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GEODESIC SPHERES AS GENERATORS OF THE HOMOTOPY GROUPS OF O, BO

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Introduction

The purpose of this paper is to show that the generators of the homotopy groups of the infinite orthogonal group O and its classifying space BO can be realized by isometric embeddings of standard euclidean spheres as totally geodesic submanifolds of finite orthogonal groups and Grassmann manifolds respectively.

This work was motivated by the observation, due to R. K. Lashof, that the existence of such representatives of homotopy classes would give a partial answer to a problem proposed by Cheeger and Gromoll in [6]. Specifically, we prove that if one adds a large enough trivial bundle to any vector bundle over any euclidean sphere, the total space of the Whitney sum will admit a complete riemannian metric of nonnegative sectional curvature.

The precise statement of the theorem and the proof are in § 8. The nonstable case is still open as far as we know, and some related questions have been discussed in [2], [7], [11], [13], [15] and [16].

The contents of this paper are essentially the author's doctoral dissertation at the University of Chicago in 1974. The author wishes to thank very sincerely his thesis advisor Professor R. K. Lashof, for his invaluable help and the University of Chicago for their support.

1. Summary

In dimensions $n = 0, 1, 3$ and 7 the generators of the homotopy groups $\pi_n(O)$ are embedded as totally geodesic submanifolds isometric to spheres of constant positive sectional curvature in small orthogonal groups. For simplicity we summarize only the case $n = 3$, all other cases being similar.

If we think of S^3 as the set of all unit quaternions, the natural inclusion of S^3 in $SO(4)$ followed by the inclusion of $SO(4)$ in $O = \varinjlim O(n)$ provides a generator of $\pi_3(O) = Z$ (see [17]).

To obtain the generators of $\pi_{8k+3}(O)$, $k = 1, 2, \cdots$ we work as follows:

According to Milnor's proof of the Bott periodicity theorem in [10, § 24] we

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may identify $O(4)$ with a certain submanifold of $O(64)$ denoted by $\Omega_8(64)$. This map is shown to be a homothety, i.e., an isometry up to a uniform scale change of the metric, and therefore maps the unit quaternions onto some totally geodesic S^3 in $\Omega_8(64)$. A certain space of paths of $\Omega_8(64)$, also defined in $O(64)$, is denoted by $\Omega_7(64)$. In fact, a chain of submanifolds of $O(64)$ is defined, each being a certain space of paths of the next satisfying

$$\Omega_8(64) \subset \Omega_7(64) \subset \cdots \Omega_1(64) \subset \Omega_0(64) = O(64) .$$

From S^3 in $\Omega_8(64)$ we define an inclusion of S^4 in $\Omega_7(64)$ through a suspension type construction, and then we replace it by another inclusion, homotopic to it, which is more suitable for the next suspension to some S^5 in $\Omega_6(64)$. Repeating this process six more times we end up with a homothetic embedding of the euclidean S^{11} as a totally geodesic submanifold of $O(64)$ which generates $\pi_{11}(O)$. Next we identify $O(64)$ with $\Omega_8(1024)$, and find S^{19} in $O(1024)$ generating $\pi_{19}(O)$.

The points which require proofs are the following:

(1) The homeomorphism

$$\lambda\colon \Omega_8(16r) \to O(r)$$

of [10] is actually a homothety.

(2) The suspension construction induces isomorphisms between the homotopy groups in question. This follows by direct application of a step in the original proof of the Bott periodicity theorem [3].

(3) The last sphere in $O(16r)$ is a totally geodesic submanifold isometric to some euclidean sphere of small radius.

These facts are first established in §§ 1, 2, 3 and then applied to prove the main result in §§ 4, 5, 6.

In [10] it is observed that $\Omega_7(16r)$ contains the grassmannian $G_{r,2r} = \dfrac{O(2r)}{O(r) \times O(r)}$ as its base point component, and therefore the above described method gives automatically the generators of the homotopy groups of BO as totally geodesic euclidean spheres isometrically embedded in grassmannians. We give an independent proof of this fact in § 7 for the following reason. The existence of isometric spheres in Grassmann manifolds was observed first by Y. C. Wong and is described in Wolf [19], without any relation to their homotopy classes. In § 7 we relate this work with [19].

1. A Homothety

In this section we recall the setting of [10, § 24] for the proof of the Bott periodicity theorem for the orthogonal group, and show that a certain homeomorphism $\lambda\colon \Omega_8(16r) \to O(r)$ is actually a homothety between these two spaces.

Let $\boldsymbol{R}^m$ be the m-dimensional euclidean space, and $O(m)$ its orthogonal group with the standard bi-invariant metric induced by the inner product $\langle A, B\rangle =$ trace AB^t, where B^t is the transpose of the matrix B, and A, B are in the Lie algebra $o(m)$ of $O(m)$ consisting of the skew-symmetric matrices. Let $\Omega_1(m) = o(m) \cap O(m)$ be the set of all matrices J in $O(m)$ such that $J^2 = -I$. Such a matrix is called a *complex structure* on $\boldsymbol{R}^m$. Two complex structures J and K are said to *anticommute* if $JK + KJ = 0$. Given fixed anticommuting complex structures $J_1, \cdots, J_{k-1}$ on $\boldsymbol{R}^m$, let $\Omega_k(m)$ be the set of all complex structures J which anticommute with $J_1, J_2, \cdots, J_{k-1}$. Note that $\Omega_k(m) \subset \Omega_{k-1}(m) \subset \cdots \subset \Omega_1(m) \subset O(m) \equiv \Omega_0(m)$.

The following lemma is proved in [10].

Lemma 1. *Each $\Omega_k(m)$ is a totally geodesic submanifold of $O(m)$. The space of manimal geodesics from J_p to $-J_p$ in $\Omega_p(m)$ is homeomorphic to $\Omega_{p+1}(m)$ for $0 \leq p < k$.*

Let $m = 16r$, $r \geq 1$, and observe that by Lemma 1, $\Omega_k(16r)$ must be a symmetric subspace G'/H' of $O(16r)$ considered as $(H \times H)/H$, where $H = O(16r)$ (see [12, Vol. II, p. 235]). In fact, if all groups below are considered with their standard bi-invariant metrices $\Omega_1(16r)$ is homothetic to $O(16r)/U(8r)$, where $U(8r)$ consists of all A in $O(16r)$ which commute with a particular J_1 in $\Omega_1(16r)$. Similarly, $\Omega_2(16r)$ is homothetic to $U(8r)/\mathrm{Sp}\,(4r)$; $\Omega_3(16r)$ has components of various dimensions, and we choose J_3 in the component which is homothetic to $G_{2r}(\boldsymbol{H}^{4r}) = \mathrm{Sp}\,(4r)/\mathrm{Sp}\,(2r) \times \mathrm{Sp}\,(2r)$ the Grassmann manifold of quaternionic $2r$-planes in the quaternionic $4r$-dimensional space $\boldsymbol{H}^{4r}$. Then $\Omega_4(16r)$ is homothetic to $\mathrm{Sp}\,(2r)$, $\Omega_4(8r)$ to $\mathrm{Sp}\,(r)$, $\Omega_5(16r)$ to $\mathrm{Sp}\,(2r)/U(2r)$, and $\Omega_6(16r)$ to $U(2r)/O(2r)$. The largest dimensional component of $\Omega_7(16r)$ is homothetic to $G_r(\boldsymbol{R}^{2r}) \equiv G_{r,2r} = O(2r)/O(r) \times O(r)$. Choosing J_7 in this component, $\Omega_8(16r)$ becomes homothetic to $O(r)$.

Now let us recall from [10] the definition of the homothety $\lambda\colon \Omega_8(16r) \to O(r)$.

