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ON THE NON-LINEAR COHOMOLOGY OF LIE EQUATIONS. III

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INTRODUCTION

This paper is a continuation of parts I and II of the same title which appeared in Acta Math. **136** (1976) 103–239, and its theme is the study of extensions of transitive Lie algebras, their realization as Lie equations on manifolds and their cohomology (linear and non-linear). We present a unified viewpoint on the solvability and non-solvability of the integrability problem; the methods used in the preceding parts of this paper to obtain solvability results are extended here to prove non-solvability. Our attention is mainly centered on abelian extensions of transitive Lie algebras, whose importance is underscored by the Jordan-Hölder decomposition of Guillemin [12].

Consider the exact sequence of topological Lie algebras

$$0 \to I \to L \to L'' \to 0 ,$$

where L, L'' are transitive Lie algebras, and I is a closed abelian ideal of L. Two questions arise, namely: how is this sequence realized by Lie equations on

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manifolds and how are the cohomologies (linear and non-linear) of the ideal I of L and of the Lie algebras L and L'' related? We attempt to resolve these questions. In addition, we show that $H^1(L, I)$ and $\tilde{H}^1(L, I)$ are isomorphic as cohomologies, and do not depend on the choice of the extension L of L'' by I but only on the structure on I of module over the Lie algebra L'' determined by the extension L. We obtain further results for such an exact sequence when we impose further conditions on L'', with no assumption on the closed ideal I: if L'' is elliptic, then $\tilde{H}^1(L, I) = 0$ if and only if $\tilde{H}^1(L) = 0$; moreover if L'' is finite-dimensional, we have an isomorphism of cohomology $\tilde{H}^1(L, I) \to \tilde{H}^1(L)$.

The first example of non-solvability of the integrability problem was given by Guillemin and Sternberg [15] and was later analyzed and generalized by Buck [23]. Following ideas of Buck [23], we use our results described above to construct a class of formally transitive Lie equations for which the integrability problem is not solvable and which includes the examples of Buck. All these examples correspond to abelian extensions of transitive Lie algebras and the non-solvability of the integrability problem for these examples arises from the local non-solvability of linear differential operators. An abelian ideal I of a transitive Lie algebra L is realized as a Lie equation determined by a linear over-determined differential operator P invariant under a transitive Lie equation. The linear or non-linear cohomology of the ideal I of L is isomorphic to the Spencer cohomology of P, which provides the obstruction to local solvability of P, and vanishes if and only if P is locally solvable. In particular, any invariant differential operator on a Lie group provides us with such a Lie equation and an abelian ideal in a transitive Lie algebra.

Following is a brief summary of the contents of the paper. The first section, § 14, is purely algebraic and is concerned with geometric modules over a transitive Lie algebra L, which are the L-modules that arise when one considers abelian extensions of L. In fact, a linearly compact L-module is a geometric L-module if and only if it satisfies the descending chain condition on closed L-submodules. This class of L-modules was first considered as filtered L-modules by Guillemin and Sternberg [29]; they proved that a module of this kind can be realized as a module of sections of a formal vector bundle. Our treatment, on the other hand, leads to a quite different realization theorem. In § 15, a definition is given of vector bundles associated to a Lie equation which generalizes the notion of vector bundles associated to a principal bundle. If E is a vector bundle associated to a formally integrable Lie equation R_k on a manifold X, sections of R_k operate as first-order linear differential operators on the sections of E; these operations are used to construct on the space $J_\infty(E)_x$ of formal sections of E at $x \in X$ a structure of a module over the Lie algebra $R_{\infty,x}$ of formal solutions of R_k at x. If R_k is formally transitive, $J_\infty(E)_x$ is a geometric $R_{\infty,x}$-module, and if $N_l \subset J_l(E)$ is a formally integrable differential equation whose space of sections is invariant under the action of the sections of the l-th prolongation R_{k+l} of R_k, the space of formal solutions $N_{\infty,x}$ of N_l at x is a closed geometric

$R_{\infty,x}$-submodule of $J_\infty(E)_x$. We generalize (Theorem 15.1 and Corollary 15.1) results of [10] concerning closed ideals of $R_{\infty,x}$ to show that, whenever X is simply connected, every closed $R_{\infty,x}$-submodule of $N_{\infty,x}$ is the space of formal solutions of a formally integrable differential equation on E of the same type as N_l. Our realization theorem (Theorem 19.3) asserts that every geometric module over a real transitive Lie algebra is isomorphic to an $R_{\infty,x}$-module of the type $N_{\infty,x}$. In § 16 the notion of the characteristic variety of a geometric module is defined; in particular the characteristic variety is defined for a transitive Lie algebra L or a closed ideal I of L, since they are geometric L-modules under the adjoint representation of L, and coincides with the characteristic variety given by Guillemin [27] (see also [28]). The main result of this section is Theorem 16.3 which asserts that, for a short exact sequence of geometric modules over a transitive Lie algebra, the characteristic variety of the middle term is the union of the characteristic varieties of the two end terms. The essential work of defining the characteristic variety of a geometric module and of proving Theorem 16.3 is largely concentrated in Proposition 16.2. The notion of ellipticity is defined for a geometric module and, as a consequence of Theorem 16.3, a transitive Lie algebra L is elliptic if and only if a closed ideal I of L and the transitive Lie algebra L/I are elliptic (Corollary 16.3). If L and I are realized as Lie equations on a manifold (as in § 10), then the characteristic varieties of these equations are completely determined by the characteristic varieties of L and I respectively. These results together with those of § 10 will be used in a future publication to give an independent proof based on the Newlander-Nirenberg theorem and the local solvability of linear analytic elliptic equations (Proposition 17.4) of the theorem of Malgrange [19] asserting that $\tilde{H}^1(L, I) = 0$ for a closed elliptic ideal I of L (see Theorems 17.1 and 17.9).

In § 17, we first give various results on the cohomology of elliptic or analytic Lie equations, which we use subsequently in our study of exact cohomology sequences, both linear and non-linear. We obtain stronger results about these sequences than those of [6] and of § 9, whenever conditions such as ellipticity or finite type are imposed on one of the equations whose cohomology appears in the sequences (Theorems 17.2, 17.5 and 17.6). As a consequence of our study in § 9 of the non-linear cohomology sequences (9.5) and (9.11), we establish the relation between lifting properties for solutions and information about the non-linear cohomology of the equations which appear in these sequences (Theorems 17.3 and 17.4). We exploit this last fact to obtain our version (Corollary 17.1) of the Kuranishi-Rodrigues theorem about lifting of solutions of analytic Lie equations. Finally, we generalize some of our results on the non-linear cohomology sequences in Theorems 17.7 and 17.8 and give their consequences concerning the cohomology of transitive Lie algebras and their closed ideals in Theorem 17.10.

In § 18 we pursue our study of abelian Lie equations and their cohomology which we started in § 11. If R_k is an integrable and formally integrable abelian

Lie equation, its structure is described (at least locally) by Theorem 18.6 and there is an isomorphism of cohomology $\tilde{H}^1(R_k) \to H^1(R_k)$; moreover, we show how certain such equations arise from vector bundles associated to Lie equations and invariant differential equations. The first part of § 19 is devoted to the construction of two classes of formally transitive and formally integrable Lie equations for which the integrability problem is not always solvable. We next use results of [10] to prove our realization theorem for geometric modules over real transitive Lie algebras. This enables us to associate to a geometric module E over a real transitive Lie algebra L a graded module $H^*(L, E)$ of linear Spencer cohomology over the graded Lie algebra of linear Spencer cohomology $H^*(L)$ of L. The remainder of the section is devoted to the study of this cohomology. In particular, if E is a closed ideal of L, this cohomology coincides with the one defined in [10] (Proposition 19.3). If L' is a transitive Lie algebra which is an abelian extension of L by E defining the given structure of L-module on E, then (Theorem 19.5) the cohomology $H^*(L', E)$ of the closed abelian ideal E of L' is isomorphic to $H^*(L, E)$ and thus does not depend on the choice of the extension. Finally, we derive the results mentioned above concerning the linear and non-linear cohomologies of such extensions, under the additional hypothesis that L is elliptic or finite-dimensional (Corollary 19.1). In § 20, we construct Lie equations which are counterexamples to the solvability of the integrability problem and which belong to the classes of such equations considered in § 19; we show how locally non-solvable invariant differential operators on Lie groups give rise to such Lie equations and that the example of Guillemin and Sternberg [15] arises in this way.

Finally, we ought to point out to the reader that all differential equations considered throughout this paper are assumed to be of order greater than or equal to one.

CHAPTER III. GEOMETRIC MODULES AND LIE EQUATIONS

14. Geometric modules over Lie algebras

Consider a field K endowed with the discrete topology and linearly compact topological vector spaces over K, i.e., those which are topological duals of vector spaces over K endowed with the discrete topology. We shall require the general facts about linearly compact topological vector spaces which are to be found in [12, § 1] and the following properties of such spaces.

Proposition 14.1. *Let E be a linearly compact topological vector space over K, and F be a closed subspace of E.*

(i) *Let*

$$\cdots \subset F^{k+1} \subset F^k \subset \cdots \subset F^1$$

be a decreasing chain of closed subspaces of E with $\bigcap_{k=1}^{\infty} F^k = F$. If U is an *open subspace of E containing F, there is an integer* k_0 *such that* $F^{k_0} \subset U$.

(ii) *There exists a closed subspace* F' *of E such that E is the topological direct sum* $F \oplus F'$.

The first part of the proposition is obtained by applying the corollary of [12, Proposition 1.5] to E/F and its subspaces F^k/F, U/F. The second part is the statement 12. (5) of [30, § 10].

Let L be a linearly compact Lie algebra over K, that is, a topological Lie algebra over K whose underlying topological vector space is linearly compact. A linearly compact L-module E is a topological L-module whose underlying topological vector space (over K) is linearly compact and for which the mapping $L \times E \to E$ determining the action of L on E is continuous. If A is a subspace of E, let

$$D_L A = \{e \in A \mid \xi \cdot e \in A \text{ for all } \xi \in L\} ,$$

and define inductively

$$D_L^1 A = D_L A , \quad D_L^k A = D_L(D_L^{k-1} A) , \quad k > 1 ;$$

set $D_L^\infty A = \bigcap_{k=1}^\infty D_L^k A$.

Proposition 14.2. (i) *If* E^0 *is an open subspace of E, there exists an open subalgebra* L^0 *of L such that*

$$L^0 \cdot E \subset E^0 .$$

(ii) *If* E^0 *is an open subspace of E, so is* $D_L E^0$.

(iii) *If A is a closed subspace of E, then* $D_L^\infty A$ *is a closed submodule of E and every submodule of L which is contained in A is contained in* $D_L^\infty A$.

The proof of this proposition is similar to those proofs given in [12, § 2] and will be omitted.

Let E^0 be an open subspace of E; set $E^{-k} = E$, $E^k = D_L^k E^0$, for $k \geq 1$. Then $L \cdot E^k \subset E^{k-1}$ and by Proposition 14.2 (ii), E^k is open and $E^{k+1} \subset E^k$ for all k. Let L^0 be an open subspace of L satisfying

$$L^0 \cdot E^0 \subset E^0 . \tag{14.1}$$

Then it is easily verified that

$$L^0 \cdot E^k \subset E^k , \qquad \text{for } k \geq 0 . \tag{14.2}$$

According to the definition of E^k and (14.2), the finite-dimensional vector space $V = L/L^0$ considered as an abelian Lie algebra has a natural representation on the graded vector space gr $E = \bigoplus_{k=-1}^\infty E^k/E^{k+1}$ and therefore also on the graded vector space

$$(\text{gr } E)^* = \bigoplus_{k=-1}^\infty (E^k/E^{k+1})^* .$$

Thus we may consider $(\mathrm{gr}\,E)^*$ as a graded module over the universal enveloping algebra of V, which is the symmetric algebra SV of V.

