^2 X = \pm\lambda(H)^2 X \text{ for all } H \in \mathfrak{a}\},$$

where we take the negative (respectively positive) sign if M is of compact (respectively noncompact) type. Then $\mathfrak{p}_\lambda = \mathfrak{p}_{-\lambda}$, $\mathfrak{h}_\lambda = \mathfrak{h}_{-\lambda}$, $\mathfrak{p}_0 = \mathfrak{a}$, and $\mathfrak{h}_0$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{h}$. If $\mathfrak{p}_\lambda \neq 0$, then λ is an $\mathfrak{a}$-root. For a suitable ordering of $\mathfrak{a}^*$, let Δ (resp. Δ^+) be the nonzero (resp. positive) $\mathfrak{a}$-roots. We have the orthogonal decompositions

$$\mathfrak{p} = \mathfrak{a} + \sum_{\lambda \in \Delta} \mathfrak{p}_\lambda, \qquad \mathfrak{h} = \mathfrak{h}_0 + \sum_{\lambda \in \Delta} \mathfrak{h}_\lambda.$$

Further [19], for each $H \in \mathfrak{a}$, ad H maps $\mathfrak{p}_\lambda$ (respectively $\mathfrak{h}_\lambda$) into $\mathfrak{h}_\lambda$ (respectively $\mathfrak{p}_\lambda$), and this map is an isomorphism or the zero map, depending on whether $\lambda(H) \neq 0$ or $\lambda(H) = 0$. Note that for any $Y \in \mathfrak{p}$, we have [9] $(\operatorname{ad} A)^2 Y = R(A, Y)A = \Pi_0(Y)$ where R is the curvature tensor.

Choose $X_i \in \mathfrak{k}$ so that $\{X_i^*\}$ at 0 forms an orthonormal basis of eigenvectors of σ with corresponding eigenvalues α_i. Let $X_{i,\lambda}$, (respectively $X'_{i,\lambda}$) be the component of X_i in $\mathfrak{p}_\lambda$ (respectively $\mathfrak{h}_\lambda$). From (2.1) we get

$$\begin{aligned} [A, X'_{i,\lambda}] &= \alpha_i X_{i,\lambda} \text{ for } \lambda \in \Delta^+, \\ 0 &= \alpha_i X_{i,0}. \end{aligned} \tag{2.3}$$

Combining with (2.2) we find

$$\begin{aligned} &d\exp(-tA)((X_i^*)_{\gamma(t)}) \\ &= \sum_{\substack{\lambda\in\Delta^+ \\ \lambda(A)\neq 0}} \left(\cos(t\lambda(A)) - \frac{\alpha_i}{\lambda(A)}\sin(t\lambda(A))\right) X_{i,\lambda} + X_{i,0} + \sum_{\substack{\lambda\in\Delta^+ \\ \lambda(A)=0}} X_{i,\lambda} \end{aligned} \tag{2.4}$$

if M is of compact type, and the analogous formula with hyperbolic functions if M is of noncompact type. Here Δ^+ denotes a system of positive roots chosen so that A is in the closure of the positive Weyl chamber, i.e., $\lambda(A) \geqslant 0$ for all $\lambda \in \Delta^+$.

Definition 2.5. N is called an amenable hypersurface if each $(X_i)_\mathfrak{p}$ lies in precisely one root space $\mathfrak{p}_{\lambda_i}$, where λ_i may be zero.

For now, we will specialize to the case where M is of rank one, i.e., $\mathfrak{a} = \mathbf{R}A$. Since $B(X_{i,0}, A) = B((X_i)_\mathfrak{p}, A) = 0$, we see that $X_{i,0} = 0$ and the last two terms in (2.4) vanish.

Now if M is of constant curvature ± 1, then there is only one positive root λ and $\lambda(A) = 1$. Thus N is always amenable.

If M is rank 1 of nonconstant curvature, then there will be two positive roots. After a change of scale, we can take these to be λ with $\lambda(A) = 1$ and

2λ. The eigenspace $\mathfrak{p}_{2\lambda}$ has dimension 1, 3, or 7 and is generated by the images of A under multiplication by the elements in the complex numbers, quaternions, or Cayley numbers whose square is $-I$. Using the double angle formula for cotangent or hyperbolic cotangent, we get

Proposition 2.6. *If M is of rank 1 of nonconstant curvature, then*

(2.6.1) $(X_i)_\mathfrak{p}$ *lies in one root space* $\mathfrak{p}_{\lambda_i}$ *iff* X_i^* *vanishes at some point on* γ, *and*

(2.6.2) N *is amenable iff the subspace generated by* $\{J\varepsilon \colon J^2 = -I, J \in \mathfrak{J}\}$ *(where* $\mathfrak{J}$ *is the algebra of endomorphisms giving the complex, quaternionic, or Cayley structure at* 0) *has a basis of principal curvature vectors.*

3. We keep the notation of §2, with no restriction on rank M. We assume N is an amenable hypersurface, so

$$(X_i)_\mathfrak{p} = X_{i,\lambda_i} \in \mathfrak{p}_{\lambda_i}, \tag{3.1}$$

where λ_i may be zero. Note by (2.3) that $\lambda_i(A) = 0$ implies $a_i = 0$. Thus we have

Proposition 3.2. *X_i^* vanishes at a point of γ iff $\lambda_i(A) \neq 0$. Further, $(X_i^*)_{\gamma(t)} = 0$ iff*

$$\begin{aligned} &\lambda_i(A)\cot(t\lambda_i(A)) = \alpha_i \quad \textit{for } M \textit{ compact},\\ &\lambda_i(A)\coth(t\lambda_i(A)) = \alpha_i \quad \textit{for } M \textit{ noncompact}. \end{aligned} \tag{3.2.1}$$

Now, for any $X \in \mathfrak{k}$, we have $X^*|\gamma = \sum c_i X_i^*|\gamma$ for constants c_i (Lemma 1.6). If X^* vanishes at $\gamma(t)$, we apply (2.4) and (3.1) to get a linear combination of $(X_i)_\mathfrak{p}$ which vanishes. Since $\{(X_i)_\mathfrak{p}\}$ is an orthonormal set, we see that $c_i \neq 0$ implies X_i^* vanishes at $\gamma(t)$. From (1.5) and (1.6), we now obtain

Theorem 3.3. *The focal points of N along γ occur precisely at points $\gamma(t)$ where t satisfies* (3.2.1) *for some $i = 1, \cdots, n$. The order of the focal point $\gamma(t)$ is the number of such i.*

Suppose now that $\gamma(t)$ is not a focal point. By (1.4) we know γ is normal to the hypersurface $K \cdot \gamma(t)$, which equals N_t if N is orientable. Let $\tilde{\varepsilon}$ be the local unit normal vector field to $K \cdot \gamma(t)$ such that

$$\tilde{\varepsilon}_{\gamma(t)} = \gamma'(t) = A^*_{\gamma(t)}, \tag{3.4}$$

and let $\tilde{\sigma}$ be the corresponding shape operator. Let e_i be the vector field along γ defined by parallel translating $(X_i^*)_0$. From [9, p. 173] we know

$$(d \exp tA)((X_i^*)_0) = (e_i)_{\gamma(t)}. \tag{3.5}$$

Since N is amenable, each $(X_i^*)_0$ is an eigenvalue of the curvature transformation Π with eigenvalue $\pm\lambda_i(A)^2$ where the negative sign is taken in the compact case. Either from the explicit solution of Jacobi's equations [8, p. 20] or directly from (2.4), we obtain

$$X_i^*|_\gamma = \begin{cases} \left(\cos(t\lambda_i(A)) - \dfrac{\alpha_i}{\lambda_i(A)}\sin(t\lambda_i(A))\right)e_i \\ \qquad \text{if } \lambda_i(A) \neq 0 \text{ and } M \text{ compact}, \\ e_i \text{ if } \lambda_i(A) = 0, \\ \left(\cosh(t\lambda_i(A)) - \dfrac{\alpha_i}{\lambda_i(A)}\sinh(t\lambda_i(A))\right)e_i \\ \qquad \text{if } \lambda_i(A) \neq 0 \text{ and } M \text{ noncompact}. \end{cases} \tag{3.6}$$