The anticommuting complex structures J_1, J_2, J_3 determine a splitting of $\boldsymbol{R}^{16r} \equiv \boldsymbol{H}^{4r}$ as a direct sum $V_1 \oplus V_2$, where V_1 is the $+1$ eigenspace of $J_1J_2J_3$ on $\boldsymbol{R}^{16r}$, and V_2 is the -1 eigenspace. By the choice of J_3 the quaternionic dimension of V_1 and also of V_2 is $2r$; $V_1 = W \oplus J_2W$, where W is the $+1$ eigenspace of $J_1J_4J_5$ on V_1, and the real dimension of W is $4r$. Moreover, $W = X \oplus J_1X$, where X is the $+1$ eigenspace of $J_2J_4J_6$ on W, and the real dimension of X is $2r$. Finally X_1 and X_2 are the -1 eigenspaces of the restriction of $J_1J_6J_7$ on X, and the real dimension of each is r.

Now given an isometry $T\colon X_1 \to X_2$ we first define

(1) $J|_{X_1} = -J_7T|_{X_1}$,

(2) $J|_{X_2} = J_7T^{-1}|_{X_2}$,

and then define J on the rest of the summands of $\boldsymbol{R}^{16r}$ described above so that it anticommutes with each of the $J_1, \cdots, J_6$ as follows: (1) and (2) determine J on X,

(3) $J(J_1x) = -J_1(Jx)$ for all x in X determines J on W,

(4) $J(J_2w) = -J_2(Jw)$ for all w in W determines J on V_1, and

(5) $J(J_3J_4v_1) = J_3J_4(Jv_1)$ for v_1 in V_1 determines J on $V_2 = J_3J_4(V_1)$.

It follows that J is in $\Omega_8(16r)$.

Let ϕ: Isom $(X_1, X_2) \to \Omega_8(16r)$ be $\phi(T) = J$ determined by (1), $\cdots$, (5). The set theoretic inverse of ϕ is ψ with

$$\psi(J) = (J_7J)|_{X_1}: X_1 \to X_2 .$$

The proof of the following is also intended to fix notation.

Lemma 2. *There is a homothety λ: $\Omega_8(16r) \to O(r)$.*

Proof. Fix J_8 in $\Omega_8(16r)$, and observe that $(J_7J_8)|_{X_1}$ is an isometry between X_1 and X_2. Therefore

$$(-J_8J_7)|_{X_2}: X_2 \to X_1 = -((J_7J_8)|_{X_1}: X_1 \to X_2)^{-1}$$

is also an isometry, and the composite $(J_8J_7)|_{X_2} \circ T \equiv c(T)$ defines a bijection c: Isom $(X_1, X_2) \to$ Isom $(X_1, X_1) = O(r)$.

Now let $\lambda = c \circ \psi$: $\Omega_8(16r) \to O(r)$. Obviously, λ is a smooth homeomorphism, $\lambda(J) = -(J_8J)|_{X_1}: X_1 \to X_1$, and $\lambda(J_8) = \mathrm{id}\,(X_1)$. The inverse λ^{-1} is also smooth since by the definition of J from a given T in $O(r)$, λ^{-1} consists of successive embeddings

$$O(r) \to O(2r) \to O(4r) \to O(8r) \to O(16r)$$

with the submanifold $\Omega_8(16r)$ as the image.

Using Assertions 1 and 2 [10, p. 137] one can now give a routine proof that this particular λ is homothetic, when $O(16r)$ and $O(r)$ are considered with their standard metrics.

Remark 3. The homothety λ of Lemma 2 as well as λ^{-1} determine a one-to-one correspondence between minimal geodesics in $\Omega_8(16r)$ and minimal geodesics in $O(r)$ mapping midpoint to midpoint. Therefore they map anticommuting complex structures to anticommuting complex structures. Indeed, if J and K are auticommuting complex structures, the geodesic $J \exp(tKJ)$ is minimal in the orthogonal group, between J and $-J$ with midpoint K.

This implies in particular the following.

Corollary 4. *For all $k \geq 0$ such that $\Omega_k(r)$ is not empty, λ^{-1} restricts to a homothety between $\Omega_k(r)$ and $\Omega_{8+k}(16r)$.*

2. A suspension

In this section we describe a morphism ϕ between the homotopy groups $\pi_j\Omega(n)$ and $\pi_{j+1}\Omega_{k-1}(n)$ induced by a naturally defined suspension of $\Omega_k(n)$ into $\Omega_{k-1}(n)$.

Consider $\Omega_{k-1}(n)$ in $O(n)$ defined as in § 1, and let J_{k-1} in $\Omega_{k-1}(n)$ define $\Omega_k(n)$. For the remainder of this section we write Ω_k, etc. in place of $\Omega_k(n)$, etc., since n is fixed. Denote by $\Lambda\Omega_{k-1}$ the space of all continuous paths in Ω_{k-1} starting at $-J_{k-1}$ with the compact—open topology, and by $\Omega\Omega_{k-1}$ the set of all elements of $\Lambda\Omega_k$ ending at J_{k-1}. Let $p\colon \Lambda\Omega_{k-1} \to \Omega_{k-1}$ be the end point map defining a fibration with fibre $\Omega\Omega_{k-1}$, where the base point of $\Omega\Omega_{k-1}$ as well as $\Lambda\Omega_{k-1}$ is the minimal geodesic $\gamma_k(t) = -J_{k-1}\exp(tJ_{k-1}J_k)$, $0 \le t \le \pi$, of Ω_{k-1} between $-J_{k-1}$ and J_k, while the base point of Ω_{k-1} is J_{k-1}.

Following [10, Lemma 24.4] there is a one-to-one map $i\colon \Omega_k \to \Omega\Omega_{k-1}$ defined as follows. For each J in Ω_k let $i(J)$ be the minimal geodesic in Ω_{k-1} from $-J_k$ to J_k with midpoint J, i.e., $-J_{k-1}\exp(tJ_{k-1}J)$, $0 \le t \le \pi$. By compactness of Ω_k, i is a homeomorphism onto its image.

The space $\Lambda\Omega_{k-1}$ is contractible, and so there is an isomorphism $\phi\colon \pi_r\Omega_k \to \pi_{r+1}\Omega_{k-1}$ for all $r \ge 0$. We want to give this ϕ an explicit form on the representatives of homotopy classes.

If D^{r+1} is the closed disc of radius π in $\boldsymbol{R}^{r+1}$, we write each nonzero element in D^{r+1} as tx for $0 < t \le \pi$ and x in the unit sphere S^r. Given a base point preserving map $a\colon (S^r, x_0) \to (\Omega_k, J_k)$, we define $a_1\colon (D^{r+1}, \partial D^{r+1}) \to (\Omega_{k-1}, J_k)$ by $a_1(tx) = -J_{k-1}\exp(tJ_{k-1}a(x))$, $0 \le t \le \pi$, where exp is the usual exponential map of $O(n)$. So $a_1(0) = -J_{k-1}$, $a_1(\pi x/2) = a(x)$ and $a_1(\pi x) = J_{k-1}$ for all x in S^r. In particular $a_1(\pi x_0/2) = J_k$, and if a is homotopic to b in Ω_k with J_k fixed throughout the homotopy, then a_1 is homotopic to b_1 in Ω_{k-1} with γ_k fixed throughout the homotopy. So we may define a map $\phi\colon \pi_r\Omega_k \to \pi_{r+1}\Omega_{k-1}$ by $\phi[a] = [a_1]$ and $\phi[0] = [0]$.

Lemma 5. *If ∂ denotes the boundary homomorphism of the fibration p defined above, and $i_*\colon \pi_r\Omega_k \to \pi_r\Omega\Omega_{k-1}$ is induced by i, then $\partial \circ \phi = i_*$ for all $r \ge 0$.*

Proof. As each $a_1(tx)$ is a minimal geodesic from $-J_{k-1}$ to J_{k-1}, a_1 maps the interior of D^{r+1} into the interior of the injectivity ball of $-J_{k-1}$ in Ω_{k-1}, and maps ∂D^{r+1} on J_{k-1}, so that ∂D^{r+1} lies on the cut locus of $-J_{k-1}$. We can lift to $\Lambda\Omega_{k-1}$ the restriction of a_1 on int (D^{r+1}) by mapping each tx, $0 \le t < \pi$, to the unique minimal geodesic in Ω_{k-1}, which joins $-J_{k-1}$ to $a_1(tx)$. Explicitly, let $a_2(tx) = -J_{k-1}\exp(sJ_{k-1}a(x))$, $0 \le s \le t$.