Proposition 14.3. *If $L^0 \subset L$ and $E^0 \subset E$ are open subspaces satisfying* (14.1) *and $V = L/L^0$, then the graded SV-module* $(\mathrm{gr}\,E)^*$ *is finitely generated.*

Proof. It suffices to show that the mapping

$$V \otimes (E^{k-1}/E^k)^* \to (E^k/E^{k+1})^*$$

defined by multiplication by V is surjective for $k \geq 0$, or therefore that the dual mapping

$$E^k/E^{k+1} \to V^* \otimes (E^{k-1}/E^k) \tag{14.3}$$

is injective for $k \geq 0$. Suppose that a belongs to the kernel of (14.3). Then if $e \in E^k$ is a representative of a, we see that $Le \subset E^k$ and hence that e belongs to E^{k+1} and $a = 0$.

The Lie algebra cohomology

$$H^*(V, \mathrm{gr}\,E) = \bigoplus_j H^j(V, \mathrm{gr}\,E)$$

of V with values in the graded V-module $\mathrm{gr}\,E$ is naturally bigraded. Let

$$\delta\colon \textstyle\bigwedge^j V^* \otimes \mathrm{gr}\,E \to \bigwedge^{j+1} V^* \otimes \mathrm{gr}\,E$$

be the coboundary operator defined by

$$\langle v_1 \wedge \cdots \wedge v_{j+1}, \delta\theta\rangle = \sum_{i=1}^{j+1} (-1)^{i+1} v_i \cdot \langle v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_{j+1}, \theta\rangle ,$$

for $\theta \in \bigwedge^j V^* \otimes \mathrm{gr}\,E$, $v_1, \cdots, v_{j+1} \in V$, where $\hat{v}_i$ indicates that v_i is to be omitted. Then

$$H^j(V, \mathrm{gr}\,E) = \bigoplus_{k=-1}^{\infty} H^{j,k}(V, \mathrm{gr}\,E) ,$$

where $H^{j,k}(V, \mathrm{gr}\,E)$ is the cohomology of the complex

$$\textstyle\bigwedge^{j-1} V^* \otimes E^{k+1}/E^{k+2} \xrightarrow{\delta} \bigwedge^j V^* \otimes E^k/E^{k+1} \xrightarrow{\delta} \bigwedge^{j+1} V^* \otimes E^{k-1}/E^k .$$

The injectivity of (14.3) implies that

$$H^{0,k}(V, \mathrm{gr}\,E) = 0 , \qquad \text{for } k \geq 0 .$$

We observe that $H^j(V, \mathrm{gr}\,E)$ is the dual of $\mathrm{Tor}_j^{SV}((\mathrm{gr}\,E)^*, K)$ and from Proposition 14.3 we deduce that it is a finite-dimensional vector space.

We consider the Lie algebra L as a linearly compact L-module via the adjoint representation of L. We set $L^{-1} = L$, $L^k = D_L^k L^0$, for $k \geq 1$. If we require that the open subspace L^0 of L satisfy the stronger condition $L^0 \cdot E \subset E^0$, it is easily seen that

$$L^j \cdot E^k \subset E^{j+k}, \qquad \text{for } j, k \geq 0 .$$

Definition 14.1. An open subspace E^0 of E is said to be fundamental if it contains no L-submodules of E except 0.

Let E^0 be a fundamental subspace of E. Then by Proposition 14.2 (iii), $\bigcap_{k=-1}^{\infty} E^k = 0$, and by Proposition 14.1 (i), $\{E^k\}_{k \geq -1}$ is a fundamental system of neighborhoods of 0. Let L^0 be an open subspace of L satisfying (14.1). If F is a closed subspace of E, set

$$\text{(14.4)} \qquad \operatorname{gr} F = \bigoplus_{k=-1}^{\infty} (F \cap E^k + E^{k+1})/E^{k+1} ;$$

this is a graded subspace of gr E.

Lemma 14.1. *Let E^0 be a fundamental subspace of E and L^0 an open subspace of L satisfying* (14.1), *and let $V = L/L^0$. If F_1, F_2 are closed subspaces of E with $F_1 \subset F_2$ and* gr F_1 = gr F_2, *then $F_1 = F_2$.*

Proof. We show by induction on k that

$$F_2 \subset F_1 + E^k .$$

This is true for $k = -1$; assume that it holds for an integer $k \geq -1$. By our hypothesis, the components of degree k of gr F_1 and gr F_2 are equal and hence

$$F_1 \cap E^k + E^{k+1} = F_2 \cap E^k + E^{k+1} ;$$

therefore we have

$$F_2 \subset F_1 + F_2 \cap E^k \subset F_1 + E^{k+1} .$$

Since F_1 is closed, we conclude that

$$F_2 \subset \bigcap_{k=-1}^{\infty} (F_1 + E^k) = F_1 .$$

The following result generalizes [12, Theorem 3.1] and its proof is the same as the one of that theorem:

Theorem 14.1. *Let E be a linearly compact L-module. Then the following properties of E are equivalent*:

(i) *E possesses a fundamental subspace*;

(ii) *E satisfies the descending chain condition on closed L-submodules.*

Proof. (ii) $\Rightarrow$ (i). Assume that (i) does not hold. Let $\{E^k\}_{k \geq 0}$ be a fundamental system of neighborhoods of 0 consisting of open subspaces of E. Then the closed submodule $F_k = D_L^\infty E^k \subset E^k$ is non-zero. According to Proposition

14.2 (iii), we obtain a descending chain of non-zero closed submodules of E

$$(14.5) \qquad E \supset F_0 \supset F_1 \supset \cdots \supset F_k \supset F_{k+1} \supset \cdots$$

satisfying $\bigcap_{k=0}^{\infty} F_k = 0$, since $\bigcap_{k=0}^{\infty} E^k = 0$; thus (14.5) does not stabilize.

(i) $\Rightarrow$ (ii). Let (14.5) be a descending chain of closed submodules of E. Let E_0 be a fundamental subspace of E. Consider the graded vector space $\operatorname{gr} F_k$ given by (14.4) and the annihilator $(\operatorname{gr} F_k)^{\perp}$ of $\operatorname{gr} F_k$ in $(\operatorname{gr} E)^*$. Let L^0 be an open subspace of L satisfying (14.1) and $V = L/L^0$. Then $\operatorname{gr} F_k$ is a V-submodule of $\operatorname{gr} E$; thus $(\operatorname{gr} F_k)^{\perp}$ is an SV-submodule of $(\operatorname{gr} E)^*$. By Proposition 14.3, $\{(\operatorname{gr} F_k)^{\perp}\}$ is an increasing sequence of submodules of a finitely generated module $\operatorname{gr} E$ over the noetherian ring SV, and thus this sequence stabilizes. Therefore so does the decreasing sequence $\{\operatorname{gr} F_k\}$ of graded subspaces of $\operatorname{gr} E$. By Lemma 14.1, the chain (14.5) stabilizes and so (ii) holds.

Definition 14.2. A linearly compact topological L-module E satisfying the properties (i) and (ii) of Theorem 14.1 is called a geometric L-module.

From Theorem 14.1, we easily deduce

Proposition 14.4. *Let F be a closed L-submodule of a linearly compact L-module E. Then E is a geometric L-module if and only if F and E/F are geometric L-modules.*

If E^0 is a fundamental subspace of a geometric L-module E, then $F^0 = F \cap E^0$ is a fundamental subspace of F and

$$F^k = D_L^k F^0 = F \cap E^k .$$

If L^0 is an open subspace of L satisfying (14.1), then $L^0 \cdot F^0 \subset F^0$.

If $\phi\colon M \to L$ is a continuous epimorphism of linearly compact Lie algebras, and E is a geometric L-module, then the M-module ϕ^*E, which is equal to E endowed with the structure of M-module given by

$$\xi \cdot e = \phi(\xi) \cdot e ,$$

for all $\xi \in M$, $e \in E$, is a geometric M-module.

Let I be a closed ideal of L. Then the adjoint action of L on I determines the structure of a linearly compact L-module on I. If I is abelian, this L-module structure of I determines on I a structure of linearly compact L/I-module. Consider the linearly compact Lie algebra L as an L-module via the adjoint representation of L; a fundamental subspace of L is obviously an open subspace of L containing no ideals of L other than 0. We say that L is a *transitive Lie algebra* if it is a geometric L-module. According to [12], such a transitive Lie algebra L possesses a fundamental subalgebra L^0; if we set $L^{-1} = L$, $L^k = D_L^k L^0$, for $k \geq 1$, then $\bigcap_{k=-1}^{\infty} L^k = 0$ and $[L^j, L^k] \subset L^{j+k}$, for $j, k \geq -1$. Moreover

$$\operatorname{gr} L = \bigoplus_{k=-1}^{\infty} L^k/L^{k+1}$$

is a graded Lie algebra; according to Proposition 14.4, a closed ideal I of L is a geometric L-module, and, if $I^k = I \cap L^k$ for $k \geq -1$, then

$$\operatorname{gr} I = \bigoplus_{k=-1}^{\infty} I^k/I^{k+1}$$

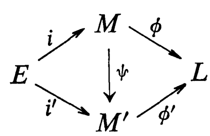
is a graded ideal of L.

We now generalize to linearly compact Lie algebras some of the standard results on extensions of Lie algebras (see [24]).

An extension M of the linearly compact Lie algebra L by E is an exact sequence of linearly compact Lie algebras over K

$$0 \longrightarrow E \xrightarrow{\ i\ } M \xrightarrow{\ \phi\ } L \longrightarrow 0\,, \tag{14.6}$$

whose mappings are continuous. Two extensions M and M' of L by E are said to be equivalent if there is a commutative diagram



where ψ is a continuous homomorphism of Lie algebras. If E is abelian, we say that M is an abelian extension of L; then the adjoint action of M on its ideal E defines by passage to the quotient a structure of linearly compact L-module on E.

Proposition 14.4 implies

Proposition 14.5. *Let M be the abelian extension* (14.6) *of the linearly compact Lie algebra L by E. Then M is a transitive Lie algebra if and only if L is a transitive Lie algebra and E is a geometric L-module.*

Let E be a linearly compact L-module. Consider the continuous Lie algebra cohomology

$$H_c^*(L, E) = \bigoplus_j H_c^j(L, E)$$

of L with values in E defined in terms of continuous cochains with values in E. We shall establish a correspondence between the abelian extensions of L by E defining the given structure of L-module on E and $H_c^2(L, E)$.

If M is the abelian extension (14.6) of L by E, by Proposition 14.1 (ii) there exists a continuous linear mapping $\sigma\colon L \to M$ such that $\phi \circ \sigma = \mathrm{id}$; then

$$\alpha(\xi, \eta) = [\sigma(\xi), \sigma(\eta)] - \sigma([\xi, \eta])\,, \tag{14.7}$$

for $\xi, \eta \in L$, belongs to E; thus α is a continuous 2-cochain on L with values in E, which, by Jacobi's identity, is easily seen to be a cocycle. The cohomology class of α depends only on the extension M.

Conversely, a continuous 2-cocycle α on L with values in E defines a structure of Lie algebra M on the linearly compact topological vector space $E \times L$ by setting

$$[(e, \xi), (f, \eta)] = (\xi \cdot f - \eta \cdot e + \alpha(\xi, \eta), [\xi, \eta]) , \tag{14.8}$$

for $e, f \in E$, $\xi, \eta \in L$. We define the exact sequence (14.6) by setting $i(e) = (e, 0)$, $\phi(e, \xi) = \xi$, for $e \in E$, $\xi \in L$. Clearly E is an abelian ideal of M, and i, ϕ are continuous Lie algebra homomorphisms, so that M is an abelian extension of L defining the given L-module structure on E. The mapping $\sigma: L \to M$, sending ξ into $(0, \xi)$, is continuous and satisfies $\phi \circ \sigma = \text{id}$. Furthermore,

$$[\sigma(\xi), \sigma(\eta)] - \sigma([\xi, \eta]) = (\alpha(\xi, \eta), 0) ,$$

for $\xi, \eta \in L$, so α is a cocycle defined by the extension M.