Note that our conventions on the sign of the curvature tensor are opposite to those in [8]. Now we compute $\tilde{\sigma}$ by (1.3) (where of course we only need to know X^* along γ) and (3.6) to obtain

Theorem 3.7. *If $\gamma(t)$ is not a focal point, then $\{(X_i^*)_{\gamma(t)}\}$ is an orthogonal basis of eigenvectors of $\tilde{\sigma}$ at $\gamma(t)$ with eigenvalues*:

$$\begin{aligned} &\lambda_i(A)\cot(\theta_i - t\lambda_i(A)) \text{ where } \lambda_i(A)\cot\theta_i = \alpha_i, \\ &\text{if } \lambda_i(A) \neq 0,\ M \text{ compact}, \\ &0 \quad \text{if } \lambda_i(A) = 0, \\ &\lambda_i(A)\frac{-1 + \dfrac{\alpha_i}{\lambda_i(A)}\coth(t\lambda_i(A))}{\coth(t\lambda_i(A)) - \dfrac{\alpha_i}{\lambda_i(A)}} \qquad \text{if } \lambda_i(A) \neq 0,\ M \text{ noncompact}. \end{aligned} \tag{3.7.1}$$

Remark 3.8. If M is noncompact and $|a_i| > |\lambda_i(A)|$, i.e., if X_i^* vanishes at some point of γ, then the third line in (3.7.1) can be written in the form of the first (with coth). Also, the first can be written in the form of the third (with cot and changing -1 to $+1$). This will be useful later.

Suppose now that $\gamma(t)$ is a focal point and $F = K \cdot \gamma(t)$. We also suppose K is compact, so F is minimal by (1.8). We will use an idea of Münzer [13] reported by Nomizu [15] to derive a generalizaton of a curious formula of Cartan. We can assume an indexing so that X_i^* is nonzero at $\gamma(t)$ iff $1 \leqslant i \leqslant f$. By the discussion after (3.2), $\{X_i^*\colon 1 \leqslant i \leqslant f\}$ forms a basis of $T_{\gamma(t)}F$. We know $\{e_1, \cdots, e_n, \gamma'\}$ at t is an orthogonal basis of $T_{\gamma(t)}M$, and from (3.6) we conclude that $\{e_1, \cdots, e_f\}$ is an orthonormal basis of $T_{\gamma(t)}F$ and $\{e_{f+1}, \cdots, e_n, \gamma'\}$ is a basis of the normal space. We can now compute the shape operator $\tilde{\sigma}$ for F with respect to the normal $\gamma'(t)$ just as we did for (3.7). Since F is minimal, each component of the mean curvature vector [12, Vol. II, p. 34] vanishes, so in particular, the trace of $\tilde{\sigma}$ at $\gamma(t)$ vanishes. Thus we obtain

Theorem 3.9. *Suppose K is compact, $\gamma(t)$ is a focal point, and X_i^* is nonzero* *at* $\gamma(t)$ *iff* $1 \leqslant i \leqslant f$. *Then the sum over* $i = 1, \cdots, f$ *of the appropriate terms in* (3.7.1) *vanishes.*

To explain why (3.9) generalizes Cartan's formula, suppose M is a space of constant curvature ± 1, so there is only one positive root λ and $\lambda(A) = 1$. Since $X_{f+1}^*, \cdots, X_n^*$ vanish at $\gamma(t)$, (3.2.1) shows that $\alpha_{f+1} = \cdots \alpha_n = \alpha$ and $\alpha = \cot(t)$ or $\alpha = \coth(t)$; further, $\alpha_i \neq \alpha$ for $i = 1, \cdots, f$, and these give all the other eigenvalues of σ. Thus (see (3.8)),

$$0 = \sum_{i=1}^{f} \frac{\mp 1 - \alpha_i \cot[h](t)}{\alpha_i - \cot[h](t)} = \sum_{i=1}^{f} \frac{\mp 1 - \alpha_i \alpha}{\alpha_i - \alpha},$$

which is Cartan's formula. Of course, we have had to assume K compact, which is a significant restriction when M is of noncompact type.

Remark 3.10. In (3.9) we only used the vanishing of one component of the mean curvature vector. We can also take the shape operator $\tilde{\sigma}_j$ of F with respect to the normal e_j, $f < j$, at $\gamma(t)$ and consider the condition Trace $\tilde{\sigma}_j = 0$. We omit this because the computation is tedious and the resulting equation does not seem geometrically meaningful. Roughly, one must compute $\langle \tilde{\sigma}_j(X_i^*), X_i^* \rangle_{\gamma(t)} = -\langle \nabla_{e_j} X_i^*, X_i^* \rangle_{\gamma(t)}$ but, since there is no formula analogous to (3.6) for X_i^* along an integral curve of e_j at $\gamma(t)$, one must use $d\exp(-tA)$ to carry the computation back to the point 0.

If K is compact and M is noncompact (hence simply-connected with nonpositive curvature [9]), then we may derive complete results without the prior assumption that N is amenable by using the fact that K must have a fixed point [12, Vol. 2, p. 111]. If the fixed point is $\tilde{p}$, we may construct the geodesic $\tilde{\gamma}$ through $\tilde{p}$ normal to N at some point $\tilde{0}$. There is $f \in K$ such that $f(0) = \tilde{0}$ and hence, perhaps after reversing the parameter, $f \circ \gamma = \tilde{\gamma}$. Thus we have a unique fixed point $\gamma(t) = f^{-1}(\tilde{p}) = \tilde{p} = \tilde{\gamma}(t)$ (this follows also from (1.8)). Thus N is in the geodesic sphere of radius t around $\tilde{p}$, and must equal the geodesic sphere since N has codimension 1. This also implies M is of rank 1 [8, pp. 59–60], and N is amenable by (2.6.1) since $X_{\gamma(t)}^* = 0$ for each $X \in \mathfrak{k}$. Finally, (3.2.1) relates t and the principal curvatures, and we have

Proposition 3.11. *If K is compact and M is noncompact, then M is of rank* 1 *and N is amenable. Further, N is the geodesic sphere of radius t around the unique fixed point p of K, which is the only focal point. If M is of constant negative curvature* -1, *then there is only one principal curvature given by* $\alpha = \coth(t)$. *If M has nonconstant curvature normalized so as to give the range* $[-4, -1]$, *then there are exactly two principal curvatures given by* $\alpha = \coth(t)$ *and* $\alpha' = 2\coth(2t)$.

Remark 3.12. Cartan had shown that a hypersurface with constant principal curvatures in the space of negative constant curvature has at most two principal curvatures. His examples with two distinct principal curvatures are of course not compact.

