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ON THE NON-LINEAR COHOMOLOGY OF LIE EQUATIONS. IV

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CHAPTER IV. ABELIAN EXTENSIONS AND COHOMOLOGY

17. Some results on cohomology

Here we bring together various results concerning cohomology, both linear and non-linear, which can be derived from the theory as it has been developed up to this point. Some of the results state conditions under which the cohomology is trivial, i.e., the linear cohomology vanishes in positive degrees and the non-linear cohomology in degree 1.

We begin by improving slightly Propositions 7.4, 7.5, 7.7 (ii) and 7.8 by making a small change in the lower bound for which the assertions hold. This is accomplished by proving Proposition 17.1.

We define the twisted δ-operator mentioned in the remark of § 7 following Proposition 7.4. Let v be a section of $T^* \otimes J_0(T)$ over X; we then have the operator

$$\delta_v\colon \textstyle\bigwedge^j T^* \otimes S^k J_0(T)^* \otimes J_0(T) \to \bigwedge^{j+1} T^* \otimes S^{k-1} J_0(T)^* \otimes J_0(T)$$

defined by

$$\delta_v w = [v, w] = [v_1, w] ,$$

where $w \in \bigwedge^j T^* \otimes S^k J_0(T)^* \otimes J_0(T)$, and v_1 is any section of $T^* \otimes J_k(T)$ over X such that $\pi_0 v_1 = v$. Let $v^* : J_0(T)^* \to T^*$ be the mapping dual to $v: T \to J_0(T)$. Then

$$\delta_v(\omega \otimes u) = (-1)^j \omega \wedge (v^* \circ \nu^{*-1} \otimes \mathrm{id}) \delta u , \tag{17.1}$$

for $\omega \in \bigwedge^j T^*$, $u \in S^k J_0(T)^* \otimes J_0(T)$. Therefore if v is the section of $T^* \otimes J_0(T)$ corresponding to ν: $T \to J_0(T)$, then $\delta_v = \delta$. Moreover the diagram

$$\begin{array}{ccc} \bigwedge^j T^* \otimes S^k J_0(T)^* \otimes J_0(T) & \xrightarrow{\delta} & \bigwedge^{j+1} T^* \otimes S^{k-1} J_0(T)^* \otimes J_0(T) \\ \big\downarrow {\scriptstyle (\nu^{-1}\circ v)^* \otimes \mathrm{id} \otimes \mathrm{id}} & & \big\downarrow {\scriptstyle (\nu^{-1}\circ v)^* \otimes \mathrm{id} \otimes \mathrm{id}} \\ \bigwedge^j T^* \otimes S^k J_0(T)^* \otimes J_0(T) & \xrightarrow{\delta_v} & \bigwedge^{j+1} T^* \otimes S^{k-1} J_0(T)^* \otimes J_0(T) \end{array} \tag{17.2}$$

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is commutative, and we thus obtain a complex

$$0 \longrightarrow S^k J_0(T)^* \otimes J_0(T) \xrightarrow{\delta_v} T^* \otimes S^{k-1} J_0(T)^* \otimes J_0(T) \xrightarrow{\delta_v} \bigwedge^2 T^* \otimes S^{k-2} J_0(T)^* \otimes J_0(T) \xrightarrow{\delta_v} \cdots \longrightarrow \bigwedge^n T^* \otimes S^{k-n} J_0(T)^* \otimes J_0(T) \longrightarrow 0$$

for $k > 0$; if $v: T \to J_0(T)$ is an isomorphism, it is exact.

Let $R_k \subset J_k(T)$ be a differential equation; then by (17.1),

$$\delta_v(g_{k+l}) \subset T^* \otimes g_{k+l-1} ,$$

for all $l \geq 1$, and thus we obtain a complex

$$0 \longrightarrow g_m \xrightarrow{\delta_v} T^* \otimes g_{m-1} \xrightarrow{\delta_v} \bigwedge^2 T^* \otimes g_{m-2} \xrightarrow{\delta_v} \cdots \longrightarrow \bigwedge^n T^* \otimes g_{m-n} \longrightarrow 0 ,$$

where $g_m = S^m J_0(T)^* \otimes J_0(T)$ for $m < k$; if $v: T \to J_0(T)$ is an isomorphism, by the commutativity of (17.2) its cohomology at $\bigwedge^j T^* \otimes g_{m-j}$ is isomorphic to $H^{m-j,j}(g_k)$.

The following proposition generalizes Propositions 7.4 and 7.5 and its proof is the same as that of [26, Proposition 3.3].

Proposition 17.1. *Let $R_k \subset J_k(T)$ be a formally integrable Lie equation, and suppose that g_{k_0} is 2-acyclic, with $k_0 \geq k$. Then, for all $m \geq k_0$, the mappings* (7.9), (7.10), (7.11) *and* (7.12) *are surjective.*

Proof. It suffices to show that (7.9) is surjective. Let $u \in Z^1(R_m)_x$, with $m \geq k_0$, $x \in X$, and choose $u' \in (\mathscr{T}^* \otimes \mathscr{R}_{m+1})_x$ such that $\pi_m u' = u$. Then $\mathscr{D}_1 u' \in \bigwedge^2 \mathscr{T}^* \otimes g_m$, and writing $v = \nu + \pi_0 u$, we have

$$\begin{aligned} \delta_v(\mathscr{D}_1 u') &= -D(Du' - \tfrac{1}{2}[u', u']) + [u, \mathscr{D}_1 u'] \\ &= \tfrac{1}{2} D[u', u'] + [u, Du'] - \tfrac{1}{2}[u, [u', u']] \\ &= [Du', u] + [u, Du'] = 0 , \end{aligned}$$

by the Jacobi identity and (1.25). Since g_m is assumed to be 2-acyclic and $v(x): T_x \to J_0(T)_x$ is an isomorphism, there is an element $u'' \in \mathscr{T}^* \otimes g_{m+1}$ satisfying $\delta_v u'' = \mathscr{D}_1 u'$. Then

$$\mathscr{D}_1(u' + u'') = Du' - \delta u'' - \tfrac{1}{2}[u', u'] - [u', u''] = \mathscr{D}_1 u' - \delta_v u'' = 0 ;$$

hence $u' + u''$ belongs to $Z^1(R_{m+1})_x$ and satisfies $\pi_m(u' + u'') = u$, that is (7.9) is surjective.

Therefore in Propositions 7.4, 7.5, 7.7 (ii), we may assume that $k_0 \geq \sup(k, 1)$ and in Proposition 7.8 we may replace $\sup(k_0, 2)$ by $\sup(k_0, 1)$. Consequently throughout § 9 we may assume that $k_0 \geq \sup(k, 1)$.

If $a \in X$, we denote by $\mathrm{id}_a = \mathrm{id}_{X,a}$ the germ of the identity mapping of X in $(\mathrm{Aut}\,(X))_a$. We say that a Lie equation $R_k \subset J_k(T)$ is of finite type if there is an integer $k_0 \geq k$ such that $g_{k_0} = 0$. The following proposition is stated without proof.

Proposition 17.2. *Let $R_k \subset J_k(T)$ be a formally integrable Lie equation of finite type. If P_k is a formally integrable finite form of R_k and $g_{k_0} = 0$, with $k_0 \geq k$, then P_k is integrable and*

$$H^j(R_k) = 0\ ,$$
$$H^0(P_k)_{m,a} = \{\mathrm{id}_a\}\ , \qquad H^1(P_k)_{m,a} = 0\ ,$$

for all $j > 0$, $m \geq k_0$, $a \in X$.

Assume that X is endowed with a structure of real-analytic manifold compatible with its structure of differentiable manifold. Let $\mathcal{O}_{X,\omega}$ be the sheaf of real-analytic real-valued functions on X. If E is a real-analytic vector bundle over X, we denote by $\mathcal{E}_\omega$ the sub-sheaf of $\mathcal{E}$ of analytic sections of E.

We next record two lemmas, of which the first is required in the proof of the second and the second is used in proving Lemma 18.2. Let $x \in X$ and set $A = \mathcal{O}_{X,\omega,x}$. If M is an A-module and $\xi \in \mathcal{T}_{\omega,x}$, an $\boldsymbol{R}$-linear mapping $D\colon M \to M$ is a ξ-derivation if

$$D(fm) = \xi f \cdot m + fDm\ ,$$

for all $f \in A$, $m \in M$. The proof of the following lemma is the same as that of [9, Lemma 8.2] and is due to Malgrange.

Lemma 17.1. *Let $\xi_1, \cdots, \xi_n \in \mathcal{T}_{\omega,x}$, and D_i be a ξ_i-derivation of an A-module M of finite type, for $i = 1, \cdots, n$. If the values $\xi_1(x), \cdots, \xi_n(x)$ of $\xi_1, \cdots, \xi_n$ at x form a basis of T_x, then M is a free A-module.*

Lemma 17.2. *Assume that X is connected. Let E be an analytic vector bundle, and let $\mathcal{F}$ be a coherent $\mathcal{O}_{X,\omega}$-submodule of $\mathcal{E}_\omega$. Assume that, for all $x \in X$, there are $\xi_1, \cdots, \xi_n \in \mathcal{T}_{\omega,x}$ and a ξ_i-derivation D_i of the $\mathcal{O}_{X,\omega,x}$-module $\mathcal{E}_{\omega,x}$ satisfying $D_i(\mathcal{F}_x) \subset \mathcal{F}_x$, for $i = 1, \cdots, n$, such that $\{\xi_1(x), \cdots, \xi_n(x)\}$ is a basis of T_x. Then there is an analytic sub-bundle F of E such that $\mathcal{F}$ is the sheaf of analytic sections of F.*

Proof. Let $\mathcal{S}$ be the coherent $\mathcal{O}_{X,\omega}$-module $\mathcal{E}_\omega/\mathcal{F}$. If $x \in X$, $\xi \in \mathcal{T}_{\omega,x}$ and D is a ξ-derivation of the $\mathcal{O}_{X,\omega,x}$-module $\mathcal{E}_{\omega,x}$ satisfying $D(\mathcal{F}_x) \subset \mathcal{F}_x$, then D induces a ξ-derivation of the $\mathcal{O}_{X,\omega,x}$-module $\mathcal{S}_x$. According to Lemma 17.1, for all $x \in X$, the $\mathcal{O}_{X,\omega,x}$-modules $\mathcal{F}_x$, $\mathcal{S}_x$ are free. Since $\mathcal{S}$ is a coherent $\mathcal{O}_{X,\omega}$-module, by the Syzygy Theorem, $\mathcal{S}$ is locally free. Therefore there is an analytic vector bundle S such that $\mathcal{S}$ is isomorphic to the sheaf of analytic sections of S. The natural mapping $\mathcal{E}_\omega \to \mathcal{S}$ is induced by an epimorphism of vector bundles $E \to S$ whose kernel is an analytic sub-bundle F of E satisfying the condition of the lemma.

We now turn to the consideration of real-analytic equations and their cohomology defined in the analytic sense and, if elliptic, in the differentiable (C^∞) sense.

Let $R_k \subset J_k(T)$ be an analytic Lie equation; assume that R_{k+l} is a vector bundle for all $l \geq 0$. Let P_{k+l} be an analytic finite form of R_{k+l}. If we place ourselves in the category of real-analytic manifolds and real-analytic mappings, then, following § 7, we can define the analytic cohomologies $H^1_\omega(P_k)_{m,a}$, $\bar{H}^1_\omega(P_k)_{m,a}$, $\hat{H}^1_\omega(P_k)_{m,a}$ and $\tilde{H}^1_\omega(R_k) = H^1_\omega(P_k)_a$, for $m \geq k$, $a \in X$. If R_k and P_k are formally integrable, $P_{k+l} = (P_k)_{+l}$, and g_{k_0} is 2-acyclic, with $k_0 \geq k$, then, according to [19, Theorems 8.5 and 8.3] and the integrability of analytic formally integrable differential equations, it follows that $\hat{H}^1_\omega(P_k)_{m,a} = 0$ for all $m \geq k_0$, $a \in X$ and hence by Proposition 7.8 we have the following

Proposition 17.3. *Let $R_k \subset J_k(T)$ be an analytic formally integrable Lie equation, and P_k be an analytic formally integrable finite form of R_k. If g_{k_0} is 2-acyclic, with $k_0 \geq k$, then*

$$H^1_\omega(P_k)_{m,a} = 0 ,$$

for all $m \geq k_0$, $a \in X$.

Assume that E is a real-analytic vector bundle. If $R_k \subset J_k(E)$ is an analytic formally integrable differential equation, there is an integer $m_1 \geq k$ such that the sub-complex

$$0 \longrightarrow (\mathscr{R}_m)_\omega \xrightarrow{D} (\mathscr{T}^* \otimes \mathscr{R}_{m-1})_\omega \xrightarrow{D} (\textstyle\bigwedge^2 \mathscr{T}^* \otimes \mathscr{R}_{m-2})_\omega \xrightarrow{D} \cdots$$
$$\longrightarrow (\textstyle\bigwedge^n \mathscr{T}^* \otimes \mathscr{R}_{m-n})_\omega \longrightarrow 0$$

of (1.7) is exact, except at $(\mathscr{R}_m)_\omega$, for all $m \geq m_1$; its cohomology at $(\mathscr{R}_m)_\omega$ is isomorphic to the sheaf $H^0_\omega(R_k)$ of analytic solutions of R_k (see [5]).

By [25, Proposition 1] and results of [5] (see also [21]), we have:

Proposition 17.4. *Assume that E is an analytic vector bundle. Let $R_k \subset J_k(E)$ be an analytic elliptic formally integrable differential equation. Then*

$$H^0(R_k) = H^0_\omega(R_k) , \quad H^j(R_k) = 0 , \quad \text{for } j > 0 .$$

The following theorem asserts in particular the result of Malgrange [19] that the non-linear Spencer cohomology of an analytic elliptic formally integrable Lie equation vanishes.

Theorem 17.1. *Let $R_k \subset J_k(T)$ be an analytic elliptic formally integrable Lie equation, and P_k an analytic finite form of R_k. Then every solution of P_k is analytic, and if P_k is formally integrable and g_{k_0} is 2-acyclic, with $k_0 \geq k$, we have*

$$H^1(P_k)_{m,a} = 0 ,$$

for all $m \geq k_0$, $a \in X$.

Proof. The first assertion is given by [9, Proposition 7.1]. If P_k is formally integrable, according to [19, Theorem 9.1] we see that $\hat{H}^1(P_k)_{m,a} = 0$, for all $m \geq k_0$, $a \in X$; the second assertion now follows from Proposition 7.8.

We continue with our treatment of linear analytic elliptic equations, but place it in the context of linear cohomology sequences for general projectable equations as developed in [6]; the final result is Theorem 17.2. We also give some complements to [6], in particular Proposition 17.5.

Let F be a vector bundle over Y, and $\varphi\colon E \to F$ be a morphism of vector bundles over $\rho\colon X \to Y$ such that the morphism $\varphi\colon E \to \rho^{-1}F$, whose kernel we denote by K, is surjective.

We consider a formally integrable differential equation $R_k \subset J_k(E; \varphi)$ satisfying the following conditions:

(A) for all $l \geq 0$, there is a differential equation $R''_{k+l} \subset J_{k+l}(F; Y)$ such that

$$\varphi(R_{k+l,a}) = R''_{k+l,\rho(a)}\,, \qquad \text{for all } a \in X\ ;$$

(B) if $\bar{R}_{k+l} = R_{k+l} \cap J_{k+l}(K)$ denotes the kernel of the epimorphism $\varphi\colon R_{k+l} \to \rho^{-1}R''_{k+l}$, the projections $\pi_{k+l}\colon \bar{R}_{k+l+m} \to \bar{R}_{k+l}$ are of constant rank, for all l, $m \geq 0$.

We now recall some facts which may be found in the paper [6]. Since $\pi_m\colon R''_{m+1} \to R''_m$ is surjective for $m \geq k$ and $R''_{m+1} \subset (R''_m)_{+1}$, there exists by the Cartan-Kuranishi prolongation theorem an integer $k_1 \geq k$ such that $(R''_{k_1})_{+l} = R''_{k_1+l}$ for all $l \geq 0$, and R''_{k_1} is a formally integrable differential equation in $J_{k_1}(F; Y)$. For all $l \geq 0$, we have $\bar{R}_{k+l} = (\bar{R}_k)_{+l}$; for $l \geq 0$ and $m \geq k$, let $\bar{R}^{(l)}_m$ be the sub-bundle $\pi_m \bar{R}_{m+l}$ of $J_m(K)$. According to [5, Theorem 1], there exist integers $m_0 \geq k$, $l_0 \geq 0$ such that $R'_{m_0} = \bar{R}^{(l_0)}_{m_0}$ is a formally integrable differential equation in $J_{m_0}(K)$, whose r-th prolongation is equal to

$$R'_{m_0+r} = \bar{R}^{(l_0)}_{m_0+r} = \bar{R}^{(l)}_{m_0+r}$$

for all $l \geq l_0$. For $m \geq k$, let

$$(\textstyle\bigwedge^j \mathscr{T}^* \otimes \mathscr{R}_m)_\varphi = (\textstyle\bigwedge^j \mathscr{T}^* \otimes \mathscr{R}_m) \cap (\textstyle\bigwedge^j \mathscr{T}^* \otimes J_m(\mathscr{E}; \varphi))_\varphi\ ;$$

for $a \in X$, with $b = \rho(a)$, the mappings

$$\varphi\colon (\textstyle\bigwedge^j \mathscr{T}^* \otimes J_m(\mathscr{E}; \varphi))_{\varphi,a} \to (\textstyle\bigwedge^j \mathscr{T}^*_Y \otimes J_m(\mathscr{F}; Y))_b \tag{17.3}$$

give us the commutative diagram

$$\begin{array}{ccccc} (\bigwedge^{j-1}\mathscr{T}^* \otimes \mathscr{R}_{m+1})_{\varphi,a} & \xrightarrow{D} & (\bigwedge^{j}\mathscr{T}^* \otimes \mathscr{R}_{m})_{\varphi,a} & \xrightarrow{D} & (\bigwedge^{j+1}\mathscr{T}^* \otimes J_{m-1}(\mathscr{E}; \varphi))_{\varphi,a} \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} \\ (\bigwedge^{j-1}\mathscr{T}^*_Y \otimes \mathscr{R}''_{m+1})_b & \xrightarrow{D} & (\bigwedge^{j}\mathscr{T}^*_Y \otimes \mathscr{R}''_{m})_b & \xrightarrow{D} & (\bigwedge^{j+1}\mathscr{T}^*_Y \otimes J_{m-1}(\mathscr{F}; Y))_b \end{array} \tag{17.4}$$

and thus determines a mapping between the cohomology $H^j_\varphi(R_k)_{m,a}$ of the top row of the diagram and the cohomology of the bottom row. For $m \geq k_1$, we therefore have a mapping

$$\varphi\colon H^j_\varphi(R_k)_{m,a} \to H^j(R''_{k_1})_{m,b} \,. \tag{17.5}$$

According to [6, Theorem 3], there is an integer $k_2 \geq k_1$ such that the natural mappings

$$H^j_\varphi(R_k)_{m,a} \to H^j(R_k)_{m,a}$$

are isomorphisms for all $m \geq k_2, j \geq 0$ and $a \in X$. These isomorphisms together with (17.5) yield mappings

$$\varphi\colon H^j(R_k)_{m,a} \to H^j(R''_{k_1})_{m,\rho(a)} \,, \tag{17.6}$$

$$\varphi\colon H^j(R_k)_a \to H^j(R''_{k_1})_{\rho(a)} \,, \tag{17.7}$$

for $m \geq k_2$, $j \geq 0$ and $a \in X$. According to [6, Theorem 3], we also have the exact sequence

$$\begin{aligned} \cdots \longrightarrow H^{j-1}(R''_{k_1})_{\rho(a)} \xrightarrow{\partial} H^j(R'_{m_0})_a \longrightarrow H^j(R_k)_a \\ \xrightarrow{\varphi} H^j(R''_{k_1})_{\rho(a)} \longrightarrow \cdots \,. \end{aligned} \tag{17.8}$$

Assume now that $\varphi\colon E \to \rho^{-1}F$ is an isomorphism. If $a \in X$ and $b = \rho(a)$, consider $\rho^*\colon T^*_{Y,b} \to T^*_a$; then $\rho^*(\bigwedge^j T^*_{Y,b} \otimes S^m T^*_{Y,b}) \otimes E_a$ is the fiber over a of a sub-bundle $(\bigwedge^j T^* \otimes S^m T^* \otimes E)_\varphi$ of $\bigwedge^j T^* \otimes S^m T^* \otimes E$, and we have a natural isomorphism

$$\varphi\colon (\textstyle\bigwedge^j T^* \otimes S^m T^* \otimes E)_{\varphi,a} \to (\bigwedge^j T^*_Y \otimes S^m T^*_Y \otimes F)_b \,.$$

According to [6, § 5], the diagram

$$\begin{array}{ccc} (S^{m+1}T^* \otimes E)_\varphi & \xrightarrow{\delta} & (T^* \otimes S^m T^* \otimes E)_\varphi \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} \\ S^{m+1}T^*_Y \otimes F & \xrightarrow{\delta} & T^*_Y \otimes S^m T^*_Y \otimes F \end{array} \tag{17.9}$$

commutes, and the diagram

$$
\begin{array}{ccc}
 & 0 & & 0 \\
 & \downarrow & & \downarrow \\
 & (S^mT^* \otimes E)_\varphi & \xrightarrow{\varphi} & S^mT_Y^* \otimes F \\
 & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon} \\
(17.10) \qquad & J_m(E;\varphi) & \xrightarrow{\varphi} & J_m(F;Y) \\
 & \downarrow{\scriptstyle \pi_{m-1}} & & \downarrow{\scriptstyle \pi_{m-1}} \\
 & J_{m-1}(E;\varphi) & \xrightarrow{\varphi} & J_{m-1}(F;Y) \\
 & \downarrow & & \downarrow \\
 & 0 & & 0
\end{array}
$$

is commutative and exact.

Consider the mapping

$$\text{(17.11)} \qquad \delta: T^* \otimes S^{m+1}T^* \to \textstyle\bigwedge^2 T^* \otimes S^m T^* \ ;$$

by (1.5), we have

$$\text{(17.12)} \qquad \langle \xi \wedge \eta, \delta u \rangle = \xi \barwedge \delta u(\eta) - \eta \barwedge \delta u(\xi) \ ,$$

for $u \in T^* \otimes S^{m+1}T^*$ and $\xi, \eta \in T$. Fix $x \in X$; denote for the moment by T^*, T_Y^* the fibers of these vector bundles over x and $\rho(x)$, and consider T_Y^* as a subspace of T^* by means of the injective mapping ρ^*. For $m \geq 0$, if the image of $u \in T^* \otimes S^{m+1}T_Y^*$ under the mapping (17.11) belongs to $\bigwedge^2 T_Y^* \otimes S^m T_Y^*$, then u is an element of $T_Y^* \otimes S^{m+1}T_Y^*$. Indeed, to verify that u belongs to $T_Y^* \otimes S^{m+1}T_Y^*$, we must show that $u(\xi) = 0$, for all $\xi \in V$. If $\xi \in V$, $\eta \in T$, then $u(\eta) \in S^{m+1}T_Y^*$ and $\xi \barwedge \delta u(\eta) = 0$; since $\langle \xi \wedge \eta, \delta u \rangle = 0$, by (17.12) we have $\eta \barwedge \delta u(\xi) = 0$. Therefore $\delta u(\xi) = 0$ and $u(\xi) = 0$.

For $m \geq k$, let $g''_m \subset S^m T_Y^* \otimes F$ be the sub-bundle with possibly varying fiber such that the sequence

$$0 \longrightarrow g''_m \xrightarrow{\varepsilon} R''_m \xrightarrow{\pi_{m-1}} J_{m-1}(F; Y)$$

is exact; for $m < k$, we set $g_m = (S^mT^* \otimes E)_\varphi$ and $g''_m = S^mT_Y^* \otimes F$. From diagram (17.10), we deduce that $g_m \subset (S^mT^* \otimes E)_\varphi$ and that

$$\varphi: g_{m,a} \to g''_{m,\rho(a)}$$

is an isomorphism for all $m \geq 0$, $a \in X$. Fix $x \in X$ and denote again for the moment by T^*, T_Y^*, g_m, g''_m the fibers of these bundles over x or $\rho(x)$. From (17.9), we obtain the commutative diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & g_{m+1} & \xrightarrow{\delta} & T^* \otimes g_m & \xrightarrow{\delta} & \bigwedge^2 T^* \otimes g_{m-1} & \xrightarrow{\delta} & \bigwedge^3 T^* \otimes g_{m-2} \\
& \uparrow \varphi^{-1} & & \uparrow \rho^*\otimes\varphi^{-1} & & \uparrow \rho^*\otimes\varphi^{-1} & & \uparrow \rho^*\otimes\varphi^{-1} \\
0 \longrightarrow & g''_{m+1} & \xrightarrow{\delta} & T^*_Y \otimes g''_m & \xrightarrow{\delta} & \bigwedge^2 T^*_Y \otimes g''_{m-1} & \xrightarrow{\delta} & \bigwedge^3 T^*_Y \otimes g''_{m-2}
\end{array}$$

whose vertical arrows are injective. Its bottom row is exact at $T^*_Y \otimes g''_m$ for $m \geq k$, and if $H^{m,2}(g_k)_x = 0$ with $m \geq k$, by the above remark concerning the mapping (17.11), it is also exact at $\bigwedge^2 T^*_Y \otimes g''_m$.

The mapping (17.3) is an isomorphism and therefore determines an isomorphism between $H^j_\varphi(R_k)_{m,a}$ and the cohomology of the bottom row of diagram (17.4).

Proposition 17.5. *Assume that $\varphi: E \to \rho^{-1}F$ is an isomorphism, and let $R_k \subset J_k(E; \varphi)$ be a formally integrable differential equation satisfying condition* (A).

(i) *The differential equation $R''_k \subset J_k(F; Y)$ is formally integrable and $R''_{k+l} = (R''_k)_{+l}$ for all $l \geq 0$.*

(ii) *If g_{k_0} is 2-acyclic, with $k_0 \geq k$, then g''_{k_0} is also 2-acyclic, the natural mapping*

$$H^1_\varphi(R_k)_{m,a} \to H^1(R_k)_{m,a}$$

is an isomorphism for all $m \geq k_0$, $a \in X$, and the mapping

$$\varphi: H^1(R_k)_{m,a} \to H^1(R''_k)_{m,\rho(a)} ,$$

which it determines, is also an isomorphism for $m \geq k_0$, $a \in X$.

Proof. (i) is given by [6, Proposition 5 (ii)]. As we have seen above, if g_{k_0} is 2-acyclic, so is g''_{k_0}; the proof of [6, Theorem 3], Lemma 3.1 and diagram (17.4) tell us that the mappings of (ii) are isomorphisms for $m \geq k_0$.

We no longer assume that $\varphi: E \to \rho^{-1}F$ is an isomorphism.

Theorem 17.2. *Assume that X, Y are real-analytic manifolds, that $\rho: X \to Y$ is an analytic submersion, that E, F are analytic vector bundles and that $\varphi: E \to F$ is an analytic morphism of vector bundles over ρ. Let $R_k \subset J_k(E; \varphi)$ be an analytic formally integrable differential equation satisfying conditions* (A) *and* (B). *If $R''_{k_1} \subset J_{k_1}(F; Y)$ is elliptic, then, for all $j \geq 1$, we have an isomorphism of cohomology*

$$H^j(R'_{m_0}) \xrightarrow{\sim} H^j(R_k) . \tag{17.13}$$

Proof. According to [6, Theorem 3], we have the exact sequence (17.8) for $j \geq 1$, and the exact and commutative diagram

$$\begin{array}{ccccccc}
H^0_\omega(R_k)_a & \xrightarrow{\varphi} & H^0_\omega(R''_{k_1})_{\rho(a)} & \longrightarrow & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
H^0(R_k)_a & \xrightarrow{\varphi} & H^0(R''_{k_1})_{\rho(a)} & \xrightarrow{\partial} & H^1(R'_{m_0})_a & \longrightarrow & H^1(R_k)_a
\end{array} \tag{17.14}$$

for all $a \in X$. Since R''_{k_1} is analytic and elliptic, by Proposition 17.4 we have $H^j(R''_{k_1})_{\rho(a)} = 0$ for $j \geq 1$, and the mapping

$$H^0_\omega(R''_{k_1})_{\rho(a)} \to H^0(R''_{k_1})_{\rho(a)}$$

is an isomorphism. Thus (17.8) gives the isomorphism (17.13) for $j \geq 2$ and the surjectivity of (17.13) for $j = 1$. From diagram (17.14), we deduce the injectivity of (17.13) for $j = 1$.

Returning to Lie equations, we now take $E = T$, $F = T_Y$, $\varphi = \rho$ and $R_k \subset J_k(T; \rho)$ to be a formally integrable Lie equation. Condition (A) is of course the same as condition (I) of § 9 and (B) the same as (III). Assume that R_k satisfies conditions (I), (II) and (III) of § 9. We shall assume as in § 9 that the order m_0 of the equation $R'_{m_0} \subset J_{m_0}(V)$ is chosen so that $m_0 \geq k_1$ and g_{m_0}, g'_{m_0}, g''_{m_0} are 2-acyclic. Let $P_k \subset Q_k(\rho)$, $P'_{m_0} \subset Q_{m_0}(V)$, $P''_{k_1} \subset Q_{k_1}(Y)$ be formally integrable finite forms of the Lie equations $R_k \subset J_k(T; \rho)$, $R'_{m_0} \subset J_{m_0}(V)$, $R''_{k_1} \subset J_{k_1}(T_Y; Y)$ respectively; we denote by P_{k+l}, P'_{m_0+l}, P''_{k_1+l} the l-th prolongations of P_k, P'_{m_0}, P''_{k_1}. Let $\bar{P}_m$ be a finite form of $\bar{R}_m$, for $m \geq k$. Consider the sequences (9.5) and (9.11) with $l \geq l_0$.

For $a \in X$, we consider the following assertions:

(i) $H^1(P'_{m_0})_{m,a} = 0$, for all $m \geq m_0$;

(ii) there exists an integer $r \geq 0$ such that, for all $m \geq m_0$ and $f'' \in H^0(P''_{k_1})_{m+r,\rho(a)}$, there is an element $f \in H^0(P_k)_{m,a}$ satisfying $\rho f = f''$;

(iii) if $m \geq m_0$ and the image of $\alpha \in H^1(P'_{m_0})_{m,a}$ in $H^1(P_k)_{m,a}$ vanishes, then $\alpha = 0$.

In § 20, we shall construct a class of Lie equations R_k satisfying conditions (I), (II) and (III) of § 9 and this assertion (ii).