In particular $a_2(\pi x_0/2) = \gamma_k|[0, \pi/2]$. Now we can lift the boundary points πx uniquely by continuity. For example, $a_2(\pi x_0/2) = \gamma_k$ and so we have defined

$$a_2\colon (D^{r+1}, \partial D^{r+1}, \pi x_0) \to (\Lambda\Omega_{k-1}, \Omega\Omega_{k-1}, \gamma_k)\,,$$

such that $p \circ a_2 = a_1$. Letting a_3 be the restiction of a_2 on $(\partial D^{r+1}, \pi x_0)$ we have that $\partial \circ \phi[a] = [a_3]$ in $\pi_r\Omega\Omega_{k-1}$. Identifying (S^r, x_0) with $(\partial D^{r+1}, \pi x_0)$ by $x \mapsto \pi x$, we observe that $a_3 = i \circ a$. In particular $\partial \circ \phi \circ [a] = i_*[a]$.

Corollary 6. *The map ϕ is a group morphism, and if for some $r \ge 0$, i_* is an isomorphism, then so is ϕ for the same r.*

Proof. Immediate, since ∂ is an isomorphism.

3. Geodesic spheres in orthogonal groups

In this section we give the structure of some totally geodesic spheres S^n in the orthogonal groups $O(r)$, and later we shall show that the structures carry the homotopy of O. Our Proposition 10 is similar to one in Wolf [19], and the proof is included here for completeness.

Let S^n be the unit sphere in the euclidean space $\boldsymbol{R}^{n+1}$ with elements $x = (x_0, \cdots, x_n)$, and let $K_0, \cdots, K_n$ be anticommuting complex structures in $O(r)$. The map $f(x) = x_0K_0 + x_1K_1 + \cdots + x_nK_n$ defines an inclusion of S^n into $\Omega_1(r)$ as it follows from the relations

$$(\Sigma_0^n x_iK_i)^t = -\Sigma_0^n x_iK_i \, , \qquad (\Sigma_0^n x_iK_i)^2 = -I \, .$$

Lemma 7. *The image $f(S^n)$, denoted by Σ^n, as a set of points, is a union of minimal geodesics of $O(r)$ between two points of the type A and $-A$. The midpoints of these geodesics form an equator Σ^{n-1}, which is also the union of a set of minimal geodesics between any two of its antipodal points. Continuing this way we end up in a closed smooth geodesic, which is minimal between any two of its antipodal points and has equator $\Sigma^0 = \{E, -E\}$, where E is in $\Omega_1(r)$.*

Remark 8. According to Lemma 7, Σ^n may be constructed by successive suspensions. Starting from Σ^0 we obtain Σ^1 then Σ^2, etc., the great circles being always smooth closed geodesics and minimal between the two vertices of each suspension. The vertices of the different suspensions are anticommuting complex structures as it follows from the fact that each Ω_i is the set of midpoints of minimal geodesics in Ω_{i-1}.

Proof of Lemma 7. For $x(x_0, \cdots, x_n)$ in S^n let t in $[0, \pi]$ be such that $\cos t = x_0$. Then $\sin t = (1 - x_0^2)^{1/2} = (x_1^2 + \cdots + x_n^2)^{1/2}$ and $f(x) = K_0(I \cos t + J \sin t)$, where

$$J = s_1\Lambda_2 + \cdots + s_n\Lambda_n \, , \quad s_i = x_i(x_1^2 + \cdots + x_n^2)^{-1/2} \, , \quad \Lambda_i = -K_0K_i \, .$$

Moreover, $s_1^2 + \cdots + s_n^2 = 1$, and $\Lambda_1, \cdots, \Lambda_n$ are anticommuting complex structures which also anticommute with K_0. Therefore J lies in $\Omega_1(r)$ and

$$f(x) = K_0 \exp(tJ) \, ,$$

i.e., $f(x)$ is a point on the minimal geodesic of $O(r)$ between K_0 and $-K_0$ which has midpoint

$$K_0J = s_1K_1 + \cdots + s_nK_n \, .$$

Since the midpoints are of the same form as the elements of Σ^n, the lemma follows by repeating the process a finite number of times.

Remark 9. Obviously, the proof and conclusion of Lemma 7 remain true if K_0 is replaced by I, i.e., if

$$f(x) = x_0 I + x_1 K_1 + \cdots + x_n K_n ,$$
$$x_0^2 + x_1^2 + \cdots + x_n^2 = 1 ,$$

and $K_1, \cdots, K_n$ are anticommuting complex structures in $O(r)$.

Proposition 10. *The image of f as in Remark 9, denoted by Σ^n, is a totally geodesic submanifold of $O(r)$ isometric to an euclidean sphere.*

Proof. Let $o(r)$ have the following conveniently normalized metric $\langle A, B\rangle = -\dfrac{1}{r} \operatorname{tr} AB$, and notice that $K_1, \cdots, K_n$ form an orthonormal basis of an n-dimensional subspace V^n of $o(r)$. As in Lemma 7 and Remark 9, Σ^n is the image of the closed disc D^n of radius π in V^n through the usual exponential map of $O(r)$. Any element of D^n may be written as tJ, where $0 \leq t \leq \pi$, J is a complex structure of the form $\Sigma_1^n s_i K_i$, $s_i = x_i(x_1^2 + \cdots + x_n^2)^{-1/2}$, and $s_1^2 + \cdots + s_n^2 = 1$, i.e., J is in the unit sphere S^{n-1} of V^n.

Any element of Σ^n may be written as $\exp(tJ) = I \cos t + J \sin t$, and since exp is periodic of period 2π when restricted to V^n, we have that $\Sigma^n = \exp(V^n)$.

Following [8, Chapter IV, § 7] we see that Σ^n is a totally geodesic submanifold of $O(r)$, and is therefore a symmetric subspace if and only if V^n is a Lie triple system, i.e., if and only if the Lie bracket $[[X, Y], Z]$ is also in V^n for X, Y and Z in V^n. Because of the following relations, we can directly verify that V^n is a Lie triple system.

If $L_{ijk} = [[K_i, K_j], K_k]$, then for distinct i, j, k we have

$$L_{iik} = 0 , \quad L_{ijj} = 4K_i , \quad L_{iji} = 0 , \quad L_{ijk} = 0 .$$

Now the sectional curvature K of a 2-dimensional plane tangent at the identity I, spanned by K_i and K_j for $i \neq j$, is

$$K = \frac{1}{4} \|[K_i, K_j]\|^2 = \|K_i K_j\|^2 = -\frac{1}{r} \operatorname{tr} (K_i K_j)^2 = \frac{1}{r} \operatorname{tr} I = 1 .$$

If R denotes the curvature tensor, then $R(X, Y)Z = [[X, Y], Z]$ in $O(r)$. The identity $\langle [[X, Y], Z], W\rangle = \langle [X, Y], [Z, W]\rangle$ and the values of L_{ijk} above imply that $\langle R(K_i, K_j)K_k, K_l\rangle$ is zero unless $k = i$ and $l = j$ (or $k = j$ and $l = i$), in which case $\langle R(K_i, K_j)K_i, K_j\rangle = 1$ (we are following the notation of [10] for the curvature tensor and sectional curvature). Since $K_1, \cdots, K_n$ form on orthonormal basis of V^n, the curvature of any plane section of V^n is equal to 1, so that the symmetric space Σ^n is of constant sectional curvature 1. From $\Sigma^n = \exp(D^n)$, where exp is injective on int D^n and collapses ∂D^n to $-I$, it follows that Σ^n is homeomorphic to the n-dimensional sphere. So Σ^n has to be isometric to the euclidean S^n since a connected riemannian manifold of constant curvature is determined up to isometry by its fundamental group (see for example Wolf [20, Cor. 2.7.2]).