Now (3.3) gives precise information on where the focal set $\mathfrak{F}$ cuts γ but does not say whether a component (i.e., orbit) in $\mathfrak{F}$ can "wrap around" and cut γ more than once (by (1.8), this must happen if there are more than two distinct focal points). However, in particular cases, where we know the root structure, we can determine the dimension of the focal set at each focal point on γ, and (1.8) implies that at most two different dimensions can occur. For example, we have

Proposition 3.13. *Suppose M is the sphere of constant curvature. Then at most two different multiplicities can occur amongst the principal curvatures.*

Of course, this was also known from the classification of Takagi-Takahashi [19]. The result does not hold in a quaternionic projective space because an X_i^* and X_j^* from two different eigenvalues can vanish at the same focal point on γ.

4. In this section, we give a complete classification of N when M is a quaternionic projective space and prove that N is always amenable. The method is based on that used by Takagi [17], [18], who gave a complete classification when M is a complex projective space. First, we observe the following easy consequence of Takagi's classification

Propositon 4.1. *Let K be a closed (hence compact) connected group of isometries of a complex projective space which has an orbit N of codimension one. Then N is amenable.*

Proof. In [18, Remark 1.1], Takagi observes that if J is the complex structure on M and ε is a normal vector to N, then $J\varepsilon$ is a direction of principal curvature. By (2.6.2), N is amenable. To prove the remark, one considers the Riemannian submersion of the sphere onto a complex projective space, pulls N back to a hypersurface of the sphere, and compares the principal curvature directions of this hypersurface with the complex structure of the ambient complex vector space via an explicit knowledge of certain root spaces. The details are similar to what we will do for a quaternionic projective space.

From now on, we deal only with the quaternionic case. $\mathbf{H}$ will denote the quaternion algebra, and we identify $\mathbf{R}^{4n+4}$ with the left quaternionic vector space $\mathbf{H}^{n+1}$. The group of left multiplications by unit quaternions will be denoted $Q(n+1)$ (isomorphic to Sp(1)) and the subgroup of SO$(4n+4)$ centralizing $Q(n+1)$ is identified with Sp$(n+1)$, which acts transitively on the unit sphere $S^{4n+3} \subset \mathbf{R}^{4n+4}$. We have the Riemannian submersion π: $S^{4n+3} \to \mathbf{P}^n\mathbf{H} = M$ given by $\pi^{-1}(\pi(x)) = \mathrm{Sp}(1) \cdot x$. Since Sp$(n+1)$ and

$Q(n+1)$ commute, we have a unique action of $\mathrm{Sp}(n+1)$ on $\mathbf{P}^n\mathbf{H}$ so that π is equivariant; this action of $\mathrm{Sp}(n+1)$ gives the identity component of the isometry group of $\mathbf{P}^n\mathbf{H}$. Let J_1, J_2, J_3 be left multiplication by the quaternions i, j, k on $\mathbf{H}^{n+1}$.

Fix points $a \in S^{4n+3}$ and $0 = \pi(a) \in \mathbf{P}^n\mathbf{H}$. Let $\mathfrak{h}(a)$ be the orthogonal complement of $\mathbf{R}a \oplus \mathbf{R}J_1a \oplus \mathbf{R}J_2a \oplus \mathbf{R}J_3a = \mathbf{R}a \oplus \operatorname{Ker} \pi_*|a$ in $\mathbf{R}^{4n+4}$. Then

$$\pi_*|a\colon \mathfrak{h}(a) \to T_0\mathbf{P}^n\mathbf{H} \text{ is a surjective isometry.} \tag{4.2}$$

Clearly, each J_l leaves $\mathfrak{h}(a)$ invariant but does not induce an operator on $T_0\mathbf{P}^n\mathbf{H}$ independent of choice of $a \in \pi^{-1}(0)$. However, the action of $Q(n+1)$ on $\mathfrak{h}(a)$ does induce a well-defined group of operators on $T_0\mathbf{P}^n\mathbf{H}$ which defines the quaternionic structure at 0.

Suppose now that K is a closed (hence compact) connected subgroup of $\mathrm{Sp}(n+1)$ such that the orbit $N = K \cdot 0 \subset M$ is of codimension one. Then $\hat{N} = \pi^{-1}(N)$ is the orbit of a under the action of $K \cdot Q(n+1)$ on $\mathbf{R}^{4n+4}$, and $\hat{N}$ is a hypersurface of S^{4n+3}. Let b be a unit vector orthogonal to $\hat{N}$ at a. Since $\hat{N}$ is invariant under $Q(n+1)$, b is in $\mathfrak{h}(a)$ and $\pi_* b = \varepsilon$ is a unit normal to N at 0. Let $\hat{\sigma}$ be the shape operator of $\hat{N}$ with respect to b at a, and σ the shape operator of N with respect to ε at 0. Then it is easy to see that

$$\langle \hat{\sigma}X, Y\rangle = \left\langle \sigma\pi_*X, \pi_*Y\right\rangle \quad \text{for } X, Y \in \mathfrak{h}(a), \text{ orthogonal to } b. \tag{4.3}$$

In particular, we have

(4.4) If $X \in \mathfrak{h}(a)$ is nonzero and $\hat{\sigma}X = \alpha X + X'$ where $\langle X', \mathfrak{h}(a)\rangle = 0$, then π_*X is an eigenvector of σ with eigenvalue α.

Now let $\hat{K}$ be the maximal compact connected subgroup of $\mathrm{SO}(4n+4)$ leaving $\hat{N}$ invariant. From the classification of Hsiang-Lawson [10], we know that either the action of $\hat{K}$ on $\mathbf{R}^{4n+4}$ is reducible or agrees (up to conjugation) with the linear isotropy action of an irreducible Riemannian symmetric pair of rank two and compact type.

Proposition 4.3. *Suppose the action of $\hat{K}$ is reducible. Then*

(4.3.1) $\mathbf{H}^{n+1} = V_1 \oplus V_2$ *as an orthogonal direct sum of quaternionic subspaces,* $\hat{N} = S_1 \times S_2$ *where S_i is the sphere of radius $r_i > 0$ in V_i,* $r_1^2 + r_2^2 = 1$.

Conversely, given a nontrivial decompostion (4.3.1), *we can let* $K = \mathrm{Sp}(n_1) \times \mathrm{Sp}(n_2)$, *where n_i is the quaternionic dimension of V_i, and the orbit of K at* 0 *is* $\pi(\hat{N})$. *In this situation, N is amenable. Writing* $r_1 = \cos\theta$, $r_2 = \sin\theta$, *the* *principal curvatures are*

$$\tan\theta \text{ on } \pi_* W_1 \text{ with multiplicity } 4(n_1 - 1),$$
$$-\cot\theta \text{ on } \pi_* W_2 \text{ with multiplicity } 4(n_2 - 1),$$
$$\tan\theta - \cot\theta \text{ on } \sum_{l=1}^{3} \mathbf{R}\pi_* J_l b \text{ with multiplicity } 3.$$

Here W_i *is the orthogonal complement of* $\{a_i, J_1a_i, J_2a_i, J_3a_i\}$ *in* V_i, *with* $a = (a_1, a_2)$, $b = (b_1, b_2)$, $a_i, b_i \in V_i$.

Note that the geodesic spheres in $\mathbf{P}^n\mathbf{H}$ occur precisely when one $n_i = 1$.

Proof. If V_1 is a proper $\hat{K}$ invariant subspace, then the orthogonal complement V_2 is also $\hat{K}$ invariant, and each V_i is a quaternionic subspace because $\hat{K} \supset Q(n + 1)$. For any $k \in \hat{K}$, we have $\hat{N} \ni ka = (ka_1, ka_2) \in S_1 \times S_2$, where S_i is the sphere in V_i through a_i, and for dimensional reasons, we have $\hat{N} = S_1 \times S_2$ with $r_i = |a_i| > 0$; cf. Takagi-Takahashi [19, p. 478].