If P_k is integrable, we now prove the implications (i) $\Rightarrow$ (ii) in Theorem 17.3 and (ii) $\Rightarrow$ (iii) in Theorem 17.4, showing how the lifting property (ii) for solutions of P''_{k_1} to solutions of P_k is related to information about the non-linear cohomology. In fact, Theorem 17.4 tells us that assertion (i) implies a lifting property for solutions of P''_{k_1} to solutions of P_k which is stronger than (ii) and which is used in Corollary 17.1 to derive our version of the Kuranishi-Rodrigues theorem [31]. Corollary 17.1 and Theorem 17.4 are required to derive further properties of the non-linear cohomology of the sequence (9.11) in Theorem 17.5, when R''_{k_1} is elliptic. All these results and Theorem 17.6 are basically consequences of our study of the sequences (9.5) and (9.11) in § 9.

Theorem 17.3. *Let $R_k \subset J_k(T; \rho)$ be a formally integrable Lie equation satisfying conditions* (I), (II) *and* (III) *of* § 9. *Assume that the finite form P_k of R_k is formally integrable and integrable. Let $m \geq m_0$, $a \in X$ and assume that $H^1(P'_{m_0})_{m,a} = 0$. Then there exists a neighborhood U of $I_{Y,m+l_0+2}(\rho(a))$ in $P''_{m+l_0+2}(\rho(a))$ such that for any germ $f'' \in \mathrm{Sol}\,(P''_{k_1})_{\rho(a)}$, with $j_{m+l_0+2}(f'')(\rho(a)) \in U$, there is $f \in \mathrm{Sol}\,(P_k)_a$ satisfying $\rho f = f''$; moreover, if $f'' \in H^0(P''_{k_1})_{m+l_0+2,\rho(a)}$, there is $f \in H^0(P_k)_{m+1,a}$ satisfying $\rho f = f''$.*

Proof. Let $m \geq m_0$ and $a \in X$; from the remarks following Lemma 9.1, we deduce the existence of a neighborhood U of $I_{Y,m+l_0+2}(\rho(a))$ in $P''_{m+l_0+2}(\rho(a))$ such that $U \subset \rho(P_{m+l_0+2}(a))$. If $f'' \in \mathrm{Sol}\,(P''_{k_1})_{\rho(a)}$ with $j_{m+l_0+2}(f'')(\rho(a)) \in U$, choose $G \in P_{m+l_0+2}(a)$ with $\rho G = j_{m+l_0+2}(f'')(\rho(a))$. Since P_k is integrable, there exists $g \in \mathrm{Sol}\,(P_k)_a$ such that $j_{m+l_0+2}(g)(a) = G$. Then $f''_1 = \rho g^{-1} \circ f''$ belongs to $\mathrm{Sol}\,(P''_{k_1})_{\rho(a)}$ and satisfies $j_{m+l_0+2}(f''_1)(\rho(a)) = I_{Y,m+l_0+2}(\rho(a))$. Since $j_1(j_{m+l_0+1}(f''_1))(\rho(a)) = \rho j_1(I_{m+l_0+1})(a)$, we see that f''_1 is an element of $H^0(P''_{k_1})_{m+l_0+1,a}$. According to our hypotheses, the element $\partial^\sharp f''_1$ of $H^1(P'_{m_0})_{m,a}$ vanishes and therefore so does the image of f''_1 in $H^1(\bar{P}_k)_{m,a}$. By Proposition 9.1, there exists $f_1 \in H^0(P_k)_{m+1,a}$ such that $\rho f_1 = f''_1$. Then the element $f = g \circ f_1$ of $\mathrm{Sol}\,(P_k)_a$ satisfies $\rho f = f''$. If $f'' \in H^0(P''_{k_1})_{m+l_0+2,\rho(a)}$, we take $g = \mathrm{id}$ and $f = f_1$.

From Theorem 17.3, we now deduce our (non-linear) version of the Kuranishi-Rodrigues theorem [31]:

Corollary 17.1. *Assume that X, Y are real-analytic manifolds, that $\rho: X \to Y$ is an analytic submersion. Let $R_k \subset J_k(T; \rho)$ be an analytic formally integrable Lie equation satisfying* (I), (II) *and* (III) *of* § 9. *Let P_k and P''_{k_1} be analytic formally integrable finite forms of R_k and R''_{k_1}; let $m \geq m_0$ and $a \in X$. Then there exists a neighborhood U of $I_{Y,m+l_0+2}(\rho(a))$ in $P''_{m+l_0+2}(\rho(a))$ such that for any analytic germ $f'' \in \mathrm{Sol}\,(P''_{k_1})_{\rho(a)}$, with $j_{m+l_0+2}(f'')(\rho(a)) \in U$, there is an analytic germ $f \in \mathrm{Sol}\,(P_k)_a$ satisfying $\rho f = f''$; moreover, if f'' is an analytic germ in $H^0(P''_{k_1})_{m+l_0+2,\rho(a)}$, there is an analytic germ f in $H^0(P_k)_{m+1,a}$ satisfying $\rho f = f''$.*

Proof. We may assume that P'_{m_0} is an analytic formally integrable finite form of R'_{m_0}; then by Proposition 17.3, $H^1_\omega(P'_{m_0})_{m,a} = 0$. Since f'' is analytic and P_k is integrable, the proof of Theorem 17.3 gives us the existence of f.

Theorem 17.4. *Let $R_k \subset J_k(T; \rho)$ be a formally integrable Lie equation satisfying conditions* (I), (II) *and* (III) *of* § 9. *Assume that the finite form P_k of R_k is formally integrable and integrable, and that there exists an integer $r \geq 0$ such that, for all $m \geq m_0$, $a \in X$ and $f'' \in H^0(P''_{k_1})_{m+r,\rho(a)}$, there is an element $f \in H^0(P_k)_{m,a}$ satisfying $\rho f = f''$. If $m \geq m_0$, $a \in X$ and the image of $\alpha \in H^1(P'_{m_0})_{m,a}$ in $H^1(P_k)_{m,a}$ vanishes, then $\alpha = 0$.*

Proof. Let $m \geq m_0$, $a \in X$ and $\alpha \in H^1(P'_{m_0})_{m,a}$; suppose that the image of α in $H^1(P_k)_{m,a}$ vanishes. According to Theorem 9.2 (i), there exists $f'' \in H^0(P''_{k_1})_{m+l_0+r+1,a}$ such that $\partial^\sharp f'' = \alpha$. Let $f \in H^0(P_k)_{m+l_0+1,a}$ with $\rho f = f''$; then the image of f'' in $H^1(\bar{P}_k)_{m+l_0,a}$ vanishes and hence so does α.

The following theorem is a non-linear analogue of Theorem 17.2:

Theorem 17.5. *Assume that X, Y are real-analytic manifolds, and that $\rho: X \to Y$ is an analytic submersion. Let $R_k \subset J_k(T; \rho)$ be an analytic formally integrable Lie equation satisfying conditions* (I), (II) *and* (III) *of* § 9. *If $R''_{k_1} \subset J_{k_1}(T_Y; Y)$ is elliptic and $m \geq m_0$, $a \in X$, then we have:*

(i) *the mapping of cohomology*

$$H^1(P'_{m_0})_{m,a} \to H^1(P_k)_{m,a} \tag{17.15}$$

is surjective;

(ii) *if the image of* $\alpha \in H^1(P'_{m_0})_{m,a}$ *in* $H^1(P_k)_{m,a}$ *vanishes, then* $\alpha = 0$;

(iii) $H^1(P'_{m_0})_a = 0$ *if and only if* $H^1(P_k)_a = 0$.

Proof. (i) We may assume that P_k and P''_{k_1} are analytic finite forms of R_k and R''_{k_1}. By Theorem 17.1, we see that $H^1(P''_{k_1})_{m,\rho(a)} = 0$; therefore, since P''_{k_1} is integrable, by Theorem 9.2 (ii) the mapping (17.15) is surjective.

(ii) Since any solution of P''_{k_1} is analytic by Theorem 17.1, Corollary 17.1 implies that the hypotheses of Theorem 17.4 hold with $r = l_0 + 1$; this last theorem gives us the result.

(iii) is a direct consequence of (i) and (ii).

If in Theorem 17.5 we replace the hypothesis that R''_{k_1} is elliptic by the stronger hypothesis that it is of finite type and remove all assumptions of real-analyticity, we obtain the stronger assertions of the following

Theorem 17.6. *Let* $R_k \subset J_k(T; \rho)$ *be a formally integrable Lie equation satisfying conditions* (I), (II) *and* (III) *of* § 9. *If* $R''_{k_1} \subset J_{k_1}(T_Y; Y)$ *is of finite type and if* $m_1 \geq m_0$ *is an integer such that* $g''_{m_1} = 0$, *then, for all* $m \geq m_1$, $l \geq l_0$, $a \in X$, *we have*:

(i) *the mapping*

$$H^1(\bar{P}_k)_{m,a} \to H^1(P_k)_{m,a}$$

is an isomorphism of cohomology;

(ii) *the mapping of cohomology*

$$H^1(P'_{m_0})_{m,a} \to H^1(P_k)_{m,a}$$

is surjective;

(iii) *if* $\alpha_1, \alpha_2 \in H^1(P'_{m_0})_{m+l,a}$ *have the same image in* $H^1(P_k)_{m+l,a}$, *then* $\pi_m\alpha_1 = \pi_m\alpha_2$ *as elements of* $H^1(P'_{m_0})_{m,a}$; *if* P_k *is integrable and the image of* $\alpha \in H^1(P'_{m_0})_{m,a}$ *in* $H^1(P_k)_{m,a}$ *vanishes, then* $\alpha = 0$;

(iv) *the mapping*

$$H^1(P'_{m_0})_a \to H^1(P_k)_a$$

is an isomorphism of cohomology.

Proof. For $m \geq m_1$, $a \in X$, by Proposition 17.2, P''_{k_1} is integrable, $H^1(P''_{k_1})_{m,\rho(a)} = 0$ and $H^0(P''_{k_1})_{m+1,a} = \{\mathrm{id}_{Y,\rho(a)}\}$. Since $\alpha^{\mathrm{id}_{Y,\rho(a)}} = \alpha$ for all $\alpha \in H^1(\bar{P}_k)_{m,a}$, Proposition 9.1 tells us that (i) holds and Theorem 9.2 (ii) gives us (ii). If $\alpha_1, \alpha_2 \in H^1(P'_{m_0})_{m+l,a}$, with $l \geq l_0$, have the same image in $H^1(P_k)_{m+l,a}$, according to Proposition 9.1 the images of α_1, α_2 in $H^1(\bar{P}_k)_{m+l,a}$ are equal; by the commutativity of (9.9), so are their images in $H^1(P'_{m_0})_{m,a}$. The second assertion of (iii) follows directly from Theorem 9.2 (i). Finally (iv) is a consequence of (i).

We now proceed to show how the above results on the sequence (9.11) can be used to derive relations between the non-linear cohomology of a pair of analytic formally integrable Lie equations R_k, $R^\sharp_k \subset J_k(T)$ on X satisfying

$$R_k \subset R_k^\sharp , \qquad [\tilde{\mathscr{R}}_{k+1}^\sharp, \mathscr{R}_k] \subset \mathscr{R}_k ,$$

if $R_k^\sharp$ is formally transitive. In particular if $R_{\infty,x}^\sharp/R_{\infty,x}$ is an elliptic transitive Lie algebra for all $x \in X$, the non-linear cohomology of either of these equations vanishes if and only if the other one does (Theorem 17.7); if these Lie algebras are finite-dimensional, we obtain a stronger result (Theorem 17.8).

Let W be an integrable sub-bundle of T. For $m \geq 0$, let

$$J_{m+1}(T; W) = \{\xi \in J_{m+1}(T) \mid [\xi, J_{m+1}(W)] \subset J_m(W)\} ,$$

$$Q_{m+1}(X; W) = \left\{ F \in Q_{m+1} \,\middle|\, \begin{array}{l} F(J_m(W)_a) = J_m(W)_b \\ \text{if } a = \text{source } F,\ b = \text{target } F \end{array} \right\} .$$

It is easily seen that $J_1(T; W)$ is a formally transitive and formally integrable Lie equation whose m-th prolongation is $J_{m+1}(T; W)$, and $Q_1(X; W)$ is a formally integrable finite form of $J_1(T; W)$ whose m-th prolongation is $Q_{m+1}(X; W)$. Moreover $J_m(W) \subset J_m(T; W)$, for $m \geq 1$.

Assume that X is connected. Let $R_k \subset J_k(T)$ be a formally transitive and formally integrable Lie equation such that

$$[\tilde{\mathscr{R}}_k, J_{k-1}(\mathscr{W})] \subset J_{k-1}(\mathscr{W}) . \tag{17.16}$$

By [10, Lemma 10.5] and Lemma 1.5, the relation (17.16) is equivalent to the inclusion $R_k \subset J_k(T; W)$. By [10, Proposition 10.3 and Lemma 10.3 (ii)] and [6, Theorem 1], we see that $\pi_m(R_{m+l} \cap J_{m+l}(W))$ is a sub-bundle of R_m for all $m \geq k$, $l \geq 0$, and we obtain a formally integrable Lie equation $N_{k_0} \subset R_{k_0}$ with $k_0 \geq k$, and an integer $l_0 \geq 0$ such that

$$N_m = \pi_m(R_{m+l} \cap J_{m+l}(W)) ,$$

for all $m \geq k_0$, $l \geq l_0$, and

$$N_\infty = R_\infty \cap J_\infty(W) ;$$

moreover, for $a \in X$, the closed ideal $N_{\infty,a}$ of $R_{\infty,a}$ is defined by a foliation in $(R_{\infty,a}, R_{\infty,a}^0)$. In particular, $J_\infty(W)_a$ is a closed ideal of $J_\infty(T; W)_a$ defined by the foliation $J_0(W)_a$ in $(J_\infty(T; W)_a, J_\infty^0(T; W)_a)$ for $a \in X$. We denote by L_a the transitive Lie algebra $J_\infty(T; W)_a/J_\infty(W)_a$ for $a \in X$; according to [10, Proposition 10.2], the image L_a^0 of $J_\infty^0(T; W)_a$ in L_a is a fundamental subalgebra of L_a. Let $L_a^\flat$ be the closed subalgebra which is the image of $R_{\infty,a}$ in L_a; then the sequence

$$0 \longrightarrow N_{\infty,a} \longrightarrow R_{\infty,a} \longrightarrow L_a^\flat \longrightarrow 0$$

is exact. Since $L_a = L_a^\flat + L_a^0$, we see that $L_a^{\flat 0} = L_a^\flat \cap L_a^0$ is a fundamental subalgebra of the transitive Lie algebra $L_a^\flat$. If $\pi_0 N_{\infty,a} = J_0(W)_a$, then $L_a^{\flat 0}$ is equal to the image of $R_{\infty,a}^0$ in $L_a^\flat$. We write

$$L_a^{\flat m} = D^m_{L_a^\flat} L_a^{\flat 0} , \qquad \text{for } m \geq 1 ,$$

$$L_a^{\flat -1} = L_a^\flat ;$$

we denote by gr $L_a^\flat$ the graded Lie algebra $\bigoplus_{m=-1}^{\infty} L_a^{\flat m}/L_a^{\flat m+1}$.

Let $P_k \subset Q_k(X; W)$ be a formally integrable finite form of R_k. If $a, b \in X$, since R_k is formally transitive, there exists $\phi \in Q_\infty(a, b)$, with $\pi_m \phi \in P_m$ for all $m \geq k$. As $\pi_m \phi \in Q_m(X; W)$ for $m \geq 1$, we have

$$\phi(J_\infty(T; W)_a) = J_\infty(T; W)_b , \quad \phi(J_\infty(W)_a) = J_\infty(W)_b , \quad \phi(R_{\infty,a}) = R_{\infty,b} .$$

Therefore ϕ determines an isomorphism $\psi\colon L_a \to L_b$ sending L_a^0 onto L_b^0 and $L_a^\flat$ onto $L_b^\flat$. Hence

$$\psi\colon (L_a^\flat, L_a^{\flat 0}) \to (L_b^\flat, L_b^{\flat 0})$$

is an isomorphism of pairs of topological Lie algebras and so induces an isomorphism

$$\text{gr } \psi\colon \text{gr } L_a^\flat \to \text{gr } L_b^\flat$$

of graded Lie algebras. In turn, this last isomorphism gives us an isomorphism of bigraded vector spaces

$$H^*(L_a^\flat/L_a^{\flat 0}, \text{gr } L_a^\flat) \to H^*(L_b^\flat/L_b^{\flat 0}, \text{gr } L_b^\flat) .$$

From these isomorphisms, we deduce the existence of an integer $k_1 \geq 1$ such that

$$H^{j,m}(L_a^\flat/L_a^{\flat 0}, \text{gr } L_a^\flat) = 0 ,$$

for all $j \geq 0$, $m \geq k_1 - 1$, $a \in X$.

Let Z be a differentiable manifold, and $\tau\colon U \to Z$ be a surjective submersion defined on an open subset U of X such that $W_{|U}$ is the bundle of vectors tangent to the fibers of τ. Then for $m \geq 1$, by Proposition 6.1 (i) we have

$$J_m(T; \tau) = J_m(T; W) , \qquad Q_m(\tau) = Q_m(X; W)$$

on U. The mapping τ determines a canonical isomorphism

$$(17.17) \qquad L_a \to J_\infty(T_Z; Z)_{\tau(a)}$$

for all $a \in U$; the image of L_a^0 under this mapping is $J_\infty^0(T_Z; Z)_{\tau(a)}$. If U and the fibers of τ are connected, by [10, Corollary 11.1 and Theorem 11.2 (i)], there is a formally transitive and formally integrable Lie equation $R_{k_2}^\flat \subset J_{k_2}(T_Z; Z)$, with $k_2 \geq k$, such that

$$\tau(R_{m,a}) = R_{m,\tau(a)}^\flat ,$$

for all $m \geq k_2$, $a \in U$. The equation $R_k \subset J_k(T_Z; Z)$ on U therefore satisfies conditions (I), (II) and (III) of § 9 with respect to the submersion τ, and the sequence

$$0 \longrightarrow N_{\infty,a} \longrightarrow R_{\infty,a} \xrightarrow{\tau} R^\flat_{\infty,\tau(a)} \longrightarrow 0$$

is exact for all $a \in U$. Let $R^\flat_m = \pi_m R^\flat_{k_2}$ for $m < k_2$, and $g^\flat_m$ be the sub-bundle of $S^m J_0(T_Z)^* \otimes J_0(T_Z)$ such that the sequence

$$0 \longrightarrow g^\flat_m \longrightarrow R^\flat_m \xrightarrow{\pi_{m-1}} R^\flat_{m-1} \longrightarrow 0$$

is exact with $m \geq 0$. Let $H^{m,j}$ denote the cohomology of the complex

$$\bigwedge^{j-1} T^*_Z \otimes g^\flat_{m+1} \xrightarrow{\delta} \bigwedge^j T^*_Z \otimes g^\flat_m \xrightarrow{\delta} \bigwedge^{j+1} T^*_Z \otimes g^\flat_{m-1} . \tag{17.18}$$

For $a \in U$, the image of $L^\flat_a$ under the mapping (17.17) is $R^\flat_{\infty,\tau(a)}$, and so this mapping determines an isomorphism of graded Lie algebras

$$\text{gr } L^\flat_a \rightarrow \text{gr } R^\flat_{\infty,\tau(a)} .$$

According to § 15, we obtain isomorphisms

$$H^{j,m-1}(L^\flat_a/L^{\flat 0}_a, \text{gr } L^\flat_a) \rightarrow H^{m,j}_{\tau(a)} ,$$

for all j, $m \geq 0$. Hence the sequence (17.18) is exact for $j \geq 0$, $m \geq k_1$. By the first remark of § 6 of [9], we may assume that $k_2 = k_1$; moreover $g^\flat_{k_1}$ is 2-acyclic.

Using the above discussion, we now derive from Theorem 12.1 and results of [10] the following

Proposition 17.6. *Assume that Y is a connected differentiable manifold. Let $R''^{\#}_{k_1} \subset J_{k_1}(T_Y; Y)$ be a formally transitive and formally integrable Lie equation, and $R''_{k_1} \subset R''^{\#}_{k_1}$ a formally integrable Lie equation such that*

$$[\tilde{\mathscr{R}}''^{\#}_{k_1+1}, \mathscr{R}''_{k_1}] \subset \mathscr{R}''_{k_1} .$$

Then there exist a connected differentiable manifold X, a surjective submersion $\rho: X \rightarrow Y$, a formally transitive and formally integrable Lie equation $R^{\#}_1 \subset J_1(T; \rho)$, a formally integrable Lie equation $R_1 \subset R^{\#}_1$ and integers $m_0 \geq k_1$, $l_0 \geq 0$ such that the following assertions hold:

(i) *the equations $R^{\#}_1$, R_1 satisfy conditions* (I), (II) *and* (III) *of* § 9 *with respect to the submersion ρ;*

(ii) *$R^{\#}_1$ is a prolongation of $R''^{\#}_{k_1}$ and R_1 is a prolongation of R''_{k_1};*

(iii) $[\tilde{\mathscr{R}}^{\#}_2, \mathscr{R}_1] \subset \mathscr{R}_1$;

(iv) *$\pi_0 \tilde{R}_1$ is an integrable sub-bundle W of T and $R^{\#}_1 \subset J_1(T; W)$, $R_1 \subset J_1(W)$;*

(v) *$g''^{\#}_{m_0}$, g''_{m_0}, $g^{\#}_{m_0}$, g_{m_0} are 2-acyclic and*

(17.19) $$\pi_m(R^\sharp_{m+l} \cap J_{m+l}(V)) = \pi_m(R_{m+l} \cap J_{m+l}(V)) = 0 ,$$

(17.20) $$\pi_m(R^\sharp_{m+l} \cap J_{m+l}(W)) = R_m ,$$

for all $m \geq m_0$, $l \geq l_0$;

(vi) *for all* $a \in X$, *the subspace* $R_{\infty,a}$ *of* $R^\sharp_{\infty,a}$ *is a closed ideal defined by the foliation* $J_0(W)_a$ *in* $(R^\sharp_{\infty,a}, R^{\sharp 0}_{\infty,a})$;

(vii) *if* $a \in X$ *and* $L^\flat_a$ *denotes the transitive Lie algebra* $R^\sharp_{\infty,a}/R_{\infty,a}$, *the image* $L^{\flat 0}_a$ *of* $R^{\sharp 0}_{\infty,a}$ *in* $L^\flat_a$ *is a fundamental subalgebra and*

$$H^{j,m}(L^\flat_a/L^{\flat 0}_a, \operatorname{gr} L^\flat_a) = 0 ,$$

for $j = 1, 2$ *and all* $m \geq m_0 - 1$; *for all* $a, b \in X$, *there are an isomorphism* $L^\flat_a \to L^\flat_b$ *of transitive Lie algebras and an isomorphism of graded Lie algebras*

$$\operatorname{gr} L^\flat_a \to \operatorname{gr} L^\flat_b ;$$

(viii) *for all* $x \in X$, *there are a neighborhood* U *of* x, *a differentiable manifold* Z, *a surjective submersion* $\tau\colon U \to Z$, *a formally transitive and formally integrable Lie equation* $R^\flat_{m_0} \subset J_{m_0}(T_Z; Z)$ *such that*:

(a) $W_{|U}$ *is the bundle of vectors tangent to the fibers of* τ;

(b) *the equation* $R^\sharp_1 \subset J_1(T; \tau)$ *on* U *satisfies conditions* (I), (II) *and* (III) *of* § 9 *with respect to the submersion* τ *and*

$$\tau(R^\sharp_{m,a}) = R^\flat_{m,\tau(a)} ,$$

for all $m \geq m_0$, $a \in U$;

(c) *for all* $a \in U$, *the sequence*

$$0 \longrightarrow R_{\infty,a} \longrightarrow R^\sharp_{\infty,a} \xrightarrow{\tau} R^\flat_{\infty,\tau(a)} \longrightarrow 0$$

is exact and the mapping τ *determines an isomorphism of pairs of topological Lie algebras*

$$(L^\flat_a, L^{\flat 0}_a) \to (R^\flat_{\infty,\tau(a)}, R^{\flat 0}_{\infty,\tau(a)}) ;$$

(d) $g^\flat_{m_0}$ *is* 2-*acyclic*.

Proof. Let $y_0 \in Y$ and set $L = R''^\sharp_{\infty,y_0}$, $L^0 = R''^{\sharp 0}_{\infty,y_0}$; by [10, Proposition 10.1], there exists an integer $k \geq k_1$ such that the closed ideal R''_{∞,y_0} of $R''^\sharp_{\infty,y_0}$ is defined by a foliation in $(L, D^k_L L^0)$. According to Theorem 12.1, there exist a connected differentiable manifold X, a surjective submersion $\rho\colon X \to Y$, a formally transitive and formally integrable ρ-projectable Lie equation $R^\sharp_1 \subset J_1(T; \rho)$ and a formally integrable ρ-projectable Lie equation $R_1 \subset R^\sharp_1$ such that (ii) and (iii) hold and such that, for all $a \in X$, with $\rho(a) = y_0$,

$$\rho\colon (R^\sharp_{\infty,a}, R^{\sharp 0}_{\infty,a}) \to (L, D^k_L L^0)$$

is an isomorphism of pairs of topological Lie algebras; moreover $\pi_0\tilde{R}_1$ is an integrable sub-bundle W of T and $R_1 \subset J_1(W; \rho)$. By [10, Proposition 10.3 and Lemma 10.3 (ii)], we see that $\pi_m(R^\#_{m+l} \cap J_{m+l}(V))$ and $\pi_m(R_{m+l} \cap J_{m+l}(V))$ are sub-bundles of $R^\#_m$ for all $m \geq 1$, $l \geq 0$, and that $V \cap W$ is a sub-bundle of T, and so (i) holds. From (ii) and [6, Theorem 1], we now obtain integers $p_1 \geq 1$, $l_1 \geq 0$ such that (17.19) holds for all $m \geq p_1$, $l \geq l_1$. From (iii), we deduce that

$$[\mathscr{R}^\#_1, J_0(\mathscr{W})] \subset J_0(\mathscr{W}) .$$

As we have seen above, this implies that $R^\#_1 \subset J_1(T; W)$, and we have a formally integrable Lie equation $N_{k_0} \subset R^\#_{k_0}$ with $k_0 \geq p_1$, and an integer $l_0 \geq l_1$ such that

$$N_m = \pi_m(R^\#_{m+l} \cap J_{m+l}(W)) ,$$

for all $m \geq k_0$, $l \geq l_0$, and

$$N_\infty = R^\#_\infty \cap J_\infty(W) .$$

Then $R_\infty \subset N_\infty$ and thus $\pi_0 N_{k_0} = J_0(W)$. If $a \in X$ satisfies $\rho(a) = y_0$, by the choice of integer k and the construction of N_{k_0}, the closed ideals $R_{\infty,a}$ and $N_{\infty,a}$ of $R^\#_{\infty,a}$ are both defined by the foliation $J_0(W)_a$ in $(R^\#_{\infty,a}, R^{\#0}_{\infty,a})$; we therefore obtain the equality $N_{\infty,a} = R_{\infty,a}$. Consequently $N_m = R_m$ for all $m \geq k_0$, and

$$R_\infty = R^\#_\infty \cap J_\infty(W) .$$

From the discussion preceding the proposition, we obtain an integer $p_2 \geq k_0$ such that (vi) and (vii) hold with m_0 replaced by p_2. Finally, let $m_0 \geq p_2$ be an integer such that $g''^\#_{m_0}$, g''_{m_0}, $g^\#_{m_0}$, g_{m_0} are 2-acyclic. Assertion (viii) follows also from the above discussion.

Remark. If $y \in Y$ and $x \in X$ satisfy $\rho(x) = y$, then the transitive Lie algebras $R''^\#_{\infty,y}/R''_{\infty,y}$ and $L^\flat_x$ are isomorphic. If $R''^\#_{\infty,y}/R''_{\infty,y}$ is finite-dimensional, by (vii) there is an integer $m_1 \geq m_0$ such that $L^{\flat m}_a/L^{\flat m+1}_a = 0$, for all $m \geq m_1 - 1$, $a \in X$; then $g^\flat_m = 0$ for $m \geq m_1$.

We continue to consider the objects of Proposition 17.6. Let $P''^\#_{k_1}$, $P''_{k_1} \subset Q_{k_1}(Y)$ be formally integrable finite forms of the Lie equations $R''^\#_{k_1}$, R''_{k_1} on Y, and

$$P^\#_1 \subset Q_1(\rho) \cap Q_1(X; W) , \qquad P_1 \subset Q_1(\rho) \cap Q_1(W)$$

be formally integrable finite forms of the Lie equations $R^\#_1$, R_1 on X. Let $y \in Y$ and $x \in X$ with $\rho(x) = y$; consider the submersion τ defined on a neighborhood U of x and the Lie equation $R^\flat_{m_0}$ on the manifold Z given by (viii). Let $P^\flat_{m_0} \subset Q_{m_0}(Z)$ be a formally integrable finite form of $R^\flat_{m_0}$. According to § 9, for $m \geq m_0$ we have the commutative diagram of cohomology

$$
\begin{array}{ccccc}
H^1(P_1)_{m,x} & \longrightarrow & H^1(P_1^\sharp)_{m,x} & \xrightarrow{\tau} & H^1(P^\flat_{m_0})_{m,\tau(x)} \\
\downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho} & & \\
H^1(P''_{k_1})_{m,y} & \longrightarrow & H^1(P''^\sharp_{k_1})_{m,y} & &
\end{array}
\tag{17.21}
$$

whose horizontal arrows in the left-hand square are induced by inclusions of Lie equations and whose top row is a complex, in view of (viii) and (17.20). Moreover the mappings ρ satisfy the assertions of Theorem 9.3 (with $m_1 = m_0$).

Now suppose that Y is endowed with a structure of a real-analytic manifold compatible with its structure of differentiable manifold and that $R''^\sharp_{k_1}$, R''_{k_1} are analytic equations. We may assume that X, Z are real-analytic manifolds, that τ is an analytic submersion and that all Lie equations considered and their finite forms are analytic. Suppose moreover that for some point $y_0 \in Y$ the transitive Lie algebra $R''^\sharp_{\infty,y_0}/R''_{\infty,y_0}$ is elliptic; by (vii) so is $L^\flat_a$, for all $a \in X$, and by (viii) so is $R^\flat_{\infty,z}$ for all $z \in Z$. Theorem 16.4 (iii) tells us that $R^\flat_{m_0}$ is an elliptic equation. For $m \geq m_0$, by Theorem 17.5 (i) the mapping $H^1(P_1)_{m,x} \to H^1(P_1^\sharp)_{m,x}$ is surjective and by Theorem 9.3 (i) so are the mappings ρ of diagram (17.21). Therefore using the commutativity of this diagram, we see that the mapping

$$
H^1(P''_{k_1})_{m,y} \to H^1(P''^\sharp_{k_1})_{m,y}
\tag{17.22}
$$

is also surjective for $m \geq m_0$. Next, let $\alpha \in H^1(P''_{k_1})_{m,y}$ with $m \geq m_0$, and assume that its image in $H^1(P''^\sharp_{k_1})_{m,y}$ vanishes. According to Proposition 17.1, choose $\alpha_1 \in H^1(P''_{k_1})_{m+l,y}$, with $l \geq l_0$, satisfying $\pi_m\alpha_1 = \alpha$; by Theorem 9.3 (i) choose $\beta \in H^1(P_1)_{m+l,x}$ satisfying $\rho\beta = \alpha_1$, and let γ be the image of β in $H^1(P_1^\sharp)_{m+l,x}$. From the commutativity of (17.21), we deduce that $\pi_m\rho\gamma = 0$; since $P''^\sharp_{k_1}$ is integrable, by Proposition 7.6 we infer that $\rho\gamma = 0$. Hence by Theorem 9.3 (ii), we have $\pi_m\gamma = 0$. Theorem 17.5 (ii) implies that $\pi_m\beta = 0$; therefore $\alpha = \rho\pi_m\beta = 0$. These facts imply that $H^1(P''_{k_1})_y = 0$ if and only if $H^1(P''^\sharp_{k_1})_y = 0$.