An extension M of L by E is inessential if there exists a closed subalgebra L' of M such that M is the topological direct sum of L' and E. An extension (14.6) is inessential if and only if there exists a continuous Lie algebra homomorphism $\sigma: L \to M$ such that $\phi \circ \sigma = \text{id}$; then the cocycle α defined by (14.7) vanishes. Finally, if α is the zero 2-cocycle on L, (14.8) gives us the semi-direct product of L and E, which is an inessential extension of L.

Thus we obtain

Theorem 14.2. *Let E be a linearly compact L-module. To each abelian extension of L by E, defining the given structure of L-module on E, corresponds a cohomology class in $H^2_c(L, E)$; this correspondence determines a bijective mapping between the equivalence classes of such extensions and $H^2_c(L, E)$. The inessential extensions form a single class and correspond to the zero element of $H^2_c(L, E)$.*

Let L be a transitive Lie algebra, and L^0 a fundamental subalgebra of L. We say that a closed ideal I of L is defined by a foliation in (L, L^0) if the only ideal I' of L satisfying

$$I \subset I' \subset I + L^0$$

is I itself.

Let M be the abelian extension (14.6) of a transitive Lie algebra L by a geometric L-module E and $\sigma: L \to M$ be a continuous linear mapping such that $\phi \circ \sigma = \text{id}$. Let α be the continuous 2-cocycle on L with values in E defined by (14.7). Let E^0 be a fundamental subspace of E, and L^0 be a fundamental subalgebra of L satisfying (14.1) and

$$\alpha(L^0 \times L^0) \subset E^0 .$$

We remark that, given σ and E^0, there always exists a fundamental subalgebra L^0 of L satisfying these conditions. Then $M^0 = E^0 + \sigma(L^0)$ is an open subalgebra of M. If $I \subset M^0$ is an ideal of M, its image in L under ϕ is contained in L^0 and so vanishes; thus I is contained in E^0 and is equal to 0, since E^0 is fundamental. Therefore M^0 is a fundamental subalgebra of M satisfying $\phi(M^0) = L^0$. By [10, Proposition 10.2] we have:

Proposition 14.6. *The open subalgebra M^0 of the transitive Lie algebra M is fundamental. The closed abelian ideal E of M is defined by a foliation in (M, M^0).*

15. Vector bundles associated to Lie equations

Let $R_k \subset J_k(T)$ be a formally integrable Lie equation, and E be a vector bundle over X. We shall identify $J_0(E)$ with E. The following definition generalizes the definition of vector bundles associated to $\tilde{J}_k(T)$ given in [9, § 3].

Definition 15.1. We say that E is associated to $\tilde{R}_k$ if, for all $\tilde{\xi} \in \Gamma(X, \tilde{R}_k)$, we have a linear differential operator

$$\mathscr{L}(\tilde{\xi}) \colon \mathscr{E} \to \mathscr{E} \tag{15.1}$$

satisfying the conditions:

(i) $\mathscr{L}(f\tilde{\xi}) = f\mathscr{L}(\tilde{\xi})$,
(ii) $\mathscr{L}(\tilde{\xi} + \tilde{\eta}) = \mathscr{L}(\tilde{\xi}) + \mathscr{L}(\tilde{\eta})$,
(iii) $\mathscr{L}([\tilde{\xi}, \tilde{\eta}]) = [\mathscr{L}(\tilde{\xi}), \mathscr{L}(\tilde{\eta})]$,
(iv) $\mathscr{L}(\tilde{\xi})fs = f\mathscr{L}(\tilde{\xi})s + (\tilde{\xi}\cdot f)s$,

for all $\tilde{\xi}, \tilde{\eta} \in \Gamma(X, \tilde{R}_k), f \in \Gamma(X, \mathscr{O}_X), s \in \mathscr{E}$.

A section s of E determines a diffeomorphism γ_s of E sending $e \in E_x$ into $e + s(x)$, where $x \in X$, and a vertical vector field

$$\mu_s = \frac{d}{dt}\gamma_{ts}\Big|_{t=0}$$

on E. If $e' = s(x)$, then $\mu_s(e)$ is equal to the image $(d/dt)(e + te')|_{t=0}$ of e' under the isomorphism

$$\mu_e \colon E_x \to V_e(E) .$$

Assume that E is associated to $\tilde{R}_k$ and let $e \in E_x$, with $x \in X$. Consider the mapping

$$\sigma_e \colon \tilde{R}_{k,x} \to T_e(E) , \tag{15.2}$$

defined by

$$\sigma_e\tilde{\xi}(x) = s_*\pi_0\tilde{\xi}(x) - \mu_e((\mathscr{L}(\tilde{\xi})s)(x)) , \tag{15.3}$$

where $\tilde{\xi}$ and s are sections of $\tilde{R}_k$ and E over a neighborhood of x satisfying $s(x) = e$. We now verify that the right-hand side of (15.3) depends only on $\tilde{\xi}(x)$ and e. According to conditions (i), (ii) and (iv) of Definition 15.1, we see that

$(\mathscr{L}(\tilde{\xi})s)(x)$ depends only on $\tilde{\xi}(x)$ and $j_1(s)(x)$; moreover, if s' is a section of E over a neighborhood of x satisfying $s'(x) = e$ and u is the element of $(T^* \otimes E)_x$ given by the exact sequence (1.1) satisfying

$$\varepsilon u = j_1(s' - s)(x) ,$$

then (iv) implies that

$$(\mathscr{L}(\tilde{\xi})s')(x) = (\mathscr{L}(\tilde{\xi})s)(x) + \pi_0\tilde{\xi}(x) \barwedge u .$$

By [4, Proposition 5.3] and the remarks following [4, Proposition 5.6], we see that

$$s'_*\pi_0\tilde{\xi}(x) = s_*\pi_0\tilde{\xi}(x) + \mu_e(\pi_0\tilde{\xi}(x) \barwedge u) .$$

From these last two relations, it follows that (15.3) is well-defined. The diagram

$$\begin{array}{ccc} \tilde{R}_{k,x} & \xrightarrow{\sigma_e} & T_e(E) \\ & \searrow^{\pi_0} & \downarrow \\ & & T_x \end{array}$$

is commutative. If $\tilde{\xi}$ is a section of $\tilde{R}_k$ over X, the vector field $\sigma(\tilde{\xi})$ on E defined by

$$\sigma(\tilde{\xi})(e) = \sigma_e(\tilde{\xi}(x)) ,$$

for $e \in E_x$, $x \in X$, is projectable onto $\pi_0\tilde{\xi}$. Using condition (iii) of Definition 15.1, it is easily seen that

$$\sigma([\tilde{\xi}, \tilde{\eta}]) = [\sigma(\tilde{\xi}), \sigma(\tilde{\eta})] , \tag{15.4}$$

for $\tilde{\xi}, \tilde{\eta} \in \Gamma(X, \tilde{R}_k)$. Moreover, if $\tilde{\xi} \in \Gamma(X, \tilde{R}_k)$ and $s \in \Gamma(X, E)$, then $\mathscr{L}(\tilde{\xi})s$ is the unique section of E such that

$$\gamma_{s*}\sigma(\tilde{\xi}) = \sigma(\tilde{\xi}) + \mu_{\mathscr{L}(\tilde{\xi})s} , \tag{15.5}$$

$$[\sigma(\tilde{\xi}), \mu_s] = \mu_{\mathscr{L}(\tilde{\xi})s} . \tag{15.6}$$

Indeed, let $e \in E_x$, with $x \in X$, and s' be a section of E over a neighborhood of x satisfying $s'(x) = e$. Then

$$\gamma_s \circ s' = s + s' , \qquad \gamma_{s*}\mu_e(e') = \mu_{e+s(x)}(e') ,$$

for $e' \in E_x$; therefore we have

$$\gamma_{s*}\sigma(\tilde{\xi})(e) = \gamma_{s*}s'_*\pi_0\tilde{\xi}(x) - \gamma_{s*}\mu_e((\mathscr{L}(\tilde{\xi})s')(x))$$

$$= (s + s')_* \pi_0 \tilde{\xi}(x) - \mu_{e+s(x)}((\mathscr{L}(\tilde{\xi})s')(x))$$
$$= \sigma(\tilde{\xi})(e + s(x)) + \mu_{e+s(x)}((\mathscr{L}(\tilde{\xi})s)(x)) ,$$

giving us formula (15.5). Equation (15.6) is a direct consequence of (15.5).

Assume that R_k is formally transitive, and that E is associated to $\tilde{R}_k$. An R_k-connection $\omega: J_0(T) \to R_k$ induces a connection ∇ in E by setting

$$\nabla_\xi s = \mathscr{L}(\tilde{\omega}(\xi))s , \qquad \text{for } \xi \in \mathscr{T}, \, s \in \mathscr{E} ,$$

where $\tilde{\omega}: T \to \tilde{R}_k$ is equal to $\nu^{-1} \circ \omega \circ \nu$. If the curvature of ω vanishes, then so does the curvature of the covariant derivative ∇ (see [9, Proposition 3.3]).

Let P_k be a finite form of R_k and $a \in X$. Assume that the projection of $P_k(a)$ onto X sending $F \in P_k(a)$ into the target of F is surjective; this condition always holds if X is connected. Then $P_k(a)$ is a principal bundle over X whose group is the set $P_k(a, a)$ of $F \in P_k$ with source $F =$ target $F = a$. If ϕ is a section of $P_k(a)$ over an open subset U of X, define $\tilde{\omega}: T \to \tilde{R}_k$ on U by

$$\tilde{\omega}(\xi) = \phi_*(\xi) \cdot \phi(x)^{-1} , \qquad \text{for } \xi \in T_x, \, x \in U . \tag{15.7}$$

Then $\omega = \nu \circ \tilde{\omega} \circ \nu^{-1}$ is an R_k-connection on U whose curvature vanishes (see [9, p. 71]).

Let E_0 be a finite-dimensional $P_k(a, a)$-module, and let E be the vector bundle

$$E = P_k(a) \times_{P_k(a,a)} E_0$$

associated to $P_k(a)$. Denote by

$$\varpi: P_k(a) \times E_0 \to E$$

the canonical projection.

For $H \in P_k$, with $x_1 =$ source H and $x_2 =$ target H, we have a mapping

$$\tau(H): P_k(a)_{x_1} \to P_k(a)_{x_2} ,$$

sending F into $H \cdot F$, and an isomorphism

$$\sigma(H): E_{x_1} \to E_{x_2} ,$$

sending $\varpi(F, e)$ into $\varpi(H \cdot F, e)$, where $F \in P_k(a)$, $e \in E_0$. If $H' \in P_k$, with source $H' = x_2$ and target $H' = x_3$, then

$$\sigma(H' \cdot H) = \sigma(H') \cdot \sigma(H) \tag{15.8}$$

as mappings from E_{x_1} to E_{x_3}. These mappings $\tau(H)$ determine isomorphisms

$$\tau_F: \tilde{R}_{k,x} \to T_F(P_k(a)) ,$$

where $F \in P_k(a)$ and $x =$ target F, and the mappings $\sigma(H)$ determine a mapping

(15.9) $$\sigma_e \colon \tilde{R}_{k,x} \to T_e(E) ,$$

where $e \in E_x$, $x \in X$. Then, if $F \in P_k(a)$, $e_0 \in E_0$, with $x = \text{target}\, F$ and $e = \varpi(F, e_0)$, we easily see that the diagram

$$\begin{array}{ccc} T_F(P_k(a)) \times T_{e_0}(E_0) & \xrightarrow{\varpi} & T_e(E) \\ \uparrow {\scriptstyle (\tau_F, 0)} & \nearrow {\scriptstyle \sigma_e} & \\ \tilde{R}_{k,x} & & \end{array}$$

commutes. If $\tilde{\xi}$ is a section of $\tilde{R}_k$ over X, then the vector field $\tau(\tilde{\xi})$ on $P_k(a)$ defined by

$$\tau(\tilde{\xi})(F) = \tau_F(\tilde{\xi}(x)) ,$$

for $F \in P_k(a)$, with $x = \text{target}\, F$, is $P_k(a, a)$-invariant; we have a vector field $\sigma(\tilde{\xi})$ on E defined by

$$\sigma(\tilde{\xi})(e) = \sigma_e(\tilde{\xi}(x)) ,$$

for $e \in E_x$, $x \in X$. Then τ is a morphism of Lie algebras from $\Gamma(X, \tilde{R}_k)$ to the algebra of projectable vector fields on $P_k(a)$, and so is σ from $\Gamma(X, \tilde{R}_k)$ to the algebra of projectable vector fields on E.