Next note that $b = (-(r_2/r_1)a_1, (r_1/r_2)a_2)$ and $W_i = \mathfrak{b}(a) \cap T_{a_i}S_i$. We know the shape operator of the sphere S_i in V_i with respect to the normal vector b_i (not of unit length!), so $\hat{\sigma}|T_{a_1}S_1 = (r_2/r_1)I$, $\hat{\sigma}|T_{a_2}S_2 = -(r_1/r_2)I$; since $W_i \subset \mathfrak{b}(a)$, (4.4) shows $\pi_* W_i$ is an eigenspace for σ with the indicated eigenvalue. On the other hand,

$$\hat{\sigma}J_l b = \left(\frac{r_2}{r_1} J_l b_1, -\frac{r_1}{r_2} J_l b_2\right) = \left(\frac{r_2}{r_1} - \frac{r_1}{r_2}\right) J_l b + J_l a.$$

Again applying (4.4), we get the last eigenvalue for σ. Finally, (2.6.2) shows N is amenable since $\sum \mathbf{R}\pi_* J_l b$ is just the image of the normal vector $\varepsilon = \pi_* b$ by the skew-involutive endomorphisms of the quaternionic structure of $\mathbf{P}^n\mathbf{H}$ at 0.

Now let (U, L) be any irreducible Riemannian symmetric pair of compact type, and let $\mathfrak{u} = \mathfrak{l} \oplus \mathfrak{p}$ be the usual decomposition of the Lie algebra. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, and let $a \in \mathfrak{a}$ be regular. Then the orbit of a under the adjoint action of L will have codimension equal to rank(U, L) in the vector space $\mathfrak{p}$. A complete description of the geometry of the situation is given in Takagi-Takahashi [19].

First we will examine the pairs (U, L) arising in the classification of quaternionic symmetric spaces with quaternionic scalar part given by Wolf [21]. Here $L = K' \cdot Q'$ where K' and Q' commute with each other and $Q' \simeq \mathrm{Sp}(1)$. Then we may identify $\mathfrak{p}$ with $\mathbf{H}^{n+1}$ so that $\mathrm{Ad}_{\mathfrak{p}}Q' = Q(n + 1)$ and $K = \mathrm{Ad}_{\mathfrak{p}}K' \subset \mathrm{Sp}(n + 1)$. The orbit $K \cdot \pi(a)$ will be a hypersurface in $\mathbf{P}^n\mathbf{H}$ iff rank$(U, L) = 2$. The only rank-two pairs which occur are $(\mathrm{SU}(n + 3), S(U_{n+1} \times U_2))$ and $(G_2, \mathrm{SO}(4))$ (see [9] for notation). The latter case may be eliminated since then the real dimension of $\mathfrak{p}$ is 8 and we would be constructing an orbit in $\mathbf{P}^1\mathbf{H} = S^4$. To describe the former case, we first define certain matrices (as in Chavel [7]). Thus E_{jk} will denote the matrix whose only nonzero entry is 1 in the jth row, kth column, and

$$(4.5)\qquad \begin{aligned} A_{jk} &= \sqrt{-1}\,(E_{jj} - E_{kk}),\\ B_{jk} &= (E_{jk} - E_{kj}),\\ C_{jk} &= \sqrt{-1}\,(E_{jk} + E_{kj}). \end{aligned}$$

Then a basis of the Lie algebra $\mathfrak{su}(n+3)$ is $\{A_{j,j+1}\colon 1 \leqslant j \leqslant n+2;\ B_{jk}, C_{jk}\colon 1 \leqslant j < k \leqslant n+3\}$. Following Wolf [21, p. 1043], it is easy to show that a basis for $\mathfrak{k}'$ is $\{Z;\ A_{j,j+1}\colon 2 \leqslant j \leqslant n+1;\ B_{jk}, C_{jk}\colon 2 \leqslant j < k \leqslant n+2\}$ where

$$Z = \frac{2}{n+3}(A_{12} + A_{13} + \cdots + A_{1,n+2}) - \frac{n+1}{n+3}A_{1,n+3},$$

a basis for $\mathfrak{q}'$ is $\{A_{1,n+3}, B_{1,n+3}, C_{1,n+3}\}$, and a basis for $\mathfrak{p}$ is $\{B_{1j}, C_{1j}, B_{j,n+3}, C_{j,n+3}\colon 2 \leqslant j \leqslant n+2\}$. Note that the imbedding of $S(U_{n+1} \times U_2)$ in $\mathrm{SU}(n+3)$ is not the most standard one. Let $E_j \in \mathbf{H}^{n+1}$ be the column vector whose only nonzero entry is 1 in the jth row. We identify $\mathfrak{p}$ with $\mathbf{H}^{n+1}$ (as real vector spaces) so that $B_{1j} = E_{j-1}$, $C_{1j} = J_1E_{j-1}$, $B_{j,n+3} = J_2E_{j-1}$, $C_{j,n+3} = J_3E_{j-1}$; then $\mathrm{ad}_{\mathfrak{p}}A_{1,n+3} = J_1$, $\mathrm{ad}_{\mathfrak{p}}B_{1,n+3} = J_2$, and $\mathrm{ad}_{\mathfrak{p}}C_{1,n+3} = J_3$ (note $\mathrm{ad}_{\mathfrak{p}}Z$ corresponds to right multiplication by i). In $\mathfrak{p}$, take $\mathfrak{a} = \mathbf{R}B_{12} \oplus \mathbf{R}B_{n+2,n+3}$, which is maximal abelian. We describe the $\mathfrak{a}$ roots as ordered pairs, where the entries are the values on B_{12} and $B_{n+2,n+3}$ respectively. We get the following six roots and corresponding root spaces:

$$\begin{aligned}
&\rho_1 = (1,0),\ \mathfrak{p}_1 = \mathrm{span}\{B_{1,k}, C_{1,k}\colon 3 \leqslant k \leqslant n+1\};\\
&\rho_2 = (0,1),\ \mathfrak{p}_2 = \mathrm{span}\{B_{j,n+3}, C_{j,n+3}\colon 3 \leqslant j \leqslant n+1\};\\
&\rho_3 = (1,-1),\ \mathfrak{p}_3 = \mathrm{span}\{B_{1,n+2} + B_{2,n+3}, C_{1,n+2} + C_{2,n+3}\};\\
&\rho_4 = (1,1),\ \mathfrak{p}_4 = \mathrm{span}\{B_{1,n+2} - B_{2,n+3}, C_{1,n+2} - C_{2,n+3}\};\\
&\rho_5 = (2,0),\ \mathfrak{p}_5 = \mathbf{R}C_{12};\\
&\rho_6 = (0,2),\ \mathfrak{p}_6 = \mathbf{R}C_{n+2,n+3}.
\end{aligned}$$

Let $a = \cos\theta B_{12} + \sin\theta B_{n+2,n+3}$, which is regular iff θ is not a multiple of $\pi/4$ (in general, a is regular iff $\rho(a) \neq 0$ for all nonzero roots). Let $\hat{N}$ be the orbit of a under the adjoint action of L, and take the normal vector $b = -\sin\theta B_{12} + \cos\theta B_{n+2,n+3}$. From Takagi-Takahashi [19], we know that the eigenspaces of the shape operator $\hat{\sigma}$ are the $\mathfrak{p}_i$ with eigenvalues $-\rho_i(b)/\rho_i(a)$. If N is the orbit of $0 = \pi(a)$ in $\mathbf{P}^n\mathbf{H}$ under the action of K (observe $K \simeq U(n+1)$), then $\pi^{-1}N = \hat{N}$.