Observe that if $o(r)$ has the metric $\langle A, B\rangle = -\operatorname{tr} AB$, then Σ^n is isometric to the euclidean sphere of radius $1/\sqrt{r}$.

Corollary 11. *If $K_1, \cdots, K_n$ are anticommuting complex structures in $O(r)$, then the set of all $x_1K_1 + \cdots + x_nK_n$ in $O(r)$ with $x_1^2 + \cdots + x_n^2 = 1$ is a totally geodesic submanifold of $O(r)$ isometric to the euclidean $(n-1)$-dimensional sphere of radius $1/\sqrt{r}$.*

Proof. Notice that left multiplication by K_1 is an isometry of $O(r)$, and apply Proposition 9. q.e.d.

We may denote now by S^n the sphere described in Remark 9, and consider its image $\tilde{S}^n$ in $\Omega_8(16r)$ through the homothety $\tilde{\lambda} = \lambda^{-1}$. Obviously $\tilde{S}^n$ is a totally geodesic submanifold of $\Omega_8(16r)$ isometric to some euclidean sphere of positive radius.

We want to show that $\tilde{S}^n$ is of the same form as S^n.

Lemma 12. *There exist anticommuting complex structures $A_1, \cdots, A_{n+1}$ in $\Omega_8(16r)$, such that $\tilde{S}^n$ is the set of all elements of $\Omega_8(16r)$ of the form*

$$a_1A_1 + \cdots + a_{n+1}A_{n+1}\ ,$$

where $a_1^2 + \cdots + a_{n+1}^2 = 1$.

Proof. According to Remark 3, $\tilde{\lambda}(I) = J_8$ and $\tilde{\lambda}(K_i) = J_{8+i}$, where $J_8, \cdots, J_{8+n}$ are anticommuting complex structures in $\Omega_8(16r)$ defining $\Omega_9(16r)$, $\Omega_{10}(16r)$, $\cdots, \Omega_{8+n+1}(16r)$. Our $\tilde{S}^n$ consists of minimal geodesics between J_8 and $-J_8$, and we can continue as in the statement of Lemma 7.

A minimal geodesic in $\Omega_8(16r)$ between J_8 and $-J_8$ has the form

$$g(t) = J_8 \exp(tB) = J_8 \cos t + J_8B \sin t\ ,$$

where B is a complex structure anticommuting with J_8, but commuting with the $J_1, \cdots, J_7$ which are used to define $\Omega_2(16r), \cdots, \Omega_8(16r)$. The midpoint of $g(t)$ is J_8B, and according to the above it may be written as

$$J_8B = J_9 \cos\theta + J_9C \sin\theta$$

for some θ in $[0, \pi]$ and some complex structure C of $O(16r)$ which anticommutes with J_9 and commutes with $J_1, \cdots, J_8$. Now $g(t)$ above, becomes

$$g(t) = aJ_8 + bJ_9 + cJ_9C,\ a^2 + b^2 + c^2 = 1\ .$$

But J_9C is the midpoint of a minimal geodesic between J_{10} and $-J_{10}$, etc. Repeating the process a finite number of times we complete the proof, where $A_1 = J_8, \cdots, A_{n+1} = J_{8+n}$. q.e.d.

Now from S^n in $\Omega_k(m)$ as in Lemma 12 we construct S^{n+1} in $\Omega_{k-1}(m)$ through the suspension ϕ of § 2. Since $\phi(A)$ is the minimal geodesic

$$-J_{k-1} \exp(tJ_{k-1}A)\ , \qquad 0 \leq t \leq \pi$$

between J_{k-1} and $-J_{k-1}$ with midpoint $J_{k-1}A$, we employ the same method as in the proof of Lemma 12 to show that S^{n+1} may be written as the set of all sums $s_1L_1 + \cdots + s_{n+2}L_{n+2}$, where $s_1^2 + \cdots + s_{n+2}^2 = 1$ and $L_1, \cdots, L_{n+2}$ are anticommuting complex structures in $\Omega_{k-1}(m)$. If $k = 1$, s^{n+1} will be of the form $x_0I + x_1L_1 + \cdots + x_{n+1}L_{n+1}$.

Thus we have proved

Corollary 13. *If S^n in $\Omega_k(m)$, $k > 0$, is of the form described in Corollary* 11, *then λ^{-1} and ϕ preserve that form.*

In conclusion, given S^n in $O(r)$ of the form described in Remark 9, we may construct S^{n+8} in $O(16r)$ of the same form by first mapping the S^n into $\Omega_8(16r)$ through λ^{-1}, then successively applying ϕ to the image and using corollary 13 in each step.

The lemma proved below will enable us to use the Bott periodicity theorem [3], [10] in the proof of the main result, while we retain the form of the homotopy generators described above.

Let $\Omega_2(n), \cdots, \Omega_k(n)$ in $\Omega_1(n)$ be defined through the anticommuting complex structures $J_1, \cdots, J_{k-1}$ and fix J_k in $\Omega_k(n)$, which is assumed nonempty. The matrices

$$\tilde{J}_i = \begin{pmatrix} J_i & 0 \\ 0 & J_i \end{pmatrix}, \qquad i = 1, \cdots, k$$

are the anticommuting complex structures in $\Omega_1(2n)$ which define $\Omega_2(2n), \cdots,$ $\Omega_k(2n)$, and $\tilde{J}_k$ is in $\Omega_k(2n)$. Let $i_r\colon \Omega_r(n) \to \Omega_r(2n)$ be the following inclusion:

$$i_r(A) = \begin{pmatrix} A & 0 \\ 0 & -J_r \end{pmatrix}$$

where $r = 0, \cdots, k$ and $J_0 = I$. This is easily seen to be well defined.

If S^q is a totally geodesic sphere in $\Omega_k(n)$ of the form $r_1A_1 + \cdots + r_{q+1}A_{q+1}$, etc., and $S^{q+1} = \phi(S^q)$ is the sphere of the same form in $\Omega_{k-1}(n)$ obtained through the suspension construction, we consider the following:

$$i_k(S^q) = \Sigma^q \text{ in } \Omega_k(2n)\,, \quad i_{k-1}(S^{q+1}) = \tilde{S}^{q+1}, \phi(\Sigma^q) = \Sigma^{q+1}\,, \quad \text{in } \Omega_{k-1}(2n)\,.$$

We have

Lemma 14. *The spheres $\tilde{S}^{q+1}$ and Σ^{q+1} belong to the same based homotopy class in $\pi_{q+1}\Omega_{k-1}(2n)$.*

Proof. Choose a continuous real valued function $h(s, t)$ for $0 \leq t \leq 1$ and $0 \leq t \leq \pi$, such that $h(0, t) = t$, $h(1, t) = 0$ and $h(s, 0) = 0$, for all s and t. Let

$$F\colon S^{q+1} \times [0, 1] \to \Omega_{k-1}(2n)$$

be defined by

$$F((A,t),s) = \begin{pmatrix} -J_{k-1}\exp(tJ_{k-1}A) & 0 \\ 0 & -J_{k-1}\exp(-h(s,t)J_{k-1}J_k) \end{pmatrix},$$

where the elements of S^{q+1} are written in the form resulting from the construction of ϕ. One can verify that F is a well defined homotopy with $\Sigma^{q+1} = F(S^{q+1} \times 0)$ and $\tilde{S}^{q+1} = F(S^{q+1} \times 1)$. Moreover F is base point preserving since $h(s, 0) = 0$.