Let ϕ be a section of P_k over an open set $U \subset X$, and assume that $\pi_0\phi$ is a diffeomorphism of U onto an open subset U' of X; then the mappings $\sigma(\phi(x))$, with $x \in U$, give us an isomorphism of vector bundles

$$\sigma(\phi) \colon E_{|U} \to E_{|U'}$$

over $\pi_0\phi$. If $\tilde{\xi} \in \tilde{R}_{k,x}$, with $x \in U$, there is a curve H_t in $P_k(x)$ such that $H_0 = I_k(x)$ and $dH_t/dt\,|_{t=0} = \tilde{\xi}$. By (2.5), we see that

$$\phi(\tilde{\xi}) = \frac{d}{dt}\phi \cdot H_t \cdot \phi(x)^{-1}\Big|_{t=0} .$$

Using this last relation and (15.8), for $e \in E_x$, we derive the equality

(15.10) $$\sigma(\phi)_*\sigma_e(\tilde{\xi}) = \sigma_{e'}(\phi(\tilde{\xi})) ,$$

where $e' = \sigma(\phi)e$.

We identify E_0 with E_a by means of the isomorphism $E_0 \to E_a$ sending e_0 into $\varpi(I_k(a), e_0)$. If $G \in P_k(a, a)$, under this identification, the automorphism of E_0 determined by G and the $P_k(a, a)$-module structure of E_0 is the same as $\sigma(G)$; moreover

$$\varpi(F, e_0) = \sigma(F)e_0 ,$$

for $F \in P_k(a)$, $e_0 \in E_0$.

To a section s of E over X corresponds the E_0-valued function $\hat{s}$ on $P_k(a)$ defined by

$$\hat{s}(F) = \sigma(F)^{-1}s(x)$$

or

$$\varpi(F, \hat{s}(F)) = s(x) ,$$

for $F \in P_k(a)$, with $x = \text{target } F$. It is easily seen that $\hat{s}$ is equivariant in the sense that

$$\hat{s}(F \cdot G) = G^{-1}\hat{s}(F) , \tag{15.11}$$

for $F \in P_k(a)$, $G \in P_k(a, a)$. Conversely, if f is an E_0-valued function on $P_k(a)$ satisfying

$$f(F \cdot G) = G^{-1}f(F) , \qquad \text{for } F \in P_k(a),\ G \in P_k(a, a) , \tag{15.12}$$

there exists a unique section s of E such that $\hat{s} = f$.

If $\tilde{\xi}$ is a section of $\tilde{R}_k$ and s is a section of E over X, the function $f = \tau(\tilde{\xi})\hat{s}$ on $P_k(a)$ satisfies (15.12), since $\tau(\tilde{\xi})$ is a right-invariant vector field on $P_k(a)$; we define $\mathscr{L}(\tilde{\xi})s$ to be the section of E corresponding to f. We thus obtain operators (15.1) which satisfy the conditions of Definition 15.1 and so E is associated to $\tilde{R}_k$. Let F_t be a curve in $P_k(x)$, with $F_0 = I_k(x)$ and $dF_t/dt\,|_{t=0} = \tilde{\xi}(x)$; set $x_t = \text{target } F_t$. Then for $F \in P_k(a)$, with target $F = x$, we have by (15.8)

$$\begin{aligned}(\tau(\tilde{\xi}) \cdot \hat{s})(F) = (\tilde{\xi}(x) \cdot F) \cdot \hat{s} &= \frac{d}{dt}\hat{s}(F_t \cdot F)\Big|_{t=0} \\ &= \frac{d}{dt}\sigma(F_t \cdot F)^{-1}s(x_t)\Big|_{t=0} \\ &= \frac{d}{dt}\sigma(F)^{-1}(\sigma(F_t)^{-1}s(x_t))\Big|_{t=0} \\ &= \sigma(F)^{-1}\frac{d}{dt}\sigma(F_t)^{-1}s(x_t)\Big|_{t=0} ,\end{aligned}$$

where we consider $\sigma(F_t)^{-1}s(x_t)$ as an element of E_x; hence

$$(\mathscr{L}(\tilde{\xi})s)(x) = \frac{d}{dt}\sigma(F_t)^{-1}s(x_t)\Big|_{t=0} . \tag{15.13}$$

If $e = s(x)$, then we have

$$\mu_e\left(\frac{d}{dt}\sigma(F_t)^{-1}s(x_t)\Big|_{t=0}\right) = s_*\pi_0\tilde{\xi}(x) - \frac{d}{dt}\sigma(F_t)\cdot s(x)\Big|_{t=0}$$
$$= s_*\pi_0\tilde{\xi}(x) - \sigma_e(\tilde{\xi}(x)) \ ;$$

it follows from (15.13) and (15.3) that the mapping (15.9) coincides with the mapping (15.2) defined in terms of the structure of associated bundle to $\tilde{R}_k$ on E.

Let ϕ be a section of P_k over an open set $U \subset X$, and assume that $\pi_0\phi$ is a diffeomorphism of U onto an open subset U' of X. If s is a section of E over U, let s' be the section $\sigma(\phi)\circ s\circ(\pi_0\phi)^{-1}$ of E over U'; then it is easily verified that

(15.14) $$\gamma_{s'}\circ\sigma(\phi) = \sigma(\phi)\circ\gamma_s\ ,$$

and hence that

(15.15) $$\sigma(\phi)_*\mu_s = \mu_{s'}\ .$$

Let ϕ be a section of $P_k(a)$ over $U \subset X$; let ω be the R_k-connection determined by (15.7), and ∇ be the connection induced by ω in E whose curvature vanishes. If s is a section of E over U and $\xi\in T_x$, $x \in U$, then

$$\nabla_\xi s = \varpi(\phi(x), (\tau(\tilde{\omega}(\xi))\cdot\hat{s})(\phi(x))) = \varpi(\phi(x), \phi_*(\xi)\cdot\hat{s})\ .$$

If $e_0\in E_0$, the section s of E over U defined by

$$s(x) = \varpi(\phi(x), e_0) = \sigma(\phi(x))e_0\ , \qquad \text{for } x\in U\ ,$$

corresponds to the E_0-valued function $\hat{s}$ on $P_k(a)$ satisfying (15.11) and

$$\hat{s}(\phi(x)) = e_0\ , \qquad \text{for } x\in U\ ;$$

if $\xi\in T_x$, $x\in U$, we therefore have $\phi_*(\xi)\cdot\hat{s} = 0$ and $\nabla_\xi s = 0$.

The vector bundle $J_{k-1}(T)$ is associated to $\tilde{J}_k(T)$ by (1.14). From the above construction, we now obtain another interpretation of the action of $\tilde{J}_k(\mathscr{T})$ on $J_{k-1}(\mathscr{T})$. Consider the finite form Q_k of $J_k(T)$. By (2.1), $J_{k-1}(T)_a$ is a $Q_k(a, a)$-module and so we have the vector bundle

$$Q_k(a)\times_{Q_k(a,a)} J_{k-1}(T)_a$$

associated to $Q_k(a)$. The mapping

(15.16) $$Q_k(a)\times_{Q_k(a,a)} J_{k-1}(T)_a \to J_{k-1}(T)\ ,$$

sending $\varpi(F,\eta)$ into $F(\eta)$, where $F\in Q_k(a)$, $\eta\in J_{k-1}(T)_a$, is an isomorphism of vector bundles; when we identify the fiber at a of the first vector bundle with $J_{k-1}(T)_a$, this mapping restricted to the fibers at a is the identity mapping of

$J_{k-1}(T)_a$. We now identify these two vector bundles by means of the isomorphism (15.16). Then for $H \in Q_k$, with x_1 = source H, x_2 = target H, the mapping

$$\sigma(H): J_{k-1}(T)_{x_1} \to J_{k-1}(T)_{x_2}$$

is equal to the usual action of H on $J_{k-1}(T)_{x_1}$ given by (2.1). Let $\tilde{\xi} \in \Gamma(X, \tilde{J}_k(T))$, $x \in X$, and let F_t be a curve in $Q_k(x)$, with $F_0 = I_k(x)$ and $dF_t/dt|_{t=0} = \tilde{\xi}(x)$; set x_t = target F_t. If $\eta \in \Gamma(X, J_{k-1}(T))$, then according to the formula (7.1) of [9], the bracket (1.14) is given by

$$(15.17) \qquad (\mathscr{L}(\tilde{\xi})\eta)(x) = \frac{d}{dt} F_t^{-1}(\eta(x_t))\Big|_{t=0} .$$

Therefore by (15.13), when we identify the two vector bundles under consideration using (15.16), for $\tilde{\xi} \in \Gamma(X, \tilde{J}_k(T))$, the two operators $\mathscr{L}(\tilde{\xi})$ on $J_{k-1}(\mathscr{T})$, the first given by (1.14) and the second, obtained by considering $J_{k-1}(T)$ as a vector bundle associated to $Q_k(a)$, are equal. Thus, if ϕ is a section of $Q_k(a)$ over $U \subset X$ and ω is the $\tilde{J}_k(T)$-connection determined by (15.7), for $\eta_0 \in J_{k-1}(T)_a$, the section η of $J_{k-1}(T)$ over U defined by

$$\eta(x) = \phi(x)(\eta_0) , \qquad \text{for } x \in U ,$$

is horizontal with respect to the covariant derivative induced by ω in $J_{k-1}(T)$.

Assume that the finite form P_k of R_k is formally integrable and denote by P_{k+l} the l-th prolongation of P_k. Let $\tilde{J}_l(P_k) \subset Q_{(l,k)}$ be the bundle of jets of order l of sections of $\mathscr{P}_k$. We have the mapping

$$(15.18) \qquad P_{k+l} \times_X J_l(E) \to J_l(E)$$

sending (H, u) into

$$H \cdot u = j_l(\sigma(\phi) \circ s \circ (\pi_0\phi)^{-1})(x') ,$$

where ϕ is a section of P_k over a neighborhood U of $x \in X$, such that $\pi_0\phi$ is a diffeomorphism of U onto an open neighborhood of x' = target $\phi(x)$ and $j_l(\phi)(x) = \lambda_l H$, and where s is a section of E over U satisfying $j_l(s)(x) = u$. By (15.8), if $H' \in P_{k+l}$, with source $H' = x'$, then

$$(H' \cdot H) \cdot u = H' \cdot (H \cdot u) .$$

Thus $J_l(E)_a$ is a $P_{k+l}(a, a)$-module, and we consider the vector bundle

$$P_{k+l}(a) \times_{P_{k+l}(a,a)} J_l(E)_a$$

associated to $P_{k+l}(a)$. The mapping

$$(15.19)\qquad P_{k+l}(a) \times_{P_{k+l}(a,a)} J_l(E)_a \to J_l(E)$$

sending $\varpi(F, u)$ into $F \cdot u$, where $F \in P_{k+l}(a)$, $u \in J_l(E)_a$, is an isomorphism of vector bundles; when we identify the fiber at a of the first vector bundle with $J_l(E)_a$, this mapping restricted to the fibers at a is the identity mapping of $J_l(E)_a$. We now identify these two vector bundles by means of the isomorphism (15.19). Then for $H \in P_{k+l}$, with $x_1 =$ source H, $x_2 =$ target H, the mapping

$$\sigma(H): J_l(E)_{x_1} \to J_l(E)_{x_2}$$

is determined by (15.18) and sends u into $H \cdot u$. Thus $J_l(E)$ is associated to $\tilde{R}_{k+l}$.