Proposition 4.6. *Let the quaternionic symmetric pair* $(\mathrm{SU}(n+3), S(U_{n+1} \times U_2))$ *induce an action of* $U(n+1)$ *on* $\mathbf{P}^n\mathbf{H}$ *as described. Let* N *be the orbit at* $0 = \pi(a)$ *where* a *is regular. Then* N *is an amenable hypersurface whose* *principal curvatures are*

$\tan\theta$ *on* $\pi_*\mathfrak{p}_1$ *with multiplicity* $2(n-1)$,

$-\cot\theta$ *on* $\pi_*\mathfrak{p}_2$ *with multiplicity* $2(n-1)$,

$\tan\theta - \cot\theta$ *on* $\mathbf{R}\pi_* J_1 b$ *with multiplicity* 1,

$2\tan 2\theta$ *on* $\mathbf{R}\pi_* J_2 b \oplus \mathbf{R}\pi_* J_3 b$ *with multiplicity* 2.

Proof. We can explicitly compute all $J_l a$, $J_l b$ and hence $\mathfrak{b}(a)$. Since $\mathfrak{p}_1$, $\mathfrak{p}_2$ are in $\mathfrak{b}(a)$, the first two principal curvatures are clear. Next $J_1 b = -\sin\theta C_{12} + \cos\theta C_{n+2,n+3} \in \mathfrak{p}_5 \oplus \mathfrak{p}_6$ and $\sigma J_1 b = (\tan\theta - \cot\theta)J_1 b - J_1 a$; applying (4.4) gives the third principal curvature, and the fourth is similar. Again, (2.6.2) shows N is amenable.

Our task from now on is to prove that the classification is already complete with (4.3) and (4.6). Thus we will be considering a compact irreducible Riemannian symmetric pair (U, L) of rank 2 such that the adjoint action of L on $\mathfrak{p}$ contains a subgroup of the form $K \cdot Q(n+1)$, and the orbits of this subgroup coincide with the orbits of $\mathrm{Ad}_{\mathfrak{p}} L$. In particular, we have elements $L_1, L_2, L_3 \in \mathfrak{l}$ so that

(4.7) $\mathrm{ad}_{\mathfrak{p}} L_i \circ \mathrm{ad}_{\mathfrak{p}} L_j = \mathrm{ad}_{\mathfrak{p}} L_k$ if $\{i, j, k\}$ is a cyclic permutation of $(1, 2, 3)$;

(4.8) $$(\mathrm{ad}_{\mathfrak{p}} L_i)^2 = -I_{\mathfrak{p}}.$$

From (4.7) we have $\mathrm{ad}_{\mathfrak{p}}[L_i, L_j] = 2\,\mathrm{ad}_{\mathfrak{p}} L_k$, so $[L_i, L_j] = 2L_k$. This shows that

(4.9) $$(\mathrm{ad}\, L_i)^2(L_j) = -4L_j \quad \text{for any } i \neq j.$$

Now choose a maximal abelian subspace $\mathfrak{t} \subset \mathfrak{l}$ with $L_1 \in \mathfrak{t}$. Then (4.8) implies $\mathfrak{t}$ is maximal abelian in $\mathfrak{u}$ (see [17, p. 497]), and $\mathfrak{t}^{\mathbf{C}}$ is a Cartan subalgebra for both $\mathfrak{u}^{\mathbf{C}}$ and $\mathfrak{l}^{\mathbf{C}}$, i.e., rank $\mathfrak{u}^{\mathbf{C}}$ = rank $\mathfrak{l}^{\mathbf{C}}$. Combining this with the conditions that $\mathrm{rank}(U, L) = 2$ and that 4 divides the dimension of $\mathfrak{p}$, we find that (U, L) must be one of the following:

$(\mathrm{SU}(3) \times \mathrm{SU}(3), \mathrm{SU}(3))$ (eliminated since $\dim \mathfrak{p} = 8$, $M = S^4$),
$(\mathrm{SU}(n+3), S(U_{n+1} \times U_2))$, $n \geq 1$,
$(G_2, \mathrm{SO}(4))$,
(4.10) $(\mathrm{SO}(8), U(4))$, $(\mathrm{SO}(10), U(5))$,
$(\mathrm{SO}(2n+4), \mathrm{SO}(2n+2) \times \mathrm{SO}(2))$,
$(E_6, \mathrm{SO}(10) + \mathbf{R})$,
$(\mathrm{Sp}(p+2), \mathrm{Sp}(p) \times \mathrm{Sp}(2))$ where $2p = n+1$, $p \geq 2$,

where the first two cases have already been discussed.

Let Δ^+ be a system of positive roots so that $-iL_1 = H_1$ is in the closure of the positive Weyl chamber. Then Δ^+ is the disjoint union of $\Delta_{\mathfrak{l}}^+$, $\Delta_{\mathfrak{p}}^+$ so

$$\mathfrak{l} = \mathfrak{t} \oplus \sum_{\alpha \in \Delta_{\mathfrak{l}}^+} (\mathfrak{u} \cap (\mathfrak{u}_\alpha^{\mathbf{C}} + \mathfrak{u}_{-\alpha}^{\mathbf{C}})), \tag{4.11}$$

$$\mathfrak{p} = \sum_{\alpha \in \Delta_{\mathfrak{p}}^+} (\mathfrak{u} \cap (\mathfrak{u}_\alpha^{\mathbf{C}} + \mathfrak{u}_{-\alpha}^{\mathbf{C}})). \tag{4.12}$$

Let $\Lambda = \{\alpha \in \Delta^+ : \alpha(H_1) = 2\}$. Then (4.8) implies

$$\alpha(H_1) = 1 \text{ for all } \alpha \in \Delta_{\mathfrak{p}}^+, \tag{4.13}$$

while (4.9) implies (using $i = 1$)

$$L_j \in \sum_{\alpha \in \Lambda} (\mathfrak{u} \cap (\mathfrak{u}_\alpha^{\mathbf{C}} + \mathfrak{u}_{-\alpha}^{\mathbf{C}})), \qquad j = 2, 3. \tag{4.14}$$

Next, let $\Phi = \{\alpha \in \Delta^+ : \alpha(H_1) = 0\}$. Since $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{l}$ [9, p. 207], each root in $\Delta_{\mathfrak{l}}^+$ is of the form $\alpha_1 \pm \alpha_2$, $\alpha_i \in \Delta_{\mathfrak{p}}^+$, so $\Delta_{\mathfrak{l}}^+ = \Phi \cup \Lambda$. Let $\Delta = -\Delta^+ \cup \Delta^+$.