Corollary 15. *Let Σ^{q+2} and $\tilde{S}^{q+2}$ in $\Omega_{k-2}(2n)$ be the respective images under ϕ of Σ^{q+1} and $\tilde{S}^{q+1}$ defined above. Then $[\Sigma^{q+2}] = [\tilde{S}^{q+2}]$ in $\pi_{q+2}\Omega_{k-1}(2n)$.*

Proof. Using the same notation for the construction ϕ and the induced morphism $\phi\colon \pi_{q+1}\Omega_{k-1}(2n) \to \pi_{q+2}\Omega_{k-2}(2n)$ we have

$$[\Sigma^{q+2}] = \phi[\Sigma^{q+1}] = \phi[\tilde{S}^{q+1}] = [\tilde{S}^{q+2}].$$

4. Generators of $\pi_{8k+3}(O)$

We begin by considering the homothety $\lambda\colon \Omega_8(64) \cong O(4)$ and the $J_1, \cdots, J_7$ in $O(64)$ defining $\Omega_i(64)$ for $i = 1, \cdots, 8$ as in § 1. We now fix J_8 in $\Omega_8(64)$ and define

$$\Omega_8(128) \subset \Omega_7(128) \subset \cdots \subset O(128)$$

through the anticommuting complex structures $\tilde{J}_i = \begin{pmatrix} J_i & 0 \\ 0 & J_i \end{pmatrix}$, $i = 1, \cdots, 7$. Then we consider the inclusions $i_k\colon \Omega_k(64) \to \Omega_k(128)$ with $i_k(A) = \begin{pmatrix} A & 0 \\ 0 & -J_k \end{pmatrix}$ for $k = 1, \cdots, 8$, and observe that i_8 viewed as an inclusion of $O(4)$ in $O(8) \cong \Omega_8(128)$ maps A to $\begin{pmatrix} A & 0 \\ 0 & -I \end{pmatrix}$.

Recall that the group of unit quaternions $S^3 = \mathrm{Sp}(1)$ is a subgroup of $SO(4)$ as follows.

Let $\sigma\colon S^3 \to SO(4)$ be defined by $\sigma(q)(r) = qr$, where the product on the right hand side is quaternionic multiplication of r in $\boldsymbol{R}^4 = \boldsymbol{H}$ by the unit quaternion q (see [17]). For example the standard complex units i, j, k in S^3 are mapped to the following anticommuting complex structures in $SO(4)$:

$$\sigma(i) = J_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\sigma(j) = J_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

$$\sigma(k) = J_3 = J_1 J_2 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

and also $\sigma(r_0 + r_1 i + r_2 j + r_3 k) = r_0 I + r_1 J_1 + r_2 J_2 + r_3 J_3$ for $r_0^2 + r_1^2 + r_2^2 + r_3^2 = 1$.

It follows from the results of the previous paragraph that $\sigma(S^3)$ is homothetic to S^3 and equal to $\exp(V^3)$, where $V^3 = \operatorname{span}\{J_1, J_2, J_3\}$ in $o(4)$.

It is known [17, § 23.6] that $\pi_3 O(4) = Z \oplus Z$ and the homotopy class of the above described σ is $(1, 0)$ with image $i_*[\sigma] = 1$ in $Z = \pi_3 O(8) = \pi_3 O$, where $i\colon O(4) \to O(8)$ is the standard inclusion. So, if Σ^3 denotes the image $i_8 \circ \lambda(S^3)$ in $\Omega_8(128)$, then $[\Sigma]^3$ is a generator of $\pi_3 \Omega_8(128) = Z$.

We now apply the suspension σ of § 2 to Σ^3 and then to $\sigma(\Sigma^3) = \Sigma^4$, etc.:

$$\begin{array}{ll} \Sigma^3 & \text{in } \Omega_8(128) \\ \downarrow{\scriptstyle\phi} & \\ \Sigma^4 & \text{in } \Omega_7(128) \\ \downarrow & \\ \vdots & \\ \Sigma^{11} & \text{in } \Omega_0(128) = O(128)\,. \end{array}$$

Each induced morphism $\phi\colon \pi_r \Omega_k \to \pi_{r+1} \Omega_{k-1}$ is an isomorphism according to Corollary 6, since $i_*\colon \pi_r \Omega_k \to \pi_r \Omega\Omega_{k-1}$ is an isomorphism as we verify below.

(1) $\pi_3 \Omega_8(128) \cong \pi_4 \Omega_7(128)$, [10, p. 146].

(2) $\Omega_6(128) \cong U(16)/O(16)$, and from [3] it follows that $\pi_k \Omega_7(128) \cong \pi_{k+1} \Omega_6(128)$, $0 \leq k \leq 7$, which we need for $k = 4$.

(3) $\Omega_5(128) = \mathrm{Sp}(16)/U(16)$, and [3] implies $\pi_k \Omega_6(128) \cong \pi_{k+1} \Omega_5(128)$, $0 < k \leq 15$, which we need for $k = 5$.

(4) $\Omega_4(128) \cong \mathrm{Sp}(16)$ and $\pi_k \Omega_5(128) \cong \pi_{k+1} \Omega_4(128)$, $0 < k \leq 32$, which we need for $k = 6$.

(5) $\pi_7 \Omega_4(128) \cong \pi_8 \Omega_3(128)$, [10], where $\Omega_3(128) \cong \mathrm{Sp}(32)/\mathrm{Sp}(16) \times \mathrm{Sp}(16)$.

(6) $\Omega_2(128) \cong U(64)/\mathrm{Sp}(32)$, and from [3] it follows that $\pi_k \Omega_3(128) \cong \pi_{k+1} \Omega_2(128)$, $0 < k \leq 63$, which we need for $k = 8$.

(7) $\Omega_1(128) \cong O(128)/U(64)$, and [3] implies that $\pi_k \Omega_2(128) \cong \pi_{k+1} \Omega_1(128)$ $0 < k \leq 124$, which we need for $k = 9$.

(8) $\pi_k \Omega_1(128) \cong \pi_{k+1} SO(128)$, $0 < k \leq 124$, which we need for $k = 10$.

Therefore Σ^{11} in $SO(128)$ generates $\pi_{11} SO(128) = \pi_{11} O = Z$.

Now we apply successive suspensions to the isometric sphere S^3 in $\Omega_8(64)$ to obtain a totally geodesic sphere S^{11} in $O(64)$ using Corollary 13 in each step.

Applying Lemma 14 and Corollary 15 in each step we conclude that $[i(S^{11})] = [\Sigma^{11}]$ in $\pi_{11}O(128)$, where $i\colon O(64) \to O(128)$ is the standard inclusion. Since i_* is an isomorphism of π_{11}'s, it follows that the totally geodesic isometric sphere S^{11} in $O(64)$ generates $\pi_{11}O$.

Now we apply the same process to $\tilde{S}^{11} = \lambda(S^{11})$ in $\Omega_8(1024)$ to obtain a generator S^{19} of $\pi_{19}O(1024) = \pi_{19}O$, as a totally geodesic submanifold of $O(1024)$, homothetic to an euclidean sphere. The verification of the corresponding steps (1)–(8) above reduces to checking the following.

If $\phi\colon \pi_q\Omega_k(n) \to \pi_{q+1}\Omega_{k-1}(n)$ is an isomorphism, then $\phi\colon \pi_{q+8}\Omega_k(16n) \to \pi_{q+9}\Omega_{k-1}(16n)$ is also an isomorphism for all $k = 1, \cdots, 8$, all q and all $n \geq 0$. This may be observed directly from the formulas of [3] and [10].

It is now obvious how by iteration of this process we may realize all generators of $\pi_{8k+3}O$ as totally geodesic euclidean spheres S^{8k+3} homothetically embedded in the orthogonal groups $O(16^{k+1} \times 4)$.

5. Generators of $\pi_{8k+7}O$

Although explicit generators of π_7O are known to homotopy theorists, we prefer to construct one using the suspension method from S^3 in $SO(4)$, since it seems convenient for the continuation of the process.