The diagram

$$(15.20)\qquad \begin{array}{ccc} P_{k+l+m} \times_X J_{l+m}(E) & \longrightarrow & J_{l+m}(E) \\ \big\downarrow{\scriptstyle (\mathrm{id}, \lambda_m)} & & \big\downarrow{\scriptstyle \lambda_m} \\ P_{k+l+m} \times_X J_m(J_l(E)) & \longrightarrow & J_m(J_l(E)) \end{array}$$

is easily seen to commute, where the top horizontal arrow is given by (15.18) with l replaced by $l + m$ and the bottom horizontal arrow is the mapping (15.18) corresponding to the vector bundle $J_l(E)$ associated to P_{k+l} with l replaced by m.

Let $N_l \subset J_l(E)$ be a differential equation such that

$$(15.21)\qquad P_{k+l} \cdot N_l \subset N_l \ .$$

Then $N_{l,a}$ is a $P_{k+l}(a, a)$-invariant subspace of $J_l(E)_a$ and the mapping (15.19) restricts to give us an isomorphism of vector bundles

$$P_{k+l}(a) \times_{P_{k+l}(a,a)} N_{l,a} \to N_l \ .$$

We thus obtain a one-to-one correspondence between the sub-bundles N_l of $J_l(E)$ satisfying (15.21) and the $P_{k+l}(a, a)$-invariant subspaces $N_{l,a}$ of $J_l(E)_a$. From the commutativity of (15.20), we deduce that

$$P_{k+l+m} \cdot N_{l+m} \subset N_{l+m} \ ,$$

for all $m \geq 0$, and hence that N_{l+m} is a vector bundle associated to $\tilde{R}_{k+l+m}$ and

$$\mathscr{L}(\xi)\mathscr{N}_{l+m} \subset \mathscr{N}_{l+m}$$

for all $\xi \in \tilde{\mathscr{R}}_{k+l+m}$.

We no longer assume that R_k is formally transitive. Let E be a vector bundle associated to $\tilde{R}_k$; we then define an operation of R_{k+l} on $J_{l+1}(E)$. In the case that $k = 1$, $R_k = J_1(T)$ and $E = J_0(T)$, and the operations (15.1) are given by (1.14), this operation reduces to (1.11) and is related to (1.14) by (1.15). Let

(15.22) $$R_k \otimes J_1(E) \to E$$

be the morphism of vector bundles sending $\xi \otimes u \in (R_k \otimes J_1(E))_x$, with $x \in X$, into

$$\xi \cdot u = (\mathscr{L}(\tilde{\xi})\pi_0 u - \tilde{\xi} \barwedge Du')(x) ,$$

where $\tilde{\xi} \in \tilde{\mathscr{R}}_{k,x}$, $u' \in J_1(\mathscr{E})_x$ satisfy $\nu\tilde{\xi}(x) = \xi$, $u'(x) = u$. According to conditions (i), (ii), (iv) of Definition 15.1 and (1.4), this mapping is well-defined and satisfies

(15.23) $$\xi \cdot \varepsilon(u) = \varepsilon(\nu^{-1}\xi \barwedge u) ,$$

for all $\xi \in R_k$, $u \in T^* \otimes E$. Conversely, given a mapping (15.22) sending $\xi \otimes u$ into $\xi \cdot u$ satisfying (15.23), then if $\tilde{\xi} \in \Gamma(X, \tilde{R}_k)$ with $\xi = \nu\tilde{\xi}$, by setting

(15.24) $$\mathscr{L}(\tilde{\xi})u = \xi \cdot u' + \tilde{\xi} \barwedge Du' ,$$

for $u' \in J_1(\mathscr{E})$ with $\pi_0 u' = u$, we obtain well-defined differential operators (15.1) satisfying conditions (i), (ii) and (iv) of Definition 15.1.

For $l \geq 0$, we have a mapping

(15.25) $$R_{k+l} \otimes J_{l+1}(E) \to J_l(E) ,$$

sending $\xi \otimes u$ into $\xi \cdot u$, namely the composition

$$R_{k+l} \otimes J_{l+1}(E) \xrightarrow{\lambda_l \otimes \mathrm{id}} J_l(R_k) \otimes J_{l+1}(E) \xrightarrow{\beta_l} J_l(E) ,$$

where the mapping β_l sends $j_l(\xi)(x) \otimes j_{l+1}(s)(x)$ into $j_l(\mathscr{L}(\tilde{\xi})s)(x)$, with $\xi \in \mathscr{R}_{k,x}$, $\tilde{\xi} = \nu^{-1}\xi$, $s \in \mathscr{E}_x$, $x \in X$. By (15.24), the mapping (15.25) gives rise to a mapping

$$R^0_{k+l} \otimes J_l(E) \to J_l(E)$$

sending $\xi \otimes u$ into $\xi \cdot u = \xi \cdot u'$, where $u' \in J_{l+1}(E)$ satisfies $\pi_l u' = u$. It is easily seen that

(15.26) $$\xi \cdot \varepsilon(u) = \varepsilon(\nu^{-1}\xi \barwedge \delta u)$$

holds for all $\xi \in R_{k+l}$, $u \in S^{l+1}T^* \otimes E$ and that

(15.27) $$[\xi, \eta] \cdot \pi_{l+1} u = \pi_{k+l}\xi \cdot (\eta \cdot u) - \pi_{k+l}\eta \cdot (\xi \cdot u) ,$$

for $\xi, \eta \in R_{k+l+1}$, $u \in J_{l+2}(E)$, by using the commutativity of (1.37). If $\tilde{\xi} \in \Gamma(X, \tilde{R}_{k+l})$, we define

$$\mathscr{L}(\tilde{\xi}) \colon J_l(\mathscr{E}) \to J_l(\mathscr{E})$$

to be the differential operator sending u into the element $\mathscr{L}(\tilde{\xi})u$ given by

(15.24), where $u' \in J_{l+1}(\mathscr{E})$ satisfies $\pi_l u' = u$ and $\xi = \nu\tilde{\xi}$. From (15.26), we see that $\mathscr{L}(\tilde{\xi})$ is well-defined and that these operators satisfy conditions (i), (ii) and (iv) of Definition 15.1; from (15.27) it follows that they also satisfy condition (iii) and thus $J_l(E)$ is associated to $\tilde{R}_{k+l}$. Moreover, we have

$$\pi_l(\xi \cdot u) = (\pi_{k+l}\xi) \cdot \pi_{l+1} u ,$$

for all $\xi \in R_{k+l+m}$, $u \in J_{l+m+1}(E)$, and

$$\pi_l(\mathscr{L}(\tilde{\xi})u) = \mathscr{L}(\pi_{k+l}\tilde{\xi})\pi_l u ,$$

for all $\tilde{\xi} \in \tilde{\mathscr{R}}_{k+l+m}$, $u \in J_{l+m}(\mathscr{E})$. Since $(R_{k+l})_{+m} = R_{k+l+m}$ and $J_l(E)$ is associated to $\tilde{R}_{k+l}$, the above shows that the vector bundle $J_m(J_l(E))$ is associated to $\tilde{R}_{k+l+m}$. If $\tilde{\xi} \in \Gamma(X, \tilde{R}_{k+l+m})$, the diagram

$$\text{(15.28)} \qquad \begin{array}{ccc} J_{l+m}(\mathscr{E}) & \xrightarrow{\mathscr{L}(\tilde{\xi})} & J_{l+m}(\mathscr{E}) \\ \downarrow{\lambda_m} & & \downarrow{\lambda_m} \\ J_m(J_l(\mathscr{E})) & \xrightarrow{\mathscr{L}(\tilde{\xi})} & J_m(J_l(\mathscr{E})) \end{array}$$

is easily seen to commute.

If F is another vector bundle over X, let

$$D: J_l(\mathscr{E}) \otimes J_m(\mathscr{F}) \to \mathscr{T}^* \otimes J_{l-1}(\mathscr{E}) \otimes J_{m-1}(\mathscr{F})$$

be the differential operator satisfying

$$\langle \xi, D(u \otimes v)\rangle = \langle \xi, Du\rangle \otimes \pi_{m-1} v + \pi_{l-1} u \otimes \langle \xi, Dv\rangle ,$$

for $\xi \in \mathscr{T}$, $u \in J_l(\mathscr{E})$, $v \in J_m(\mathscr{F})$; by (1.4), this operator is well-defined. For $l \geq 1$, the commutativity of the left-hand square of the diagram

$$\text{(15.29)} \qquad \begin{array}{ccccc} \mathscr{R}_{k+l} \otimes J_{l+1}(\mathscr{E}) & \xrightarrow{\lambda_l \otimes \mathrm{id}} & J_l(\mathscr{R}_k) \otimes J_{l+1}(\mathscr{E}) & \xrightarrow{\beta_l} & J_l(\mathscr{E}) \\ \downarrow{D} & & \downarrow{D} & & \downarrow{D} \\ \mathscr{T}^* \otimes \mathscr{R}_{k+l-1} \otimes J_l(\mathscr{E}) & \xrightarrow{\mathrm{id} \otimes \lambda_{l-1} \otimes \mathrm{id}} & \mathscr{T}^* \otimes J_{l-1}(\mathscr{R}_k) \otimes J_l(\mathscr{E}) & \xrightarrow{\mathrm{id} \otimes \beta_{l-1}} & \mathscr{T}^* \otimes J_{l-1}(\mathscr{E}) \end{array}$$

follows from [26, Proposition 1.4]. If $\tilde{\xi} \in \tilde{\mathscr{R}}_{k,x}$, $s \in \mathscr{E}_x$, with $x \in X$ and $\xi = \nu\tilde{\xi}$, then

$$D(j_l(\xi) \otimes j_{l+1}(s)) = 0 ,$$

$$D\beta_l(j_l(\xi) \otimes j_{l+1}(s)) = D(j_l(\mathscr{L}(\tilde{\xi})s)) = 0 .$$

Moreover if $u \in J_l(\mathscr{R}_k) \otimes J_{l+1}(\mathscr{E})$, $f \in \mathscr{O}_X$, then by (1.4) we have

$$D(f\beta_l u) - fD(\beta_l u) = df \otimes \pi_{l-1}\beta_l u = df \otimes \beta_{l-1}(\pi_{l-1} \otimes \pi_l)u$$
$$= (\mathrm{id} \otimes \beta_{l-1})(D(fu) - fDu) ,$$

from which we infer the commutativity of the right-hand square of (15.29) and hence of the whole diagram. The compositions of the horizontal arrows of diagram (15.29) are the mappings induced by (15.25). We define the morphism of vector bundles

$$(15.30) \qquad (\textstyle\bigwedge^i T^* \otimes R_{k+l}) \otimes (\bigwedge^j T^* \otimes J_{l+1}(E)) \to \bigwedge^{i+j} T^* \otimes J_l(E)$$

sending $v \otimes w$ into $v \cdot w$ by setting

$$(\alpha \otimes \xi) \cdot (\beta \otimes u) = (\alpha \wedge \beta) \otimes (\xi \cdot u) ,$$

for $\alpha \in \bigwedge^i T^*$, $\beta \in \bigwedge^j T^*$, $\xi \in R_{k+l}$ and $u \in J_{l+1}(E)$. For $l \geq 1$, if $u \in \bigwedge^i \mathscr{T}^* \otimes \mathscr{R}_{k+l}$, $v \in \bigwedge^j \mathscr{T}^* \otimes J_{l+1}(\mathscr{E})$, then we have

$$(15.31) \qquad D(u \cdot v) = (Du) \cdot \pi_l v + (-1)^i (\pi_{k+l-1} u) \cdot Dv .$$