We no longer assume that the equations $R''^\sharp_{k_1}$, R''_{k_1} are analytic. We now suppose that for some point $y_0 \in Y$ the Lie algebra $R''^\sharp_{\infty,y_0}/R''_{\infty,y_0}$ is finite-dimensional; according to the remark following Proposition 17.6, there is an integer $m_1 \geq m_0$ depending only on $R''^\sharp_{k_1}$ and R''_{k_1} such that $g^\flat_m = 0$ for $m \geq m_1$. By Theorem 17.6 (ii), the above argument concerning the surjectivity of (17.22) shows that this mapping is surjective for $m \geq m_1$. Let $\alpha_1, \alpha_2 \in H^1(P''_{k_1})_{m+l,y}$, where $m \geq m_1$, $l = 2l_0 + 1$, have the same image in $H^1(P''^\sharp_{k_1})_{m+l,y}$; we shall now show that $\pi_m\alpha_1 = \pi_m\alpha_2$. Indeed, according to Theorem 9.3 (i) choose $\beta_1, \beta_2 \in H^1(P_1)_{m+l,x}$ satisfying $\rho\beta_1 = \alpha_1$, $\rho\beta_2 = \alpha_2$. By the commutativity of (17.21), the images γ_1, γ_2 of β_1, β_2 in $H^1(P_1^\sharp)_{m+l,x}$ verify $\rho\gamma_1 = \rho\gamma_2$, and so by Theorem 9.3 (ii) we have $\pi_{m+l_0}\gamma_1 = \pi_{m+l_0}\gamma_2$. Therefore $\pi_{m+l_0}\beta_1$, $\pi_{m+l_0}\beta_2$ have the same image in $H^1(P_1^\sharp)_{m+l_0,x}$; from Theorem 17.6 (iii), we deduce that $\pi_m\beta_1 = \pi_m\beta_2$ and hence that $\pi_m\alpha_1 = \pi_m\alpha_2$. The injectivity of the mapping $H^1(P''_{k_1})_y \to H^1(P''^\sharp_{k_1})_y$ is an immediate consequence of the property of the mappings (17.22) we have just verified. To prove that it is surjective, it suffices by the Mittag-Leffler theorem (see [1, § 3, No. 5, Corollary 2]) to show that if $(\beta_m) \in H^1(P''^{\#}_{k_1})_y$, with $\beta_m \in H^1(P''^{\#}_{k_1})_{m,y}$, $m \geq k_1$, then, for all $m \geq m_1$ and all $r \geq m + 2l_0 + 1$ and for $\alpha \in H^1(P''_{k_1})_{m+2l_0+1,y}$ whose image in $H^1(P''^{\#}_{k_1})_{m+2l_0+1,y}$ is equal to β_{m+2l_0+1}, there exists $\alpha' \in H^1(P''_{k_1})_{r,y}$ whose image in $H^1(P''^{\#}_{k_1})_{r,y}$ is equal to β_r and which satisfies $\pi_m\alpha' = \pi_m\alpha$. To verify that this condition is satisfied, we choose $\alpha' \in H^1(P''_{k_1})_{r,y}$ whose image in $H^1(P''^{\#}_{k_1})_{r,y}$ is equal to β_r. Then $\pi_{m+2l_0+1}\alpha'$ and α have the same image β_{m+2l_0+1} in $H^1(P''^{\#}_{k_1})_{m+2l_0+1,y}$. Hence by the above, $\pi_m\alpha' = \pi_m\alpha$. Finally, if $P''^{\#}_{k_1}$ is integrable and the image of $\alpha \in H^1(P''_{k_1})_{m,y}$, with $m \geq m_1$, vanishes in $H^1(P''^{\#}_{k_1})_{m,y}$, by Proposition 17.1 choose $\alpha_1 \in H^1(P''_{k_1})_{m+l,y}$, with $l = 2l_0 + 1$, satisfying $\pi_m\alpha_1 = \alpha$. Then the image β_1 of α_1 in $H^1(P''^{\#}_{k_1})_{m+l,y}$ satisfies $\pi_m\beta_1 = 0$. By Proposition 7.6, we see that $\beta_1 = 0$. Thus the two elements α_1 and 0 of $H^1(P''_{k_1})_{m+l,y}$ have the same image in $H^1(P''^{\#}_{k_1})_{m+l,y}$; therefore $\alpha = \pi_m\alpha_1 = 0$.

We state the above results as the two following theorems:

Theorem 17.7. *Assume that X is a connected real-analytic manifold. Let $R^{\#}_k$ be an analytic formally transitive and formally integrable Lie equation, and let $R_k \subset R^{\#}_k$ be a formally integrable Lie equation such that*

$$[\tilde{\mathscr{R}}^{\#}_{k+1}, \mathscr{R}_k] \subset \mathscr{R}_k .$$

Let $P^{\#}_k$ and P_k be formally integrable finite forms of $R^{\#}_k$ and R_k respectively. If $x \in X$ and $R^{\#}_{\infty,x}/R_{\infty,x}$ is an elliptic transitive Lie algebra, then there is an integer $m_0 \geq k$ such that, for all $m \geq m_0$, $a \in X$, we have:

(i) *the mapping of cohomology*

$$H^1(P_k)_{m,a} \to H^1(P^{\#}_k)_{m,a}$$

is surjective;

(ii) *if the image of $\alpha \in H^1(P_k)_{m,a}$ vanishes in $H^1(P^{\#}_k)_{m,a}$, then $\alpha = 0$;*

(iii) *$H^1(P_k)_a = 0$ if and only if $H^1(P^{\#}_k)_a = 0$.*

Theorem 17.8. *Assume that X is connected. Let $R^{\#}_k$ be a formally transitive and formally integrable Lie equation, and let $R_k \subset R^{\#}_k$ be a formally integrable Lie equation such that*

$$[\tilde{\mathscr{R}}^{\#}_{k+1}, \mathscr{R}_k] \subset \mathscr{R}_k .$$

Let $P^{\#}_k$ and P_k be formally integrable finite forms of $R^{\#}_k$ and R_k respectively. If $x \in X$ and $R^{\#}_{\infty,x}/R_{\infty,x}$ is finite-dimensional, then there are integers $m_1 \geq k$, $l_1 \geq 1$ such that, for all $m \geq m_1$, $l \geq l_1$, $a \in X$, we have:

(i) *the mapping of cohomology*

$$H^1(P_k)_{m,a} \to H^1(P^{\#}_k)_{m,a}$$

is surjective;

(ii) *if* $\alpha_1, \alpha_2 \in H^1(P_k)_{m+l,a}$ *have the same image in* $H^1(P_k^\sharp)_{m+l,a}$, *then* $\pi_m\alpha_1 = \pi_m\alpha_2$; *if* $P_k^\sharp$ *is integrable and the image of* $\alpha \in H^1(P_k)_{m,a}$ *in* $H^1(P_k^\sharp)_{m,a}$ *vanishes, then* $\alpha = 0$;

(iii) *the mapping of cohomology*

$$H^1(P_k)_a \to H^1(P_k^\sharp)_a$$

is an isomorphism of cohomology.

Remark. Let $R'_k \subset R_k^\sharp$ be a formally integrable Lie equation satisfying

$$[\tilde{\mathscr{R}}_{k+1}^\sharp, \mathscr{R}'_k] \subset \mathscr{R}'_k\ , \qquad R_k \subset R'_k\ .$$

Then in Theorems 17.7 and 17.8, we may replace the equation $R_k^\sharp$ by R'_k.

We now give consequences of some results of this section concerning the cohomology of transitive Lie algebras and their closed ideals.

Theorem 17.9. *Let L be a real transitive Lie algebra, and I a closed elliptic ideal of L. Then*

$$H^j(L, I) = 0 \quad for\ j > 0\ , \quad \tilde{H}^1(L, I) = 0\ .$$

Proof. By [9, Corollary 6.1] and [10, Theorem 10.1], there exist a formally transitive and formally integrable analytic Lie equation $R_k^\sharp \subset J_k(T)$ on a connected analytic manifold X, a point $x \in X$, and a formally integrable Lie equation $R_k \subset R_k^\sharp$ such that $[\tilde{\mathscr{R}}_{k+1}^\sharp, \mathscr{R}_k] \subset \mathscr{R}_k$ and $(R_{\infty,x}^\sharp, R_{\infty,x})$ and (L, I) are isomorphic as pairs of topological Lie algebras. By Theorem 16.4 (iii), R_k is an elliptic equation; therefore from Proposition 17.4 and Theorem 17.1, we obtain the desired vanishing of cohomology.

Theorem 17.10. *Let* $\phi\colon L \to L''$ *be an epimorphism of real transitive Lie algebras, and* $I \subset L$, $I'' \subset L''$ *be closed ideals of L and L″ such that* $\phi(I) = I''$. *Let I′ be the closed ideal of L which is the kernel of* $\phi\colon I \to I''$. *Assume that I″ is an elliptic ideal of L″. Then we have an isomorphism of cohomology*

$$H^j(L, I') \to H^j(L, I)\ , \qquad for\ j > 0\ ,$$

and a mapping of cohomology

$$\tilde{H}^1(L, I') \to \tilde{H}^1(L, I)\ . \tag{17.23}$$

If the image of $\alpha \in \tilde{H}^1(L, I')$ *under the mapping* (17.23) *vanishes, then* $\alpha = 0$; *moreover,* $\tilde{H}^1(L, I') = 0$ *if and only if* $\tilde{H}^1(L, I) = 0$. *If I″ is finite-dimensional, the mapping* (17.23) *is an isomorphism of cohomology.*

Proof. We apply Theorem 10.1 to $\phi\colon L \to L''$ and to the ideals I, I' of L and I'' of L'', and consider the various objects and relations connecting them whose existence is asserted by that theorem. We may assume that the kernels of $\pi_{k-1}\colon N_k \to J_{k-1}(T)$, $\pi_{k-1}\colon N'_k \to J_{k-1}(T)$ and $\pi_{k_1-1}\colon N''_{k_1} \to J_{k_1-1}(T_Y; Y)$ are 2-acyclic. Let $P_k \subset Q_k(\rho)$, $P'_k \subset Q_k(V)$ and $P''_{k_1} \subset Q_{k_1}(Y)$ be formally integrable analytic finite forms of $N_k \subset J_k(T; \rho)$, $N'_k \subset J_k(V)$ and $N''_{k_1} \subset J_{k_1}(T_Y; Y)$ respectively. By Theorem 16.4 (iii), N''_{k_1} is an elliptic equation; if I'' is finite-dimensional, then N''_{k_1} is of finite type. Theorem 17.2, Theorem 17.5 (ii) and (iii) and Theorem 17.6 (iv) give us the desired result.

Corollary 17.2. *Let $\phi: L \to L''$ be an epimorphism of real transitive Lie algebras, and let J be the kernel of ϕ. Assume that L'' is elliptic. Then we have an isomorphism of cohomology*

$$H^j(L, J) \to H^j(L) , \qquad \text{for } j > 0 ,$$

and a mapping of cohomology

$$\tilde{H}^1(L, J) \to \tilde{H}^1(L) . \tag{17.24}$$

If the image of $\alpha \in \tilde{H}^1(L, J)$ under the mapping (17.24) *vanishes, then $\alpha = 0$; moreover, $\tilde{H}^1(L, J) = 0$ if and only if $\tilde{H}^1(L, I) = 0$. If L'' is finite-dimensional, the mapping* (17.24) *is an isomorphism of cohomology.*

18. The cohomology and structure of abelian Lie equations

We begin by recalling the construction of abelian Lie equations given at the beginning of § 11 in the case where $Z = Y$ and σ is the identity mapping of Y.

Let X be an affine bundle A over Y, whose associated vector bundle we denote by F, and let $\rho: X \to Y$ be the projection of the affine bundle A onto Y. If V is the integrable sub-bundle of T of vectors tangent to the fibers of ρ, we have a canonical morphism of vector bundles $\lambda: V \to F$ over ρ such that the corresponding mapping

$$\lambda: V \to \rho^{-1}F \tag{18.1}$$

is an isomorphism of vector bundles over X (see [4, Proposition 3.6]). A section f of F over Y determines a diffeomorphism $\gamma_f: X \to X$ sending x into $x + f(\rho(x))$ and a vector field

$$\mu_f = \frac{d}{dt}\gamma_{tf}\Big|_{t=0}$$

on X, which is a section of $\mathscr{V}_\lambda$. If f_1, f_2 are sections of F over Y, then

$$\gamma_{f_1} \circ \gamma_{f_2} = \gamma_{f_2} \circ \gamma_{f_1} = \gamma_{f_1+f_2} , \tag{18.2}$$

$$[\mu_{f_1}, \mu_{f_2}] = 0 . \tag{18.3}$$

The mapping

$$\lambda: J_k(V; \lambda) \to J_k(F; Y)$$

induced by (18.1) is a morphism of vector bundles over ρ sending $j_k(\mu_f)(x)$ into $j_k(f)(y)$, where $x \in X$ and $y = \rho(x)$, such that the corresponding mapping

$$\lambda\colon J_k(V;\lambda) \to \rho^{-1}J_k(F;Y)$$

is an isomorphism of vector bundles over X. Then by (18.3), we have

$$[J_k(V;\lambda), J_k(V;\lambda)] = 0 , \tag{18.4}$$

and $J_1(V;\lambda)$ is a formally integrable abelian Lie equation.

The image $Q_k(V;\lambda)$ of the injective mapping

$$\gamma\colon \rho^{-1}J_k(F;Y) \to Q_k(V) ,$$

sending $(x, j_k(f)(y))$, with $y = \rho(x)$, into $j_k(\gamma_f)(x)$, is a sub-bundle of $Q_k(V)$ and a finite form of $J_k(V;\lambda)$. We set $\tilde{\mathscr{Q}}_k(V;\lambda) = \tilde{\mathscr{Q}}_k \cap \mathscr{Q}_k(V;\lambda)$. Let

$$\alpha\colon Q_k(V;\lambda) \to J_k(V;\lambda) ,$$
$$\beta\colon Q_k(V;\lambda) \to J_k(F;Y)$$

be the mappings sending $j_k(\gamma_f)(x)$ into $j_k(\mu_f)(x)$ and $j_k(f)(y)$ respectively, where $y = \rho(x)$. Then the induced mapping

$$\beta\colon Q_k(V;\lambda) \to \rho^{-1}J_k(F;Y)$$

sends $j_k(\gamma_f)(x)$ into $(x, j_k(f)(y))$ and $\beta = \lambda \circ \alpha$.

We have

$$j_{k+1}(\gamma_{f_1})(x)(j_k(\mu_{f_2})(x)) = j_k(\mu_{f_2})(x + f_1(y)) ,$$

so if $\phi \in Q_{k+1}(V;\lambda)$, the diagram

$$\begin{array}{ccc} J_k(V;\lambda)_a & \xrightarrow{\ \phi\ } & J_k(V;\lambda)_c \\ \big\downarrow \lambda & & \big\downarrow \lambda \\ J_k(F;Y)_b & \xrightarrow{\ \mathrm{id}\ } & J_k(F;Y)_b \end{array} \tag{18.5}$$

commutes, where $a =$ source ϕ, $c =$ target ϕ and $b = \rho(a)$. If $\phi \in \tilde{\mathscr{Q}}_{k+1}(V;\lambda)_a$, $u \in (\bigwedge^j \mathscr{T}^* \otimes J_k(\mathscr{V};\lambda))_{\lambda,a}$, and λu is the element v of $(\bigwedge^j \mathscr{T}^*_Y \otimes J_k(\mathscr{F};Y))_b$, where $b = \rho(a)$, then, since $\pi_0\phi \in \tilde{\mathscr{Q}}_0(V)$, we see that $\phi^{-1}(u)$ is the unique element of $(\bigwedge^j \mathscr{T}^* \otimes J_k(\mathscr{V};\lambda))_{\lambda,c}$ satisfying $\lambda(\phi^{-1}(u)) = v$, where $c =$ source $\pi_0\phi(a)^{-1}$ and $\rho(c) = b$. In particular if $\pi_0\phi(a) = a$, then $\phi^{-1}(u) = u$.

We shall identify $J_0(F;Y)$ with F. If $u \in T^* \otimes J_k(V;\lambda)$, then $u \in (T^* \otimes J_k(V;\lambda))^\wedge$ if and only if the element $\lambda + \lambda(\pi_0 u)$ of $V^* \otimes_X F$ is invertible, where $\lambda(\pi_0 u)$ is defined by

$$\lambda(\pi_0 u)(\xi) = \lambda\pi_0 u(\xi) , \qquad \text{for } \xi \in V .$$

Consequently

$$(\mathcal{T}^* \otimes J_k(\mathcal{V}; \lambda))_\lambda \subset (\mathcal{T}^* \otimes J_k(\mathcal{V}; \lambda))^\wedge . \tag{18.6}$$

By [4, Proposition 5.1], it is easily verified that the diagram

$$\begin{array}{ccc} T^* \otimes \tilde{J}_k(V; \lambda) & \xrightarrow{\partial^{-1}} & J_1(Q_k(V; \lambda)) \\ \Big\downarrow{\scriptstyle \mathrm{id}\otimes\nu} & & \Big\downarrow{\scriptstyle J_1(\alpha)} \\ T^* \otimes J_k(V; \lambda) & \xrightarrow{\varepsilon} & J_1(J_k(V; \lambda)) \end{array}$$

is commutative, where the mapping ∂^{-1} sends $u \in (T^* \otimes \tilde{J}_k(V; \lambda))_x$, with $x \in X$, into $j_1(I_k)(x) + u$. Let $\phi \in \tilde{\mathcal{Q}}_{k+1}(V; \lambda)_x$ with $x \in X$; if $\phi(x) = j_{k+1}(\gamma_s)(x)$, where s is a section of F over Y and $x' = \gamma_s(x)$, by (2.27) and (1.2) we obtain

$$\begin{aligned} \varepsilon(\mathcal{D}\phi)(x) &= J_1(\alpha)\cdot\partial^{-1}\cdot(\mathrm{id}\otimes\nu^{-1})(\mathcal{D}\phi)(x) \\ &= J_1(\alpha)((\lambda_1\phi(x)^{-1})\cdot j_1(\pi_k\phi)(x)) \\ &= J_1(\alpha)(j_1(j_k(\gamma_{-s}))(x')\cdot j_1(\pi_k\phi)(x)) \\ &= j_1(\alpha(j_k(\gamma_{-s})\cdot\pi_k\phi))(x) \\ &= j_1(j_k(\mu_{-s}) + \alpha(\pi_k\phi))(x) \\ &= j_1(\alpha(\pi_k\phi))(x) - j_1(j_k(\mu_s))(x) \\ &= (\varepsilon D\alpha(\phi))(x) . \end{aligned}$$

We have thus shown that the left-hand square of the diagram

$$\begin{array}{ccccc} \tilde{\mathcal{Q}}_{k+1}(V;\lambda) & \xrightarrow{\mathcal{D}} & (\mathcal{T}^*\otimes J_k(\mathcal{V};\lambda))^\wedge & \xrightarrow{\mathcal{D}_1} & \bigwedge^2\mathcal{T}^*\otimes J_{k-1}(\mathcal{V};\lambda) \\ \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle \mathrm{id}} \\ J_{k+1}(\mathcal{V};\lambda) & \xrightarrow{D} & \mathcal{T}^*\otimes J_k(\mathcal{V};\lambda) & \xrightarrow{D} & \bigwedge^2\mathcal{T}^*\otimes J_{k-1}(\mathcal{V};\lambda) \end{array} \tag{18.7}$$

is commutative; the commutativity of the right-hand square is a consequence of (18.4), and $\phi \in \mathcal{Q}_{k+1}(V; \lambda)$ belongs to $\tilde{\mathcal{Q}}_{k+1}(V; \lambda)$ if and only if $D\alpha(\phi)$ belongs to $(\mathcal{T}^* \otimes J_k(\mathcal{V}; \lambda))^\wedge$ (see Proposition 11.1). From (18.6) and [6, Proposition 4 (ii)], it follows that

$$\mathcal{Q}_{k+1}(V; \lambda)_\beta \subset \tilde{\mathcal{Q}}_{k+1}(V; \lambda) ,$$

and that, for $a \in X$ with $b = \rho(a)$, the diagram

$$\begin{array}{ccccc} \mathcal{Q}_{k+1}(V;\lambda)_{\beta,a} & \xrightarrow{\mathcal{D}} & (\mathcal{T}^*\otimes J_k(\mathcal{V};\lambda))_{\lambda,a} & \xrightarrow{\mathcal{D}_1} & (\bigwedge^2\mathcal{T}^*\otimes J_{k-1}(\mathcal{V};\lambda))_{\lambda,a} \\ \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \lambda} & & \Big\downarrow{\scriptstyle \lambda} \\ J_{k+1}(\mathcal{F};Y)_b & \xrightarrow{D} & (\mathcal{T}_Y^*\otimes J_k(\mathcal{F};Y))_b & \xrightarrow{D} & (\bigwedge^2\mathcal{T}_Y^*\otimes J_{k-1}(\mathcal{F};Y))_b \end{array} , \tag{18.8}$$

whose vertical arrows are bijective, is commutative. Moreover, from (18.2) we deduce that if $\phi, \psi \in \mathscr{Q}_{k+1}(V; \lambda)_{\beta,a}$, then $\phi \cdot \psi \in \mathscr{Q}_{k+1}(V; \lambda)_{\beta,a}$ and

$$\beta(\phi \cdot \psi) = \beta(\phi) + \beta(\psi) .$$

If $u \in (\mathscr{T}^* \otimes J_k(\mathscr{V}; \lambda))_{\lambda,a}$, $\phi \in \tilde{\mathscr{Q}}_{k+1}(V; \lambda)_a$ with $\pi_0\phi(a) = a$, then as $\phi^{-1}(u) = u$, we have

$$u^{\phi} = u + \mathscr{D}\phi = u + D\alpha(\phi) . \tag{18.9}$$

Thus if $\phi \in \mathscr{Q}_{k+1}(V; \lambda)_{\beta,a}$, then by Lemma 3.1, u^{ϕ} belongs to $(\mathscr{T}^* \otimes J_k(\mathscr{V}; \lambda))_{\lambda}$.

The first statement of the following lemma should be compared with Lemma 6.5 and the second with Proposition 6.4 (ii). Here we consider the mapping

$$\lambda\colon T^* \otimes J_k(V; \lambda) \to V^* \otimes_X J_k(F; Y) .$$

Lemma 18.1. (i) *Let* $\phi \in \tilde{\mathscr{Q}}_{k+1}(V; \lambda)$ *and* $u \in \mathscr{T}^* \otimes J_k(\mathscr{V}; \lambda)$. *Then* $\lambda(u) = 0$ *if and only if* $\lambda(u^{\phi}) = \pi_k \cdot d_{X/Y}\beta(\phi)$.

(ii) *Let* $u_1, u_2 \in (\mathscr{T}^* \otimes J_k(\mathscr{V}; \lambda))_{\lambda,a}$ *and* $\phi \in \tilde{\mathscr{Q}}_{k+1}(V; \lambda)_a$, *with* $a \in X$ *and* $\pi_0\phi(a) = a$. *If* $u_2 = u_1^{\phi}$, *then* $\phi \in \mathscr{Q}_{k+1}(V; \lambda)_{\beta}$.

Proof. (i) By the commutativity of (18.5) and (3.2), by (18.7) and the fact that $f = \pi_0\phi$ preserves V,

$$\lambda(u^{\phi}) = \lambda(u) \circ f + \lambda(D\alpha(\phi)) = \lambda(u) \circ f + \pi_k \cdot d_{X/Y}\beta(\phi) ,$$

as elements of $\mathscr{V}^* \otimes J_k(\mathscr{F}; Y)_X$. Now $\lambda(u) = 0$ if and only if $\lambda(u) \circ f = 0$, which is equivalent to $\lambda(u^{\phi}) = \pi_k \cdot d_{X/Y}\beta(\phi)$.

(ii) By (18.9), $D\alpha(\phi)$ belongs to $(\mathscr{T}^* \otimes J_k(\mathscr{V}; \lambda))_{\lambda}$; Lemma 3.1 implies that $\alpha(\phi) \in J_{k+1}(\mathscr{V}; \lambda)_{\lambda}$ and hence that $\phi \in \mathscr{Q}_{k+1}(V; \lambda)_{\beta}$.

Let $N_k \subset J_k(F; Y)$ be a formally integrable differential equation. Let $R_{k+l} \subset J_{k+l}(V; \lambda)$ be the inverse image of $\rho^{-1}N_{k+l}$ under the isomorphism $\lambda\colon J_{k+l}(V; \lambda) \to \rho^{-1}J_{k+l}(F; Y)$. According to [6, Proposition 5 (ii)], $R_{k+l} = (R_k)_{+l}$ for $l \geq 0$, and R_k is formally integrable. Let $k_0 \geq k$ be an integer such that g_{k_0} is 2-acyclic. By Proposition 17.5 (ii), the natural mapping

$$H^1_{\lambda}(R_k)_{m,a} \to H^1(R_k)_{m,a} \tag{18.10}$$

is an isomorphism for all $m \geq k_0$, $a \in X$, and so determines an isomorphism

$$\lambda\colon H^1(R_k)_{m,a} \longrightarrow H^1(N_k)_{m,\rho(a)} \tag{18.11}$$

for all $m \geq k_0$, $a \in X$. Moreover, according to [6, Theorem 3] the mapping

$$\lambda\colon H^*(R_k)_a \to H^*(N_k)_{\rho(a)} , \qquad \text{for } a \in X ,$$

given by (17.7), is an isomorphism.

By (18.4), we have

$$[R_{k+l}, R_{k+l}] = 0 \ , \qquad \text{for all } l \geq 0 \ ;$$

therefore by [19, Proposition 4.4], R_k is an abelian Lie equation, and the graded Lie algebra $H^*(R_k)_a$ is abelian for $a \in X$. Let $P_{k+l} = \alpha^{-1}(R_{k+l})$; by (18.2), P_{k+l} is a groupoid. If $a \in X$ and f is a section of F over a neighborhood of $b = \rho(a)$ such that $j_{k+l}(f)(b) \in N_{k+l}$, then the element of $\tilde{R}_{k+l,a}$

$$\tilde{j}_{k+l}(\mu_f)(a) = \frac{d}{dt} j_{k+l}(\gamma_{tf})(a)\Big|_{t=0}$$

belongs to $V_{I_{k+l}(a)}(P_{k+l})$, since $j_{k+l}(\gamma_{tf})(a) \in P_{k+l}$. Thus $\tilde{R}_{k+l,a} \subset V_{I_{k+l}(a)}(P_{k+l})$; as the dimensions of these vector spaces are equal, we see that P_{k+l} is a finite form of R_{k+l}. It can easily be seen that $P_{k+l} = (P_k)_{+l}$ and that P_k is a formally integrable finite form of R_k.

For $m \geq k$, let

$$Z^1_\lambda(R_m) = Z^1(R_m) \cap (\mathscr{T}^* \otimes \mathscr{R}_m)_\lambda \ ;$$

then by (18.7)

$$Z^1_\lambda(R_m) = \{u \in (\mathscr{T}^* \otimes \mathscr{R}_m)_\lambda \mid Du = 0\} \ .$$

For $a \in x$, let

$$\mathscr{P}_{m,\beta} = \mathscr{P}_m \cap \mathscr{Q}_m(V; \lambda)_\beta \ ,$$
$$\mathscr{P}^{\cdot}_{m,\beta,a} = \mathscr{P}_{m,\beta,a} \cap \tilde{\mathscr{P}}^{\cdot}_{m,a} \ .$$

For $m \geq k$, $a \in X$, according to (18.8) and (18.9), the group $\mathscr{P}^{\cdot}_{m+1,\beta,a}$ operates on $Z^1_\lambda(R_m)_a$ and the set of orbits

$$H^1_\lambda(P_k)_{m,a} = Z^1_\lambda(R_m)_a / \mathscr{P}^{\cdot}_{m+1,\beta,a}$$

under the right operations of the group $\mathscr{P}^{\cdot}_{m+1,\beta,a}$ on $Z^1_\lambda(R_m)_a$ is the quotient of the vector space $Z^1_\lambda(R_m)_a$ by its subspace

$$\{Du \mid u \in \mathscr{R}_{m+1,\lambda,a}, \ u(a) = 0\} \ .$$

The cohomology $H^1_\lambda(P_k)_{m,a}$ is therefore a vector space. We have the mapping of cohomology

$$H^1_\lambda(P_k)_{m,a} \to H^1(P_k)_{m,a} \tag{18.12}$$

which sends the class of $u \in Z^1_\lambda(R_m)_a$ in $H^1_\lambda(P_k)_{m,a}$ into the orbit $\{u^F \mid F \in \tilde{\mathscr{P}}^{\cdot}_{m+1,a}\}$.

The proof of the following theorem is analogous to that of Theorem 9.1, although it is considerably simpler.

Theorem 18.1. *Let $a \in X$ with $b = \rho(a)$ and $m \geq k_0$. The mapping* (18.12) *is* *an isomorphism of cohomology. Moreover, if* $u \in Z^1(R_m)_a$, *there exists* $F \in \tilde{\mathscr{P}}^{\cdot}_{m+1,a}$ *such that* $u^F(a) = 0$ *and* $u^F \in Z^1_\lambda(R_m)_a$.

Proof. If $u_1, u_2 \in Z^1_\lambda(R_m)_a$ and if $\phi \in \tilde{\mathscr{P}}^{\cdot}_{m+1,a}$ satisfy $u_1^\phi = u_2$, then by Lemma 18.1 (ii), ϕ belongs to $\mathscr{P}^{\cdot}_{m+1,\beta,a}$ and so (18.12) is injective.