First recall that $\Omega_4(8k)$ is homothetic to $\mathrm{Sp}\,(k)$ for $k \geq 1$. For example $\mathrm{Sp}\,(1) = S^3$ is homothetic to $\Omega_4(8)$, and $\mathrm{Sp}\,(2)$ is homothetic to $\Omega_4(16)$. Next observe that there exist exactly seven anticommuting complex structures in $O(8)$ (left Cayley multiplication by each of the standard complex units in $\boldsymbol{R}^8$). This follows from the steps defining the J_k's in § 1, as there is just one choice for J_7, namely, $J_7|_X = J_6J_1|_X$ or $-J_6J_1|_X$, X being a one-dimensional subspace of $\boldsymbol{R}^8$. We apply now the suspension ϕ to $\Omega_4(8) = S^3$ to obtain S^4 in $\Omega_3(8)$, then S^5 in $\Omega_2(8)$, S^6 in $\Omega_1(8)$ and S^7 in $O(8)$. According to Corollary 13, $S^7 = \exp(V^7)$ where V^7 is the span of the $J_1, \cdots, J_7$ in $o(8)$, and this S^7 is a totally geodesic euclidan sphere of radius $1/\sqrt{7}$ in $SO(8)$.

Observe now that

$$\pi_7SO(8) \cong \pi_7(S^7) \oplus \pi_7SO(7) \cong Z \oplus Z$$

by the parallelizability of S^7, and we want to show that the S^7 in $SO(8)$ described above is a generator $(1, 0)$ in $Z \oplus Z$ which projects to a generator of $Z \cong \pi_7O$ through the map induced by the standard inclusion of $O(8)$ in $O(16)$.

First observe that $a \mapsto \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}$ is an inclusion of $\mathrm{Sp}\,(1)$ in $\mathrm{Sp}\,(2)$ which generates $\pi_3\,\mathrm{Sp}\,(2) = Z$ as follows from the fact that the homogenious quotient of the standard inclusion $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ is diffeomorphic to S^7.

The inclusion $a \mapsto \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}$ induces $i_4\colon \Omega_4(8) \mapsto \Omega_4(16)$, $i_4(A) = \begin{pmatrix} A & 0 \\ 0 & -J_4 \end{pmatrix}$

for some fixed element J_4 in $\Omega_4(8)$, as in the beginning of § 4. Now we suspend $i_4(S^3) = \Sigma^3$, which generates $\pi_3\Omega_4(16) = Z$ to a Σ^4 in $\Omega_3(16)$, etc., and finally to a Σ^7 in $O(16)$. Using Bott's theorem II [3] in each step, one may verify that ϕ induces isomorphisms of the homotopy groups $\pi_3\Omega_4(16) \cong \pi_4\Omega_3(16), \cdots, \pi_6\Omega_1(16) \cong \pi_7 O(16)$ and therefore $[\Sigma^7] = 1$ or -1 in $Z \cong \pi_7 O(16) \cong \pi_7 O$. Using Corollary 15 in each step we conclude that $i_*[S^7]$ generates $\pi_7 O(16)$, where S^7 is the sphere of radius $1/\sqrt{7}$ in $SO(8)$ described above.

The process of obtaining S^{15} in $O(128)$, as a totally geodesic submanifold isometric to a sphere and generating $\pi_{15}O$, is now familiar and is illustrated below:

$$
\begin{tikzcd}
O(8) \arrow[rrrrr, "i"] \arrow[dd, "\lambda"'] & & & & & O(16) \arrow[dd, "\lambda"] \\
& S^7 \arrow[rrr, mapsto, "i"] \arrow[d, "\cong"'] & & & \Sigma^7 \arrow[d, "\cong"] & \\
\Omega_8(128) \arrow[r, phantom, "\supset"] & S^7 \arrow[rrr, mapsto, "i_8"] \arrow[d, mapsto] & & & \Sigma^7 \arrow[r, phantom, "\subset"] \arrow[d, mapsto] & \Omega_8(256) \\
\Omega_7(128) \arrow[r, phantom, "\supset"] & S^8 \arrow[rr, mapsto, "i_7"] \arrow[d, mapsto] & & \tilde{\Sigma}^8 \arrow[r, phantom, "\sim"] & \Sigma^8 \arrow[r, phantom, "\subset"] \arrow[d, mapsto] & \Omega_7(256) \\
& \vdots & & & \vdots & \\
O(128) \arrow[r, phantom, "\supset"] & S^{15} \arrow[rr, mapsto, "i_0"] & & \tilde{\Sigma}^{15} \arrow[r, phantom, "\sim"] & \Sigma^{15} \arrow[r, phantom, "\subset"] & O(256)
\end{tikzcd}
$$

Here λ is the homothety of § 1, and the horizontal maps $i_8, \cdots, i_0$ are the inclusions introduced in § 3. The vertical arrows $S^i \mapsto S^{i+1}$ and $\Sigma^i \mapsto \Sigma^{i+1}$ indicate that S^{i+1}, Σ^{i+1} are obtained from S^i, Σ^i by the suspension ϕ. The Σ^i's are generators of $\pi_i\Omega_j(256)$ as follows from the application of Bott's theorem II [3] in each step, while the S^i's are totally geodesic euclidean spheres in $\Omega_j(128)$. The base-point preserving homotopies $\tilde{\Sigma}^i \sim \Sigma^i$ are given by Lemma 14.

Repeating the process one obtains generators of $\pi_{8k+7}O$ as S^{8k+7} homothetically embedded euclidean spheres in $O(16^k \times 8)$, as totally geodesic submanifolds for all $k \geq 0$.

6. Generators of $\pi_{8k+1}O$ and of $\pi_{8k}O$

It is known [17, § 22.8] that the standard inclusion of $SO(2) = S^1$ in $SO(4)$ provides a geodesic loop at I in $SO(4)$ which generates $\pi_1 SO(4) \cong \pi_1 O \cong Z_2$. To apply the method of the last two sections we first obtain $\tilde{J}_1, \cdots, \tilde{J}_7$ in $\Omega_1(64)$, define $\Omega_i(64)$ through them and then we embedd $\Omega_i(32)$ in $\Omega_i(64)$, $i = 0, \cdots, 8$ in the same way as in §§ 4 and 5.

Restricting to the connected component $\Omega_8'(32)$ of $\Omega_8(32)$, which is homothetic to $SO(2)$ through λ, we have the commutative

$$\begin{array}{ccc} \Omega_8'(32) & \xrightarrow{i} & \Omega_8'(64) \\ {\scriptstyle\lambda}\downarrow & & \downarrow \\ SO(2) & \longrightarrow & SO(4)\,. \end{array}$$

We apply ϕ successively to $\Omega_8'(32)$ and to $\Sigma^1 = i(\Omega_8'(32))$, the generator of $\pi_1\Omega_8(64) = \pi_1 O$. Using Corollary 13, Lemma 14 and Bott's theorem II we obtain a generator of $\pi_9 O$ as a totally geodesic euclidean sphere in $O(32)$. Repeating the process one realizes a generator of $\pi_{8k+1}O = Z_2$ as a homothetically embedded totally geodesic euclidean S^{8k+1} in $O(16^k \times 2)$.

In order to obtain the generators of $\pi_{8k+1}O$ we begin with the grassmannian manifold $G_{5,10}$ of 5-dimensional planes in $\boldsymbol{R}^{10}$, which is homothetic to $\Omega_7(80)$ and has $\pi_1 G_{5,10} = \pi_0 O(5) = Z_2$. From the version of Bott's theorem in [10] it follows that the inclusion $i\colon \Omega_8(80) \to \Omega\Omega_7(80)$ induces a bijection $i_*\colon \pi_0\Omega_8(80) \to \Omega_0\Omega\Omega_7(80)$, and therefore $\phi\colon \pi_0\Omega_8(80) \to \pi_1\Omega_7(80)$ is also a bijection.

We may construct now a smooth geodesic loop S^1 in $\Omega_7(80)$, generating the fundamental group as follows.