In fact, because of (1.4) it suffices to verify this formula for $i = j = 0$, and in this case it follows from the commutativity of (15.29). We now verify the formula

$$(15.32) \qquad \mathscr{L}(\pi_{k+l}\tilde{\xi})(\eta \cdot u) = (\mathscr{L}(\tilde{\xi})\eta) \cdot u + \eta \cdot \mathscr{L}(\tilde{\xi})u ,$$

for $\tilde{\xi} \in \tilde{\mathscr{R}}_{k+l+1}$, $\eta \in \mathscr{R}_{k+l}$, $u \in J_{l+1}(\mathscr{E})$. Indeed, if $\eta' \in \mathscr{R}_{k+l+1}$, $u \in J_{l+2}(\mathscr{E})$ satisfy $\pi_{k+l}\eta' = \eta$, $\pi_{l+1}u' = u$ and $\xi = \nu\tilde{\xi}$, by (15.24), (15.27), (15.31) and (1.15) we have

$$\mathscr{L}(\pi_{k+l}\tilde{\xi})(\eta \cdot u) = \pi_{k+l}\xi \cdot (\eta' \cdot u') + \tilde{\xi} \barwedge D(\eta' \cdot u')$$
$$= [\xi, \eta'] \cdot u + \eta \cdot (\xi \cdot u') + (\tilde{\xi} \barwedge D\eta') \cdot u + \eta \cdot (\tilde{\xi} \barwedge Du')$$
$$= (\mathscr{L}(\tilde{\xi})\eta) \cdot u + \eta \cdot \mathscr{L}(\tilde{\xi})u .$$

In the case $k = 1$, $R_k = J_1(T)$ and $E = J_0(T)$, and the operations (15.1) are given by (1.14), the mapping (15.30) coincides with the bracket (1.19) and formula (15.31) with (1.25); moreover, (15.32) follows from the Jacobi identity for $\bigwedge \check{J}_\infty(\mathscr{T})^* \otimes \check{J}_\infty(\mathscr{T})$.

If E is the vector bundle associated to P_k considered at the beginning of this section, then the structure of vector bundle associated to $\tilde{R}_{k+l}$ on $J_l(E)$ determined by (15.19) coincides with the one obtained by the above discussion from the structure of vector bundle associated to $\tilde{R}_k$ on E given by (15.13).

Sometimes, we shall encounter the situation where E is associated to $\tilde{R}_k$ and there is an integer $l \geq 0$ such that the mapping (15.25) factors through

$\pi_k \otimes \mathrm{id}\colon R_{k+l} \otimes J_{l+1}(E) \to R_k \otimes J_{l+1}(E)$, giving rise to a mapping

$$R_k \otimes J_{l+1}(E) \to J_l(E) .$$

Then the mapping (15.25), with l replaced by $l+m$, factors through $\pi_{k+m} \otimes \mathrm{id}$ and gives us a mapping

$$R_{k+m} \otimes J_{l+m+1}(E) \to J_{l+m}(E) .$$

Thus for $\tilde{\xi} \in \Gamma(X, \tilde{R}_{k+m})$, we obtain a differential operator

$$\mathscr{L}(\tilde{\xi})\colon J_{l+m}(\mathscr{E}) \to J_{l+m}(\mathscr{E})$$

and $J_{l+m}(E)$ is associated to R_{k+m}.

The vector bundle E^* is also associated to $\tilde{R}_k$ if we set

$$\langle s, \mathscr{L}(\tilde{\xi})\alpha\rangle = \mathscr{L}(\tilde{\xi})\langle s, \alpha\rangle - \langle \mathscr{L}(\tilde{\xi})s, \alpha\rangle ,$$

for $\tilde{\xi} \in \Gamma(X, \tilde{R}_k)$, $s \in \mathscr{E}$, $\alpha \in \mathscr{E}^*$. If F is another vector bundle associated to $\tilde{R}_k$, the vector bundle $E \otimes F$ is associated to $\tilde{R}_k$ if we set

$$\mathscr{L}(\tilde{\xi})(e \otimes f) = \mathscr{L}(\tilde{\xi})e \otimes f + e \otimes \mathscr{L}(\tilde{\xi})f ,$$

for $\tilde{\xi} \in \Gamma(X, \tilde{R}_k)$, $e \in \mathscr{E}, f \in \mathscr{F}$. The l-th symmetric product $S^l E$ of E considered as a sub-bundle of $\bigotimes^l E$ is stable under the operations $\mathscr{L}(\tilde{\xi})$, for $\tilde{\xi} \in \Gamma(X, \tilde{R}_k)$, and so is also associated to $\tilde{R}_k$.

Since $J_0(T)$ is associated to $\tilde{J}_1(T)$ and

$$(\mathscr{L}(\tilde{\xi})\zeta)(x) = \nu[\tilde{\eta}, \tilde{\zeta}](x) ,$$

for $\tilde{\xi} \in \Gamma(X, \tilde{J}_1(T))$, $\zeta \in \Gamma(X, J_0(T))$ and $x \in X$, where $\tilde{\zeta} = \nu^{-1}\zeta$ and $\tilde{\xi}(x) = \tilde{j}_1(\tilde{\eta})(x)$, with $\tilde{\eta} \in \mathscr{T}_x$, we see that

$$(\mathscr{L}(\tilde{\xi})\alpha)(x) = \nu^{*-1}(\mathscr{L}(\tilde{\eta})\nu^*\alpha)(x) , \tag{15.33}$$

for $\alpha \in \Gamma(X, J_0(T)^*)$. Let

$$\bar{\delta}\colon S^{l+1}J_0(T)^* \to J_0(T)^* \otimes S^l J_0(T)^*$$

be the mapping $(\nu^{*-1} \otimes \nu^{*-1}) \circ \delta \circ \nu^*$; then

$$\mathscr{L}(\tilde{\xi})\bar{\delta}u = \bar{\delta}\mathscr{L}(\tilde{\xi})u , \tag{15.34}$$

for all $u \in S^{l+1}J_0(\mathscr{T})^*$.

The bundle $J_0(T)$ is associated to $\tilde{R}_k$ if we define

$$\mathscr{L}(\tilde{\xi})\eta = \mathscr{L}(\pi_1\tilde{\xi})\eta ,$$

for $\tilde{\xi} \in \Gamma(X, \tilde{R}_k)$, $\eta \in J_0(\mathscr{T})$; thus the vector bundle $S^l J_0(T)^* \otimes E$ is associated to $\tilde{R}_k$ by the above constructions.

Lemma 15.1. *Let E be a vector bundle associated to $\tilde{R}_k$. If $\tilde{\xi} \in \Gamma(X, \tilde{R}_{k+l})$, the diagram*

$$\begin{array}{ccc} S^l J_0(\mathscr{T})^* \otimes \mathscr{E} & \xrightarrow{\mathscr{L}(\pi_k\tilde{\xi})} & S^l J_0(\mathscr{T})^* \otimes \mathscr{E} \\ \downarrow{\scriptstyle \varepsilon\circ(\nu^*\otimes \mathrm{id})} & & \downarrow{\scriptstyle \varepsilon\circ(\nu^*\otimes \mathrm{id})} \\ J_l(\mathscr{E}) & \xrightarrow{\mathscr{L}(\tilde{\xi})} & J_l(\mathscr{E}) \end{array}$$

commutes.

Proof. We proceed by induction on l. First, we verify the lemma for $l = 1$; it suffices to show that

$$\mathscr{L}(\tilde{\xi})\varepsilon(df \otimes s) = \varepsilon(df \otimes \mathscr{L}(\pi_k\tilde{\xi})s + \nu^*\mathscr{L}(\pi_1\tilde{\xi})\nu^{*-1}df \otimes s)\ , \tag{15.35}$$

if $s \in \Gamma(X, E)$ and f is a function on X. We set

$$u = j_2(fs) - fj_2(s)\ .$$

Then

$$\pi_1 u = \varepsilon(df \otimes s)\ , \qquad Du = -df \otimes j_1(s)\ .$$

If $x \in X$ and $\lambda_1\nu\tilde{\xi}(x) = j_1(\eta)(x)$ with $\eta \in \mathscr{R}_{k,x}$, and if we set $\tilde{\eta} = \nu^{-1}\eta$, we have $\nu\pi_1\tilde{\xi}(x) = j_1(\pi_0\tilde{\eta})(x)$ and

$$\begin{aligned} (\mathscr{L}(\tilde{\xi})\varepsilon(df \otimes s))(x) &= \nu\tilde{\xi}(x)\cdot u(x) + \pi_0\tilde{\xi}(x) \barwedge Du \\ &= j_1(\mathscr{L}(\tilde{\eta})(fs))(x) - f(x)\cdot j_1(\mathscr{L}(\tilde{\eta})s)(x) \\ &\quad - \langle \pi_0\tilde{\xi}(x), df\rangle j_1(s)(x) \\ &= j_1(f\mathscr{L}(\tilde{\eta})s)(x) - f(x)\cdot j_1(\mathscr{L}(\tilde{\eta})s)(x) \\ &\quad + j_1((\mathscr{L}(\tilde{\eta})f)s)(x) - (\mathscr{L}(\pi_0\tilde{\eta})f)(x)\cdot j_1(s)(x) \\ &= \varepsilon(df \otimes \mathscr{L}(\tilde{\eta})s + d\mathscr{L}(\pi_0\tilde{\eta})f \otimes s)(x) \\ &= \varepsilon(df \otimes \mathscr{L}(\pi_k\tilde{\xi})s + \mathscr{L}(\pi_0\tilde{\eta})df \otimes s)(x)\ , \end{aligned}$$

which gives us (15.35) at $x \in X$ by (15.33). Now suppose that the lemma holds for an $l \geq 1$. Since $\lambda_1\colon J_{l+1}(E) \to J_1(J_l(E))$ is injective, from the commutativity of (15.28) with $m = 1$ and of the diagram

$$\begin{array}{ccccc} S^{l+1}T^* \otimes E & \xrightarrow{\delta} & T^* \otimes S^l T^* \otimes E & \xrightarrow{\mathrm{id}\otimes\varepsilon} & T^* \otimes J_l(E) \\ \downarrow{\scriptstyle \varepsilon} & & & & \downarrow{\scriptstyle \varepsilon} \\ J_{l+1}(E) & & \xrightarrow{\lambda_1} & & J_1(J_l(E)) \end{array}$$

of [3, § 3], and from the formula (15.34), the lemma for $l + 1$ follows easily.