Let $u \in Z^1(R_m)_a$; then since g_m is 2-acyclic, by [5, Theorem 2] there exists $u_1 \in Z^1(R_{m+2})_a$ such that $\pi_m u_1 = u$. By Lemma 7.1, there exists $\phi_1 \in \tilde{\mathscr{P}}^{\cdot}_{m+3,a}$ such that $u_1^{\phi_1}(a) = 0$. We set $u_2 = u_1^{\phi_1}$; then $Du_2 = \mathscr{D}_1 u_2 = 0$, and the element $w = \pi_{m+1}\lambda(u_2)$ of $(\mathscr{V}^* \otimes \mathscr{N}_{m+1,X})_a$ satisfies $w(a) = 0$ and $d_{X/Y} w = 0$, by the commutativity of diagram (3.2). There exists $\bar{v} \in \mathscr{N}_{m+1,X,a}$ such that $j_1(\bar{v})(a) = 0$ and $d_{X/Y}\bar{v} = w$. Choose $v \in \mathscr{R}_{m+2,a}$ satisfying $\lambda(\pi_{m+1}v) = \bar{v}$ and $j_1(v)(a) = 0$. If $\phi_2 = \alpha^{-1}(v) \in \mathscr{P}_{m+2,a}$, since $j_1(\phi_2)(a) = j_1(I_{m+2})(a)$, we see that ϕ_2 belongs to $\tilde{\mathscr{P}}^{\cdot}_{m+2,a}$ and that $(\mathscr{D}\phi_2)(a) = 0$. Set $u_3 = (\pi_{m+1}u_2)^{\phi_2^{-1}}$. As $u_2(a) = 0$, we have $u_3(a) = 0$ and

$$\lambda(u_3^{\phi_2}) = w = \pi_{m+1} \cdot d_{X/Y}\beta(\phi_2) ;$$

it follows from Lemma 18.1 (i) that $\lambda(u_3) = 0$ or equivalently that

$$u_3 \in F^1_1(J_{m+1}(\mathscr{V}; \lambda)) .$$

Since $Du_3 = \mathscr{D}_1 u_3 = 0$, by [6, Proposition 4 (i)] we know that $u_4 = \pi_m u_3$ belongs to $(\mathscr{T}^* \otimes J_m(\mathscr{V}; \lambda))_\lambda$. Finally, we note that $u_4 = u^\phi$ and $u_4(a) = 0$, where $\phi = \pi_{m+1}\phi_1 \cdot \pi_{m+1}\phi_2^{-1} \in \tilde{\mathscr{P}}^{\cdot}_{m+1,a}$. Hence $u_4 \in Z^1_\lambda(R_m)_a$ belongs to the same cohomology class in $H^1(P_k)_{m,a}$ as u, showing that (18.12) is surjective and completing the proof of the theorem.

We have a mapping of vector spaces

$$H^1_\lambda(P_k)_{m,a} \to H^1_\lambda(R_k)_{m,a} , \tag{18.13}$$

for $m \geq k$, $a \in X$, which is clearly surjective. By means of the isomorphisms (18.12), (18.10) and (18.11), for $a \in X$ with $b = \rho(a)$, and $m \geq k_0$, we obtain surjective mappings of cohomology

$$H^1(P_k)_{m,a} \to H^1(R_k)_{m,a} , \tag{18.14}$$

$$H^1(P_k)_{m,a} \to H^1(N_k)_{m,b} ; \tag{18.15}$$

by Proposition 7.5, these mappings give rise to surjective mappings of cohomology

$$H^1(P_k)_a \to H^1(R_k)_a , \tag{18.16}$$

$$H^1(P_k)_a \to H^1(N_k)_b . \tag{18.17}$$

Theorem 18.2. *Let* $a \in X$ *and* $b = \rho(a)$. *Assume that* N_k *is integrable.*

(i) *For* $m \geq k_0$, *the mappings* (18.14)-(18.17) *are isomorphisms of cohomology.*

(ii) *If* $m \geq k$ *and* $u \in (\mathcal{T}^* \otimes \mathcal{R}_m)_a^\wedge$ *satisfies* $Du = 0$, *then the cohomology class of* u *in* $H^1(P_k)_{m,a}$ *vanishes if and only if the cohomology class of* u *in* $H^1(R_k)_{m,a}$ *vanishes.*

Proof (cf. Proposition 11.2). Let $u \in (\mathcal{T}^* \otimes \mathcal{R}_m)_a$, with $m \geq k$, satisfy $u = Dv$ for some $v \in \mathcal{R}_{m+1,a}$. Then $\lambda v(a) \in N_{m+1,b}$, and we can write $\lambda v(a) = j_{m+1}(f)(b)$ for some solution f of N_k over a neighborhood of b. We see that $\xi = \mu_f$ is a λ-projectable section of V over a neighborhood of a which is a solution of R_k and satisfies $j_{m+1}(\xi)(a) = v(a)$. If we also denote by ξ the germ of ξ in $\mathcal{V}_a$, clearly $j_{m+1}(\xi) \in \mathcal{R}_{m+1,\lambda,a}$ and $v_1 = v - j_{m+1}(\xi)$ belongs to $\mathcal{R}_{m+1,a}$ and satisfies $v_1(a) = 0$ and $Dv_1 = u$. If v belongs to $\mathcal{R}_{m+1,\lambda,a}$ so does v_1, showing that (18.13) is injective for all $m \geq k$; this last fact implies (i). By the commutativity of (18.7), if $u \in (\mathcal{T}^* \otimes \mathcal{R}_m)_a^\wedge$, the equations $Dv_1 = u$, $v_1(a) = 0$, with $v_1 \in \mathcal{R}_{m+1,a}$, are equivalent to $\mathcal{D}\phi = u$, $\phi(a) = I_{m+1}(a)$, with $\phi = \alpha^{-1}(v_1) \in \tilde{\mathcal{P}}_{m+1,a}$, and thus (ii) holds.

It follows from Theorem 18.2 (i) that the mappings

$$\pi_m \colon H^1(P_k)_{m+1,a} \to H^1(P_k)_{m,a}$$

are isomorphisms of cohomology for all $m \geq k_0$, $a \in X$.

We shall now construct the formally transitive and formally integrable Lie equation $A_2 \subset J_2(T)$ corresponding to the pseudogroup of transformations of X whose restriction to a fiber of ρ is an affine mapping of that fiber to another. For $x \in X$, we shall endow $J_\infty(F; Y)_{\rho(x)}$ with the structure of a geometric module over the transitive Lie algebra $A_{\infty,x}$.

Let $\{f_1, \cdots, f_r\}$ be a frame for F and $\sigma\colon Y \to X$ a section of ρ over an open subset U of Y. Then, for $x \in \rho^{-1}(U)$, we can write

$$x = \sigma(\rho(x)) + \sum_{i=1}^{r} x^i f_i \,,$$

thus defining functions $x^1, \cdots, x^r$ on $\rho^{-1}(U)$. Let $(y^1, \cdots, y^q)$ be a system of coordinates on U; we write for simplicity $y^j = y^j \circ \rho$. Clearly $(x^1, \cdots, x^r, y^1, \cdots, y^q)$ is a system of coordinates for X on $\rho^{-1}(U)$ and $\mu_{f_i} = \partial/\partial x^i$, for $1 \leq i \leq r$. If $f = \sum_{i=1}^r c^i f_i$ is a section of F over U, then

$$\mu_f = \sum_{i=1}^{r} (c^i \circ \rho)\mu_{f_i} = \sum_{i=1}^{r} (c^i \circ \rho)\frac{\partial}{\partial x^i} \tag{18.18}$$

on $\rho^{-1}(U)$.

Let $\xi \in J_{m+1}(T)_x$ where $x \in \rho^{-1}(U)$; there exist functions $a^1, \cdots a^r, b^1, \cdots, b^q$ on a neighborhood of x such that

$$\xi = \sum_{i=1}^{r} j_{m+1}\Big(a^i \frac{\partial}{\partial x^i}\Big)(x) + \sum_{l=1}^{q} j_{m+1}\Big(b^l \frac{\partial}{\partial y^l}\Big)(x) \,.$$

For $1 \leq j \leq r$, we have

$$\left[\xi, j_{m+1}\left(\frac{\partial}{\partial x^j}\right)(x)\right]$$
$$= \sum_{i=1}^{r} j_m\left(\left[a^i \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]\right)(x) + \sum_{l=1}^{q} j_m\left(\left[b^l \frac{\partial}{\partial y^l}, \frac{\partial}{\partial x^j}\right]\right)(x)$$
$$= -\sum_{i=1}^{r} j_m\left(\frac{\partial a^i}{\partial x^j} \frac{\partial}{\partial x^i}\right)(x) - \sum_{l=1}^{q} j_m\left(\frac{\partial b^l}{\partial x^j} \frac{\partial}{\partial y^l}\right)(x) .$$

If

(18.19) $$[\xi, j_{m+1}(\mu_{f_j})(x)] \in J_m(V; \lambda) , \qquad \text{for } 1 \leq j \leq r ,$$

there exist sections $f_{(j)} = \sum_{i=1}^{r} c_j^i f_i$ of F over U such that

$$\left[\xi, j_{m+1}\left(\frac{\partial}{\partial x^j}\right)(x)\right] = -j_m(\mu_{f_{(j)}})(x) = -\sum_{i=1}^{r} j_m\left((c_j^i \circ \rho) \frac{\partial}{\partial x^i}\right)(x) ,$$

by (18.18). We deduce that

$$j_m\left(\frac{\partial a^i}{\partial x^j}\right)(x) = j_m(c_j^i \circ \rho)(x) , \qquad j_m\left(\frac{\partial b^l}{\partial x^j}\right)(x) = 0 ,$$

for $1 \leq i, j \leq r$, $1 \leq l \leq q$; hence we can find functions $d^1, \cdots, d^r, \bar{b}^1, \cdots, \bar{b}^q$ defined on U such that

$$j_{m+1}(a^i)(x) = j_{m+1}\left(d^i \circ \rho + \sum_{j=1}^{r} x^j \cdot (c_j^i \circ \rho)\right)(x) ,$$
$$j_{m+1}(b^l)(x) = j_{m+1}(\bar{b}^l \circ \rho)(x) ,$$

for $1 \leq i \leq r$, $1 \leq l \leq q$, and

(18.20) $$\xi = j_{m+1}\left(\sum_{i=1}^{r} \left(d^i \circ \rho + \sum_{j=1}^{r} x^j \cdot (c_j^i \circ \rho)\right) \frac{\partial}{\partial x^i} + \sum_{l=1}^{q} (\bar{b}^l \circ \rho) \frac{\partial}{\partial y^l}\right)(x) .$$

Moreover, if $f = \sum_{s=1}^{r} e^s f_s$ is a section of F over U, then

$$[\xi, j_{m+1}(\mu_f)(x)] = j_m(\mu_{f'})(x) ,$$

where f' is the section

$$f' = \sum_{i=1}^{r} \left(\sum_{l=1}^{q} \bar{b}^l \frac{\partial e^i}{\partial y^l} - \sum_{j=1}^{r} c_j^i e^j\right) f_i$$

of F over U. Thus, if we set

$$A_{m+1} = \{\xi \in J_{m+1}(T) \mid [\xi, J_{m+1}(V; \lambda)] \subset J_m(V; \lambda)\}$$

for $m \geq 1$, then ξ belongs to $A_{m+1,x}$, where $x \in \rho^{-1}(U)$, if and only if (18.19) holds, or equivalently if we can write ξ in the form (18.20), where c^i_j, d^i, $\bar{b}^l$ are functions on U. It is easily verified that A_2 is a formally transitive and formally integrable Lie equation, with $A_{2+l} = (A_2)_{+l}$ and $A_{2+l} \subset J_{2+l}(T; \rho)$ for $l \geq 0$; moreover

$$A_1 = \pi_1 A_2 = J_1(T; \rho) , \qquad A_0 = \pi_0 A_2 = J_0(T) ,$$

and A_2 is ρ-projectable, with

$$\rho(A_{m,x}) = J_m(T_Y; Y)_{\rho(x)} ,$$

for all $m \geq 1$, $x \in X$. For $m \geq 0$, we have $J_m(V; \lambda) \subset A_m$, and so $J_\infty(V; \lambda)_x$ is a closed abelian ideal of the transitive Lie algebra $A_{\infty,x}$ for $x \in X$. Let $B_m \subset Q_m(\rho)$ be the bundle of m-jets of ρ-projectable diffeomorphisms f of X whose restriction to the fiber $\rho^{-1}(y)$ is an affine mapping from $\rho^{-1}(y)$ to $\rho^{-1}(\rho(f(x)))$, where $x \in X$ and $y = \rho(x)$. Then B_2 is a formally integrable finite form of A_2, with $B_{2+l} = (B_2)_{+l}$ for $l \geq 0$.

Let

$$(18.21) \qquad A_{m+1} \times_Y J_{m+1}(F; Y) \to J_m(F; Y)$$

be the mapping sending (ξ, u) into $\xi \cdot u = \lambda[\xi, \lambda_x^{-1} u]$, where $x \in X$, $\xi \in A_{m+1,x}$, $u \in J_{m+1}(F; Y)_{\rho(x)}$ and λ_x^{-1} is the inverse of the isomorphism $\lambda: J_{m+1}(V; \lambda)_x \to J_{m+1}(F; Y)_{\rho(x)}$. If $x \in X$ and $y = \rho(x)$, then (18.21) induces a mapping

$$(18.22) \qquad A_{\infty,x} \otimes J_\infty(F; Y)_y \to J_\infty(F; Y)_y ,$$

which endows $J_\infty(F; Y)_y$ with the structure of a module over the Lie algebra $A_{\infty,x}$. We see that

$$A_{\infty,x} \cdot J^m_\infty(F; Y)_y \subset J^{m-1}_\infty(F; Y)_y , \qquad \text{for } m \geq 1 ,$$
$$A^0_{\infty,x} \cdot J^m_\infty(F; Y)_y \subset J^m_\infty(F; Y)_y , \qquad \text{for } m \geq 0 ,$$

and since A_2 is formally transitive,

$$J^m_\infty(F; Y)_y = D^m_{A_{\infty,x}} J^0_\infty(F; Y)_y , \qquad \text{for } m \geq 1 .$$

It follows that $J_\infty(F; Y)_y$ is a linearly compact $A_{\infty,x}$-module and, by Proposition 14.2 (iii), that $J^0_\infty(F; Y)_y$ is a fundamental subspace of $J_\infty(F; Y)_y$. Thus $J_\infty(F; Y)_y$ is a geometric $A_{\infty,x}$-module.

The following theorem gives the essential ingredients in the construction of certain Lie equations derived from abelian Lie equations; this theorem and the following lemma, namely Lemma 18.2, will be used in § 19.

Theorem 18.3. *Let $R''_q \subset J_q(T_Y; Y)$ be a formally integrable Lie equation. Assume that F is associated to $\tilde{R}''_q$ and that*

(18.23) $$R''_{q+k} \cdot N_{k+1} \subset N_k \ .$$

For all $m \geq h$, let $\mathscr{R}^\flat_m$ be an $\mathscr{O}_X$-submodule of $\mathscr{A}_m$ satisfying the following conditions:

(a) *for all $m \geq h$, we have*

$$\pi_m(\mathscr{R}^\flat_{m+1}) = \mathscr{R}^\flat_m \ , \quad D(\mathscr{R}^\flat_{m+1}) \subset \mathscr{T}^* \otimes \mathscr{R}^\flat_m \ , \quad [\mathscr{R}^\flat_{m+1}, \mathscr{R}^\flat_{m+1}] \subset \mathscr{R}^\flat_m \ ;$$

(b) *if $x \in X$ and $R^\flat_{m,x}$ denotes the image of the mapping $\mathscr{R}^\flat_{m,x} \to A_{m,x}$ sending $u \in \mathscr{R}^\flat_{m,x}$ into the value $u(x)$ of u at x, where $m \geq h$, we have*

$$\rho(R^\flat_{m,x}) = R''_{m,\rho(x)} \ , \qquad \text{for } m \geq \sup(h, q) \ ;$$

(c) *for all $x \in X$ with $y = \rho(x)$, if $R^\flat_{\infty,x} = \varprojlim R^\flat_{m,x}$, the diagram*

(18.24) $$\begin{array}{ccc} R^\flat_{\infty,x} \otimes J_\infty(F; Y)_y & \longrightarrow & J_\infty(F; Y)_y \\ \Big\downarrow \rho \otimes \mathrm{id} & & \Big\downarrow \mathrm{id} \\ R''_{\infty,y} \otimes J_\infty(F; Y)_y & \longrightarrow & J_\infty(F; Y)_y \end{array}$$

commutes, where the top horizontal arrow is the restriction of (18.22) *and the bottom horizontal arrow is given by the $R''_{\infty,y}$-module structure of $J_\infty(F; Y)_y$;*

(d) *for all $m \geq \sup(h, k)$, there is a sub-bundle $R^\sharp_m$ of A_m such that*

$$\mathscr{R}^\sharp_m = \mathscr{R}_m + \mathscr{R}^\flat_m \ .$$

Then there exists an integer $p \geq \sup(h, k)$ such that $R^\sharp_p$ is a formally integrable ρ-projectable Lie equation satisfying

$$R^\sharp_{p+l} = (R^\sharp_p)_{+l} \ , \qquad \text{for all } l \geq 0 \ ,$$

(18.25) $$[\tilde{\mathscr{R}}^\sharp_{p+1}, \mathscr{R}_p] \subset \mathscr{R}_p \ ,$$

(18.26) $$[R^\flat_{\infty,x}, R_{\infty,x}] \subset R_{\infty,x} \ ,$$

and $R^\flat_{\infty,x}$ is a closed Lie subalgebra of $R^\sharp_{\infty,x}$ for all $x \in X$, and

(18.27) $$\rho(R^\sharp_{m,x}) = R''_{m,\rho(x)} \ ,$$

for all $m \geq \sup(p, q)$, $x \in X$.

Assume moreover that the following condition holds:

(e) *for all $x \in X$, the mapping*

$$\rho : R^\flat_{\infty,x} \to R''_{\infty,\rho(x)}$$

is an isomorphism.

Then:

(i) *for all* $x \in X$, *the linearly compact Lie algebra* $R^\sharp_{\infty,x}$ *is the semi-direct product of its closed subalgebra* $R^\flat_{\infty,x}$ *and the linearly compact* $R^\flat_{\infty,x}$*-module* $R_{\infty,x}$ *and is the inessential abelian extension*

$$(18.28) \qquad 0 \longrightarrow R_{\infty,x} \longrightarrow R^\sharp_{\infty,x} \xrightarrow{\rho} R''_{\infty,\rho(x)} \longrightarrow 0$$

of the linearly compact Lie algebra $R''_{\infty,\rho(x)}$ *by the linearly compact* $R''_{\infty,\rho(x)}$*-module* $R_{\infty,x}$;

(ii) *for all* $x \in X$, *with* $y = \rho(x)$, *the diagram*

$$\begin{array}{ccc} R^\sharp_{\infty,x} \otimes J_\infty(F;Y)_y & \longrightarrow & J_\infty(F;Y)_y \\ \big\downarrow \rho\otimes\mathrm{id} & & \big\downarrow \mathrm{id} \\ R''_{\infty,y} \otimes J_\infty(F;Y)_y & \longrightarrow & J_\infty(F;Y)_y \end{array}$$

commutes, where the top horizontal arrow is the restriction of (18.22) *and sends* $R^\sharp_{\infty,x} \otimes N_{\infty,y}$ *into* $N_{\infty,y}$, *and the bottom horizontal arrow is given by the* $R''_{\infty,y}$*-module structure of* $J_\infty(F;Y)_y$;

(iii) *if* R''_q *is formally transitive and* $\pi_0\colon N_k \to F$ *is surjective, then* $R^\sharp_p$ *is formally transitive and* $R_{\infty,x}$ *is defined by the foliation* $J_0(V)_x$ *in* $(R^\sharp_{\infty,x}, R^{\sharp 0}_{\infty,x})$, *for all* $x \in X$.

Proof. From (a), we infer that

$$\pi_m(R^\flat_{m+1,x}) = R^\flat_{m,x}\ , \qquad [R^\flat_{\infty,x}, R^\flat_{\infty,x}] \subset R^\flat_{\infty,x}\ ,$$

for $m \geq h$, $x \in X$, and that $R^\flat_{\infty,x}$ is a closed Lie subalgebra of $A_{\infty,x}$. From (a) and (d), it follows that

$$R^\sharp_{m,x} = R_{m,x} + R^\flat_{m,x}\ , \quad R^\sharp_{m+1} \subset (R^\sharp_m)_{+1}\ , \quad \pi_m(R^\sharp_{m+1}) = R^\sharp_m\ ,$$

for all $m \geq \sup(h,k)$, $x \in X$. The Cartan-Kuranishi prolongation theorem (see [5, Theorem 1]) gives us an integer $p \geq \sup(h,k,2)$ such that $R^\sharp_{p+l} = (R^\sharp_p)_{+l}$ for all $l \geq 0$. Then $R^\sharp_p \subset A_p$ is a formally integrable differential equation in $J_p(T)$. From (18.23) and the commutativity of (18.24), we deduce (18.26); for $x \in X$, we have $\pi_{p+1}(R^\flat_{\infty,x}) = R^\flat_{p+1,x}$ and hence

$$[R^\flat_{p+1,x}, R_{p+1,x}] \subset R_{p,x}\ .$$

Thus by (a), we have

$$[R^\sharp_{p+1}, R_{p+1}] \subset R_p\ , \qquad [R^\sharp_{p+1}, R^\sharp_{p+1}] \subset R^\sharp_p\ .$$

Therefore by [19, Proposition 4.4], $R^\sharp_p$ is a Lie equation, and by Lemma 1.5, (18.25) holds. Since $R_p \subset J_p(V; \lambda)$, by (b), we see that $R_p^\#$ is ρ-projectable and satisfies (18.27). Now assume that condition (e) also holds. To show that (18.28) is exact, it suffices to prove that

$$(18.29) \qquad R^\#_{\infty,x} \cap J_\infty(V)_x \subset R_{\infty,x} ,$$

for $x \in X$. In fact, fix $x \in X$; for $m \geq h$ and $l \geq 0$, set

$$R_m^{(l)} = \pi_m(R^\flat_{m+l,x} \cap J_{m+l}(V)_x), \qquad \bar{R}_m = \bigcap_{l \geq 0} R_m^{(l)} .$$

Then $\pi_m(\bar{R}_{m+1}) = \bar{R}_m$ for $m \geq h$, and since $\rho\colon R^\flat_{\infty,x} \to R''_{\infty,\rho(x)}$ is an isomorphism, we have

$$\varprojlim \bar{R}_m = \varprojlim (R^\flat_{m,x} \cap J_m(V)_x) = 0 .$$

Hence $\bar{R}_m = 0$ for all $m \geq h$. Since $R_m^{(l+1)} \subset R_m^{(l)}$ and these are finite-dimensional vector spaces, for each $m \geq h$ there is an integer $l_m \geq 0$ such that $R_m^{(l_m)} = 0$ or

$$(18.30) \qquad \pi_m(R^\flat_{m+l_m,x} \cap J_{m+l_m}(V)_x) = 0 .$$

Let $\xi \in R^\#_{\infty,x} \cap J_\infty(V)_x$, and for $m \geq \sup(h, k)$ let $l = l_m$; we have $\pi_{m+l}\xi \in R^\#_{m+l,x}$ and we can write $\pi_{m+l}\xi = \eta + \zeta$, with $\eta \in R_{m+l,x}$ and $\zeta \in R^\flat_{m+l,x}$. Since $\pi_{m+l}\xi \in J_{m+l}(V)_x$, we see that $\zeta \in J_{m+l}(V)_x$. Now (18.30) implies that $\pi_m\zeta = 0$ and hence that $\pi_m\xi \in R_m$. Therefore $\xi \in R_{\infty,x}$ and so (18.29) holds. The remaining assertions of (i) are consequences of the exactness of (18.28) and the fact that $\rho\colon R^\flat_{\infty,x} \to R''_{\infty,\rho(x)}$ is an isomorphism of linearly compact Lie algebras for $x \in X$. Finally, (ii) follows from (i) and (c), and (iii) from the exactness of (18.28) and [10, Proposition 10.2].

If we are in the category of real-analytic manifolds and real-analytic mappings, if $\pi_0\colon N_k \to F$ is surjective and the equation R''_q of Theorem 18.3 is formally transitive, then the following lemma shows, under an additional assumption of coherence, that condition (d) of that theorem is implied by conditions (a)-(c).

Lemma 18.2. *Assume that Y is connected and is endowed with a structure of real-analytic manifold compatible with its structure of differentiable manifold, that A is an analytic affine bundle over Y and that $\pi_0\colon N_k \to F$ is surjective. Let $R''_q \subset J_q(T_Y; Y)$ be an analytic formally transitive and formally integrable Lie equation. Assume that F is associated to $\tilde{R}''_q$, that the mapping $R''_q \otimes J_1(F) \to F$ is analytic and that* (18.23) *holds. For all $m \geq h$, let $R^\flat_{m,\omega}$ be a coherent $\mathcal{O}_{X,\omega}$-submodule of $\mathcal{A}_{m,\omega}$ satisfying the following conditions*:

(i) *for all $m \geq h$, we have*

$$\pi_m(\mathcal{R}^\flat_{m+1,\omega}) = \mathcal{R}^\flat_{m,\omega} , \qquad D(\mathcal{R}^\flat_{m+1,\omega}) \subset \mathcal{T}^*_\omega \otimes \mathcal{R}^\flat_{m,\omega} ,$$
$$[\mathcal{R}^\flat_{m+1,\omega}, \mathcal{R}^\flat_{m+1,\omega}] \subset \mathcal{R}^\flat_{m,\omega} ;$$

(ii) *if* $x \in X$ *and* $R^\flat_{m,x}$ *denotes the image of the mapping* $\mathscr{R}^\flat_{m,\omega,x} \to A_{m,x}$ *sending* $u \in \mathscr{R}^\flat_{m,\omega,x}$ *into the value* $u(x)$ *of* u *at* x, *where* $m \geq h$, *we have*

$$\rho(R^\flat_{m,x}) = R''_{m,\rho(x)}\ , \qquad \textit{for } m \geq \sup(h, q)\ ,$$

and condition (c) *of Theorem* 18.3 *holds.*

Then for all $m \geq \sup(h, k)$, *there is an analytic sub-bundle* $R^\sharp_m \subset A_m$ *such that*

$$\mathscr{R}^\sharp_{m,\omega} = \mathscr{R}_{m,\omega} + \mathscr{R}^\flat_{m,\omega}\ .$$

Proof. The hypotheses imply that N_k, and hence also R_k, are analytic equations. For $m \geq h$, we write $\tilde{\mathscr{R}}^\flat_{m,\omega} = \nu^{-1}\mathscr{R}^\flat_{m,\omega}$; for $m \geq \sup(h, k)$,

$$\mathscr{R}^\sharp_{m,\omega} = \mathscr{R}_{m,\omega} + \mathscr{R}^\flat_{m,\omega}$$

is a coherent $\mathscr{O}_{X,\omega}$-submodule of $\mathscr{A}_{m,\omega}$ and, if $\tilde{\mathscr{R}}^\sharp_{m,\omega} = \nu^{-1}\mathscr{R}^\sharp_{m,\omega}$, we verify that

$$[\tilde{\mathscr{R}}^\sharp_{m+1,\omega}, \mathscr{R}^\sharp_{m,\omega}] \subset \mathscr{R}^\sharp_{m,\omega}\ . \tag{18.31}$$

First, since R_k is a Lie equation, we have

$$[\tilde{\mathscr{R}}_{m+1,\omega}, \mathscr{R}_{m,\omega}] \subset \mathscr{R}_{m,\omega}\ , \qquad \text{for } m \geq k\ .$$

From (i), using (1.15) we infer that

$$[\tilde{\mathscr{R}}^\flat_{m+1,\omega}, \mathscr{R}^\flat_{m,\omega}] \subset \mathscr{R}^\flat_{m,\omega}\ , \qquad \text{for } m \geq h\ .$$

Next, from (18.23) and the commutativity of (18.24), for $m \geq \sup(h, k)$, $x \in X$, we deduce

$$[R^\flat_{m+1,x}, R_{m+1,x}] \subset R_{m,x}\ ,$$

since $\pi_m(R^\flat_{\infty,x}) = R^\flat_{m,x}$, and

$$[\mathscr{R}^\flat_{m+1,\omega}, \mathscr{R}_{m+1,\omega}] \subset \mathscr{R}_{m,\omega}\ .$$

Therefore by (1.15) and (i),

$$[\tilde{\mathscr{R}}^\flat_{m+1,\omega}, \mathscr{R}_{m,\omega}] \subset \mathscr{R}_{m,\omega}\ , \qquad [\tilde{\mathscr{R}}_{m+1,\omega}, \mathscr{R}^\flat_{m,\omega}] \subset \mathscr{R}^\sharp_{m,\omega}\ ,$$

for $m \geq \sup(h, k)$, and so (18.31) holds. If $m \geq \sup(h, k)$, $x \in X$, choose $\xi_1, \cdots, \xi_r \in \tilde{\mathscr{R}}_{m+1,\omega,x}$ and $\xi_{r+1}, \cdots, \xi_n \in \tilde{\mathscr{R}}^\flat_{m+1,\omega,x}$ such that $\{\pi_0\xi_1(x), \cdots, \pi_0\xi_r(x)\}$ is a basis of V_x and $\{\rho\pi_0\xi_{r+1}(x), \cdots, \rho\pi_0\xi_n(x)\}$ is a basis of $T_{Y,\rho(x)}$. Then $\{\pi_0\xi_1(x), \cdots, \pi_0\xi_n(x)\}$ is a basis of T_x, and $\mathscr{L}(\xi_i)$ is a $\pi_0\xi_i$-derivation of $\mathscr{A}_{m,\omega,x}$ with

$$\mathscr{L}(\xi_i)(\mathscr{R}^\sharp_{m,\omega,x}) \subset \mathscr{R}^\sharp_{m,\omega,x}\ ,$$

by (18.31), for $i = 1, \cdots, n$. Since X is connected, Lemma 17.2 gives us the desired sub-bundle $R^\#_m$ of A_m.

Remark. Let $R^\flat_h \subset A_h$ be a formally integrable and ρ-projectable Lie equation, with $h \geq 2$, which is a prolongation of the equation R''_q of Theorem 18.3 satisfying the following condition:

(d′) $R_m + R^\flat_m$ is a sub-bundle of A_m, for all $m \geq \sup(h, k)$.

Then the $\mathcal{O}_X$-submodules $\mathcal{R}^\flat_{h+l} = (\mathcal{R}^\flat_h)_{+l}$ of $\mathcal{A}_{h+l}$, with $l \geq 0$, satisfy conditions (a), (b), (d) and (e) of Theorem 18.3. If the category is the real-analytic one, if Y is connected and $\pi_0: N_k \to F$ is surjective, if F is associated to $\tilde{R}''_q$ and (18.23) holds, and if R''_q is formally transitive and condition (c) of Theorem 18.3 holds, then by Lemma 18.2 condition (d′) is satisfied.

Remark. In Theorem 18.3, if we do not consider the vector bundle F and the equation N_k, and we replace the affine bundle X over Y, the abelian Lie equation R_k and A_m by any manifold X fibered over Y, a formally integrable Lie equation $R_k \subset J_k(V)$ and $J_m(T; \rho)$ respectively, and the hypotheses that (18.24) is commutative and that $\pi_0 N_k = F$ by the relations (18.26) and $\pi_0 \tilde{R}_k = V$ respectively, then the proof of Theorem 18.3 can be modified to show that all its conclusions hold, other than (ii) and the fact that (18.28) is an abelian extension. A similar remark is valid for Lemma 18.2.