Let $\alpha_1\colon \{1, 0\} \to O(5)$ be such that $\alpha_1(0) = I$, $\alpha_1(1) = -I$, generating $\pi_0 O(5)$, and $\alpha_0 = \lambda^{-1} \circ \alpha_1$, where λ^{-1} is the homothety $O(5) \to \Omega_8(80)$. Now $\alpha_0(0) = J_8$, $\alpha_0(1) = -J_8$ and $[\alpha_0] = 1$ in $\pi_0\Omega_8(80) = Z_2$. We apply the suspension ϕ to α_0 to obtain $\alpha\colon S^1 \to \Omega_7(80)$, whose image consists of two minimal geodesics from J_7 to $-J_7$, determined by their midpoints J_8 and $-J_8$. Therefore their initial tangent vectors are collinear and have opposite orientations, and so are their final tangent vectors. Thus the image $\alpha(S^1)$ is $J_7 \exp(V^1)$, smooth geodesic loop at J_7, where V^1 is the 1-dimensional subspace of $o(80)$ generated by J_8, and the exponential is the usual one of $O(80)$.

We apply now iterated suspensions to $\alpha(S^1)$ to obtain homothetic euclidean spheres $\phi(\alpha) = S^2$ in $\Omega_6(80), \cdots, S^8$ in $O(80)$, each totally geodesic in the corresponding $\Omega_i(80)$ and $[S^8] = 1$ in $\pi_8 O(80) \cong Z_2$.

Repeating the process we obtain S^{8k} as totally geodesic euclidean spheres in $O(16^k \times 5)$ generating $\pi_{8k} O \cong Z_2$ for all $k \geq 0$.

7. Generators of $\pi_{q+1} BO$

For each S^q in $O(r)$ generator of $\pi_q O$ constructed in the previous sections we have $\lambda(S^q)$ in $\Omega_8(16r)$ and $\phi\lambda(S^q) = S^{q+1}$ in $\Omega_7(16r)$. Applying Lemma 14 one sees that $i_*[S^{q+1}]$ generates $\pi_{q+1}\Omega_7(32r)$ which is equal to $\pi_{q+1}BO$, where $\Omega_7(32r)$ is homothetic to the grassmannian $G_{2r,4r}$ and i is the inclusion. The results of § 2 assert that S^{q+1} is a totally geodesic submanifold of $\Omega_7(16r)$ homothetic to the euclidean sphere.

In this section we intend to indicate what amounts to the same construction described above but without direct use of the homothety between the grassmannians and the Ω_7's. Our reasons for doing so are that the present work began by considering *Y-C.* Wong's "Isoclinic spheres in grassmann manifolds" as they are described by Wolf in [19] without connection to homotopy.

Let S^q be the generator described at the beginning of this section. According to the preceeding results, $S^q = \exp(D^q)$, where D^q is the closed disc of radius π centered at the origin of $V^q = \operatorname{span}\{J_1, \cdots, J_q\}$ in $o(r)$, and $J_1, \cdots, J_q$ are anticommuting complex structures in $o(r)$. Now we define $\Lambda_i = \begin{pmatrix} 0 & J_i \\ J_i & 0 \end{pmatrix}$ for $i = 1, \cdots, q$ and $\Lambda_{q+1} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ in $o(2r)$, and let V^{q+1} be the subspace of $o(2r)$ generated by $\Lambda_1, \cdots, \Lambda_{q+1}$, and D_1^{q+1} the closed disc of radius $\pi/2$ in V^{q+1} centered at the origin.

Observe that the exponential of $O(2r)$ is a diffeomorphism when restricted to D^{q+1} and the image $D^{q+1} = \exp(D_1^{q+1})$ is a totally geodesic submanifold of $O(2r)$ with totally geodesic boundary equal to

$$(S^q)' \equiv \{s_1\Lambda_1 + \cdots + s_{q+1}\Lambda_{q+1} | s_1^2 + \cdots + s_{q+1}^2 = 1\} .$$

Now let $G_{r,2r} = O(2r)/O(r) \times O(r)$ be the homogeneous quotient, where the denominator is included in $O(2r)$ diagonally, and we denote the obvious projection by $p\colon O(2r) \to G_{r,2r}$. Let $B = p(I_{2r})$ and $B^\perp = p(\Lambda_{q+1})$ in $G_{r,2r}$. To justify the notation we notice that B and $B^\perp$ determine an orthogonal splitting of $\boldsymbol{R}^{2r} = \boldsymbol{R}_1^r \oplus \boldsymbol{R}_2^r = B \oplus B^\perp$.

One may prove now the following lemma using a short argument similar to the ones in § 3.

Lemma 16. (a) *The map p is a diffeomorphism when restricted to the interior of D^{q+1},*

(b) *the boundary $(S^q)'$ of D^{q+1} lies in $p^{-1}(B^\perp)$,*

(c) *$p(D^{q+1})$ is a totally geodesic submanifold of $G_{r,2r}$ in its homogeneous metric, which is homothetic to a euclidean sphere S^{q+1}.*

The proof is ommitted.

Consider now the based fibration

$$(p^{-1}(B^\perp); \Lambda_{q+1}) \to (O(2r); \Lambda_{q+1}) \to (G_{r,2r}; B^\perp) ,$$

and notice that by the above construction we have $\partial[S^{q+1}] = [(S^q)']$, where ∂ is the usual boundary homomorphism. All fibres of p are totally geodesic in $O(2r)$ and isometric to $O(r) \times O(r) = p^{-1}(B)$. An isometry from $(p^{-1}(B^\perp); \Lambda_{q+1})$ to $(O(r) \times O(r); I)$ is given by left multiplication by $-\Lambda_{q+1}$. (This corresponds to lifting horizontally the geodesic $p \circ \exp(-t\Lambda_{q+1})$ from each point of $p^{-1}(B^\perp)$). The induced isomorphism between based homotopy groups sends $[(S^q)']$ to $[\bar{S}^q]$ in $\pi_q(O(r) \times O(r); I_{2r})$, where

$$\bar{S}^q = \left\{ s_0 I + \Sigma_1^q s_i \begin{pmatrix} -J_i & 0 \\ 0 & J_i \end{pmatrix} \middle| s_0^2 + \cdots + s_q^2 = 1 \right\}.$$

Lemma 17. *If p_2: $O(r) \times O(r) \to O(r)$ is the projection to the second factor and i': $O(r) \to O$ the usual inclusion then $i'_* p_*[\bar{S}^q]$ generates $\pi_q O$.*

Proof. Notice that $p_2(\bar{S}^q) = S^q$, the generator we started out in this section. q.e.d.

We consider now the following maps of fibrations and the induced maps on their homotopy sequence diagrams:

$$\begin{tikzcd}
O(r) \times O(r) \arrow[r, "p_2"] \arrow[d] & O(r) \arrow[r, "i'"] \arrow[d] & O \arrow[d] \\
O(2r) \arrow[r] \arrow[d] & O(2r)/O(r) \arrow[r] \arrow[d] & EO \arrow[d] \\
G_{r,2r} \arrow[r, phantom, "="] & G_{r,2r} \arrow[r, "i"] & BO
\end{tikzcd}$$

$$\begin{tikzcd}
\pi_{q+1} G_{r,2r} \arrow[r, "1"] \arrow[d, "\partial"] & \pi_{q+1} G_{r,2r} \arrow[r, "i_*"] \arrow[d] & \pi_{q+1} BO \arrow[d, "\partial", "\cong"'] \\
\pi_q(O(r) \times O(r)) \arrow[r, "p_{2*}"] & \pi_q O(r) \arrow[r, "i'_*"] & \pi_q O
\end{tikzcd}$$

The commutativity of the last diagram and the fact that $i'_* p_{2*}\partial[S^{q+1}]$ imply now that $i_*[S^{q+1}]$ generates $\pi_{q+1}BO$, and therefore we have

Corollary 18. *A generator of each nonzero homotopy group $\pi_{q+1}BO$ may be realized as a homothetically embedded totally geodesic euclidean sphere S^{q+1} in a sufficiently large grassmannian $G_{r,2r}$.*

8. Bundles with nonnegative curvature

In this section we give a partial answer to the following problem proposed in [6].