For each $\alpha \in \Delta$, choose $H_\alpha \in i\mathfrak{t}$, $X_\alpha \in \mathfrak{u}_\alpha^{\mathbf{C}}$ so that

$$\begin{aligned}
&B(H, H_\alpha) = \alpha(H) \quad \text{for } H \in i\mathfrak{t}, \text{ where } B \text{ is the Killing form,}\\
&[X_\alpha, X_{-\alpha}] = H_\alpha,\ [H, X_\alpha] = \alpha(H)X_\alpha \quad \text{for } H \in i\mathfrak{t},\\
&[X_\alpha, X_\beta] = 0 \quad \text{if } \alpha + \beta \neq 0, \alpha + \beta \notin \Delta,\\
&[X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta} \quad \text{if } \alpha + \beta \in \Delta,\\
&\text{where } N_{\alpha,\beta} = -N_{\beta,\alpha} = -N_{-\alpha,-\beta} \neq 0,\\
&Y_\alpha = X_\alpha - X_{-\alpha},\ Z_\alpha = i(X_\alpha + X_{-\alpha}) \text{ are in } \mathfrak{u}
\end{aligned} \tag{4.15}$$

(see [9, Theorem 5.5, p. 151; Lemma 3.1, pp. 219–220; and Definition, p. 332]). Let

$$L_2 = \sum_{\alpha \in \Lambda} (l_\alpha X_\alpha + l_{-\alpha} X_{-\alpha}) \quad \text{where } l_{-\alpha} = -\bar{l}_\alpha. \tag{4.16}$$

From $[L_2, [L_2, L_1]] = -4L_1 = -4iH_1$, we get

$$\sum l_\alpha \bar{l}_\beta N_{\alpha,-\beta} X_{\alpha-\beta} = 0, \text{ sum over distinct } \alpha, \beta \in \Lambda \text{ with } \alpha - \beta \in \Delta, \tag{4.17}$$

$$\sum l_\alpha \bar{l}_\alpha H_\alpha = H_1, \quad \text{sum over } \alpha \in \Lambda. \tag{4.18}$$

Let $\mathfrak{q}$ be the algebra generated by L_1, L_2, L_3 in $\mathfrak{l}$, and let $\mathfrak{z}(\mathfrak{q})$ be the centralizer of $\mathfrak{q}$ in $\mathfrak{l}$. For $Z \in \mathfrak{l}$, it is easy to see that

$$[Z, L_1] = 0 \quad \text{iff } Z \in \mathfrak{t} \oplus \sum_{\alpha \in \Phi} (\mathfrak{u} \cap (\mathfrak{u}_\alpha^{\mathbf{C}} + \mathfrak{u}_{-\alpha}^{\mathbf{C}})). \tag{4.19}$$

Clearly, the adjoint action on $\mathfrak{p}$ of the analytic subgroup corresponding to $\mathfrak{z}(\mathfrak{q})$, respectively $\mathfrak{q}$, contains K, respectively $Q(n + 1)$.

Suppose $\mathfrak{a} \subset \mathfrak{p}$ is maximal abelian and $a \in \mathfrak{a}$ is regular. Then

$$\{Z \in \mathfrak{l}: [Z, a] = 0\} = \{Z \in \mathfrak{l}: [Z, \mathfrak{a}] = 0\} = \mathfrak{z}(\mathfrak{a}). \tag{4.20}$$

The tangent space at a of the orbit of a under the adjoint action of the group L can be identified with $[\mathfrak{l}, a]$ (see [19] or [17]). By our assumption, the adjoint action of the analytic subgroup whose Lie algebra is $\mathfrak{q} + \mathfrak{z}(\mathfrak{q})$ must give the same orbit, so $[\mathfrak{l}, a] = [\mathfrak{q}, a] + [\mathfrak{z}(\mathfrak{q}), a]$. Thus we have

$$\mathfrak{l} = \mathfrak{q} + \mathfrak{z}(\mathfrak{q}) + \mathfrak{z}(\mathfrak{a}). \tag{4.21}$$

Let $\mathfrak{l}_\Lambda = \Sigma_{\alpha \in \Lambda} (\mathfrak{u} \cap (\mathfrak{u}^{\mathbf{C}}_\alpha + \mathfrak{u}^{\mathbf{C}}_{-\alpha})) \subset \mathfrak{l}$, and let P: $\mathfrak{l} \to \mathfrak{l}_\Lambda$ be the orthogonal projection. From (4.19) we have $P(\mathfrak{z}(\mathfrak{q})) = 0$, so (4.21) implies $P(\mathfrak{q} + \mathfrak{z}(\mathfrak{a})) = P(\mathfrak{l}) = \mathfrak{l}_\Lambda$. Of course, $\dim P(\mathfrak{q}) = 2$, so we have

$$2 + \dim P(\mathfrak{z}(\mathfrak{a})) \geqslant \dim \mathfrak{l}_\Lambda. \tag{4.22}$$

We learned the following device from Wolf [20]. Let Π be a system of simple roots for Δ^+, and let Π_0, Π_1, Π_2 be the intersection of Π with Φ, $\Delta^+_\mathfrak{p}$, Λ, respectively. Let μ be a maximal root in Δ^+, so $\mu = \Sigma_{\pi \in \Pi} m_\pi \pi$ where the coefficients m_π are positive integers. Thus

$$2 = \mu(H_1) = \sum_{\pi \in \Pi_1} m_\pi + 2 \sum_{\pi \in \Pi_2} m_\pi.$$

Note Π_1 is nonempty because $\Delta^+_\mathfrak{p}$ is so. Thus the only possibilities are

$$\Pi_2 \text{ empty}, \quad \Pi_1 = \{\rho\}, \quad m_\rho = 2, \tag{4.23}$$

$$\Pi_2 \text{ empty}, \quad \Pi_1 = \{\rho, \sigma\}, \quad m_\rho = m_\sigma = 1. \tag{4.24}$$

Examining the root diagram and maximal root for each $\mathfrak{u}$ occurring in (4.10) (see [1] or [11]), we can determine which choices of Π_1 lead to a $\mathfrak{p}$ (defined by (4.12)), which contains a maximal abelian subspace of dimension two. Of course, we do not need to consider $\mathfrak{u} = \mathfrak{su}(n + 3)$ or $\mathfrak{g}_2$, although it is easy to see that in those cases, we do end up with the appropriate quaternionic symmetric space.

Case 1. $\mathfrak{u} = \mathfrak{so}(2n + 4)$.

With $l = n + 2$, we can take $\Delta^+ = \{\varepsilon_i \pm \varepsilon_j: 1 \leqslant i < j \leqslant l\}$, and $\Pi = \{\pi_i: 1 \leqslant i \leqslant l\}$ where $\pi_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leqslant i < l$ and $\pi_l = \varepsilon_{l-1} + \varepsilon_l$. As usual, we can identify the ε_i with the standard Euclidean basis vectors in $\mathbf{R}^l$. The maximal root is

$$\mu = \pi_1 + 2\pi_2 + \cdots + 2\pi_{l-2} + \pi_{l-1} + \pi_l = \varepsilon_1 + \varepsilon_2.$$

The only choices of Π_1 leading to a rank-two pair are $\Pi_1 = \{\pi_1, \pi_l\}$ and $\Pi_1 = \{\pi_{l-1}, \pi_l\}$.

Subcase 1a. $\Pi_1 = \{\pi_1, \pi_l\}$.

A positive root is in $\Delta^+_\mathfrak{p}$ (respectively, Λ) iff its expression in terms of simple roots contains precisely one of π_1, π_l (respectively, both π_1 and π_l). Thus

$$\Delta_{\mathfrak{p}}^+ = \{\varepsilon_1 - \varepsilon_j : 2 \leqslant j \leqslant l;\ \varepsilon_i + \varepsilon_j : 2 \leqslant i < j \leqslant l\},$$
$$\Phi = \{\varepsilon_i - \varepsilon_j : 2 \leqslant i < j \leqslant l\}, \quad \Lambda = \{\varepsilon_1 + \varepsilon_j : 2 \leqslant j \leqslant l\}.$$

If $l \geqslant 6$, the $\mathfrak{p}$ contains the commuting vectors $\{Y_{\varepsilon_1 - \varepsilon_2}, Y_{\varepsilon_3 + \varepsilon_4}, Y_{\varepsilon_5 + \varepsilon_6}\}$ so $\operatorname{rank}(\mathfrak{u}, \mathfrak{l}) \geqslant 3$.