We now assume that X is an open subset of the affine bundle A over Y, and that the surjective submersion $\rho: X \to Y$ is the restriction of the projection of A onto Y.

The following theorem is a partial converse of Theorem 18.3; this is made more explicit after its proof. It shows how, under certain assumptions, a formally transitive and formally integrable Lie equation $R^\#_p \subset J_p(T)$, with $p \geq k$, satisfying $R_p \subset R^\#_p$ and (18.25) gives rise to a Lie equation R''_{q_0} on Y to which the vector bundle F is associated in such a way that

$$R''_{q_0+k} \cdot N_{k+1} \subset N_k \ .$$

Theorem 18.4. *Assume that $\pi_0: N_k \to F$ is surjective and that N_k is integrable. Let $R^\#_p \subset J_p(T)$ be a formally integrable Lie equation, with $p \geq k$, satisfying*

$$[\tilde{\mathcal{R}}^\#_{p+1}, \mathcal{R}_p] \subset \mathcal{R}_p \ .$$

(i) *For all $l \geq 0$, we have*

$$R^\#_{p+l} \subset A_{p+l} \ .$$

(ii) *If $x \in X$, the subspace $R^\#_{\infty,x} \cap J_\infty(V; \lambda)_x$ of $R^\#_{\infty,x}$ is a closed abelian ideal. If X is connected and $R^\#_p$ is formally transitive, and if*

$$R^\#_{\infty,x} \cap J_\infty(V; \lambda)_x = R^\#_{\infty,x} \cap J_\infty(V)_x \tag{18.32}$$

for some $x \in X$, *then the equality*

$$R^\sharp_\infty \cap J_\infty(V;\lambda) = R^\sharp_\infty \cap J_\infty(V) \tag{18.33}$$

holds.

(iii) *Assume that* X *and the fibers of* ρ *are connected and that* $R^\sharp_p$ *is formally transitive; then* $R^\sharp_p$ *is* ρ*-projectable. Let* $R''_q \subset J_q(T;Y)$ *be the formally transitive and formally integrable Lie equation such that*

$$\rho(R^\sharp_{m,x}) = R''_{m,\rho(x)} \tag{18.34}$$

holds for all $m \geq \sup(p,q)$ *and* $x \in X$. *If* $R_p \subset R^\sharp_p$, *and if* (18.33) *holds and* $R^\sharp_p$ *is integrable, then there exists an integer* $q_0 \geq q$ *such that* F *is associated to* $\tilde{R}''_{q_0}$,

$$R''_{q_0+k}\cdot N_{k+1} \subset N_k \ ,$$

and assertion (ii) *of Theorem* 18.3 *holds.*

Proof. (i) We set

$$R^\sharp_m = \pi_m R^\sharp_p \ , \qquad \text{for } 0 \leq m \leq p \ ,$$
$$R_m = \pi_m R_k \ , \quad N_m = \pi_m N_k \ , \qquad \text{for } 0 \leq m \leq k \ .$$

We have $\lambda(R_{m,a}) = N_{m,\rho(a)}$ for all $m \geq 0$, $a \in X$. Let $y \in Y$. Since N_k is integrable, there exists a frame $\{f_1, \cdots, f_r\}$ for F consisting of solutions of N_k over a neighborhood U of y; then $\{\mu_{f_1}, \cdots, \mu_{f_r}\}$ is a frame for V consisting of solutions of R_k over $\rho^{-1}(U)$. By Lemma 1.5,

$$[R^\sharp_{m+1}, R_{m+1}] \subset R_m \ , \qquad \text{for all } m \geq 0 \ ; \tag{18.35}$$

therefore any element $\xi \in R^\sharp_{m+1,x}$, with $x \in \rho^{-1}(y)$, satisfies (18.19) and thus belongs to A_{m+1} if $m \geq 1$. Therefore

$$R^\sharp_m \subset A_m$$

for $m \geq 2$.

(ii) The first assertion is a consequence of (i) and (18.4). Assume that X is connected. By [10, Lemma 10.3 (ii)], $R^\sharp_m$, R_m and N_m are vector bundles for all $m \geq 0$. Let $l_0 \geq 0$, $p_0 \geq 1$ be the integers and $R'_m \subset R^\sharp_m$ be the Lie equations given by [5, Theorem 1] and [10, Proposition 10.3 (ii)] satisfying

$$\begin{gathered} R'_m = \pi_m(R^\sharp_{m+l_0} \cap J_{m+l_0}(V)) = \pi_m(R^\sharp_\infty \cap J_\infty(V)) \ , \\ R'_{m+r} \subset (R'_m)_{+r} \ , \qquad \pi_m R'_{m+r} = R'_m \ , \\ [\tilde{\mathscr{R}}^\sharp_{m+1}, \mathscr{R}'_m] \subset \mathscr{R}'_m \ , \\ R'_{p_0+r} = (R'_{p_0})_{+r} \ , \end{gathered} \tag{18.36}$$

for all $m, r \geq 0$. From (i) and Lemma 1.5, it follows that

(18.37) $$[\mathscr{R}^{\#}_{m+1}, J_m(\mathscr{V}; \lambda)] \subset J_m(\mathscr{V}; \lambda) ,$$

for $m \geq 1$. If (18.32) holds, then $R'_{m,x} \subset J_m(V; \lambda)_x$; by [10, Lemma 10.3 (i)], relations (18.36) and (18.37) imply that $R'_m \subset J_m(V; \lambda)$ for $m \geq 0$ and that

$$R^{\#}_{\infty} \cap J_{\infty}(V) \subset J_{\infty}(V; \lambda) .$$

(iii) By [10, Corollary 11.1] and (i), $R^{\#}_p$ is ρ-projectable; then $R''_m = \pi_m R''_q$ is a formally transitive Lie equation on Y, and (18.34) holds for all $m \geq 0$ and $x \in X$. From (18.35), we obtain a mapping

$$R^{\#}_{m+1} \times_Y N_{m+1} \to N_m ,$$

which is the restriction of (18.21), and a mapping

$$R^{\#}_{\infty,x} \otimes N_{\infty,\rho(x)} \to N_{\infty,\rho(x)} ,$$

for $x \in X$, which is the restriction of (18.22). Assume that (18.33) holds. For $x \in X$, with $y = \rho(x)$, and $m \geq 0$, consider the mappings

(18.38) $$\begin{aligned} &R''_{m+l_0+1,y} \otimes N_{m+1,y} \to N_{m,y} , \\ &R''_{m+l_0+1,y} \otimes J_{m+1}(F; Y)_y \to J_m(F; Y)_y , \end{aligned}$$

sending $\xi \otimes u$ into $\xi \cdot u = \pi_{m+1}\xi' \cdot u$, where $\xi' \in R^{\#}_{m+l_0+1,x}$ satisfies $\rho\xi' = \xi$. If $\xi'' \in R^{\#}_{m+l_0+1,x}$ satisfies $\rho\xi'' = \xi$, then $\xi' - \xi'' \in R^{\#}_{m+l_0+1} \cap J_{m+l_0+1}(V)$, and $\pi_{m+1}(\xi' - \xi'')$ belongs to R'_{m+1} and hence to $J_{m+1}(V; \lambda)$; by (18.4)

$$\pi_{m+1}(\xi' - \xi'') \cdot u = 0 , \qquad \text{for } u \in J_{m+1}(F; Y)_y ,$$

and so the mappings (18.38) are well-defined. If $R_p \subset R^{\#}_p$, we now show that the mappings (18.38) depend only on y and not on the choice of the point x of the fiber $\rho^{-1}(y)$. Indeed, let $P^{\#}_p$ be a formally integrable finite form of $R^{\#}_p$, whose l-th prolongation we denote by $P^{\#}_{p+l}$. Then, for $m \geq p$, the intersection $P_m \cap P^{\#}_m$ is a neighborhood of I_m in P_m. Since the fibers of ρ are connected and $\pi_0\colon R_k \to J_0(V)$ is surjective, given $a, b \in X$ with $\rho(a) = \rho(b)$, we see that there exists $\phi \in P_{m+l_0+2} \cap P^{\#}_{m+l_0+2}$ with source $\phi = a$ and target $\phi = b$; we have

$$\phi(R^{\#}_{m+l_0+1,a}) = R^{\#}_{m+l_0+1,b} ,$$

and $\phi \in Q_{m+l_0+2}(V; \lambda)$. If $\xi \in R''_{m+l_0+1,\rho(a)}$, $u \in J_{m+1}(F; Y)_{\rho(a)}$, $\xi' \in R^{\#}_{m+l_0+1,a}$, $\eta \in J_{m+1}(V; \lambda)_a$ satisfy $\rho(\xi') = \xi$, $\lambda(\eta) = u$, then by the commutativity of (18.5),

$$\begin{aligned} \pi_{m+1}\xi' \cdot u &= \lambda[\pi_{m+1}\xi', \eta] = \lambda(\pi_{m+1}\phi)([\pi_{m+1}\xi', \eta]) \\ &= \lambda[\pi_{m+1}(\phi(\xi')), (\pi_{m+2}\phi)(\eta)] = (\pi_{m+1}\phi(\xi')) \cdot u , \end{aligned}$$

since $\lambda(\pi_{m+2}\phi)(\eta) = u$. As the element $\phi(\xi')$ of $R^{\#}_{m+l_0+1,b}$ satisfies $\rho\phi(\xi') = \xi$, we see that the mappings (18.38) do not depend on $x \in \rho^{-1}(y)$ for $m \geq p$, and hence also for all $m \geq 0$. Thus the diagram

$$\begin{array}{ccc} R^{\sharp}_{m+l_0+1} \otimes J_{m+1}(V; \lambda) & \longrightarrow & J_m(V; \lambda) \\ \Big\downarrow{\scriptstyle \rho \otimes \lambda} & & \Big\downarrow{\scriptstyle \lambda} \\ R''_{m+l_0+1} \otimes J_{m+1}(F; Y) & \longrightarrow & J_m(F; Y) \end{array} \tag{18.39}$$

is commutative, where the top horizontal arrow sends $\xi \otimes \eta$ into $[\xi, \eta] = [\pi_{m+1}\xi, \eta]$, and the bottom horizontal arrow is (18.38); we deduce that

$$[\mathscr{R}^{\sharp}_{m+l_0+1,\rho}, J_{m+1}(\mathscr{V}; \lambda)_{\lambda}] \subset J_m(\mathscr{V}; \lambda)_{\lambda} \; . \tag{18.40}$$

To complete the proof of (iii), we now verify that the mappings (18.38) satisfy the following properties:

(a) for all $\xi \in R''_{m+l_0+1}$, $u \in S^{m+1}T^*_Y \otimes F$,

$$\xi \cdot \varepsilon(u) = \varepsilon(\nu^{-1}\xi \wedge \delta u) \; ;$$

(b) for all $\xi, \eta \in R''_{m+l_0+2}$, $u \in J_{m+2}(F; Y)$,

$$[\xi, \eta] \cdot \pi_{m+1}u = \pi_{m+l_0+1}\xi \cdot (\eta \cdot u) - \pi_{m+l_0+1}\eta \cdot (\xi \cdot u) \; ;$$

(c) the diagram

$$\begin{array}{ccc} R''_{m+l_0+1} \otimes J_{m+1}(F; Y) & \longrightarrow & J_m(F; Y) \\ \Big\downarrow{\scriptstyle \lambda_m \otimes \mathrm{id}} & & \Big\downarrow{\scriptstyle \mathrm{id}} \\ J_m(R''_{l_0+1}; Y) \otimes J_{m+1}(F; Y) & \longrightarrow & J_m(F; Y) \end{array} \tag{18.41}$$

commutes, where the top horizontal arrow is (18.38), and the bottom horizontal arrow sends $j_m(\xi)(y) \otimes j_{m+1}(s)(y)$ into $j_m(\xi \cdot j_1(s))(y)$, with $\xi \in \mathscr{R}''_{l_0+1,y}$, $s \in \mathscr{F}_y$ and $y \in Y$.

Indeed, if $\xi \in R''_{m+l_0+1,y}$, $u \in (S^{m+1}T^*_Y \otimes F)_y$ with $y \in Y$, choose $x \in \rho^{-1}(y)$ and $\xi' \in R^{\sharp}_{m+l_0+1,x}$, $u' \in (S^{m+1}T^* \otimes V)_{\lambda,x}$ satisfying $\rho\xi' = \xi$ and $\lambda u' = u$; then by (1.15) and the commutativity of the diagrams (17.9) and (17.10) with $E = V$ and $\varphi = \lambda$, we have

$$\begin{aligned} \xi \cdot \varepsilon(u) &= \lambda[\pi_{m+1}\xi', \varepsilon u'] = \lambda\varepsilon(\nu^{-1}\xi' \wedge \delta u') \\ &= \varepsilon\lambda(\nu^{-1}\xi' \wedge \delta u') = \varepsilon(\nu^{-1}\rho\xi' \wedge \delta\lambda u') = \varepsilon(\nu^{-1}\xi \wedge u) \; , \end{aligned}$$

and so (a) holds. Next, if $\xi, \eta \in R''_{m+l_0+2,y}$, $u \in J_{m+2}(F; Y)_y$ and $\xi', \eta' \in R^{\sharp}_{m+l_0+2,x}$, $u' \in J_{m+2}(V; \lambda)_x$, with $x \in X$ and $y = \rho(x)$, satisfy $\rho\xi' = \xi$, $\rho\eta' = \eta$ and $\lambda u' = u$, then by (6.5), $\rho[\xi', \eta'] = [\xi, \eta]$ and by the Jacobi identity,

$$
\begin{aligned}
[\xi, \eta]\cdot\pi_{m+1}u &= \pi_{m+1}[\xi', \eta']\cdot\pi_{m+1}u = \lambda[\pi_{m+1}[\xi', \eta'], \pi_{m+1}u'] \\
&= \lambda([\pi_{m+1}\xi', [\pi_{m+2}\eta', u']] - [\pi_{m+1}\eta', [\pi_{m+2}\xi', u']]) \\
&= \pi_{m+1}\xi'\cdot\lambda[\pi_{m+2}\eta', u'] - \pi_{m+1}\eta'\cdot\lambda[\pi_{m+2}\xi', u'] \\
&= \pi_{m+1}\xi'\cdot(\pi_{m+2}\eta'\cdot u) - \pi_{m+1}\eta'\cdot(\pi_{m+2}\xi'\cdot u) \\
&= \pi_{m+l_0+1}\xi\cdot(\eta\cdot u) - \pi_{m+l_0+1}\eta\cdot(\xi\cdot u) ,
\end{aligned}
$$

and (b) is verified. Finally, by [9, Proposition 5.4] we have

$$\lambda_m(R^{\#}_{l_0+m+1}) \subset J_m(R^{\#}_{l_0+1}) , \qquad \lambda_m(R''_{l_0+m+1}) \subset J_m(R''_{l_0+1}; Y) ,$$

and so diagram (18.41) is well-defined; in fact, since $R^{\#}_p$ is integrable

$$\lambda_m(R^{\#}_{l_0+m+1}) \subset J_m(R^{\#}_{l_0+1}; \rho) .$$

Consider the diagram

$$
\begin{array}{ccccc}
R^{\#}_{m+l_0+1}\otimes J_{m+1}(V;\lambda) & \xrightarrow{\lambda_m\otimes\mathrm{id}} & J_m(R^{\#}_{l_0+1};\rho)\otimes J_{m+1}(V;\lambda) & \longrightarrow & J_m(V;\lambda) \\
\downarrow{\scriptstyle \rho\otimes\lambda} & & \downarrow{\scriptstyle \rho\otimes\lambda} & & \downarrow{\scriptstyle \lambda} \\
R''_{m+l_0+1}\otimes J_{m+1}(F;Y) & \xrightarrow{\lambda_m\otimes\mathrm{id}} & J_m(R''_{l_0+1};Y)\otimes J_{m+1}(F;Y) & \longrightarrow & J_m(F;Y) ,
\end{array}
$$

whose second top horizontal arrow sends $j_m(\xi)(x)\otimes j_{m+1}(\eta)(x)$ into $j_m([\xi, j_1(\eta)])(x)$, with $\xi \in \mathscr{R}^{\#}_{l_0+1,\rho,x}$, $\eta \in \mathscr{V}_{\lambda,x}$, $x \in X$, and is well-defined by (18.40), and whose second bottom horizontal arrow is the bottom horizontal arrow of diagram (18.41). The left-hand square is clearly commutative, and the right-hand one commutes because of the commutativity of (18.39) with $m = 0$. The composition of the arrows of the top row is equal to the top arrow of diagram (18.39). Therefore, by the commutativity of (18.39), the composition of the arrows of the bottom row is equal to the bottom arrow of (18.39), and we have proved (c).

If $\tilde{\xi} \in \Gamma(Y, \tilde{R}''_{m+l_0+1})$, we define

$$\mathscr{L}(\tilde{\xi}) \colon J_m(\mathscr{F}; Y) \to J_m(\mathscr{F}; Y)$$

to be the differential operator sending u into the element $\mathscr{L}(\tilde{\xi})u$ given by (15.24), where $u' \in J_{m+1}(\mathscr{F}; Y)$ satisfies $\pi_m u' = u$ and $\xi = \nu\tilde{\xi}$. From properties (a) and (b), it follows that $J_m(F; Y)$ is associated to $\tilde{R}''_{m+l_0+1}$. If $q_0 = \sup(q, l_0 + 1)$, then $J_m(F; Y)$ is associated to $\tilde{R}''_{q_0+m}$ by setting

$$\mathscr{L}(\tilde{\xi})u = \mathscr{L}(\pi_{m+l_0+1}\tilde{\xi})u ,$$

for $\tilde{\xi} \in \Gamma(Y, \tilde{R}''_{q_0+m})$, $u \in J_m(\mathscr{F}; Y)$. Property (c) implies that these operators $\mathscr{L}(\tilde{\xi})$ acting on $J_m(\mathscr{F}; Y)$ are precisely the ones arising from the action of $\tilde{\mathscr{R}}''_{q_0}$ on $\mathscr{F}$. The remaining properties of this action are immediate consequences of those of the mappings (18.38).

Remark. If $R_p^\#$ is formally transitive and $R_p \subset R_p^\#$, then, for all $x \in X$, the Lie algebras $R_{\infty,x}$ and $R_{\infty,x}^\# \cap J_\infty(V;\lambda)_x$ are closed abelian ideals of the transitive Lie algebra $R_{\infty,x}^\#$, and

$$R_{\infty,x} \subset R_{\infty,x}^\# \cap J_\infty(V;\lambda)_x .$$

Under these conditions, since $\pi_0 R_\infty = J_0(V)$, if $x \in X$ and $R_{\infty,x}$ is defined by the foliation $J_0(V)_x$ in $(R_{\infty,x}^\#, R_{\infty,x}^{\#0})$, then (18.32) holds.

Theorem 18.4 is a partial converse of Theorem 18.3. Indeed, let $X = A$ and $R_q'' \subset J_q(T_Y; Y)$ be a Lie equation on a connected manifold Y, and for all $m \geq h$ let $\mathscr{R}_m^\flat$ be an $\mathscr{O}_X$-submodule of $\mathscr{A}_m$ satisfying conditions (a)-(e) of Theorem 18.3. Assume moreover that N_k, $R_m^\#$ are integrable for $m \geq \sup(h, k)$, that $\pi_0 \colon N_k \to F$ is surjective and that R_q'' is formally transitive. Then the formally transitive Lie equation $R_p^\# = R_p + R_p^\flat$ given by Theorem 18.3 satisfies (18.25) and (18.33). Therefore all the assumptions in Theorem 18.4 are satisfied; the Lie equation R_{q_0}'' on Y, obtained from Theorem 18.4 to which F is associated, is none other than a prolongation of our original equation R_q''.

The following theorem describes the structures of graded module induced in the cohomology corresponding to the equations of Theorems 18.3 and 18.4.

Theorem 18.5. *Let $R_p^\# \subset A_p$, $R_q'' \subset J_q(T_Y; Y)$ be formally integrable Lie equations, with $p \geq k$, satisfying*

$$R_p \subset R_p^\# , \quad [\tilde{\mathscr{R}}_{p+1}^\#, \mathscr{R}_p] \subset \mathscr{R}_p , \quad \rho(R_{m,x}^\#) = R_{m,\rho(x)}'' ,$$

for all $m \geq \sup(p, q)$ and $x \in X$. Assume that the sequence (18.28) *is exact for all $x \in X$, that F is associated to $\tilde{R}_q''$, and that* (18.23) *and assertion* (ii) *of Theorem* 18.3 *hold. Then for $x \in X$, the linearly compact Lie algebra $R_{\infty,x}^\#$ is an abelian extension of $R_{\infty,\rho(x)}''$ by the linearly compact $R_{\infty,\rho(x)}''$-module $N_{\infty,\rho(x)}$. Moreover, if $R_p^\#$ satisfies condition* (III) *of* § 9, *the mapping*

$$\rho\colon H^*(R_p^\#)_x \to H^*(R_q'')_{\rho(x)} , \tag{18.42}$$

given by (17.7), *is a morphism of graded Lie algebras, and $H^*(R_k)_x$ is a graded $H^*(R_p^\#)_x$-module, and $H^*(N_k)_{\rho(x)}$ a graded $H^*(R_q'')_{\rho(x)}$-module; if $\lambda\colon H^*(R_k)_x \to H^*(N_k)_{\rho(x)}$ is the isomorphism given by* (17.7), *we have*

$$\lambda(\alpha\cdot\beta) = \rho\alpha\cdot\lambda\beta , \tag{18.43}$$

for all $\alpha \in H^*(R_p^\#)_x$, $\beta \in H^*(R_k)_x$.

Proof. Since $\lambda\colon R_{\infty,x} \to N_{\infty,\rho(x)}$ is an isomorphism for $x \in X$, the first assertion is a direct consequence of the hypotheses. The structures on the Spencer cohomologies of graded Lie algebras or of graded modules over these graded Lie algebras are given by § 15. That the mapping (18.42) is a morphism of graded Lie algebras follows from (6.10). Assertion (ii) of Theorem 18.3 implies that the diagram

$$\begin{array}{ccc} R^{\#}_{q+m} \otimes J_{m+1}(V;\lambda) & \longrightarrow & J_m(V;\lambda) \\ \Big\downarrow \rho\otimes\lambda & & \Big\downarrow \lambda \\ R''_{q+m} \otimes J_{m+1}(F;Y) & \longrightarrow & J_m(F;Y) \end{array}$$

is commutative, where the top horizontal arrow sends $\xi \otimes \eta$ into $[\xi, \eta] = [\pi_{m+1}\xi, \eta]$. If $u \in \bigwedge^i T^* \otimes R^{\#}_{q+m}$ belongs to $F^i_i(J_{q+m}(T); \rho)$, and $v \in \bigwedge^j T^* \otimes J_{m+1}(V; \lambda)$ belongs to $F^j_j(J_{m+1}(V); \lambda)$ with $q + m \geq p + 1$, then we see that the element $[u, v] = [\pi_{m+1}u, v]$ of $\bigwedge^{i+j} T^* \otimes J_m(V; \lambda)$ satisfies

$$[u, v] \in F^{i+j}_{i+j}(J_m(V); \lambda) , \qquad \lambda[u, v] = \rho u \cdot \lambda v ,$$

where λ is the mapping

$$\lambda\colon F^l_l(J_r(V); \lambda) \to \bigwedge^l T^*_Y \otimes J_r(F; Y) ,$$

with $l = i$ and $r = m + 1$, or $l = i + j$ and $r = m$, and where the product of $\rho u \in \bigwedge^i T^*_Y \otimes R''_{q+m}$ and λv is given by (15.30). We deduce that, if $u \in (\bigwedge^i \mathscr{T}^* \otimes \mathscr{R}^{\#}_{q+m})_\rho$ and $v \in (\bigwedge^j \mathscr{T}^* \otimes J_{m+1}(\mathscr{V}; \lambda))_\lambda$, then $[u, v] \in (\bigwedge^{i+j} \mathscr{T}^* \otimes J_m(\mathscr{V}; \lambda))_\lambda$ and

$$\lambda[u, v] = \rho u \cdot \lambda v .$$

For $m \geq p$, $x \in X$, we therefore obtain the commutative diagram

$$\begin{array}{ccc} H^*_\rho(R^{\#}_p)_{q+m,x} \otimes H^*_\lambda(R_k)_{m+1,x} & \longrightarrow & H^*_\lambda(R_k)_{m,x} \\ \Big\downarrow \rho\otimes\lambda & & \Big\downarrow \lambda \\ H^*(R''_q)_{q+m,\rho(x)} \otimes H^*(N_k)_{m+1,\rho(x)} & \longrightarrow & H^*(N_k)_{m,\rho(x)} \end{array}$$

whose horizontal arrows are induced by the bracket (1.19) and the mapping (15.30), and whose vertical arrows are given by (17.5). By [6, Theorem 3], there is an integer $m_0 \geq p$ such that the mappings

$$H^*_\lambda(R_k)_{m,x} \to H^*(R_k)_{m,x} , \qquad H^*_\rho(R^{\#}_p)_{m,x} \to H^*(R^{\#}_p)_{m,x}$$

are isomorphisms for all $m \geq m_0$; by means of these isomorphisms and the above commutative diagram, we deduce (18.43).

We now suppose throughout the remainder of this section that X is again an arbitrary manifold and that $\rho\colon X \to Y$ is a surjective submersion. We no longer assume that R_k is the abelian Lie equation constructed from the differential equation N_k.

The first part of the following theorem generalizes Theorem 11.1 when the equation N_k of Theorem 11.1 vanishes. This theorem implies that under certain assumptions an integrable abelian Lie equation $R_k \subset J_k(T)$ is locally of the type of the examples considered above.

Theorem 18.6. *Let $R_k \subset J_k(T)$ be an integrable and formally integrable abelian Lie equation such that $\pi_0 \tilde{R}_k$ is a sub-bundle V of T. Let $k_0 \geq k$ be an integer such that g_{k_0} is 2-acyclic. Then, for all $x_0 \in X$, with X replaced if necessary by a neighborhood of x_0, there exist a manifold Y, a surjective submersion $\rho: X \to Y$, an affine bundle A over Y whose associated vector bundle we denote by F, a diffeomorphism $\varphi: X \to A$ over Y of X onto an open subset of A, and an integrable and formally integrable differential equation $N_k \subset J_k(F; Y)$ such that, if we identify X with its image in A under the mapping φ, the following assertions hold*:

(i) *V is the bundle of vectors tangent to the fibers of ρ;*

(ii) *if $\lambda: V \to F$ is the canonical morphism over ρ given by the structure of affine bundle of A, we have $R_{k+l} \subset J_{k+l}(V; \lambda)$ for all $l \geq 0$;*

(iii) *the morphism $\lambda: J_{k+l}(V; \lambda) \to J_{k+l}(F; Y)$ induced by $\lambda: V \to F$ gives an isomorphism*

$$\lambda: R_{k+l,a} \to N_{k+l,\rho(a)} ,$$

for all $l \geq 0$ and $a \in X$, and $\pi_0 N_k = F$;

(iv) *if $\alpha: Q_k(V; \lambda) \to J_k(V; \lambda)$ is the isomorphism given by the structure of affine bundle of A, and P_k is the formally integrable finite form $\alpha^{-1}(R_k)$ of R_k, then the mapping λ induces isomorphisms of cohomology*

$$H^*(R_k)_a \to H^*(N_k)_b ,$$
$$H^1(P_k)_{m,a} \to H^1(R_k)_{m,a} \to H^1(N_k)_{m,b} ,$$
$$H^1(P_k)_a \to H^1(R_k)_a \to H^1(N_k)_b ,$$

for all $m \geq k_0$, $a \in X$, with $b = \rho(a)$.

Furthermore, let $R_k^\# \subset J_k(T)$ be a formally transitive and formally integrable Lie equation such that

$$R_k \subset R_k^\# , \qquad [\tilde{\mathscr{R}}_{k+1}^\#, \mathscr{R}_k] \subset \mathscr{R}_k .$$

Then, with X still replaced by this neighborhood of x_0 considered as a subset of A, we have:

(v) *for all $l \geq 0$,*

$$R_{k+l}^\# \subset A_{k+l} ,$$

and $R_k^\#$ is ρ-projectable;

(vi) *if $R_k^\#$ is integrable and the closed ideal R_{∞,x_0} of $R_{\infty,x_0}^\#$ is defined by a foliation in $(R_{\infty,x_0}^\#, R_{\infty,x_0}^{\#0})$ and if $R_q'' \subset J_q(T_Y; Y)$ is the formally transitive and formally integrable Lie equation such that*

$$\rho(R_{m,x}^\#) = R''_{m,\rho(x)} ,$$

for all $m \geq \sup(k, q)$ and $x \in X$, there exists an integer $q_0 \geq q$ such that F is associated to $\tilde{R}''_{q_0}$,

$$R''_{q_0+k}\cdot N_{k+1}\subset N_k\ ,$$

and such that assertion (ii) *of Theorem* 18.3 *holds, and for all* $x\in X$, *with* $y=\rho(x)$, *the diagram*

$$\begin{array}{ccc} R^{\#}_{\infty,x}\otimes R_{\infty,x} & \longrightarrow & R_{\infty,x} \\ \big\downarrow{\scriptstyle \rho\otimes\lambda} & & \big\downarrow{\scriptstyle \lambda} \\ R''_{\infty,y}\otimes N_{\infty,y} & \longrightarrow & N_{\infty,y} \end{array} \tag{18.44}$$

commutes, where the top horizontal arrow is given by the bracket (1.11) *and the bottom horizontal arrow is given by the* $R''_{\infty,y}$*-module structure of* $N_{\infty,y}$; *moreover the conclusions of Theorem* 18.5 *are valid.*

Proof. The existence of the objects described in the theorem satisfying (i)-(iii) follows from Theorem 11.1 (with $N_k=0$, $Z=Y$ and σ the identity mapping of Y). We may assume that the neighborhood of x_0 and the fibers of ρ are connected. Then, in combination with Theorem 18.5, Theorem 18.2 (i) gives us (iv) and Theorem 18.4 together with the remark which follows it implies (v) and (vi).

Remark. In Theorem 18.6, one may take A to be the vector bundle F considered as an affine bundle over Y.

In the two following propositions R_k denotes the Lie equation of § 17 satisfying conditions (I), (II) and (III) of § 9, and P_k is a formally integrable finite form of R_k. The equation $R'_{m_0}\subset J_{m_0}(V)$ obtained from R_k satisfies

$$[\tilde{\mathscr{R}}_{m_0+1},\mathscr{R}'_{m_0}]\subset\mathscr{R}'_{m_0}$$

and so if X is connected, by [10, Lemma 10.3 (ii)], $\pi_0\tilde{R}'_{m_0}$ is a sub-bundle of T. If in Theorems 17.5 and 17.6, R'_{m_0} is integrable and abelian, the following two propositions show that its non-linear cohomology can be replaced by its linear cohomology.