Lemma 15.2. *Assume that X is connected and that R_k is formally transitive. Let E be a vector bundle associated to $\tilde{R}_k$, and let $N_l \subset J_l(E)$ be a differential equation such that*

$$\mathscr{L}(\tilde{\xi})\mathscr{N}_l \subset \mathscr{N}_l\,, \qquad \textit{for all } \tilde{\xi} \in \tilde{\mathscr{R}}_{k+l}\,.$$

Then N_{l+m} is a vector bundle for all $m \geq 0$, and $N_m = \pi_m N_l$ is a sub-bundle of $J_m(E)$ for $0 \leq m \leq l - 1$. Moreover, if $\pi_l: N_{l+1} \to N_l$ is surjective, then $N_{m+1} \subset (N_m)_{+1}$, for $0 \leq m \leq l - 1$; the sub-bundle $F = \pi_0 N_l$ is associated to $\tilde{R}_k$ and

$$\mathscr{L}(\tilde{\xi})J_m(\mathscr{F}) \subset J_m(\mathscr{F})\,, \qquad \textit{for all } \tilde{\xi} \in \tilde{\mathscr{R}}_{k+m}\,,$$
$$N_l \subset J_l(F)\,.$$

Proof. First let ω be an R_{k+l+m}-connection on a simply connected open subset of X whose curvature vanishes. Since N_l is associated to $\tilde{R}_{k+l}$, the bundles $J_{l+m}(E)$, $J_m(J_l(E))$, $J_m(N_l)$ are associated to $\tilde{R}_{k+l+m}$ and we consider the covariant derivatives induced by ω in these vector bundles. Since X is connected and diagram (15.28) commutes, applying [9, Proposition 3.2] to the injective mappings $\lambda_m: J_{l+m}(E) \to J_m(J_l(E))$ and $J_m(N_l) \to J_m(J_l(E))$, we see that N_{l+m} is a sub-bundle of $J_{l+m}(E)$. Next, let ω' be an R_{k+l}-connection on a simply connected open subset of X whose curvature vanishes. The bundles N_l and $J_m(E)$, with $m < l$, are associated to $\tilde{R}_{k+l}$, and we consider the covariant derivatives induced by ω' in these vector bundles. Since X is connected, applying [9, Proposition 3.2] to $\pi_m: N_l \to J_m(E)$, we deduce that N_m is a sub-bundle of $J_m(E)$. If $\pi_l: N_{l+1} \to N_l$ is surjective and $u \in \mathscr{N}_{m+1}$, with $0 \leq m \leq l - 1$, then $Du \in \mathscr{T}^* \otimes \mathscr{N}_m$; hence $N_{m+1} \subset (N_m)_{+1}$ and $N_l \subset J_l(F)$.

Remark. If X is an analytic manifold, R_k is an analytic formally transitive Lie equation, and E is an analytic vector bundle associated to $\tilde{R}_k$, then a differential equation $N_l \subset J_l(E)$ satisfying the hypothesis of Lemma 15.2 is analytic.

The following lemma is easily verified (see Lemma 1.5):

Lemma 15.3. *Let E be a vector bundle associated to $\tilde{R}_k$ and let $N_l \subset J_l(E)$ be a formally integrable differential equation. Then the following assertions are equivalent:*

(a) $\mathscr{L}(\tilde{\xi})\mathscr{N}_l \subset \mathscr{N}_l$, *for all* $\tilde{\xi} \in \tilde{\mathscr{R}}_{k+l}$;

(b) $\mathscr{L}(\tilde{\xi})\mathscr{N}_{l+m} \subset \mathscr{N}_{l+m}$, *for all* $\tilde{\xi} \in \tilde{\mathscr{R}}_{k+l+m}$ *and all* $m \geq 0$;

(c) $R_{k+l} \cdot N_{l+1} \subset N_l$;

(d) $R_{k+l+m} \cdot N_{l+m+1} \subset N_{l+m}$, *for all* $m \geq 0$.

For $x \in X$, according [10, § 9] and (1.25), the bracket (1.19) determines a structure of graded Lie algebra on the Spencer cohomology $H^*(R_k)_x$. If condition (d) of the above lemma holds, by (15.31) and (15.27) the mapping (15.30) determines on $H^*(N_l)_x$ a structure of graded module over the graded Lie algebra $H^*(R_k)_x$; in particular, we see that, if $\xi \in \operatorname{Sol}(R_k)$ and $s \in \operatorname{Sol}(N_l)$, then

$\mathscr{L}(\tilde{j}_k(\xi))s$ is an element of Sol (N_l). Therefore, if $R'_k \subset R_k$ is a formally integrable Lie equation satisfying

$$[\tilde{\mathscr{R}}_{k+1}, \mathscr{R}'_k] \subset \mathscr{R}'_k ,$$

we obtain structures of graded Lie algebra and of graded $H^*(R_k)_x$-module on $H^*(R'_k)_x$, for $x \in X$; the mapping

$$\iota: H^*(R'_k)_x \to H^*(R_k)_x ,$$

induced by the inclusion $R'_k \subset R_k$, is a morphism of graded Lie algebras which intertwines $H^*(R'_k)_x$ and $H^*(R_k)_x$ in the sense that

$$\iota(\alpha)\cdot\beta = [\alpha, \beta] , \qquad \iota(\gamma\cdot\alpha) = [\gamma, \iota(\alpha)] ,$$

for $\alpha, \beta \in H^*(R'_k)_x$, $\gamma \in H^*(R_k)_x$ (see [10]).

Let $N_l \subset J_l(E)$ be a formally integrable differential equation and $x \in X$. We consider $N_{\infty,x}$ as the linearly compact topological vector space $\varprojlim N_{l+m,x}$ over $\boldsymbol{R}$, where $N_{l+m,x}$ is endowed with the discrete topology. The kernel $N^m_{\infty,x}$ of the projection $\pi_m: N_{\infty,x} \to J_m(E)_x$ is an open subspace of $N_{\infty,x}$, and by Proposition 14.1 (i), $\{N^m_{\infty,x}\}$ is a fundamental system of neighborhoods of 0. In particular, $J_\infty(E)_x$ is a linearly compact topological vector space and $N_{\infty,x}$ is a closed subspace of $J_\infty(E)_x$. We set $N^m_{\infty,x} = N_{\infty,x}$ for $m < 0$ and $N_m = \pi_m(N_l)$ for $m < l$, and let h_m be the sub-bundle of $S^mT^* \otimes E$ with possibly varying fiber such that the sequence

$$0 \longrightarrow h_m \xrightarrow{\ \varepsilon\ } N_m \xrightarrow{\ \pi_{m-1}\ } N_{m-1} \longrightarrow 0$$

is exact. From the equality $\pi_m N^{m-1}_{\infty,x} = \varepsilon(h_{m,x})$, we obtain a surjective mapping

$$\varepsilon^{-1}\cdot\pi_m: N^{m-1}_{\infty,x} \to h_{m,x} ,$$

which sends $u \in N^{m-1}_{\infty,x}$ into the unique element v of $h_{m,x}$ satisfying $\varepsilon(v) = \pi_m u$, and whose kernel is $N^m_{\infty,x}$. This mapping therefore induces an isomorphism

$$\psi: N^{m-1}_{\infty,x}/N^m_{\infty,x} \to h_{m,x} , \qquad \text{for } m \geq 0 .$$

For $m \geq l$, the mappings

$$T \otimes h_{m+1} \to h_m ,$$

sending $\xi \otimes u$ into $\xi \,\overline{\wedge}\, \delta u$, gives us a natural representation of T_x regarded as an abelian Lie algebra on the graded vector space

$$\bigoplus_{m=l}^{\infty} h_{m,x} .$$

Therefore we may consider the graded vector space

$$M_x = \bigoplus_{m=l}^{\infty} h^*_{m,x} \tag{15.36}$$

as a graded module over the symmetric algebra ST_x of T_x; in fact, this structure of ST_x-module on M_x is the one obtained according to [5, Lemma 1] from the mappings $\delta\colon h_{m+1} \to T^* \otimes h_m$ and the complexes (1.8).

If we consider the Lie equation $R_k \subset J_k(T)$, then $R_{\infty,x}$ endowed with the topology defined above is a linearly compact Lie algebra over $\boldsymbol{R}$. Assume that E is associated to $\tilde{R}_k$ and that

$$R_{k+l} \cdot N_{l+1} \subset N_l \, .$$

By Lemma 15.3, N_l is associated to $\tilde{R}_{k+l}$ and the mappings (15.25) endow $J_\infty(E)_x$ and $N_{\infty,x}$ with structures of modules over $R_{\infty,x}$. We see that

$$R_{\infty,x} \cdot J^m_\infty(E)_x \subset J^{m-1}_\infty(E)_x \, , \qquad \text{for } m \geq 1 \, ,$$

and by (15.26) that

$$R^0_{\infty,x} \cdot J^m_\infty(E)_x \subset J^m_\infty(E)_x \, , \qquad \text{for } m \geq 0 \, .$$

Therefore $J_\infty(E)_x$ is a linearly compact $R_{\infty,x}$-module and $N_{\infty,x}$ is a closed $R_{\infty,x}$-submodule of $J_\infty(E)_x$.

Assume moreover that R_k is formally transitive. Then $R_{\infty,x}$ is a transitive Lie algebra and by (15.26)

$$J^m_\infty(E)_x = D^m_{R_{\infty,x}} J^0_\infty(E)_x \, , \qquad N^m_{\infty,x} = D^m_{R_{\infty,x}} N^0_{\infty,x} \, ,$$

for $m \geq 1$; by Proposition 14.2 (iii), $J^0_\infty(E)_x$ and $N^0_{\infty,x}$ are fundamental subspaces of $J_\infty(E)_x$ and $N_{\infty,x}$ respectively. Thus $J_\infty(E)_x$ is a geometric $R_{\infty,x}$-module, and $N_{\infty,x}$ is a closed geometric $R_{\infty,x}$-submodule of $J_\infty(E)_x$. In § 19, we shall show that every geometric module over a real transitive Lie algebra is isomorphic to a geometric module of the type $N_{\infty,x}$.

From Lemma 15.2, it follows that

$$\delta(h_{m+1}) \subset T^* \otimes h_m \, , \qquad \text{for all } m \geq 0 \, .$$

For $m \geq 0$, the mappings

$$T \otimes h_{m+1} \to h_m \, , \tag{15.37}$$

sending $\xi \otimes u$ into $\xi \barwedge \delta u$, give us a representation of the abelian Lie algebra T_x on the graded vector space

$$\bigoplus_{m=0}^{\infty} h_{m,x} \, ,$$

which thus becomes a graded T_x-submodule of $\bigoplus_{m=0}^{\infty} S^m T_x^* \otimes E_x$. Formula (15.26) implies that the diagram

$$\begin{array}{ccc} R_{\infty,x} \otimes N_{\infty,x}^{m-1} & \longrightarrow & N_{\infty,x}^{m-2} \\ \big\downarrow {\scriptstyle \nu^{-1}\cdot\pi_0 \otimes \varepsilon^{-1}\cdot\pi_m} & & \big\downarrow {\scriptstyle \varepsilon^{-1}\cdot\pi_{m-1}} \\ T_x \otimes h_{m,x} & \longrightarrow & h_{m-1,x} \end{array} \tag{15.38}$$

is commutative, where the top horizontal arrow is given by the $R_{\infty,x}$-module structure of $N_{\infty,x}$, and the bottom horizontal arrow is the mapping (15.37) with m replaced by $m-1$. Now $R_{\infty,x}^0$ is a fundamental subalgebra of $R_{\infty,x}$, and $N_{\infty,x}^0$ is a fundamental subspace of $N_{\infty,x}$ satisfying

$$R_{\infty,x}^0 \cdot N_{\infty,x}^0 \subset N_{\infty,x}^0 \ ;$$

we identify T_x with the quotient $R_{\infty,x}/R_{\infty,x}^0$ via the exact sequence

$$0 \longrightarrow R_{\infty,x}^0 \longrightarrow R_{\infty,x} \xrightarrow{\nu^{-1}\cdot\pi_0} T_x \longrightarrow 0 \ . \tag{15.39}$$