If $l = 4$, (4.13) implies H_1 has a unique expression involving H_α, $\alpha \in \Lambda$, namely

$$2H_1 = H_{\varepsilon_1 + \varepsilon_2} + H_{\varepsilon_1 + \varepsilon_3} + H_{\varepsilon_1 + \varepsilon_4}.$$

Then (4.17) and (4.18) imply that

$$l_\alpha \bar{l}_\alpha = \tfrac{1}{2} \text{ for } \alpha \in \Lambda, \quad l_\alpha \bar{l}_\beta = 0 \text{ for distinct } \alpha, \beta \in \Lambda.$$

Treating each l_α as a vector of length $\sqrt{2}/2$ in $\mathbf{R}^2$, we get a contradiction. If $l = 5$, we find H_1 is uniquely determined by the conditions

$$\tfrac{1}{3}\varepsilon_1(H_1) = \varepsilon_2(H_1) = \cdots = \varepsilon_5(H_1) = \tfrac{1}{2},$$

and so

$$2H_1 = \tfrac{1}{2}H_{\varepsilon_1 + \varepsilon_2} + H_{\varepsilon_1 + \varepsilon_3} + H_{\varepsilon_1 + \varepsilon_4} + H_{\varepsilon_1 + \varepsilon_5} - \tfrac{1}{2}H_{\varepsilon_1 - \varepsilon_2}.$$

One checks that H_1 cannot be written as a linear combination of H_α, $\alpha \in \Lambda$, contradicting (4.18).

Subcase 1b. $\Pi_1 = \{\pi_{l-1}, \pi_l\}$.

Computing as before, we have

$$\Delta_{\mathfrak{p}}^+ = \{\varepsilon_i \pm \varepsilon_l : 1 \leqslant i < l\}, \quad \Phi = \{\varepsilon_i - \varepsilon_j : 1 \leqslant i < j < l\},$$
$$\Lambda = \{\varepsilon_i + \varepsilon_j : 1 \leqslant i < j < l\}.$$

Let $\mathfrak{a} = \mathbf{R}Y_{\varepsilon_1 - \varepsilon_l} \oplus \mathbf{R}Y_{\varepsilon_1 + \varepsilon_l}$ which is maximal abelian in $\mathfrak{p}$. One computes

$$\mathfrak{z}(\mathfrak{a}) = \operatorname{span}\left\{\sqrt{-1}\, H_{\varepsilon_i \pm \varepsilon_j}, Y_{\varepsilon_i \pm \varepsilon_j}, Z_{\varepsilon_i \pm \varepsilon_j} : 1 < i < j < l\right\}.$$

Using the orthogonal projection $P\colon \mathfrak{l} \to \mathfrak{l}_\Lambda$, we see

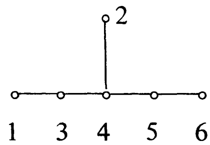
$$\dim P\mathfrak{z}(\mathfrak{a}) = (l-2)(l-3), \quad \dim \mathfrak{l}_\Lambda = (l-1)(l-2).$$

Then (4.22) gives $2 \geqslant (l-2)((l-1)-(l-3)) = 2(l-2)$, so $l \leqslant 3$. However, $l \geqslant 2$, the case with $l = 2$ is reducible, and for $l = 3$ we note

$$\mathrm{SO}(6)/\mathrm{SO}(4) \times \mathrm{SO}(2) \simeq \mathrm{SU}(4)/S(U_2 \times U_2).$$

Case 2. $\mathfrak{u} = \mathfrak{e}_6$.

A system of simple roots is $\Pi = \{\pi_i \colon 1 \leqslant i \leqslant 6\}$ with root diagram



Each π_i can be written in terms of the standard Euclidean basis vectors in $\mathbf{R}^8$, but this is complicated [1, p. 261], [11, p. 65]. The maximal root is

$$\mu = \pi_1 + 2\pi_2 + 2\pi_3 + 3\pi_4 + 2\pi_5 + \pi_6.$$

The only choice of Π_1 leading to a rank-two pair is $\Pi_1 = \{\pi_1, \pi_6\}$. A positive root is in $\Delta_{\mathfrak{p}}^+$ (respectively, Λ) iff its expression in terms of simple roots contains precisely one of π_1, π_6 (respectively, both π_1 and π_6). One finds $\Lambda = \{\lambda_1, \cdots, \lambda_8\}$ where

$$\lambda_1 = (1\ 2\ 2\ 3\ 2\ 1),\quad \lambda_2 = (1\ 1\ 2\ 3\ 2\ 1),\quad \lambda_3 = (1\ 1\ 2\ 2\ 2\ 1),$$
$$\lambda_4 = (1\ 1\ 2\ 2\ 1\ 1),\quad \lambda_5 = (1\ 1\ 1\ 2\ 2\ 1),\quad \lambda_6 = (1\ 1\ 1\ 2\ 1\ 1),$$
$$\lambda_7 = (1\ 1\ 1\ 1\ 1\ 1),\quad \lambda_8 = (1\ 0\ 1\ 1\ 1\ 1).$$

Here the 6-tuples give the coefficients with respect to the simple roots in order. Here and in the calculations which follow, we use the tables in [1, p. 260] which give all roots and their expression in terms of the simple roots (in a slightly different notation).

Similarly, $\Phi = \{\phi_1, \cdots, \phi_{12}\}$ where

$$\phi_1 = (0\ 1\ 0\ 0\ 0\ 0),\quad \phi_2 = (0\ 1\ 0\ 1\ 0\ 0),\quad \phi_3 = (0\ 0\ 0\ 1\ 0\ 0),$$
$$\phi_4 = (0\ 0\ 1\ 0\ 0\ 0),\quad \phi_5 = (0\ 0\ 1\ 1\ 0\ 0),\quad \phi_6 = (0\ 0\ 1\ 1\ 1\ 0),$$
$$\phi_7 = (0\ 1\ 1\ 1\ 0\ 0),\quad \phi_8 = (0\ 1\ 1\ 1\ 1\ 0),\quad \phi_9 = (0\ 1\ 0\ 1\ 1\ 0),$$
$$\phi_{10} = (0\ 0\ 0\ 1\ 1\ 0),\quad \phi_{11} = (0\ 1\ 1\ 2\ 1\ 0),\quad \phi_{12} = (0\ 0\ 0\ 0\ 1\ 0).$$

Let $\mathfrak{a} = \mathbf{R}Y_{\pi_1} \oplus \mathbf{R}Y_{\pi_6}$ which is maximal abelian in $\mathfrak{p}$. It is easy to see that

$$(4.25)\qquad \mathfrak{z}(\mathfrak{a}) = (\mathfrak{z}(\mathfrak{a}) \cap \mathfrak{t}) \oplus \left(\mathfrak{z}(\mathfrak{a}) \cap \sum_{\pm\alpha\in\Delta_{\mathfrak{l}}^+} \mathbf{C}X_\alpha\right).$$