Proposition 18.1. *Under the hypotheses of Theorem* 17.5, *if* R'_{m_0} *is an abelian Lie equation, then for all* $m\geq m_0$, $a\in X$ *we have*:

(i) *a surjective mapping of cohomology*

$$H^1(R'_{m_0})_{m,a}\to H^1(P_k)_{m,a}\ ;$$

(ii) *if the image of* $\alpha\in H^1(R'_{m_0})_{m,a}$ *vanishes in* $H^1(P_k)_{m,a}$, *then* $\alpha=0$;

(iii) $H^1(R'_{m_0})_a=0$ *if and only if* $H^1(P_k)_a=0$.

Proof. By Theorem 18.6 (iv), we have isomorphisms of cohomology

$$H^1(R'_{m_0})_{m,a}\to H^1(P'_{m_0})_{m,a}$$

for all $m\geq m_0$, $a\in X$. From these isomorphisms and Theorem 17.5, the assertions of the proposition follow.

Proposition 18.2. *Under the hypotheses of Theorem* 17.6, *if R'_{m_0} is an integrable abelian Lie equation, then we have isomorphisms of cohomology*

$$H^1(R'_{m_0})_{m,a} \to H^1(P_k)_{m,a} \,, \qquad H^1(R'_{m_0})_a \to H^1(P_k)_a \,,$$

for all $m \geq m_1$, $a \in X$.

Proof. By Theorem 18.6 (iv), we have isomorphisms of cohomology

$$H^1(R'_{m_0})_{m,a} \to H^1(P'_{m_0})_{m,a}$$

and the mappings

$$\pi_m : H^1(P'_{m_0})_{m+1,a} \to H^1(P'_{m_0})_{m,a}$$

are isomorphisms of cohomology, for all $m \geq m_0$, $a \in X$. From these isomorphisms and Theorem 17.6 (ii) and (iii), we obtain the desired isomorphisms.

The final two theorems of this section are consequences of Theorems 17.7 and 17.8, and assert that, if the equation R_k of these last theorems is integrable and abelian, its non-linear cohomology can be replaced by its linear cohomology; the proofs, being similar to those of Propositions 18.1 and 18.2 respectively, will be omitted. These two theorems as well as the preceding two propositions will be used in § 19 and § 20 to derive results on the non-vanishing of non-linear cohomology.

Theorem 18.7. *Assume that X is a connected real-analytic manifold. Let $R^\sharp_k$ be an analytic formally transitive and formally integrable Lie equation, and let $R_k \subset R^\sharp_k$ be a formally integrable abelian Lie equation such that*

$$[\tilde{\mathscr{R}}^\sharp_{k+1}, \mathscr{R}_k] \subset \mathscr{R}_k \,.$$

Let $P^\sharp_k$ be a formally integrable finite form of $R^\sharp_k$. If $x \in X$ and $R^\sharp_{\infty,x}/R_{\infty,x}$ is an elliptic transitive Lie algebra, then there is an integer $m_0 \geq k$ such that, for all $m \geq m_0$, $a \in X$, we have:

(i) *a surjective mapping of cohomology*

$$H^1(R_k)_{m,a} \to H^1(P^\sharp_k)_{m,a} \,;$$

(ii) *if the image of $\alpha \in H^1(R_k)_{m,a}$ vanishes in $H^1(P^\sharp_k)_{m,a}$, then $\alpha = 0$;*

(iii) $H^1(R_k)_a = 0$ *if and only if* $H^1(P^\sharp_k)_a = 0$.

Theorem 18.8. *Assume that X is connected. Let $R^\sharp_k$ be a formally transitive and formally integrable Lie equation, and let $R_k \subset R^\sharp_k$ be an integrable and formally integrable abelian Lie equation such that*

$$[\tilde{\mathscr{R}}^\sharp_{k+1}, \mathscr{R}_k] \subset \mathscr{R}_k \,.$$

Let $P^\sharp_k$ be a formally integrable finite form of $R^\sharp_k$. If $x \in X$ and $R^\sharp_{\infty,x}/R_{\infty,x}$ is finite-dimensional, then there is an integer $m_1 \geq k$ such that, for all $m \geq m_1$, $a \in X$, we have isomorphisms of cohomology

$$H^1(R_k)_{m,a} \to H^1(P_k^\sharp)_{m,a}\,, \qquad H^1(R_k)_a \to H^1(P_k^\sharp)_a\,.$$

Remark. Let $R'_k \subset R_k^\sharp$ be a formally integrable Lie equation satisfying

$$[\tilde{\mathscr{R}}^\sharp_{k+1}, \mathscr{R}'_k] \subset \mathscr{R}'_k\,, \qquad R_k \subset R'_k\,.$$

Then in Theorems 18.7 and 18.8, we may replace the Lie equation $R_k^\sharp$ by R'_k.

19. The cohomology and realization of geometric modules

Let F be a vector bundle over Y and X be the vector bundle F considered as an affine bundle over Y, and let $\rho\colon X \to Y$ be the projection of this vector bundle F onto Y. Let $R''_q \subset J_q(T_Y; Y)$ be a formally integrable Lie equation. Assume that F is associated to $\tilde{R}''_q$.

Consider the morphism σ of Lie algebras from $\Gamma(Y, \tilde{R}''_q)$ to the algebra of ρ-projectable vector fields on X defined at the beginning of § 15 and determined by the mappings (15.2)

$$\sigma_x\colon \tilde{R}''_{q,y} \to T_x\,,$$

for $x \in X$ with $y = \rho(x)$. If $\tilde{\xi}$ is a section of $\tilde{R}''_q$ over Y, then, by (15.6), $\sigma(\tilde{\xi})$ is a solution of A_2 and $\rho\sigma(\tilde{\xi}) = \pi_0\tilde{\xi}$. For $x \in X$ with $y = \rho(x)$, we thus obtain a mapping

$$\sigma_x\colon J_m(\tilde{R}''_q; Y)_y \to A_{m,x}\,,$$

sending $j_m(\tilde{\xi})(y)$ into $j_m(\sigma(\tilde{\xi}))(x)$, where $\tilde{\xi} \in \tilde{\mathscr{R}}''_{q,y}$; then by (15.4)

$$\sigma_x[\xi, \eta] = [\sigma_x\xi, \sigma_x\eta]\,, \tag{19.1}$$

for $\xi, \eta \in J_m(\tilde{R}''_q; Y)_y$, where the bracket on the left-hand side is given by (1.33). These mappings give us a morphism of vector bundles over X

$$\sigma\colon \rho^{-1}J_m(\tilde{R}''_q; Y) \to A_m\,.$$

We also denote by σ_x the composition

$$R''_{q+m,y} \xrightarrow{\tilde{\lambda}_m} J_m(\tilde{R}''_q; Y)_y \xrightarrow{\sigma_x} A_{m,x}\,;$$

by the commutativity of (1.37) and (19.1), we have

$$\sigma_x[\xi, \eta] = [\sigma_x\xi, \sigma_x\eta]\,, \tag{19.2}$$

for $\xi, \eta \in R''_{q+m,y}$. These mappings give us a morphism of vector bundles over X

$$\sigma\colon \rho^{-1}R''_{q+m} \to A_m\,. \tag{19.3}$$

The diagram

$$\begin{array}{ccc} \mathscr{A}_{m+1} & \xrightarrow{D} & \mathscr{T}^* \otimes \mathscr{A}_m \\ \uparrow{\scriptstyle \sigma} & & \uparrow{\scriptstyle \mathrm{id}\otimes\sigma} \\ J_{m+1}(\tilde{\mathscr{R}}''_q;\, Y)_X & \xrightarrow{D} & \mathscr{T}^* \otimes J_m(\tilde{\mathscr{R}}''_q;\, Y)_X \end{array} \tag{19.4}$$

is commutative, since

$$D(j_{m+1}(\tilde{\xi}) \circ \rho) = 0 \;, \qquad D(\sigma(j_{m+1}(\tilde{\xi}) \circ \rho)) = D(j_{m+1}(\sigma(\tilde{\xi}))) = 0 \;,$$

for $\tilde{\xi} \in \tilde{\mathscr{R}}''_q$, and, by (3.5) and (1.4),

$$\begin{aligned} (\mathrm{id} \otimes \sigma)(D(fu)) &= df \otimes \sigma\pi_m u + f(\mathrm{id} \otimes \sigma)Du \;, \\ D(f\sigma u) &= df \otimes \pi_m \sigma u + fD(\sigma u) \;, \end{aligned}$$

for $f \in \mathscr{O}_X$, $u \in J_{m+1}(\tilde{\mathscr{R}}''_q;\, Y)_X$. By [26, Proposition 1.4], the diagram

$$\begin{array}{ccc} J_{q+m+1}(\mathscr{T}_Y;\, Y)_X & \xrightarrow{D} & \mathscr{T}^* \otimes J_{q+m}(\mathscr{T}_Y;\, Y)_X \\ \downarrow{\scriptstyle \bar{\lambda}_{m+1}} & & \downarrow{\scriptstyle \mathrm{id}\otimes\bar{\lambda}_m} \\ J_{m+1}(\tilde{J}_q(\mathscr{T}_Y;\, Y);\, Y)_X & \xrightarrow{D} & \mathscr{T}^* \otimes J_m(\tilde{J}_q(\mathscr{T}_Y;\, Y);\, Y)_X \end{array} \tag{19.5}$$

commutes. From the commutativity of diagrams (19.4) and (19.5), we see that the diagram

$$\begin{array}{ccc} \mathscr{A}_{m+1} & \xrightarrow{D} & \mathscr{T}^* \otimes \mathscr{A}_m \\ \uparrow{\scriptstyle \sigma} & & \uparrow{\scriptstyle \mathrm{id}\otimes\sigma} \\ \mathscr{R}''_{q+m+1,X} & \xrightarrow{D} & \mathscr{T}^* \otimes \mathscr{R}''_{q+m,X} \;, \end{array} \tag{19.6}$$

whose bottom arrow is the restriction of the top arrow of (19.5) (see [26, § 2]), is also commutative.

For $x \in X$ with $y = \rho(x)$, define the mapping

$$\sigma_x \colon (\textstyle\bigwedge^j T^*_Y \otimes R''_{q+m})_y \to (\textstyle\bigwedge^j T^* \otimes A_m)_x$$

sending u into the element $\sigma_x u$ given by the formula

$$(\sigma_x u)(\xi_1 \wedge \cdots \wedge \xi_j) = \sigma_x(u(\rho\xi_1 \wedge \cdots \wedge \rho\xi_j)) \;,$$

for $\xi_1, \cdots, \xi_j \in T_x$; then $\sigma_x u \in F^j_j(J_m(T);\, \rho)_x$ and

$$\rho(\sigma_x u) = \pi_m u \;.$$

It is easily seen that

$$\sigma_x(T_Y^* \otimes R''_{q+m})_y^\wedge \subset (T^* \otimes A_m)_x^\wedge \ . \tag{19.7}$$

If $u \in (\bigwedge^i T_Y^* \otimes R''_{q+m+1})_y$, $v \in (\bigwedge^j T_Y^* \otimes R''_{q+m+1})_y$, then by (19.2) we have

$$\sigma_x[u, v] = [\sigma_x u, \sigma_x v] \ . \tag{19.8}$$

We obtain a mapping

$$\sigma_x : (\bigwedge^j \mathscr{T}_Y^* \otimes \mathscr{R}''_{q+m})_y \to (\bigwedge^j \mathscr{T}^* \otimes \mathscr{A}_m)_{\rho,x} \tag{19.9}$$

such that

$$\rho(\sigma_x(u)) = \pi_m u \ , \tag{19.10}$$

for $u \in (\bigwedge^j \mathscr{T}_Y^* \otimes \mathscr{R}''_{q+m})_y$. From the commutativity of (19.6) and [26, Proposition 1.2], we infer that the diagram

$$\begin{array}{ccc} (\bigwedge^j \mathscr{T}^* \otimes \mathscr{A}_{m+1})_{\rho,x} & \xrightarrow{D} & (\bigwedge^{j+1} \mathscr{T}^* \otimes \mathscr{A}_m)_{\rho,x} \\ \uparrow \sigma_x & & \uparrow \sigma_x \\ (\bigwedge^j \mathscr{T}_Y^* \otimes \mathscr{R}''_{q+m+1})_y & \xrightarrow{D} & (\bigwedge^{j+1} \mathscr{T}_Y^* \otimes \mathscr{R}''_{q+m})_y \end{array} \tag{19.11}$$

is commutative, and from (19.8) that

$$\sigma_x[u, v] = [\sigma_x u, \sigma_x v] \ , \tag{19.12}$$

for all $u \in (\bigwedge^i \mathscr{T}_Y^* \otimes \mathscr{R}''_{q+m})_y$, $v \in (\bigwedge^j \mathscr{T}_Y^* \otimes \mathscr{R}''_{q+m})_y$. If $\xi'' \in \mathrm{Sol}\,(R''_q)_y$, then

$$\xi = \sigma_x(\xi'') = \nu^{-1}\sigma_x(j_q(\xi''))$$

belongs to $\mathrm{Sol}\,(A_2)_x$ and satisfies

$$\rho\xi = \xi'' \ .$$

If $\eta'' \in \mathrm{Sol}\,(R''_q)_y$, we have

$$\sigma_x[\xi'', \eta''] = [\sigma_x\xi'', \sigma_x\eta''] \ . \tag{19.13}$$

From the commutativity of (19.11) and (19.12), we obtain the formula

$$\sigma_x(\mathscr{D}_1 u) = \mathscr{D}_1(\sigma_x u) \ , \tag{19.14}$$

for $u \in (\mathscr{T}_Y^* \otimes \mathscr{R}''_{q+m})_y$.

The image $R_m^\flat$ of the morphism (19.3) is a sub-bundle of A_m with possibly varying fiber. We denote by $\mathscr{R}_m^\flat$ the sub-sheaf of $\mathscr{A}_m$ which is the image of the mapping of sheaves induced by the morphism (19.3). For all $x \in X$, the image of the mapping $\mathscr{R}^\flat_{m,x} \to A_{m,x}$ sending u into the value $u(x)$ of u at x is equal to $R^\flat_{m,x}$. We now verify that the $\mathcal{O}_X$-submodules $\mathscr{R}^\flat_m$ of $\mathscr{A}_m$ satisfy conditions (a)-(c) and (e) of Theorem 18.3, with $h = 0$. In fact, since R''_q is formally integrable, we have

$$\pi_m(\mathscr{R}^\flat_{m+1}) = \mathscr{R}^\flat_m , \qquad \text{for } m \geq 0 ,$$

and from the relation (19.2) we deduce that

$$[\mathscr{R}^\flat_{m+1}, \mathscr{R}^\flat_{m+1}] \subset \mathscr{R}^\flat_m , \qquad \text{for } m \geq 0 .$$

The commutativity of (19.6) implies that

$$D(\mathscr{R}^\flat_{m+1}) \subset \mathscr{T}^* \otimes \mathscr{R}^\flat_m , \qquad \text{for } m \geq 0 .$$

It is easily seen that, for $x \in X$ with $y = \rho(x)$, the diagram

$$\begin{array}{ccc} R^\flat_{m,x} & & \\ \uparrow{\scriptstyle \sigma_x} & \searrow{\scriptstyle \rho} & \\ R''_{m+q,y} & \xrightarrow{\pi_m} & R''_{m,y} \end{array}$$

commutes. Thus

$$\rho(R^\flat_{m,x}) = R''_{m,y} , \qquad \text{for } m \geq 0 ,$$

and ρ induces an isomorphism

$$\rho \colon R^\flat_{\infty,x} \to R''_{\infty,y} , \tag{19.15}$$

and σ_x an isomorphism

$$\sigma_x \colon R''_{\infty,y} \to R^\flat_{\infty,x} ,$$

which is equal to the inverse of (19.15). Finally, for $x \in X$ with $y = \rho(x)$ and $m \geq 0$, the diagram

$$\begin{array}{ccc} J_{m+1}(\tilde{R}''_q; Y)_y \otimes J_{m+1}(F; Y)_y & \longrightarrow & J_m(F; Y)_y \\ \Big\downarrow{\scriptstyle \sigma_x \otimes \mathrm{id}} & & \Big\downarrow{\scriptstyle \mathrm{id}} \\ A_{m+1,x} \otimes J_{m+1}(F; Y)_y & \longrightarrow & J_m(F; Y)_y \end{array}$$

is commutative, where the top horizontal arrow sends $j_{m+1}(\xi)(y) \otimes j_{m+1}(s)(y)$ into $j_m(\mathscr{L}(\xi)s)(y)$, with $\xi \in \tilde{\mathscr{R}}''_{q,y}$, $s \in \mathscr{F}_y$, and the bottom horizontal arrow is given by (18.21). In fact, if $\tilde{\xi} \in \tilde{\mathscr{R}}''_{q,y}$, $s \in \mathscr{F}_y$, then by (15.6)

$$\begin{aligned}
&\sigma_x(j_{m+1}(\tilde{\xi})(y)) \cdot j_{m+1}(s)(y) \\
&\quad = j_{m+1}(\sigma(\tilde{\xi}))(x) \cdot j_{m+1}(s)(y) = \lambda j_m([\sigma(\tilde{\xi}), \mu_s])(x) \\
&\quad = \lambda j_m(\mu_{\mathscr{L}(\tilde{\xi})s})(x) = j_m(\mathscr{L}(\tilde{\xi})s)(y) .
\end{aligned}$$

Thus the diagram

$$\begin{array}{ccc}
R''_{q+m+1,y} \otimes J_{m+1}(F;\, Y)_y & \longrightarrow & J_m(F;\, Y)_y \\
\downarrow{\scriptstyle \sigma_x \otimes \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\
A_{m+1,x} \otimes J_{m+1}(F;\, Y)_y & \longrightarrow & J_m(F;\, Y)_y
\end{array}$$

commutes, where the top horizontal arrow sends $\xi \otimes u$ into $\pi_{q+m}\xi \cdot u$, and the bottom horizontal arrow is given by (18.21). We deduce that the diagram

$$\begin{array}{ccc}
R''_{\infty,y} \otimes J_\infty(F;\, Y)_y & \longrightarrow & J_\infty(F;\, Y)_y \\
\downarrow{\scriptstyle \sigma_x \otimes \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\
A_{\infty,x} \otimes J_\infty(F;\, Y)_y & \longrightarrow & J_\infty(F;\, Y)_y
\end{array}$$

commutes, where the top horizontal arrow is given by the $R''_{\infty,y}$-module structure of $J_\infty(F;\, Y)_y$ and the bottom horizontal arrow is (18.22); since the mapping (19.15) is the inverse of $\sigma_x \colon R''_{\infty,y} \to R^\flat_{\infty,x}$, the diagram (18.24) is commutative, completing the verification of these conditions of Theorem 18.3.

For $x \in X$, let β_x^{-1} denote the inverse of the mapping

$$\beta \colon Q_{m+1}(V;\, \lambda)_x \to J_{m+1}(F;\, Y)_{\rho(x)} ,$$

and λ_x^{-1} the inverse of the isomorphism

$$\lambda \colon J_m(V;\, \lambda)_x \to J_m(F;\, Y)_{\rho(x)} .$$

If $a \in X$ with $y = \rho(a)$, and $\zeta \in R''_{q+m,y}$, $u \in J_{m+1}(F;\, Y)_y$, then $\zeta \cdot u \in J_m(F;\, Y)_y$ and, if we set $b = a + \pi_0 u$, we have $\rho(b) = y$; the elements $\beta_a^{-1}u$ of $Q_{m+1}(V;\, \lambda)_a$ and $\sigma_a\zeta$ of $R^\flat_{m,a}$ satisfy

$$(\beta_a^{-1}u)(\sigma_a\zeta) = \sigma_b\zeta + \lambda_b^{-1}(\zeta \cdot u) . \tag{19.16}$$

Indeed, let $\tilde{\xi}$ be a section of $\tilde{R}''_q$, and s a section of F over a neighborhood of y satisfying $j_m(\tilde{\xi})(y) = \bar{\lambda}_m\zeta$ and $j_{m+1}(s)(y) = u$; by (15.5) we have

$$\begin{aligned}
(\beta_a^{-1}u)(\sigma_a\zeta) &= j_{m+1}(\gamma_s)(a)(\sigma_a\zeta) = j_m(\gamma_{s*}\sigma(\tilde{\xi}))(b) \\
&= j_m(\sigma(\tilde{\xi}))(b) + j_m(\mu_{\mathscr{L}(\tilde{\xi})s})(b) \\
&= \sigma_b\zeta + \lambda_b^{-1} j_m(\mathscr{L}(\tilde{\xi})s)(y) = \sigma_b\zeta + \lambda_b^{-1}(\zeta \cdot u) .
\end{aligned}$$

Let $N_k \subset J_k(F; Y)$ be a formally integrable differential equation such that (18.23) holds. Let $R_k \subset J_k(V; \lambda)$ be the formally integrable abelian Lie equation whose l-th prolongation R_{k+l} is the inverse image of $\rho^{-1}N_{k+l}$ under the isomorphism

$$\lambda: J_{k+l}(V; \lambda) \to \rho^{-1}J_{k+l}(F; Y) .$$

If $P_{k+l} = \alpha^{-1}(R_{k+l})$, then P_k is a formally integrable finite form of R_k with $(P_k)_{+l} = P_{k+l}$. Since R_k is a Lie equation and by (19.16) and (18.23), we have

$$\phi(R_{m,a}) = R_{m,b} , \tag{19.17}$$

$$\phi(R^\flat_{m,a}) \subset R_{m,b} + R^\flat_{m,b} , \tag{19.18}$$

for all $m \geq k$ and $\phi \in P_{m+1}$ with source $\phi = a$, target $\phi = b$.

For $m \geq k$, let $R^\sharp_m$ denote the image of the morphism of vector bundles

$$R_m \oplus \rho^{-1}R''_{q+m} \to A_m , \tag{19.19}$$

sending (u, v) into $u + \sigma v$, where $u \in R_m$, $v \in \rho^{-1}R''_{q+m}$; then

$$R^\sharp_{m,a} = R_{m,a} + R^\flat_{m,a} ,$$

for $a \in X$. From (19.17) and (19.18), we deduce that

$$\phi(R^\sharp_{m,a}) = R^\sharp_{m,b} , \tag{19.20}$$

for all $\phi \in P_{m+1}$ with source $\phi = a$, target $\phi = b$.

Proposition 19.1. *Assume that Y is connected and endowed with the structure of a real-analytic manifold compatible with its structure of differentiable manifold, and that F is an analytic vector bundle. Let $R''_q \subset J_q(T_Y; Y)$ be an analytic formally transitive and formally integrable Lie equation. Assume that F is associated to $\tilde{R}''_q$, that the mapping $R''_q \otimes J_1(F) \to F$ is analytic, and that $\pi_0: N_k \to F$ is surjective. Then $R^\sharp_m$ is an analytic vector bundle for all $m \geq k$.*

Proof. For $m \geq k$, let $\mathscr{R}''_{q+m,X,\omega}$ denote the $\mathscr{O}_{X,\omega}$-module of analytic sections of $\rho^{-1}R''_{q+m}$, and let $\mathscr{R}^\flat_{m,\omega}$ be the coherent $\mathscr{O}_{X,\omega}$-submodule of $\mathscr{A}_{m,\omega}$ which is the image of the mapping

$$\sigma: \mathscr{R}''_{q+m,X,\omega} \to \mathscr{A}_{m,\omega} .$$

Clearly, for $x \in X$, the image of the mapping $\mathscr{R}^\flat_{m,\omega} \to A_{m,x}$ sending u into the value $u(x)$ of u at x is equal to $R^\flat_{m,x}$. By the above discussion of the sheaves $\mathscr{R}^\flat_m$, we see that conditions (i) and (ii) of Lemma 18.2 hold with $h = 0$. Therefore from Lemma 18.2 we deduce that $R^\sharp_m$ is an analytic sub-bundle of A_m satisfying

$$\mathscr{R}^{\#}_{m,\omega} = \mathscr{R}_{m,\omega} + \mathscr{R}^{\flat}_{m,\omega} ,$$

for $m \geq k$.

We assume now that $R^{\#}_m$ is a vector bundle for all $m \geq k$. Clearly

$$\mathscr{R}^{\#}_m = \mathscr{R}_m + \mathscr{R}^{\flat}_m ,$$

for $m \geq k$, and so conditions (a)-(e) of Theorem 18.3 are satisfied with $h = 0$. Let $p \geq k$ be the integer given by that theorem such that $R^{\#}_p$ is a formally integrable Lie equation with

$$R^{\#}_{p+l} = (R^{\#}_p)_{+l} , \qquad \text{for } l \geq 0 .$$

Then by Theorem 18.3, all the hypotheses of Theorem 18.5 other than condition (III) of § 9 for $R^{\#}_p$ are verified. If $\pi_0: N_k \to F$ is surjective and R''_q is formally transitive, then $R^{\#}_p$ is formally transitive and by Theorem 18.3 it satisfies conditions (I) and (II) of § 9; if moreover Y is connected, by [10, Proposition 10.3 and Lemma 10.3 (ii)] it also satisfies condition (III) of § 9. For $x \in X$ with $y = \rho(x)$, the linearly compact Lie algebra $R^{\#}_{\infty,x}$ is the inessential abelian extension (18.28) of the Lie algebra $R''_{\infty,y}$ by $R_{\infty,x}$, which is split by the homomorphism $\sigma_x: R''_{\infty,y} \to R^{\#}_{\infty,x}$. Therefore, if $L^{\#}_y$ denotes the semi-direct product of $R''_{\infty,y}$ and the linearly compact $R''_{\infty,y}$-module $N_{\infty,y}$, the mapping $\phi_x: L^{\#}_y \to R^{\#}_{\infty,x}$, sending (u, ξ) into $\lambda_x^{-1}u + \sigma_x\xi$, where $u \in N_{\infty,y}$, $\xi \in R''_{\infty,y}$ and λ_x^{-1} is the inverse of the isomorphism $\lambda: J_\infty(V; \lambda)_x \to J_\infty(F; Y)_y$, is an isomorphism of linearly compact Lie algebras; furthermore the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N_{\infty,y} & \longrightarrow & L^{\#}_y & \longrightarrow & R''_{\infty,y} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \lambda_x^{-1}} & & \downarrow{\scriptstyle \phi_x} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & R_{\infty,x} & \longrightarrow & R^{\#}_{\infty,x} & \xrightarrow{\rho} & R''_{\infty,y} & \longrightarrow & 0
\end{array}$$

is commutative and exact, and its vertical arrows are isomorphisms. Thus

$$\phi_x: (L^{\#}_y, N_{\infty,y}) \to (R^{\#}_{\infty,x}, R_{\infty,x})$$

is an isomorphism of pairs of topological Lie algebras.

Suppose moreover that $R^{\#}_p$ also satisfies condition (III) of § 9. By § 15 and Theorem 18.5, for $x \in X$ the Spencer cohomologies $H^*(R_k)_x$, $H^*(R^{\#}_p)_x$, $H^*(R''_q)_{\rho(x)}$ are graded Lie algebras, $H^*(R_k)_x$ is abelian and a graded $H^*(R^{\#}_p)_x$-module, and $H^*(N_k)_{\rho(x)}$ is a graded $H^*(R''_q)_{\rho(x)}$-module; the mappings (18.42) and

$$\iota: H^*(R_k)_x \to H^*(R^{\#}_p)_x ,$$

induced by the inclusion $R_p \subset R^{\#}_p$, are morphisms of graded Lie algebras and ι intertwines $H^*(R_k)_x$ and $H^*(R^{\#}_p)_x$; moreover the relation (18.43) holds.

For $m \geq p$ and $x \in X$ with $y = \rho(x)$, the image of the mapping (19.9) belongs to $(\bigwedge^j \mathscr{T}^* \otimes \mathscr{R}_m^\#)_{\rho,x}$; by the commutativity of (19.11) and (19.10), this mapping induces a mapping

$$\sigma_x \colon H^j(R_q'')_{q+m,y} \to H_\rho^j(R_p^\#)_{m,x}$$

such that the diagram

$$\begin{array}{ccc} H^j(R_q'')_{q+m,y} & \xrightarrow{\sigma_x} & H_\rho^j(R_p^\#)_{m,x} \\ \downarrow{\scriptstyle \pi_m} & \swarrow{\scriptstyle \rho} & \\ H^j(R_q'')_{m,y} & & \end{array}$$

commutes, where the mapping ρ is given by (17.5). By means of [6, Theorem 3] we obtain a mapping

$$\sigma_x \colon H^*(R_q'')_y \to H^*(R_p^\#)_x \tag{19.21}$$

such that $\rho\sigma_x$ is the identity mapping of $H^*(R_q'')_y$, where ρ denotes the mapping (18.42). Because of (19.12), the mapping (19.21) is a morphism of graded Lie algebras.

By the exactness of the sequence (18.28), the formally integrable Lie equation obtained from the vector bundles $R_m^\# \cap J_m(V)$, with $m \geq \sup(p, q)$, by means of [6, Theorem 1] is equal to R_{k+l} for some $l \geq 0$. According to [6, Theorem 3], the sequence

$$\begin{aligned} \cdots \longrightarrow H^{j-1}(R_q'')_y \xrightarrow{\partial} H^j(R_k)_x \xrightarrow{\iota} H^j(R_p^\#)_x \\ \xrightarrow{\rho} H^j(R_q'')_y \longrightarrow \cdots, \end{aligned} \tag{19.22}$$

given by (17.8) with $x \in X$ and $y = \rho(x)$, is exact. The properties of the mappings (19.21) imply that the mappings ∂ of the sequence (19.22) are equal to zero, and hence that the graded Lie algebra $H^*(R_p^\#)_x$ is the inessential abelian extension of the graded Lie algebra $H^*(R_q'')_y$ by $H^*(R_k)_x$, which is split by the morphism (19.21). Therefore, for $x \in X$ with $y = \rho(x)$, if $H_y^\#$ denotes the semidirect product of the graded Lie algebra $H^*(R_q'')_y$ and the graded $H^*(R_q'')_y$-module $H^*(N_k)_y$, the mapping $\Phi_x \colon H_y^\# \to H^*(R_p^\#)_x$, sending (α, β) into $\lambda_x^{-1}\alpha + \sigma_x\beta$, where $\alpha \in H^*(N_k)_y$, $\beta \in H^*(R_q'')_y$ and λ_x^{-1} is the inverse of the isomorphism $\lambda \colon H^*(R_k)_x \to H^*(N_k)_y$ given by (17.7), is an isomorphism of graded Lie algebras; furthermore the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^*(N_k)_y & \longrightarrow & H_y^\# & \longrightarrow & H^*(R_q'')_y & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \lambda_x^{-1}} & & \downarrow{\scriptstyle \Phi_x} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & H^*(R_k)_x & \xrightarrow{\iota} & H^*(R_p^\#)_x & \xrightarrow{\rho} & H^*(R_q'')_y & \longrightarrow & 0 \end{array} \tag{19.23}$$

is commutative and exact, and its vertical arrows are isomorphisms. Thus

$$\Phi_x\colon (H^{\sharp}_y, H^*(N_k)_y) \to (H^*(R^{\sharp}_p)_x, H^*(R_k)_x)$$

is an isomorphism of pairs of graded Lie algebras.