Do all vector bundles over spheres admit complete riemannian metrics of nonnegative sectional curvature?

In what follows "metric" means "complete riemannian metric", and K denotes the sectional curvature.

We begin with the abelian group $[V(S^n)]$ of stable classes of real vector bundles over the sphere S^n for arbitrary n as it is defined in [9], and recall that $[V(S^n)]$ is isomorphic as an abelian group to $\pi_n BO$. A natural isomorphism is defined as follows.

If ξ is in $V(S^n)$ with fibre $\boldsymbol{R}^k$, then ξ is the pull-back $f^*(\gamma_{m,k})$ of the vector bundle $\gamma_{m,k}$, described below, for some $m > 1$, by a smooth map f: $S^n \to G_{m,m+k}$

$\equiv O(m+k)/O(m) \times O(k)$. The Lie group $O(m) \times O(k)$ acts linearly on $\boldsymbol{R}^k$ from the left by its second factor, as $(A, B)x = Bx$, and we denote by $E_{m,k}$ the quotient of the diagonal action of $O(m) \times O(k)$ on the product $O(m+k) \times \boldsymbol{R}^k$, i.e.,

$$(C, x)(A, B) = \left(C\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, B^{-1}x\right).$$

We denote the elements of $E_{m,k}$ by $[C, x]$ and the projection $[C, x] \mapsto [C]$ in $G_{m,m+k}$ by p. The map p defines the vector bundle $\gamma_{m,k}$ with fibre $\boldsymbol{R}^k$. If we denote the inclusion $G_{m,m+k} \to BO$ by i, define $[i \circ f]$ in $\pi_n BO$, and recall [9] that the correspondence $[\xi] \leftrightarrow [i \circ f]$ determines an isomorphism between $[V(S^n)]$ and $\pi_n BO$.

If ξ is in $V(S^n)$ and $\pi_n BO = 0$, then there is a trivial bundle in the stable class of ξ, say $S^n \times \boldsymbol{R}^q$, and the product metric on the total space has $K \geq 0$.

If $\pi_n BO$ is nonzero, let α be the embedding $S^n \to SO(2r)/SO(r) \times SO(r)$ asserted by Corollary 18. Now we recall [12, Vol. II, p. 235] that each totally geodesic submanifold of a symmetric space G/H is itself a symmetric space and may be written as G'/H', where G' is a closed connected subgroup of G and $H' = G' \cap H$. Therefore $G'/H' = \alpha(S^n)$ is a totally geodesic sphere homothetic to S^n in $G_{r,2r}$, and the inclusion $\alpha\colon G'/H' \to G_{r,2r} \to BO$ generates $\pi_n BO$.

Theorem. *For every class of stable vector bundles over a sphere there is a representative whose total space admits a metric of nonnegative sectional curvature.*

Proof. STEP I. A representative of $[\xi]$. The total space $E(\alpha)$ of $\alpha^*(\gamma_{r,r})$ is $G' \times_{H'} V$, where $V = \boldsymbol{R}^r$, and H' acts diagonally on the product as a subgroup of $O(r) \times O(r)$. This is easily verified by noticing that the map $[g', v]' \mapsto [g', v]$ from $G' \times_{H'} V$ to $G \times_H V$ is a well defined bundle morphism which covers the inclusion $g'H' \mapsto g'H$ of G'/H' in G/H. Therefore we have the following pull back diagram:

$$\begin{tikzcd}
V \arrow[d, dashed] & V \arrow[d, dashed] & \\
G' \times_{H'} V \arrow[r] \arrow[d] & G \times_H V \arrow[d, "p"] & \\
G'/H' \arrow[r] & G/H \arrow[r] & BO
\end{tikzcd}$$

where the composite inclusion at the bottom is α. We give $G' \times_{H'} V$ the following metric.

Let G' have a subgroup metric from $SO(2r)$, with $K \geq 0$, and let $V = \boldsymbol{R}^r$ have the euclidean flat metric. When $G' \times V$ is given, the product metric H'

acts on it freely and by isometries inducing a quotient metric on $G' \times_{H'} V$ with $K \geq 0$ by B. O'Neill's theorem [14] on riemannian submersions.

STEP II. Representatives of $s[\xi]$, $s = 2, 3, \cdots$. The homotopy class of $s\alpha$ in $\pi_n BO$ corresponds to the stable class $s[\xi] = [\oplus_1^s \xi_i]$ in $[V(S^n))]$ where $\oplus_1^s \xi_i$ is the Whitney sum of s copies of ξ. A bundle in the class $[\oplus_1^s \xi_i]$ is the pull-back by the diagonal Δ: $G'/H' \to (G'/H')^s$, where $\Delta(g'H') = (g'H', \cdots, g'H')$, of the cartesian product bundle ξ^s, $V^s \cdots (G' \times_{H'} V)^s \to (G'/H')^s$. The total space of ξ^s may be written as $(G' \times V)^s/(H')^s$, where each coordinate of $(H')^s$ acts diagonally on the corresponding coordinate of $(G' \times V)^s$. We should therefore exhibit a vector bundle over G'/H', whose total space we denote by $E(s\alpha)$, with fibre V^s and a vector bundle morphism δ from $E(s\alpha)$ to $(G' \times V)^s/(H')^s$ which covers Δ.

To do this we first consider the space $G' \times (H')^{s-1} \times V^s$ and the following action of $(H')^s$ on it:

$$\begin{aligned}(g, v_1, h_1, v_2, h_2, \cdots, h_{s-1}, v_s)(\bar{h}_0, \bar{h}_1, \cdots, \bar{h}_{s-1}) \\ = (g\bar{h}_0, \bar{h}_0^{-1}v_1, \bar{h}_0^{-1}h_1\bar{h}_1, \bar{h}_1^{-1}v_2, \bar{h}_0^{-1}h_2\bar{h}_2, \cdots, \bar{h}_{s-1}^{-1}v_s) .\end{aligned}$$

This is a free diagonal action by isometries with respect to the obvious product metric on $G' \times (H')^{s-1} \times V^s$ and we claim that the quotient of this action is $E(s\alpha)$.

Before we proceed, notice that $E(s\alpha)$ with the quotient riemannian metric has $K \geq 0$, by O'Neill's theorem [14].

To define the map δ consider first the map

$$\hat{\delta}: G' \times (H')^{s-1} \times V^s \to (G' \times V)^s$$

defined by

$$\hat{\delta}(g, v_1, h_1, v_2, h_2, \cdots, h_{s-1}, v_s) = ((g, v_1), (gh_1, v_2), \cdots, (gh_{s-1}, v_s)) ,$$

and observe that $\hat{\delta}$ is $(H')^s$-equivariant, and therefore it covers a well defined map δ. One may now check easily that δ is a V^s-vector bundle morphism which covers the diagonal Δ.

STEP III. Representatives of $-s[\xi]$, $s = 1, 2, \cdots$. For the values of n such that $\pi_n BO \cong Z$ we remark that each generator α of $\pi_n BO$ and corresponding α: $S^n \to G_{r,2r}$ there is a naturally defined β: $S^n \to G_{r.2r}$ given by reversing the orientation, which is also a homothety onto a totally geodesic submanifold of $G_{r,2r}$ and such that $[\alpha] + [\beta] = 0$ in $\pi_n BO$. Repeating Steps I and II above we see that for each $-s[\xi]$, $s = 1, 2, \cdots$, there is a representative $E(s\beta)$ in $V(S^n)$ which admits a metric with $K \geq 0$. q.e.d.

As a representative bundle of the zero class in $\pi_n BO$ a trivial bundle suffices for the purpose of this theorem. In [5] it is shown that the tangent bundle TS^n admits a metric with $K \geq 0$ for all n.

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