For

$$(4.26)\qquad X = \sum_{\alpha\in\Delta_{\mathfrak{l}}^+} (y_\alpha Y_\alpha + z_\alpha Z_\alpha)$$

one finds $[X, \mathfrak{a}] = 0$ iff

$$(4.27)\qquad \sum_\alpha y_\alpha(N_{\alpha,\pi}Y_{\alpha+\pi} - N_{\alpha,-\pi}Y_{\alpha-\pi}) + \sum_\alpha z_\alpha(N_{\alpha,\pi}Z_{\alpha+\pi} - N_{\alpha,-\pi}Z_{\alpha-\pi}) = 0$$

for $\pi = \pi_1, \pi_6$, sum on $\alpha \in \Delta_{\mathfrak{l}}^+$, where we let $N_{\alpha,\beta} = 0$ if $\alpha + \beta$ is not a root. If $\alpha \pm \pi$ is a root, it must be in $\Delta_{\mathfrak{p}}$. If $\alpha \in \Lambda$, then $\alpha + \pi$ is never a root, and

$$\alpha - \pi_1 \text{ is a root iff } \alpha = \lambda_5, \lambda_6, \lambda_7, \lambda_8,$$
$$\alpha - \pi_6 \text{ is a root iff } \alpha = \lambda_4, \lambda_6, \lambda_7, \lambda_8.$$

If $\alpha \in \Phi$, then $\alpha - \pi$ is never a root, and

$$\alpha + \pi_1 \text{ is a root iff } \alpha = \phi_4, \phi_5, \phi_6, \phi_7, \phi_8, \phi_{11},$$
$$\alpha + \pi_2 \text{ is a root iff } \alpha = \phi_6, \phi_8, \phi_9, \phi_{10}, \phi_{11}, \phi_{12}.$$

Thus all terms in (4.27) are independent and we conclude that X is in the sum of the root spaces corresponding to $\lambda_1, \lambda_2, \lambda_3, \phi_1, \phi_2, \phi_3$. Again using the orthogonal projection $P\colon \mathfrak{l} \to \mathfrak{l}_\Lambda$, we have

$$\dim P\mathfrak{z}(\mathfrak{a}) = 6, \quad \dim \mathfrak{l}_\Lambda = 16,$$

which contradicts (4.22).

Case 3. $\mathfrak{u} = \mathfrak{sp}(p + 2)$.

With $l = p + 2 \geqslant 4$, we can take $\Delta^+ = \{2\varepsilon_i\colon 1 \leqslant i \leqslant l;\ \varepsilon_i \pm \varepsilon_j\colon 1 \leqslant i < j \leqslant l\}$ and $\Pi = \{\pi_i = \varepsilon_i - \varepsilon_{i+1}\colon 1 \leqslant i < l;\ \pi_l = 2\varepsilon_l\}$. The maximal root is

$$\mu = 2\pi_1 + 2\pi_2 + \cdots + 2\pi_{l-1} + \pi_l = 2\varepsilon_1,$$

so only (4.23) arises. If $\Pi_1 = \{\pi_r\}$ for some $r < l$, then

$$\begin{aligned} \Delta_{\mathfrak{p}}^+ &= \{\varepsilon_i \pm \varepsilon_j\colon 1 \leqslant i \leqslant r < j \leqslant l\}, \\ \Lambda &= \{\varepsilon_i + \varepsilon_j\colon 1 \leqslant i < j \leqslant r;\ 2\varepsilon_i\colon 1 \leqslant i \leqslant r\}, \\ \Phi &= \{\varepsilon_i - \varepsilon_j\colon 1 \leqslant i < j \leqslant r \text{ or } r < i < j \leqslant l; \\ &\qquad \varepsilon_i + \varepsilon_j\colon r < i < j \leqslant l;\ 2\varepsilon_i\colon r < i \leqslant l\}. \end{aligned}$$

We will also need to consider

$$\begin{aligned} \Delta_1^+ &= \{\varepsilon_1 \pm \varepsilon_2, 2\varepsilon_1, 2\varepsilon_2\},\ \Delta_2^+ = \Delta_{\mathfrak{l}}^+ \smallsetminus \Delta_1^+, \\ \Delta_3^+ &= \{2\varepsilon_i\colon 2 < i \leqslant l - 2;\ \varepsilon_i \pm \varepsilon_j\colon 2 < i < j \leqslant l - 2\}. \end{aligned}$$

The only choices of r leading to a rank-2 pair are $r = 2$ and $r = l - 2$. In either case, a maximal abelian subspace of $\mathfrak{p}$ is given by

$$\mathfrak{a} = \mathbf{R}Y_{\varepsilon_1 - \varepsilon_{l-1}} \oplus \mathbf{R}Y_{\varepsilon_2 - \varepsilon_l}.$$

We have the same situation as in (4.25), (4.26), and (4.27), and find that the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{z}(\mathfrak{a})$ is spanned by the real parts of the root spaces corresponding to roots in $\pm \Delta_3^+$ together with vectors of the form

$$aY_{2\varepsilon_1} + bY_{2\varepsilon_{l-1}},\ aZ_{2\varepsilon_1} + bZ_{2\varepsilon_{l-1}},\quad cY_{2\varepsilon_2} + dY_{2\varepsilon_l},\ cZ_{2\varepsilon_2} + dZ_{2\varepsilon_l}.$$

This complication is caused by the fact that all terms in (4.27) are not independent, as opposed to the situation in Case 2.

If $r = l - 2 > 2$, then we use the projection P and find

$$\dim P\mathfrak{z}(\mathfrak{a}) \leqslant 2\big((l-4) + \tfrac{1}{2}(l-4)(l-5) + 2\big),$$
$$\dim \mathfrak{l}_\Lambda = (l-2)(l-1),$$

and (4.22) gives a contradiction.

If $r = 2$, we must use a different projection and compute $\mathfrak{z}(\mathfrak{q})$ more carefully. Let $\mathfrak{l}'$ be the sum of the real parts of root spaces corresponding to roots in $\pm\Delta_1^+$, and P': $\mathfrak{l} \to \mathfrak{l}'$ the orthogonal projection. Note $\Lambda \subset \Delta_1^+$, $\Delta_2^+ \subset \Phi$ and for $\alpha_i \in \Delta_i^+$, $\alpha_1 \pm \alpha_2$ is never a root. Thus ((4.14) and (4.19))

$$\sum_{\alpha \in \Delta_2^+} (\mathbf{R}Y_\alpha \oplus \mathbf{R}Z_\alpha) \subset \mathfrak{z}(\mathfrak{q}).$$

So to compute $P'\mathfrak{z}(\mathfrak{q})$, it suffices to consider the projection of

$$\{X \in \mathfrak{t} \oplus \mathbf{R}Y_{\varepsilon_1-\varepsilon_2} \oplus \mathbf{R}Z_{\varepsilon_1-\varepsilon_2}\colon [X, L_2] = 0\}. \tag{4.28}$$

Clearly, $\dim P'\mathfrak{z}(\mathfrak{q}) \leqslant 2$, and we get a contradiction by explicitly writing down the equations for two elements in (4.28) to project into $Y_{\varepsilon_1-\varepsilon_2}$ and $Z_{\varepsilon_1-\varepsilon_2}$. Thus $\dim P'\mathfrak{z}(\mathfrak{q}) \leqslant 1$. Further, $\dim \mathfrak{l}' = 8$ and $\dim P'\mathfrak{z}(\mathfrak{a}) \leqslant 4$. From (4.21) we have

$$\dim P'(\mathfrak{q}) + \dim P'(\mathfrak{z}(\mathfrak{q})) + \dim P'(\mathfrak{z}(\mathfrak{a})) \geqslant \dim \mathfrak{l}',$$

which is impossible. Summarizing, we have

Proposition 4.29. *If M is a quaternionic projective space and N is a hypersurface which is the orbit of a closed connected group of isometries of M, then N is amenable and is given by Proposition* 4.3 *or* 4.6, *where $M \neq S^4$ is assumed.*

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