At this point we turn to the consideration of the sequence of non-linear cohomology which is analogous to (19.22). Let $P''_q \subset Q_q(Y)$ and $P^{\sharp}_p \subset Q_p$ be formally integrable finite forms of R''_q and $R^{\sharp}_p$ whose l-th prolongations we denote by P''_{q+l} and $P^{\sharp}_{p+l}$. Let $m_0 \geq p$ be an integer such that g_{m_0}, $g^{\sharp}_{m_0}$, g''_{m_0} are 2-acyclic. If $R^{\sharp}_p$ satisfies conditions (II) and (III) of § 9 and N_k is integrable, then by Theorems 18.3 and 18.2 (i) and by § 9 we have the sequence of cohomology

$$(19.24) \qquad H^1(N_k)_{m,y} \longrightarrow H^1(P^{\sharp}_p)_{m,x} \xrightarrow{\ \rho\ } H^1(P''_q)_{m,y}\ ,$$

for all $m \geq m_0$ and $x \in X$ with $y = \rho(x)$. If moreover P''_q is integrable, Theorem 9.2 (ii) asserts that the sequence (19.24) is exact. Furthermore the mapping ρ of sequence (19.24) is surjective. Indeed, if $u \in Z^1(R''_m)_y$, by Proposition 17.1 we choose $u_1 \in Z^1(R''_{m+q})_y$ such that $\pi_m u_1 = u$; then according to (19.7), (19.14) and (19.10), $\sigma_x(u_1)$ belongs to $Z^1_\rho(R^{\sharp}_p)_{m+q,x}$ and satisfies $\rho\sigma_x(u_1) = u$.

We summarize some of the above results as:

Theorem 19.1. *Suppose that $R^{\sharp}_m$ is a vector bundle for all $m \geq k$.*

(i) *The hypotheses* (a)-(e) *of Theorem* 18.3 *with $h = 0$ and of Theorem* 18.5, *other than condition* (III) *of* § 9 *for $R^{\sharp}_p$, hold.*

(ii) *For $x \in X$ with $y = \rho(x)$, the linearly compact Lie algebra $R^{\sharp}_{\infty,x}$ is isomorphic to the semi-direct product of $R''_{\infty,y}$ and the linearly compact $R''_{\infty,y}$-module $N_{\infty,y}$; if $R^{\sharp}_p$ satisfies condition* (III) *of* § 9, *the graded Lie algebra $H^*(R^{\sharp}_p)_x$ is isomorphic to the semi-direct product of the graded Lie algebra $H^*(R''_q)_y$ and the graded $H^*(R''_q)_y$-module $H^*(N_k)_y$.*

(iii) *If $R^{\sharp}_p$ satisfies condition* (II) *of* § 9, *the mapping of cohomology*

$$\rho\colon H^1(P^{\sharp}_p)_{m,x} \to H^1(P''_q)_{m,\rho(x)}$$

is surjective for all $m \geq m_0$, $x \in X$.

From the above discussion and Propositions 18.1 and 18.2, we derive the following:

Theorem 19.2. (i) *If the hypotheses of Proposition* 19.1 *hold and R''_q is elliptic, then $R^{\sharp}_p$ is a formally transitive and formally integrable Lie equation, and $H^1(P^{\sharp}_p)_x = 0$ if and only if $H^1(N_k)_{\rho(x)} = 0$, for $x \in X$.*

(ii) *If R''_q is formally transitive and of finite type, N_k is an integrable differential equation, $\pi_0\colon N_k \to F$ is surjective, and $R^{\sharp}_m$ is a vector bundle for all $m \geq k$, then $R^{\sharp}_p$ is a formally transitive and formally integrable Lie equation, and we have an isomorphism of cohomology*

$$H^1(N_k)_{\rho(x)} \to H^1(P_p^\sharp)_x \ ,$$

for $x \in X$.

Proof. If the hypotheses of (i) hold, by Proposition 19.1 so do those of Proposition 18.1; on the other hand, the hypotheses of (ii) imply those of Proposition 18.2. The conclusions of the theorem follow from these last two propositions.

Theorem 19.2 gives us two classes of formally transitive and formally integrable Lie equations $R_p^\sharp$, obtained from (i) or (ii), for which the second fundamental theorem does not always hold; indeed, if $H^1(N_k) \neq 0$, the non-linear cohomology of $R_p^\sharp$ does not vanish. The first class is related to the examples considered by Buck [20]. In § 20 we shall construct Lie equations belonging to these classes.

Henceforth we shall identify two graded modules of linear cohomology over a graded Lie algebra which are isomorphic.

Although a special case of results which follow, we first make some observations about a closed ideal I of a real transitive Lie algebra L. By [9, Corollary 6.1] and [10, Theorem 10.1], there exist an analytic manifold X, a point $x \in X$, a formally transitive and formally integrable analytic Lie equation $R_k \subset J_k(T)$, and a formally integrable Lie equation $R'_{k_1} \subset R_{k_1}$, with $k_1 \geq k$, such that

$$[\tilde{\mathscr{R}}_{k_1+1}, \mathscr{R}'_{k_1}] \subset \mathscr{R}'_{k_1} \ ,$$

and $(R_{\infty,x}, R'_{\infty,x})$ and (L, I) are isomorphic as pairs of topological Lie algebras. According to § 15, we have structures of graded Lie algebras on $H^*(L) = H^*(R_k)_x$ and $H^*(L, I) = H^*(R'_{k_1})_x$ and of graded $H^*(L)$-module on $H^*(L, I)$, and a morphism

$$\iota \colon H^*(L, I) \to H^*(L)$$

of graded Lie algebras induced by the inclusion $R'_{k_1} \subset R_{k_1}$, which intertwines $H^*(L, I)$ and $H^*(L)$ in the sense that

$$\iota(\alpha)\cdot\beta = [\alpha, \beta] \ , \qquad \iota(\gamma\cdot\alpha) = [\gamma, \iota(\alpha)] \ ,$$

for $\alpha, \beta \in H^*(L, I)$, $\gamma \in H^*(L)$. Using Proposition 17.6 and formula (6.10), we see easily that, without changing the graded Lie algebra and module structures on $H^*(L)$ and $H^*(L, I)$ and their relationship, we may suppose that there is an analytic surjective submersion $\rho \colon X \to Y$ such that the Lie equation R_k is ρ-projectable and $R'_\infty = R_\infty \cap J_\infty(V)$; under these additional assumptions, by [10, formulas (9.11) and (9.10)], the morphism of graded Lie algebras ι and the graded $H^*(L)$-module structure on $H^*(L, I)$ coincide with the ones given by [10, Theorem 13.1 (iii)], which are well-defined. From [10, Theorem 13.1] we obtain

Proposition 19.2. *Let I be a closed ideal of a real transitive Lie algebra L. Then the structure of graded $H^*(L)$-module on $H^*(L, I)$ and the morphism*

$$\iota\colon H^*(L, I) \to H^*(L)$$

of graded Lie algebras, which intertwines $H^*(L, I)$ and $H^*(L)$, are well-defined up to automorphisms of these graded Lie algebras, and depend only on the isomorphism class of (L, I) as a pair of topological Lie algebras.

Let $L'' = L/I$ and $\phi\colon L \to L''$ be the natural epimorphism of transitive Lie algebras. If $R''_{k_1} \subset J_{k_1}(T_Y; Y)$ is the formally transitive and formally integrable analytic Lie equation, with $k_1 \geq k$, such that

$$\rho(R_{m,a}) = R''_{m,\rho(a)} ,$$

for all $m \geq k_1$, $a \in X$, then the well-defined morphism of graded Lie algebras

$$\phi\colon H^*(L) \to H^*(L'') ,$$

induced by ϕ and given by [10, Theorem 13.1 (ii)], is equal to

$$\rho\colon H^*(R_k)_x \to H^*(R''_{k_1})_{\rho(x)} ,$$

up to automorphisms of these graded Lie algebras.

Let E be a geometric module over a real transitive Lie algebra L. Consider a transitive Lie algebra L' which is an abelian extension

$$0 \longrightarrow E \longrightarrow L' \xrightarrow{\phi} L \longrightarrow 0 \tag{19.25}$$

of L by E, defining the given structure of L-module on E. Let L'^0 be a fundamental subalgebra of L' such that the ideal E of L' is defined by a foliation in (L', L'^0).

According to [9, Corollary 6.1] and [10, Theorem 10.1], there exist an analytic connected manifold X, a point $x \in X$, a formally transitive and formally integrable analytic Lie equation $R'_k \subset J_k(T)$, a formally integrable analytic Lie equation $R_k \subset R'_k$, and an isomorphism of transitive Lie algebras $\psi'\colon L' \to R'_{\infty,x}$ such that

$$[\tilde{\mathscr{R}}'_{k+1}, \mathscr{R}_k] \subset \mathscr{R}_k , \quad \psi'(E) = R_{\infty,x} , \quad \psi'(L'^0) = R'^0_{\infty,x} .$$

By [10, Lemma 10.3 (ii)], $V = \pi_0\tilde{R}_k$ is a sub-bundle of T and by Lemmas 1.5 and 11.3, R_k is an abelian Lie equation. Moreover, $R_{\infty,x}$ is defined by the foliation $J_0(V)_x$ in $(R'_{\infty,x}, R'^0_{\infty,x})$. We now apply Theorem 18.6 to R_k and R'_k. Replacing X if necessary by a neighborhood of x, we obtain an analytic manifold Y, an analytic surjective submersion $\rho\colon X \to Y$, an analytic vector bundle F over Y, a formally transitive and formally integrable analytic Lie equation $R''_q \subset J_q(T_Y; Y)$, and a formally integrable differential equation $N_k \subset J_k(F; Y)$ such that $\rho\colon X \to Y$ can be identified with an open fibered submanifold of the vector bundle F, considered as an affine bundle A over Y, and all the assertions of Theorem 18.6 hold with $q_0 = q$. Then, if $y = \rho(x)$, there is an isomorphism of transitive Lie algebras $\psi: L \to R''_{\infty,y}$ such that the exact diagram

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ E & \xrightarrow{\psi'} & R_{\infty,x} \\ \downarrow & & \downarrow \\ L' & \xrightarrow{\psi'} & R'_{\infty,x} \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \rho} \\ L & \xrightarrow{\psi} & R''_{\infty,y} \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

is commutative. We set $\eta = \lambda \circ \psi'$, where λ is the isomorphism $J_\infty(V; \lambda)_x \to J_\infty(F; Y)_y$. From the commutativity of (18.44), we deduce that the diagram

$$\begin{array}{ccc} L \otimes E & \longrightarrow & E \\ \downarrow{\scriptstyle \psi\otimes\eta} & & \downarrow{\scriptstyle \eta} \\ R''_{\infty,y} \otimes N_{\infty,y} & \longrightarrow & N_{\infty,y} \end{array} \tag{19.26}$$

commutes, where the horizontal arrows are given by the L-module structure of E and the $R''_{\infty,y}$-module structure of $N_{\infty,y}$. From the above discussion, we obtain the following realization theorem for geometric modules over real transitive Lie algebras, a formal version of which was given in [29]; namely, we show that every such geometric module is isomorphic to one of the type considered in § 15.

Theorem 19.3. *Let E be a geometric module over a real transitive Lie algebra L; let $L^0 \subset L$ be a fundamental subalgebra of L, and $E^0 \subset E$ be a fundamental subspace of E such that*

$$L^0 \cdot E^0 \subset E^0 .$$

Then there exist an analytic manifold Y, a point $y \in Y$, an analytic formally transitive and formally integrable Lie equation $R''_q \subset J_q(T_Y; Y)$, an analytic vector bundle F over Y associated to $\tilde{R}''_q$, an analytic formally integrable linear differential equation $N_k \subset J_k(F; Y)$, an isomorphism of transitive Lie algebras $\psi: L \to R''_{\infty,y}$, and an isomorphism of topological vector spaces $\eta: E \to N_{\infty,y}$ such that

$$\pi_0 N_k = F\,, \qquad R''_{q+k}\cdot N_{k+1} \subset N_k\,,$$
$$\psi(L^0) = R''^0_{\infty,y}\,, \qquad \eta(E^0) = N^0_{\infty,y}\,,$$

$N_{\infty,y}$ is a closed geometric $R''_{\infty,y}$-submodule of $J_\infty(F; Y)_y$ and the diagram (19.26) *commutes.*

Proof. Let L' be the abelian extension (19.25) of L by E. Let $\sigma\colon L \to L'$ be a continuous linear mapping such that $\phi\circ\sigma = \mathrm{id}$. Assume that the continuous 2-cocycle α on L with values in E defined by (14.7) satisfies $\alpha(L^0\times L^0) \subset E^0$. In particular, we may take L' to be the semi-direct product of L and E and σ to be the mapping sending ξ into $(0,\xi)\in E\times L$; in this case $\alpha = 0$. Then by Proposition 14.6, $L'^0 = E^0 + \sigma(L^0)$ is a fundamental subalgebra of L', and the ideal E of L' is defined by a foliation in (L', L'^0). Consider the objects we have associated above to (19.25) and L'^0. The isomorphism $\lambda\colon J_\infty(V;\lambda)_x \to J_\infty(F; Y)_y$ satisfies $\lambda(J^0_\infty(V;\lambda)_x) = J^0_\infty(F; Y)_y$; thus $\eta(E^0) = N^0_{\infty,y}$ and, since $\pi_0 N_k = F$, the mapping η induces an isomorphism $E/E^0 \to F_y$. As $\phi(L'^0) = L^0$, we have $\psi(L^0) \subset R''^0_{\infty,y}$, and so ψ induces a surjective mapping $L/L^0 \to J_0(T_Y)_y$. Because

$$\dim L/L^0 = \dim L'/L'^0 - \dim E/E^0 = \dim X - \operatorname{rank} F = \dim Y\,,$$

this mapping is an isomorphism and hence $\psi(L^0) = R''^0_{\infty,y}$.

Let E be a geometric module over a real transitive Lie algebra L. According to Theorem 19.3, there exist a formally transitive and formally integrable analytic Lie equation $R''_q \subset J_q(T_Y; Y)$ on an analytic manifold Y, a point $y\in Y$, an analytic vector bundle F over Y associated to $\tilde{R}''_q$, an analytic formally integrable differential equation $N_k \subset J_k(F; Y)$, an isomorphism of transitive Lie algebras $\psi\colon L \to R''_{\infty,y}$, and an isomorphism of topological vector spaces $\eta\colon E \to N_{\infty,y}$ such that

$$R''_{q+k}\cdot N_{k+1} \subset N_k$$

and the diagram (19.26) commutes. Then $H^*(L)$ is the graded Lie algebra $H^*(R''_q)_y$. We define the linear Spencer cohomology of the geometric L-module E to be the graded $H^*(L)$-module

$$H^*(L, E) = \bigoplus_{j\geq 0} H^j(L, E)\,, \qquad \text{with } H^j(L, E) = H^j(N_k)_y\,,$$

given by § 15. We now show that this cohomology is well-defined.

Theorem 19.4. *Let E be a geometric module over a real transitive Lie algebra L.*

(i) *The graded $H^*(L)$-module $H^*(L, E)$ of linear Spencer cohomology of E is well-defined and depends only on the isomorphism class of E as a topological L-module.*

(ii) *If $L^\#$ is the semi-direct product of L and E, the graded Lie algebra $H^*(L^\#)$ is equal to the semi-direct product of $H^*(L)$ and the $H^*(L)$-module $H^*(L, E)$.*

Proof. Consider the objects we have just associated to the L-module E. Replacing F by $\pi_0 N_k$ and Y by the connected component of y, by Lemma 15.2 we may suppose that $\pi_0 N_k = F$. Let X be the vector bundle F, and consider the mapping (19.3) and the abelian Lie equation R_k on X obtained from N_k. According to Proposition 19.1, the image $R^\#_m$ of the morphism of vector bundles (19.19) over X is a vector bundle for $m \geq k$. Theorems 19.1 and 18.3 give us the formally transitive and formally integrable analytic Lie equation $R^\#_p \subset J_p(T)$, with $p \geq k$, whose l-th prolongation is $R^\#_{p+l}$ and which satisfies conditions (I), (II) and (III) of § 9; moreover they tell us that (18.25) holds and, for $x \in X$, with $y = \rho(x)$, that $R^\#_{\infty,x}$ is isomorphic to the semi-direct product of $R''_{\infty,y}$ and the $R''_{\infty,y}$-module $N_{\infty,y}$, and that $H^*(R^\#_p)_x$ is isomorphic to the semi-direct product of $H^*(R''_q)_y$ and the $H^*(R''_q)_y$-module $H^*(N_k)_y$. If $L^\#$ is the semi-direct product of L and E, and $\phi\colon L^\# \to L$ is the natural projection, the linear Spencer cohomologies of the closed ideal E of $L^\#$ and of $L^\#$ are given by

$$H^*(L^\#, E) = H^*(R_k)_x\,, \qquad H^*(L^\#) = H^*(R^\#_p)_x\,,$$

with $x \in X$, and the morphism $\phi\colon H^*(L^\#) \to H^*(L)$ of graded Lie algebras induced by ϕ is equal to $\rho\colon H^*(R^\#_p)_x \to H^*(R''_q)_{\rho(x)}$, with $x \in X$, up to automorphisms of these graded Lie algebras. Since the linear Spencer cohomology $H^*(L^\#, E)$ of the closed ideal E of $L^\#$ is well-defined by Proposition 19.2 as a graded $H^*(L^\#)$-module, from the commutativity of diagram (19.23) it follows that $H^*(L, E) = H^*(N_k)_y$, with $y \in Y$, is well-defined as a graded $H^*(L)$-module and is equal to $H^*(L^\#, E)$. The remaining assertions of the theorem now hold by [10, Theorem 13.1 (i)].

The following proposition is an immediate consequence of Theorem 19.4 (i) and the definitions of the Spencer cohomologies involved.

Proposition 19.3. *If I is a closed ideal of the real transitive Lie algebra L, the graded $H^*(L)$-module $H^*(L, I)$ of linear Spencer cohomology of the ideal I of L is equal to the graded $H^*(L)$-module of linear Spencer cohomology of I considered as a geometric L-module.*

Theorem 19.5. *Let E be a geometric module over a real transitive Lie algebra L. Let L′ be the transitive Lie algebra which is the abelian extension* (19.25) *of L by E, defining the given structure of L-module on E. If $\phi\colon H^*(L') \to H^*(L)$ is the morphism of graded Lie algebras induced by $\phi\colon L' \to L$, there is an isomorphism of graded vector spaces*

$$\lambda\colon H^*(L', E) \to H^*(L, E)$$

such that

$$\lambda(\alpha \cdot \beta) = \phi(\alpha) \cdot \lambda(\beta)\,,$$

for all $\alpha \in H^*(L')$, $\beta \in H^*(L', E)$. *Moreover, we have isomorphisms of cohomology*

$$\tilde{H}^1(L', E) \to H^1(L', E) , \qquad \tilde{H}^1(L', E) \to H^1(L, E) ,$$

and a mapping of cohomology

$$H^1(L, E) \to \tilde{H}^1(L') . \tag{19.27}$$

Proof. Let L'^0 be a fundamental subalgebra of L' such that the ideal E of L' is defined by a foliation in (L', L'^0). Consider the objects which we associated above to the abelian extension (19.25) and to L'^0. Then we have the equalities of Spencer cohomologies

$$H^*(L') = H^*(R'_k)_x , \qquad H^*(L', E) = H^*(R_k)_x ,$$
$$H^*(L) = H^*(R''_q)_y , \qquad H^*(L, E) = H^*(N_k)_y ,$$

and the morphism $\phi: H^*(L') \to H^*(L)$ of graded Lie algebras induced by ϕ is equal to $\rho: H^*(R'_k)_x \to H^*(R''_q)_y$ up to automorphisms of these graded Lie algebras. The desired results now follow from Theorems 18.5 and 18.6 (iv).

Thus if L' is the abelian extension (19.25) of the transitive Lie algebra L by E, the Spencer cohomology $H^*(L', E)$ of the closed abelian ideal E of L' depends only on the geometric L-module E and not on the choice of the extension (19.25) of L by E.

Applying Theorems 18.7 (ii) and (iii) and 18.8 to the above equations R_k and R'_k, we obtain the following:

Corollary 19.1. *Let L be an elliptic real transitive Lie algebra, and let L′ be the transitive Lie algebra which is the abelian extension* (19.25) *of L by the geometric L-module E.*

(i) *If the image of* $\alpha \in H^1(L, E)$ *under the mapping* (19.27) *vanishes, then* $\alpha = 0$; *moreover* $H^1(L, E) = 0$ *if and only if* $\tilde{H}^1(L') = 0$.

(ii) *If L is finite-dimensional, the mapping* (19.27) *is an isomorphism of cohomology.*

The corollary also follows from Corollary 17.2. Let I be a closed ideal of L' containing E; in the corollary, we may replace L' by I and L by the image of I in L.

From the corollary we deduce that if $H^1(L, E) \neq 0$, then $\tilde{H}^1(L') \neq 0$, from which fact we shall obtain a class of abelian extensions of transitive Lie algebras, whose non-linear cohomology does not vanish.

Proposition 19.4. *Let* $\phi: L \to L''$ *be an epimorphism of real transitive Lie algebras, and E a geometric L″-module. If* $\phi: H^*(L) \to H^*(L'')$ *is the morphism of graded Lie algebras induced by* ϕ, *there is an isomorphism of graded vector spaces*

$$\phi: H^*(L, \phi^*E) \to H^*(L'', E)$$

such that

$$\phi(\alpha \cdot \beta) = \phi(\alpha) \cdot \phi(\beta) ,$$

for all $\alpha \in H^*(L)$, $\beta \in H^*(L, \phi^*E)$.

Proof. Let $L^\#$ be the semi-direct product of L and ϕ^*E, and $L''^\#$ be the semi-direct product of L'' and E; then the epimorphism of transitive Lie algebras $\phi^\#: L^\# \to L''^\#$, which is equal to id $\times\ \phi$, induces an isomorphism of the closed ideal ϕ^*E of $L^\#$ onto the closed ideal E of $L^\#$. From [10, Corollary 13.1 (ii)], we obtain an isomorphism of graded vector spaces

$$\phi^\#: H^*(L^\#, \phi^*E) \to H^*(L''^\#, E) ;$$

if we apply [10, Theorem 13.1 (iv)] to the commutative and exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \phi^*E & \longrightarrow & L^\# & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi^\#} & & \downarrow{\scriptstyle \phi^\#} & & \downarrow{\scriptstyle \phi} & & \\ 0 & \longrightarrow & E & \longrightarrow & L''^\# & \longrightarrow & L'' & \longrightarrow & 0 \end{array}$$

of topological Lie algebras, we see that there is a commutative diagram of graded Lie algebras

$$\begin{array}{ccc} H^*(L^\#) & \longrightarrow & H^*(L) \\ \downarrow{\scriptstyle \phi^\#} & & \downarrow{\scriptstyle \phi} \\ H^*(L''^\#) & \longrightarrow & H^*(L'') \end{array}$$

such that

$$\phi^\#(\alpha \cdot \beta) = \phi^\#(\alpha) \cdot \phi^\#(\beta) ,$$

for all $\alpha \in H^*(L^\#)$, $\beta \in H^*(L^\#, \phi^*E)$. By means of Theorem 19.5, we now deduce the proposition.

Theorem 19.6. *Let L be a real transitive Lie algebra, and*

$$0 \longrightarrow E' \xrightarrow{\alpha} E \xrightarrow{\beta} E'' \longrightarrow 0$$

an exact sequence of geometric L-modules, whose mappings are continuous. Then we have an exact sequence

$$\cdots \longrightarrow H^j(L, E') \xrightarrow{\alpha} H^j(L, E) \xrightarrow{\beta} H^j(L, E'') \xrightarrow{\partial} H^{j+1}(L, E') \longrightarrow \cdots$$

of Spencer cohomology.

Proof. Let $L^\#$ be the semi-direct product of L and E, and $L''^\#$ the semi-direct product of L and E''. Then β determines an epimorphism of transitive

Lie algebras $\beta^{\#}: L^{\#} \to L''^{\#}$, which is equal to id $\times$ β, and α a monomorphism of topological Lie algebras $\alpha^{\#}: E' \to L^{\#}$, which is equal to (id, 0) and which allows us to identify E' with a closed ideal of $L^{\#}$. If we apply [10, Theorem 13.1 (iii)] to the commutative and exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E' & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & E'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E' & \xrightarrow{\alpha^{\#}} & L^{\#} & \xrightarrow{\beta^{\#}} & L''^{\#} & \longrightarrow & 0 \end{array}$$

of topological Lie algebras, we obtain the exact sequence

$$\cdots \longrightarrow H^j(L^{\#}, E') \xrightarrow{\alpha^{\#}} H^j(L^{\#}, E) \xrightarrow{\beta^{\#}} H^j(L''^{\#}, E'') \xrightarrow{\partial} H^{j+1}(L^{\#}, E') \longrightarrow \cdots .$$

According to Propositions 19.3 and 19.4, if $\phi: L^{\#} \to L$ is the natural projection, we have the isomorphisms of Spencer cohomologies

$$H^*(L^{\#}, E') \to H^*(L^{\#}, \phi^*E') \to H^*(L, E') .$$

From these isomorphisms, Theorem 19.5 and the above exact sequence, we obtain the desired exact sequence of Spencer cohomology.

Let $\phi: L \to L''$ be an epimorphism of real transitive Lie algebras, $I \subset L$, $I'' \subset L''$ be closed ideals of L and L'' such that $\phi(I) = I''$. Let I' be the kernel of $\phi: I \to I''$. Applying Theorem 19.6 to the exact sequence

$$0 \longrightarrow I' \longrightarrow I \xrightarrow{\phi} \phi^*I'' \longrightarrow 0$$

of geometric L-modules, from Proposition 19.4 we recover the exact sequence of Spencer cohomology of [10, Theorem 13.1 (iii)].

20. Counterexamples to the integrability problem

In this section, we give examples of Lie equations of the type of the equation $R_p^{\#}$ of Theorem 19.1 and determine special properties of these examples.

Let $R_q'' \subset J_q(T_Y; Y)$ be a formally transitive and formally integrable Lie equation. Let $y_0 \in Y$ and let $P_q'' \subset Q_q(Y)$ be a formally integrable finite form of R_q'', whose m-th prolongation we denote by P_{q+m}''. Assume that the projection of $P_q''(y_0)$ onto Y sending $p \in P_q''(y_0)$ into the target of p is surjective. Then $P_q''(y_0)$ is a principal bundle over Y whose group is $G'' = P_q''(y_0, y_0)$. Let F_0 be a finite-dimensional G''-module, and consider the vector bundle

$$F = P_q''(y_0) \times_{G''} F_0$$

associated to $P''_q(y_0)$ and to $\tilde{R}''_q$. Let X be the vector bundle F considered as an affine bundle over Y, and $\rho: X \to Y$ the projection of this vector bundle F onto Y.

According to § 15, to each section ϕ of P''_q over an open set $U \subset Y$, for which $\pi_0\phi$ is a diffeomorphism of U onto an open subset U' of Y, corresponds an isomorphism of vector bundles

$$\sigma(\phi): F_{|U} \to F_{|U'}$$

over $\pi_0\phi$. Then $\sigma(\phi)$ is a solution of the finite form B_2 of A_2. Let $\tilde{J}_l(P''_{q+m}; Y) \subset Q_{(l,q+m)}(Y)$ denote the bundle of jets of order l of sections of $\tilde{\mathscr{P}}''_{q+m}$. For $x \in X$ with $y = \rho(x)$, we obtain a mapping

$$\sigma: \tilde{J}_m(P''_q; Y)_y \to B_{m,x}$$

sending $j_m(\phi)(y)$ into $j_m(\sigma(\phi))(x)$, where $\phi \in \tilde{\mathscr{P}}''_{q,y}$. The compositions

$$P''_{q+m,y} \xrightarrow{\lambda_m} \tilde{J}_m(P''_q; Y)_y \xrightarrow{\sigma} B_{m,x} ,$$

with $x \in X$ and $y = \rho(x)$, give us a morphism of fibered manifolds over X

$$\sigma: \rho^{-1}P''_{q+m} \to B_m . \tag{20.1}$$

By (15.8), for $a \in X$, with $y = \rho(a)$, and $\phi \in P''_{q+m,y}$, $\psi \in P''_{q+m,\rho(b)}$, where $b = \sigma(\pi_q\psi)a$, we have

$$\sigma_b\psi \cdot \sigma_a\phi = \sigma_a(\psi \cdot \phi) . \tag{20.2}$$

We thus obtain a mapping

$$\sigma_x: \mathscr{P}''_{q+m,y} \to \mathscr{B}_{m,x} , \tag{20.3}$$

for $x \in X$ with $y = \rho(x)$, such that

$$\sigma_x(\tilde{\mathscr{P}}''_{q+m,y}) \subset \tilde{\mathscr{B}}_{m,x} ,$$

and

$$\sigma_x(\tilde{\mathscr{P}}''^{\cdot}_{q+m,y}) \subset \tilde{\mathscr{B}}^{\cdot}_{m,x} .$$

If $f'' \in \mathrm{Sol}\,(P''_q)_y$, then

$$f = \sigma_x(f'') = \sigma_x(j_q(f''))$$

belongs to $\mathrm{Sol}\,(B_2)_x$ and satisfies

$$\rho f = f'' . \tag{20.4}$$

If $\tilde{J}_l(B_m) \subset Q_{(l,m)}$ is the sub-groupoid of l-jets of sections of $\tilde{\mathscr{B}}_m$, we have the mapping

$$\sigma_x \colon \tilde{J}_l(P''_{q+m};\, Y)_y \to \tilde{J}_l(B_m)_x ,$$

for $x \in X$, with $y = \rho(x)$, sending $j_l(\psi)(y)$ into $j_l(\sigma_x\psi)(x)$, where $\psi \in \tilde{\mathscr{P}}''_{q+m,y}$; it is easily verified that the diagram

$$\text{(20.5)} \qquad \begin{array}{ccc} P''_{q+l+m,y} & \xrightarrow{\sigma_x} & B_{l+m,x} \\ \downarrow{\scriptstyle \lambda_l} & & \downarrow{\scriptstyle \lambda_l} \\ \tilde{J}_l(P''_{q+m};\, Y)_y & \xrightarrow{\sigma_x} & \tilde{J}_l(B_m)_x \end{array}$$

is commutative.

As F is associated to $\tilde{R}''_q$, we consider the mapping (19.3) and we write

$$\sigma = \nu^{-1} \circ \sigma \circ \nu \colon \rho^{-1}\tilde{R}''_{q+m} \to \tilde{A}_m .$$

We identify $\rho^{-1}T(P''_{q+m}/Y)$ with $V(\rho^{-1}P''_{q+m})$. By (20.2), for $a \in X$ and $\psi \in P''_{q+m}$, with source $\psi = \rho(a)$, target $\psi = y$ and $b = \sigma(\pi_q\psi)a$, the diagram

$$\text{(20.6)} \qquad \begin{array}{ccc} \tilde{A}_{m,b} & \xrightarrow{\sigma_a\psi} & V_{\sigma_a\psi}(B_m) \\ \uparrow{\scriptstyle \sigma_b} & & \uparrow{\scriptstyle \sigma_*} \\ \tilde{R}''_{q+m,y} & \xrightarrow{\psi} & T_\psi(P''_{q+m}/Y) \end{array}$$

is commutative, where $\sigma_a\psi$ and ψ operate on the right. Also if $x \in X$, with $y = \rho(x)$, and $\phi \in \tilde{\mathscr{P}}''_{q+m,y}$, the diagram

$$\text{(20.7)} \qquad \begin{array}{ccc} \tilde{A}_{m,x} & \xrightarrow{\sigma_x\phi} & V_{\sigma_x\phi(x)}(B_m) \\ \uparrow{\scriptstyle \sigma_x} & & \uparrow{\scriptstyle \sigma_*} \\ \tilde{R}_{q+m,y} & \xrightarrow{\phi} & T_{\phi(y)}(P''_{q+m}/Y) \end{array}$$

commutes, where $\sigma_x\phi$ and ϕ operate on the left. From the commutativity of (20.6), (20.7) and (20.5), by (2.6), if $a \in X$, with $y = \rho(a)$, for $\psi \in P''_{q+m+1,y}$, $\xi \in R''_{q+m,y}$ we have

$$\text{(20.8)} \qquad (\sigma_a\psi)(\sigma_a\xi) = \sigma_b(\psi(\xi)) ,$$

where $b = \sigma(\pi_q\psi)a$.

For $x \in X$, with $y = \rho(x)$, and $\psi \in \tilde{\mathscr{P}}''_{q+m+1,y}$, we have

$$\text{(20.9)} \qquad \sigma_x(\mathscr{D}\psi) = \mathscr{D}(\sigma_x\psi) ,$$

where σ_x on the left-hand side is the mapping (19.9) with $j = 1$. Indeed, to prove (20.9) it suffices to show that

$$(\pi_0\sigma_x\xi) \barwedge \mathscr{D}(\sigma_x\psi) = \sigma_x(\pi_0\xi \barwedge \mathscr{D}\psi) ,$$

for $\xi \in \tilde{R}''_{q+m,y}$; by (2.28) and the commutativity of (20.7) and (20.5) we have

$$\begin{aligned}(\pi_0\sigma_x\xi) \barwedge \mathscr{D}(\sigma_x\psi) &= \nu((\lambda_1\sigma_x\psi(x)^{-1}\cdot\sigma_x\pi_{q+m}\psi\cdot\sigma_x\xi - \sigma_x\xi) \\ &= \nu((\lambda_1\sigma_x\psi(x)^{-1}\cdot\sigma_*\pi_{q+m}\psi\cdot\xi - \sigma_x\xi) \\ &= \nu(\sigma_x(\lambda_1\psi(x))^{-1}\cdot\pi_{q+m}\psi\cdot\xi - \sigma_x\xi) \\ &= \sigma_x(\pi_0\xi \barwedge \mathscr{D}\psi) .\end{aligned}$$

If $a \in X$ satisfies $\rho(a) = y$, for $\psi \in \tilde{\mathscr{P}}''_{q+m+1,y}$, $u \in (\mathscr{T}^*_Y \otimes \mathscr{R}''_{q+m})_z$ with target $\psi(y) = z$ and $b = \sigma(\pi_q\psi)a$, we have $\rho(b) = z$ and, by (20.9) and (20.8),

$$\sigma_a(u^\psi) = (\sigma_b u)^{\sigma_a\psi} . \tag{20.10}$$

We denote by $P^\flat_m$ the image of the mapping (20.1). If $a \in X$, with $y = \rho(a)$, and $\psi \in P''_{q+m+1,y}$, $\zeta \in R''_{q+m,y}$, then the elements $\sigma_a\psi$ of $P^\flat_{m+1,a}$ and $\sigma_a(\zeta)$ of $R^\flat_{m,a}$ satisfy

$$(\sigma_a\psi)(\sigma_a\zeta) = \sigma_b(\psi(\zeta)) , \tag{20.11}$$

where $b = \sigma(\pi_q\psi)a$. Indeed, let ϕ be a section of P''_q over a neighborhood U of y such that $\pi_0\phi$ is a diffeomorphism of U onto an open subset U' of Y, and let $\tilde{\xi}$ be a section of $\tilde{R}''_q$ over U satisfying $j_{m+1}(\phi)(y) = \lambda_{m+1}\psi$ and $j_m(\tilde{\xi})(y) = \bar{\lambda}_m\zeta$. Then $\phi(\tilde{\xi})$ is a section of $\tilde{R}''_q$ over U' and $j_m(\phi(\tilde{\xi}))((\pi_0\phi)(y)) = \bar{\lambda}_m\psi(\zeta)$. Therefore by (15.10)

$$\begin{aligned}(\sigma_a\psi)(\sigma_a\zeta) &= j_{m+1}(\sigma(\phi))(a)(j_m(\sigma(\tilde{\xi}))(a)) \\ &= j_m(\sigma(\phi)_*\sigma(\tilde{\xi}))(\sigma(\phi)a) \\ &= j_m(\sigma(\phi(\tilde{\xi})))(\sigma(\phi)a) = \sigma_b(\psi(\zeta)) .\end{aligned}$$

Hence

$$\psi(R^\flat_{m,a}) = R^\flat_{m,b} , \tag{20.12}$$

for all $\psi \in P^\flat_{m+1}$, with source $\psi = a$, target $\psi = b$.

If $a \in X$, with $y = \rho(a)$, and $\psi \in P''_{q+m+1,y}$, $u \in J_m(F; Y)_y$, then $\pi_{q+m}\psi\cdot u \in J_m(F; Y)_z$, where $z =$ target ψ; the elements $\sigma_a\psi$ of $P^\flat_{m+1,a}$ and $\lambda_a^{-1}u$ of $J_m(V; \lambda)_a$ satisfy

$$(\sigma_a\psi)(\lambda_a^{-1}u) = \lambda_b^{-1}(\pi_{q+m}\psi\cdot u) , \tag{20.13}$$

where $b = \sigma(\pi_q\psi)a$. In fact, let ϕ be a section of P''_q over a neighborhood U of y such that $\pi_0\phi$ is a diffeomorphism of U onto an open subset U' of Y, and let

s be a section of F over U satisfying $j_{m+1}(\phi)(y) = \lambda_{m+1}\psi$ and $j_m(s)(y) = u$; if s' is the section $\sigma(\phi)\circ s\circ(\pi_0\phi)^{-1}$ of F over U', then $\pi_{q+m}\psi\cdot u = j_m(s')(z)$ and by (15.15)

$$\begin{aligned}(\sigma_a\psi)(\lambda_a^{-1}u) &= j_{m+1}(\sigma(\phi))(a)(j_m(\mu_s)(a)) \\ &= j_m(\sigma(\phi)_*\mu_s)(\sigma(\phi)a) \\ &= j_m(\mu_{s'})(\sigma(\phi)a) \\ &= \lambda_b^{-1}(j_m(s')(z)) = \lambda_b^{-1}(\pi_{q+m}\psi\cdot u)\ .\end{aligned}$$

If $a\in X$, with $y = \rho(a)$, and $\psi\in P''_{q+m,y}$, $u\in J_m(F;\,Y)_y$, and if we set $b = a + \pi_0 u$ and $c = \sigma(\pi_q\psi)a$, then $\rho(b) = y$ and $\rho(c) = $ target ψ; the elements $\sigma_b\psi$ of $P^\flat_{m,b}$ and $\beta_a^{-1}u$ of $Q_m(V;\,\lambda)_a$ satisfy

$$\sigma_b\psi\cdot\beta_a^{-1}u = \beta_c^{-1}(\psi\cdot u)\cdot\sigma_a\psi\ . \tag{20.14}$$

In fact, let ϕ be a section of P''_q over a neighborhood U of y such that $\pi_0\phi$ is a diffeomorphism onto an open subset U' of Y, and let s be a section of F over U verifying $j_m(\phi)(y) = \lambda_m\psi$ and $j_m(s)(y) = u$; if s' is the section $\sigma(\phi)\circ s\circ(\pi_0\phi)^{-1}$ of F over U', then $\psi\cdot u = j_m(s')(\rho(c))$ and by (15.14)

$$\begin{aligned}\sigma_b\psi\cdot\beta_a^{-1}u &= j_m(\sigma(\phi))(b)\cdot j_m(\gamma_s)(a) \\ &= j_m(\gamma_{s'})(\sigma(\phi)a)\cdot j_m(\sigma(\phi))(a) \\ &= \beta_c^{-1}(\psi\cdot u)\cdot\sigma_a\psi\ .\end{aligned}$$

Let $N_k\subset J_k(F;\,Y)$ be a formally integrable differential equation such that $\pi_0\colon N_k\to F$ is surjective and (18.23) holds. Let $R_k\subset J_k(V;\,\lambda)$ be the formally integrable abelian Lie equation whose l-th prolongation R_{k+l} is the inverse image of $\rho^{-1}N_{k+l}$ under the isomorphism

$$\lambda\colon J_{k+l}(V;\,\lambda)\to\rho^{-1}J_{k+l}(F;\,Y)\ .$$

If $P_{k+l} = \alpha^{-1}(R_{k+l})$, then P_k is a formally integrable finite form of R_k with $(P_k)_{+l} = P_{k+l}$. For $m\geq k$, let $R^\sharp_m$ denote the image of the mapping (19.19).

Proposition 20.1. *If*

$$P''_{q+k}\cdot N_k\subset N_k\ , \tag{20.15}$$

then $R^\sharp_m$ is a vector bundle for all $m\geq k$.

Proof. Condition (20.15) implies (18.23) according to § 15. Since

$$P''_{q+k+l}\cdot N_{k+l}\subset N_{k+l}\ ,$$

for all $l\geq 0$, we see from (20.13) that

$$\psi(R_{m,a}) = R_{m,b}\ ,$$

for all $m \geq k$ and $\psi \in P^{b}_{m+1}$, with source $\psi = a$, target $\psi = b$, and hence by (20.12) that

$$\psi(R^{\#}_{m,a}) = R^{\#}_{m,b} \ . \tag{20.16}$$

By our hypothesis on $P''_q(y_0)$ and the fact that the fibers of X are connected, given $a, b \in X$, there exist $\psi \in P''_{q+m+1}$ with source $\psi = \rho(a)$, target $\psi = \rho(b)$ and $\phi \in P_{m+1}$ with source $\phi = \sigma(\pi_q \psi)a$ and target $\phi = b$. Then by (20.16) and (19.20), we have

$$(\phi \cdot \sigma_a \psi)(R^{\#}_{m,a}) = R^{\#}_{m,b} \ ,$$

showing that $R^{\#}_m$ is a vector bundle.

We now assume that $R^{\#}_m$ is a vector bundle for all $m \geq k$. Let $p \geq k$ be the integer given by Theorem 18.3 such that $R^{\#}_p$ is a formally transitive and formally integrable Lie equation with

$$R^{\#}_{p+l} = (R^{\#}_p)_{+l} \ , \qquad \text{for } l \geq 0 \ .$$

If Y is connected or if (20.15) holds, then by results of [10] or the proof of Proposition 20.1 the equation $R^{\#}_p$ satisfies condition (III) of § 9.

Let $P^{\#}_p$ be a formally integrable finite form of $R^{\#}_p$ whose l-th prolongation we denote by $P^{\#}_{p+l}$. Let $m \geq p$; since $R^{b}_m \subset R^{\#}_m$ and diagram (20.6) commutes, we see that

$$\sigma_*(V_{(a,\psi)}(\rho^{-1}P''_{q+m})) \subset \tilde{R}^{\#}_{m,b} \cdot \sigma_a \psi \ ,$$

for $\psi \in P''_{q+m}$, $a, b \in X$, with source $\psi = \rho(a)$ and $b = \sigma(\pi_q \psi)a$. Since $P^{\#}_m$ is a finite form of $R^{\#}_m$ and the image of the section $I_{Y,q+m} \circ \rho$ of $\rho^{-1}P''_{q+m}$ under the mapping (20.1) is equal to the section I_m of $P^{\#}_m$, there is an open neighborhood U of the section $I_{Y,q+m} \circ \rho$ in $\rho^{-1}P''_{q+m}$ such that $\sigma(U) \subset P^{\#}_m$. Therefore for all $x \in X$, with $y = \rho(x)$, we have

$$\sigma_x(\mathscr{P}''^{\boldsymbol{\cdot}}_{q+m,y}) \subset \mathscr{P}^{\#\boldsymbol{\cdot}}_{m,x} \ , \tag{20.17}$$

$$\sigma_x(H^0(P''_q)_{q+m,y}) \subset H^0(P^{\#}_p)_{m,x} \ ; \tag{20.18}$$

if $f'' \in H^0(P''_q)_{q+m,y}$, then by (20.18) and (20.4), $\sigma_x(f'')$ belongs to $H^0(P^{\#}_p)_{m,x}$ and satisfies

$$\rho\sigma_x(f'') = f'' \ .$$

Thus if $R^{\#}_p$ satisfies condition (III) of § 9 and $P^{\#}_p$ is integrable, the hypotheses of Theorem 17.4 hold for $R^{\#}_p$, with $r = q$ and $m_0 = p$.

Let $m_0 \geq p$ be an integer such that $g_{m_0}, g^{\#}_{m_0}, g''_{m_0}$ are 2-acyclic. If $R^{\#}_p$ satisfies condition (III) of § 9 and if N_k is integrable, we consider the sequence of cohomology (19.24) for $m \geq m_0$ and $x \in X$, with $y = \rho(x)$. If moreover $P_p^\#$ is integrable, Theorem 17.4 tells us that, if the image of $\alpha \in H^1(N_k)_{m,x}$ in $H^1(P_p^\#)_{m,x}$ vanishes, then $\alpha = 0$. For $m \geq p$ and $x \in X$ with $y = \rho(x)$, the mappings (19.9) and (20.3) induce, according to (19.14), (19.7), (20.10), (20.17) and (19.10), a mapping of cohomology

$$\sigma_x\colon H^1(P_q'')_{q+m,y} \to H^1_\rho(P_p^\#)_{m,x}$$

such that the diagram

$$\begin{array}{ccc} H^1(P_q'')_{q+m,y} & \xrightarrow{\sigma_x} & H^1_\rho(P_p^\#)_{m,x} \\ \big\downarrow \pi_m & \swarrow \rho & \\ H^1(P_q'')_{m,y} & & \end{array} \tag{20.19}$$

commutes. By means of Theorem 9.1, for $m \geq m_0$ we obtain a mapping

$$\sigma_x\colon H^1(P_q'')_{q+m,y} \to H^1(P_p^\#)_{m,x}$$

such that $\rho\sigma_x$ is equal to the projection π_m of diagram (20.19), where ρ denotes the mapping of the sequence (19.24). Hence by Proposition 17.1, it follows that the mapping ρ of sequence (19.24) is surjective. One verifies easily that the diagram

$$\begin{array}{ccc} H^1(P_q'')_{q+l+m,y} & \xrightarrow{\sigma_x} & H^1(P_p^\#)_{l+m,x} \\ \big\downarrow \pi_{q+m} & & \big\downarrow \pi_m \\ H^1(P_q'')_{q+m,y} & \xrightarrow{\sigma_x} & H^1(P_p^\#)_{m,x} \end{array}$$

is commutative for $l \geq 0$; we thus obtain a mapping of cohomology

$$\sigma_x\colon H^1(P_q'')_y \to H^1(P_p^\#)_x$$

such that $\rho\sigma_x$ is the identity mapping of $H^1(P_q'')_y$, where ρ denotes the mapping of cohomology

$$\rho\colon H^1(P_p^\#)_x \to H^1(P_q'')_y \,. \tag{20.20}$$

It follows that (20.20) is a surjective mapping.

We now summarize some of the above results and obtain part (iii) of the following theorem as a consequence of (i), (ii) and the exactness of (19.24).

Theorem 20.1. *Assume that $R_m^\#$ is a vector bundle for $m \geq k$. Let $m \geq m_0$ and $x \in X$. The following assertions hold*:

(i) *The mapping of cohomology*

$$\rho\colon H^1(P_p^\#)_x \to H^1(P_q'')_{\rho(x)}$$

is surjective.

(ii) *If* $R_p^\#$ *satisfies condition* (III) *of* § 9, N_k *and* $P_p^\#$ *are integrable, and the image of* $\alpha \in H^1(N_k)_{m,\rho(x)}$ *in* $H^1(P_p^\#)_{m,x}$ *vanishes, then* $\alpha = 0$.

(iii) *If* $R_p^\#$ *satisfies condition* (III) *of* § 9, *and* N_k, $P_p^\#$ *and* P_q'' *are integrable, then* $H^1(N_k)_{\rho(x)} = 0$ *and* $H^1(P_q'')_{\rho(x)} = 0$ *if and only if* $H^1(P_p^\#)_x = 0$.

Theorem 20.1 (ii) gives us another class of formally transitive and formally integrable Lie equations $R_p^\#$ for which the second fundamental theorem does not always hold; indeed, if $H^1(N_k) \neq 0$, the non-linear cohomology of $R_p^\#$ does not vanish.

Remark. For $m \geq k$, let $P_m \times_Y P''_{q+m}$ be the set of all $(\phi, \psi) \in P_m \times P''_{q+m}$ satisfying $\rho(\text{source } \phi) = \text{source } \psi$, and consider the mapping

$$\Phi : P_m \times_Y P''_{q+m} \to B_m ,$$

sending (ϕ, ψ) into $\sigma_a \psi \cdot \phi$, where $a = \text{target } \phi$. If (20.15) holds, by Proposition 20.1 and Theorem 18.3, $R_m^\#$ is a Lie equation; then using the relation (20.14), it can be shown that the image $P_m^\#$ of Φ is a differentiable sub-groupoid of B_m and a finite form of $R_m^\#$. Furthermore by (20.9) and Proposition 7.2, $P_{m+1}^\# \subset (P_m^\#)_{+1}$ and $P_p^\#$ is a formally integrable finite form of $R_p^\#$ whose l-th prolongation is $P_{p+l}^\#$. If N_k and P_q'' are integrable, so is $P_p^\#$; if $x \in X$ and $f'' \in \text{Sol}\,(P_q'')_{\rho(x)}$, then $\sigma_x(f'')$ belongs to $\text{Sol}\,(P_p^\#)_x$.

Assume that Y is a Lie group G and that y_0 is the identity element of G; let $\mathfrak{g}$ be the Lie algebra of G with the bracket defined in terms of right-invariant vector fields on G. Let

$$\iota : \mathfrak{g} \to \Gamma(Y, T_Y)$$

be the homomorphism of Lie algebras sending ξ into the right-invariant vector field $\hat{\xi}$ on Y whose value at y_0 is equal to ξ. We denote by R_m'' the image of the morphism of vector bundles

$$\iota_m : Y \times \mathfrak{g} \to J_m(T_Y; Y) ,$$

sending (y, ξ) into $j_m(\hat{\xi})(y)$. We have $R_0'' = J_0(T_Y)$ and

$$\pi_m : R''_{m+1} \to R''_m$$

is an isomorphism of vector bundles for $m \geq 0$. Clearly

$$[R''_{m+1}, R''_{m+1}] \subset R''_m , \qquad R''_{m+1} \subset (R''_m)_{+1} ,$$

and therefore R_1'' is a formally transitive and formally integrable analytic Lie equation of finite type such that

$$(R_1'')_{+m} = R''_{m+1} , \qquad \text{for } m \geq 0 .$$

The mapping ι_∞ determines, for $y \in Y$, an isomorphism of Lie algebras of $\mathfrak{g}$ with the transitive Lie algebra $R''_{\infty,y}$.

The image P''_m of the morphism of fibered manifolds over Y

$$\iota\colon Y \times G \to Q_m(Y) , \tag{20.21}$$

sending (y, g) into the m-jet at y of the left-translation of Y by g, is an analytic sub-groupoid of $Q_m(Y)$ and a finite form of R''_m. Moreover P''_1 is formally integrable and of finite type with

$$(P''_1)_{+m} = P''_{m+1} ,$$

and

$$\pi_m\colon P''_{m+1} \to P''_m$$

is bijective for $m \geq 0$. For $y \in Y$, we see that $P''_m(y)$ is a principal bundle with structure group $\{I_{Y,m}(y)\}$, and the mapping (20.21) determines a bijective mapping

$$\iota_y\colon G \to P''_m(y) .$$

Assume that the vector bundle F is a G-bundle, that is, possesses the structure of a G-space such that $g\colon F \to F$ is a morphism of vector bundles over the left-translation $g\colon Y \to Y$, for $g \in Y$. Then F has a natural trivialization

$$Y \times F_{y_0} \to F ,$$

which sends (g, f) into $g \cdot f$, and thus F is an analytic vector bundle. We consider F as a vector bundle associated to the principal bundle $P''_1(y_0)$ by means of the mapping

$$\iota_{y_0} \times \mathrm{id}\colon Y \times F_{y_0} \to P''_1(y_0) \times F_{y_0} .$$

The diagram

$$\begin{array}{ccc} G \times F & \longrightarrow & F \\ \Big\downarrow{\scriptstyle \iota \times \mathrm{id}} & & \Big\downarrow{\scriptstyle \mathrm{id}} \\ P''_1 \times_Y F & \longrightarrow & F \end{array}$$

is easily seen to commute, where the top horizontal arrow is given by the G-bundle structure of F, and the bottom horizontal arrow is determined by the structure on F of vector bundle associated to $P''_1(y_0)$. For $g \in G$, we have an endomorphism of $\Gamma(Y, F)$ sending s into $g \cdot s \cdot g^{-1}$ and a morphism of vector bundles

$$g\colon J_m(F;\, Y) \to J_m(F;\, Y)$$

over the left-translation $g\colon Y \to Y$ defined by

$$g \cdot j_m(s)(y) = j_m(g \cdot s \cdot g^{-1})(gy) \;,$$

where s is a section of F over Y and $y \in Y$; thus $J_m(F;\, Y)$ is endowed with the structure of a G-bundle. Then the diagram

$$\text{(20.22)} \qquad \begin{array}{ccc} G \times J_m(F;\, Y) & \longrightarrow & J_m(F;\, Y) \\ \Big\downarrow{\scriptstyle \iota \times \mathrm{id}} & & \Big\downarrow{\scriptstyle \mathrm{id}} \\ P''_{m+1} \times_Y J_m(F;\, Y) & \longrightarrow & J_m(F;\, Y) \end{array}$$

also commutes, where the top horizontal arrow is given by the G-bundle structure of $J_m(F;\, Y)$, and the bottom horizontal arrow is determined by the structure on F of vector bundle associated to $P''_1(y_0)$.

We say that a differential equation $N_k \subset J_k(F;\, Y)$ is G-invariant if N_k is a G-invariant sub-bundle of $J_k(F;\, Y)$. For such an equation, there exist a G-vector bundle F' over Y and a G-morphism of vector bundles $\varphi\colon J_k(F;\, Y) \to F'$ such that $\ker \varphi = N_k$. Moreover, the differential operator

$$P = \varphi \circ j_k\colon \Gamma(Y, F) \to \Gamma(Y, F')$$

is G-invariant in the sense that it commutes with the induced action of G on $\Gamma(Y, F)$ and $\Gamma(Y, F')$. Conversely, given G-vector bundles F, F' over Y and a G-invariant linear differential operator $P\colon \mathscr{F} \to \mathscr{F}'$ of order k, there is a G-morphism of vector bundles $\varphi\colon J_k(F;\, Y) \to F'$ such that $P = \varphi \circ j_k$, and $N_k = \ker \varphi$ is a G-invariant differential equation.

Let $N_k \subset J_k(F;\, Y)$ be a G-invariant differential equation; then N_k is an analytic equation, and N_{k+l} is a G-invariant sub-bundle of $J_{k+l}(F;\, Y)$. In view of the commutativity of (20.22), we have

$$P''_{k+l+1} \cdot N_{k+l} \subset N_{k+l} \;;$$

moreover for $y \in Y$, if we identify $\mathfrak{g}$ with $R''_{\infty,y}$ by means of the mapping ι_∞, the $R''_{\infty,y}$-module structure on $N_{\infty,y}$ coincides with the natural $\mathfrak{g}$-module structure on $N_{\infty,y}$ obtained from the G-invariance of N_k. Assume now that N_k is formally integrable and that $\pi_0\colon N_k \to F$ is surjective; then N_k is integrable. Let $R_k \subset J_k(V;\, \lambda)$ be the inverse image of $\rho^{-1}N_k$ under λ. By Proposition 20.1 and Theorem 18.3, we obtain from R''_1 and R_k the formally transitive and formally integrable Lie equation $R^\#_p$. Then by Theorem 19.1 (ii), for $x \in X$ the transitive Lie algebra $R^\#_{\infty,x}$ is isomorphic to the semi-direct product of $\mathfrak{g}$ and the $\mathfrak{g}$-module $N_{\infty,\rho(x)}$. Let $P^\#_p$ be a formally integrable finite form of $R^\#_p$, and let

$m_0 \geq p$ be an integer such that g_{m_0} and $g^{\#}_{m_0}$ are 2-acyclic. From Theorem 19.2 (ii) and Proposition 18.2, we obtain

Theorem 20.2. *Let Y be a Lie group G, and F a G-invariant vector bundle. Let $N_k \subset J_k(F; Y)$ be a formally integrable G-invariant differential equation such that $\pi_0 \colon N_k \to F$ is surjective. Then $R^{\#}_p$ is a formally transitive and formally integrable Lie equation and we have isomorphisms of cohomology*

$$H^1(N_k)_{m,y} \longrightarrow H^1(P^{\#}_p)_{m,x} \ ,$$
$$H^1(N_k)_y \longrightarrow H^1(P^{\#}_p)_x \ ,$$

for all $m \geq m_0$, $x \in X$, with $y = \rho(x)$.

Thus a formally integrable G-invariant differential equation $N_k \subset J_k(F; Y)$, such that $\pi_0 \colon N_k \to F$ is surjective, gives rise to a formally transitive and formally integrable Lie equation $R^{\#}_p$ that belongs to the three classes of Lie equations of Theorems 19.2 and 20.1 (ii) for which the integrability problem is not always solvable; in fact, if $H^1(N_k) \neq 0$, the non-linear cohomology of $R^{\#}_p$ does not vanish.

More generally, to any G-invariant differential operator on Y corresponds a Lie equation belonging to these classes, as we now proceed to show. Let F' be a G-vector bundle over Y, and $P \colon \mathscr{F} \to \mathscr{F}'$ a G-invariant linear differential operator of order k. If $\varphi \colon J_k(F; Y) \to F'$ is the G-morphism of vector bundles such that $P = \varphi \circ j_k$ and N_k is the G-invariant differential equation $\ker \varphi$, then N_{k+l} is a vector bundle for all $l \geq 0$, and the mappings $\pi_{k+l} \colon N_{k+l+m} \to N_{k+l}$ are of constant rank for all $l, m \geq 0$. According to [5, Theorem 1], there exist a formally integrable differential equation $N'_{k_0} \subset J_{k_0}(F; Y)$, with $k_0 \geq k$, and an integer $l_0 \geq 0$ such that

$$N'_{k_0+r} = \pi_{k_0+r} N_{k_0+l_0+r} \ ,$$

for all $r \geq 0$, and

$$N'_\infty = N_\infty \ .$$

By [5, Theorem 3], there is a vector bundle F'' over Y and a linear differential operator $Q \colon \mathscr{F}' \to \mathscr{F}''$ of order l such that the sequence

$$\mathscr{F} \xrightarrow{P} \mathscr{F}' \xrightarrow{Q} \mathscr{F}'' \tag{20.23}$$

is formally exact in the sense that the sequences of vector bundles

$$J_{k+l+m}(F; Y) \xrightarrow{p_{l+m}(\varphi)} J_{l+m}(F'; Y) \xrightarrow{p_m(\psi)} J_m(F''; Y)$$

are exact for all $m \geq 0$, where $p_{l+m}(\varphi)$, $p_m(\psi)$ are the morphisms of vector bundles satisfying

$$j_{l+m} \circ P = p_{l+m}(\varphi) \circ j_{k+l+m} \ , \qquad j_m \circ Q = p_m(\psi) \circ j_{l+m} \ .$$

The differential operator Q is the compatibility condition for P; by [5, Proposition 8], we have the equality of Spencer cohomologies

$$H^*(N_k) = H^*(N'_{k_0}) ,$$

and by [5, Theorem 3] the cohomology $H^1(N_k)$ is isomorphic to the cohomology of the complex (20.23). The vector bundle F'' can be chosen to be a G-vector bundle and the differential operator Q to be G-invariant. If F_0 is the G-invariant sub-bundle $\pi_0 N'_{k_0}$ of F, then by Lemma 15.2 we see that $N'_{k_0} \subset J_{k_0}(F_0; Y)$ is a formally integrable G-invariant differential equation in F_0 whose cohomology $H^1(N'_{k_0})$ is isomorphic to that of the complex (20.23). Let X be the vector bundle F_0, and $R^\#_p \subset J_p(T)$ be the formally transitive and formally integrable Lie equation constructed from F_0, N'_{k_0} and R''_1 by Theorem 18.3; for $x \in X$, the transitive Lie algebra $R^\#_{\infty,x}$ is isomorphic to the semi-direct product of $\mathfrak{g}$ and the $\mathfrak{g}$-module $N_{\infty,\rho(x)}$. Since Theorem 20.2 gives us an isomorphism of cohomology

$$H^1(N_k)_y \to \tilde{H}^1(R^\#_p)_x ,$$

for all $x \in X$ with $y = \rho(x)$, we thus obtain a formally transitive and formally integrable Lie equation $R^\#_p$ on X, whose non-linear cohomology at $x \in X$ is isomorphic to the cohomology of the complex (20.23) at $\rho(x)$. If the differential operator P is not locally solvable, that is, the complex (20.23) is not exact, the second fundamental theorem does not hold for the Lie equation $R^\#_p$, and we have thus constructed counterexamples to the integrability problem.

Finally, we point out how the counterexample of Guillemin and Sternberg [15] arises in this way. Let Y be the Lie group $SU(2)$, and let $\{\eta_1, \eta_2, \eta_3\}$ be a basis for the Lie algebra of left-invariant vector fields on Y such that the relations

$$[\eta_i, \eta_j] = \eta_l$$

hold for all cyclic permutations (i, j, l) of $(1, 2, 3)$. Under the standard identification of Y with the three-dimensional sphere S^3 imbedded in C^2, the differential operator $\bar{\partial}_b \colon \mathcal{O}_Y \to \mathcal{O}_Y$ determined by the complex vector field $\eta_1 + \sqrt{-1}\,\eta_2$ on Y coincides essentially with the tangential Cauchy-Riemann operator on the real hypersurface S^3 of C^2, which is the locally non-solvable operator of H. Lewy. The example of Guillemin and Sternberg [15] is the pseudogroup corresponding to the formally transitive and formally integrable Lie equation $R^\#_1$ of order one on $Y \times C$ obtained by the above procedure from the invariant differential operator $\bar{\partial}_b$ on Y. By Theorem 20.2, the non-linear cohomology of $R^\#_1$ does not vanish.

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