According to § 14, the graded vector space

$$\operatorname{gr} N_{\infty,x} = \bigoplus_{m=-1}^{\infty} N_{\infty,x}^m / N_{\infty,x}^{m+1}$$

has the structure of a T_x-module. The commutativity of (15.38) implies that the mapping

$$\psi\colon \operatorname{gr} N_{\infty,x} \to \bigoplus_{m=0}^{\infty} h_{m,x}$$

is an isomorphism of T_x-modules. The dual mapping

$$\psi^*\colon \bigoplus_{m=0}^{\infty} h_{m,x}^* \to (\operatorname{gr} N_{\infty,x})^* \tag{15.40}$$

is therefore an isomorphism of graded ST_x-modules. The natural structure of ST_x-module on $\bigoplus_{m=0}^{\infty} h_{m,x}^*$ is the same as the one obtained according to [5, Lemma 1] from the mappings $\delta\colon h_{m+1} \to T^* \otimes h_m$. Set $h_m = 0$ for $m < 0$; the diagram

$$\begin{array}{ccccc} \bigwedge^{j-1} T_x^* \otimes (N_{\infty,x}^m / N_{\infty,x}^{m+1}) & \xrightarrow{\delta} & \bigwedge^{j} T_x^* \otimes (N_{\infty,x}^{m-1} / N_{\infty,x}^{m}) & \xrightarrow{\delta} & \bigwedge^{j+1} T_x^* \otimes (N_{\infty,x}^{m-2} / N_{\infty,x}^{m-1}) \\ \big\downarrow {\scriptstyle \mathrm{id}\otimes\psi} & & \big\downarrow {\scriptstyle \mathrm{id}\otimes\psi} & & \big\downarrow {\scriptstyle \mathrm{id}\otimes\psi} \\ \bigwedge^{j-1} T_x^* \otimes h_{m+1,x} & \xrightarrow{\delta} & \bigwedge^{j} T_x^* \otimes h_{m,x} & \xrightarrow{\delta} & \bigwedge^{j+1} T_x^* \otimes h_{m-1,x} \end{array}$$

is easily seen to be commutative, where the mappings δ of the top row are the coboundary operators for the Lie algebra cohomology

$$\delta\colon \bigwedge T_x^* \otimes \operatorname{gr} N_{\infty,x} \to \bigwedge T_x^* \otimes \operatorname{gr} N_{\infty,x}$$

considered in § 14. If $H_x^{m,j}$ denotes the cohomology of the bottom row of this diagram, the mapping ψ induces an isomorphism

$$\psi\colon H^{j,m-1}(T_x, \operatorname{gr} N_{\infty,x}) \to H_x^{m,j} ,$$

for all $j, m \geq 0$, and hence an isomorphism of graded vector spaces

$$\psi\colon H^j(T_x, \operatorname{gr} N_{\infty,x}) \to \bigoplus_{m=0}^{\infty} H_x^{m,j} .$$

We consider the formally transitive Lie equation R_k and let $R_m = \pi_m R_k$ for $m < k$. Let g_m be the sub-bundle of $S^m J_0(T)^* \otimes J_0(T)$ with possibly varying fiber such that the sequence

$$0 \longrightarrow g_m \longrightarrow R_m \xrightarrow{\pi_{m-1}} R_{m-1} \longrightarrow 0$$

is exact. Since $[R_x^m, R_x^p] \subset R_x^{m+p}$, for all $m, p \geq -1$, the graded vector space

$$\operatorname{gr} R_{\infty,x} = \bigoplus_{m=-1}^{\infty} R_{\infty,x}^m / R_{\infty,x}^{m+1}$$

is a graded Lie algebra. The natural isomorphism

$$\operatorname{gr} R_{\infty,x} \to \bigoplus_{m=0}^{\infty} g_{m,x} ,$$

sending the class of $u \in R_{\infty,x}^{m-1}$ in $R_{\infty,x}^{m-1}/R_{\infty,x}^m$ into $\pi_m u \in g_{m,x}$, gives us a structure of graded Lie algebra on $\bigoplus_{m=0}^{\infty} g_{m,x}$ such that

$$[g_{m,x}, g_{p,x}] \subset g_{m+p-1,x} .$$

If $R'_k \subset R_k$ is a formally integrable Lie equation satisfying

$$[\tilde{\mathscr{R}}_{k+1}, \mathscr{R}'_k] \subset \mathscr{R}'_k ,$$

then R'_k is associated to $\tilde{R}_{k+1}$ and $R'_{\infty,x}$ is a closed ideal of the transitive Lie algebra $R_{\infty,x}$, for $x \in X$; moreover $\operatorname{gr} R'_{\infty,x}$ is a graded ideal of $\operatorname{gr} R_{\infty,x}$.

Assume that the finite form P_k of the formally transitive Lie equation R_k is formally integrable, and denote by P_{k+l} its l-th prolongation. Let $\bar{h}_l$ be the sub-bundle $(\nu^{*-1} \otimes \mathrm{id})h_l$ of $S^l J_0(T)^* \otimes E$ with possibly varying fiber.

Lemma 15.4. *If X is connected and $a, b \in X$, there exists $F \in P_k$, with source $F = a$, target $F = b$, and an isomorphism $\varphi: E_a \to E_b$ such that*

$$(\pi_1 F \otimes \varphi)(\bar{h}_{l,a}) = \bar{h}_{l,b} .$$

Proof. Let ϕ be a section of $P_{k+l}(a)$ over a simply connected neighborhood U of $a \in X$, and ω be the R_{k+l}-connection on U determined by (15.7) whose curvature vanishes. Consider the covariant derivatives ∇ on U with vanishing curvatures which are induced by ω in the vector bundles $J_0(T)^*$, E, N_l and $J_{l-1}(E)$ associated to $\tilde{R}_{k+l}$. By [9, Proposition 3.1], the covariant derivatives ∇ give us unique isomorphisms

$$\theta_x: J_0(T)^*_a \to J_0(T)^*_x , \qquad \varphi_x: E_a \to E_x ,$$

for all $x \in X$, such that the sections $x \mapsto \theta_x(\alpha)$ of $J_0(T)^*$ and $x \mapsto \varphi_x(e)$ of E over U, with $\alpha \in J_0(T)^*_a$, $e \in E_a$, are horizontal with respect to ∇. In fact, by the construction of ω and ∇ given above, we have $\theta_x = \pi_1\phi(x)^{-1}$, for $x \in U$. By Lemma 15.2, we have the exact sequence

$$0 \longrightarrow \bar{h}_l \xrightarrow{\varepsilon\circ(\nu^*\otimes \mathrm{id})} N_l \xrightarrow{\pi_{l-1}} J_{l-1}(E)$$

of vector bundles over X; by [9, Proposition 3.2] and Lemma 15.1, we see that $\bar{h}_l$ is stable under the covariant derivative in $S^l J_0(T)^* \otimes E$ on U induced by the covariant derivatives ∇ in $J_0(T)^*$ and E. Thus by [9, Proposition 3.2], we have

$$(\pi_1\phi(x)^{-1} \otimes \varphi_x)(\bar{h}_{l,a}) = \bar{h}_{l,x} .$$

The desired result therefore holds, for $b \in U$, with $F = \pi_k\phi(b)$; since X is connected, it holds for all $a, b \in X$.

We continue to assume that the Lie equation R_k is formally transitive, that E is associated to $\tilde{R}_k$ and that

$$R_{k+l} \cdot N_{l+1} \subset N_l .$$

For $x \in X$, let L_x denote the transitive Lie algebra which is the semi-direct product of $R_{\infty,x}$ and $J_\infty(E)_x$; then L_x is the abelian extension

$$0 \to J_\infty(E)_x \to L_x \to R_{\infty,x} \to 0$$

of $R_{\infty,x}$. By Proposition 14.6, we see that

$$L^0_x = J^0_\infty(E)_x \times R^k_{\infty,x}$$

is a fundamental subalgebra of L_x and that the closed ideal $J_\infty(E)_x$ of L_x is defined by a foliation in (L_x, L^0_x); it is easily verified that

$$L_x^m = D_{L_x}^m L_x^0 = J_\infty^m(E)_x \times R_{\infty,x}^{k+m} ,$$

for $m \geq 1$. We may identify L_x/L_x^m with $J_m(E)_x \times R_{k+m,x}$. Let V_x be the abelian Lie algebra L_x/L_x^0 and $L_x^{-1} = L_x$. Then $N_{\infty,x} \subset J_\infty(E)_x$ is a closed abelian ideal of L_x and

$$N_{\infty,x} \cap L_x^m = N_{\infty,x}^m , \qquad \text{for } m \geq -1 ;$$

moreover gr $N_{\infty,x}$ is a V_x-submodule of the graded V_x-module

$$\operatorname{gr} L_x = \bigoplus_{m=-1}^{\infty} L_x^m / L_x^{m+1} .$$

According to Proposition 14.3 and [6, Proposition 1], there is an integer $l_0 \geq l$ such that

$$(15.41) \qquad \begin{aligned} H^{j,m}(V_x, \operatorname{gr} L_x) &= 0 , \\ H^{1,m}(V_x, \operatorname{gr} N_{\infty,x}) &= 0 , \end{aligned}$$

for $j = 1, 2$ and all $m \geq l_0$, $x \in X$.

Proposition 15.1. *Assume that X is connected and that R_k is formally transitive. Let E be a vector bundle associated to $\tilde{R}_k$, and let $N_l \subset J_l(E)$ be a differential equation such that*

$$R_{k+l} \cdot N_{l+1} \subset N_l .$$

If $a, b \in X$, there exist $\phi \in Q_\infty(a, b)$ and an isomorphism $\psi\colon J_\infty(E)_a \to J_\infty(E)_b$ of topological vector spaces such that

$$\psi(J_\infty^m(E)_a) = J_\infty^m(E)_b , \qquad \textit{for } m \geq 0 ,$$
$$\phi(R_{\infty,a}) = R_{\infty,b} , \qquad \psi(N_{\infty,a}) = N_{\infty,b}$$

and the diagram

$$(15.42) \qquad \begin{array}{ccc} R_{\infty,a} \otimes J_\infty(E)_a & \longrightarrow & J_\infty(E)_a \\ \downarrow{\scriptstyle \phi\otimes\psi} & & \downarrow{\scriptstyle \psi} \\ R_{\infty,b} \otimes J_\infty(E)_b & \longrightarrow & J_\infty(E)_b \end{array}$$

commutes, where the horizontal arrows are given by the $R_{\infty,x}$-module structure of $J_\infty(E)_x$, with $x = a$ or b.

Proof. Let $m \geq l$ and ω an R_{k+m+2}-connection whose curvature vanishes on a simply connected neighborhood U of $a \in X$. The vector bundles $J_{m+j}(E)$, R_{k+m+j} are associated to $\tilde{R}_{k+m+2}$, for $j = 0, 1$, and so we may consider the covariant derivatives ∇ with vanishing curvature induced by ω in these vector bundles and $R_{k+m} \otimes J_{m+1}(E)$. By [9, Proposition 3.1], we obtain isomorphisms

$$\phi_{m+j,x}: R_{k+m+j,a} \to R_{k+m+j,x} ,$$
$$\psi_{m+j,x}: J_{m+j}(E)_a \to J_{m+j}(E)_x ,$$

for all $x \in U$ and $j = 0, 1$, such that

$$\pi_{k+m}\phi_{m+1,x} = \phi_{m,x}\pi_{k+m} , \qquad \pi_m\psi_{m+1,x} = \psi_{m,x}\pi_m ,$$

and the sections $x \mapsto \phi_{m+j,x}(\xi)$ of R_{k+m+j} and $x \mapsto \psi_{m+j,x}(u)$ of $J_{m+j}(E)$ over U, with $\xi \in R_{k+m+j,a}$, $u \in J_{m+j}(E)_a$, are horizontal with respect to ∇. By (15.32), we may apply [9, Proposition 3.2] to the mapping (15.25) with l replaced by m and these covariant derivatives to deduce that the diagram

$$(15.43) \qquad \begin{array}{ccc} R_{k+m,a} \otimes J_{m+1}(E)_a & \longrightarrow & J_m(E)_a \\ \big\downarrow{\scriptstyle \phi_{m,x}\otimes\psi_{m+1,x}} & & \big\downarrow{\scriptstyle \psi_{m,x}} \\ R_{k+m,x} \otimes J_{m+1}(E)_x & \longrightarrow & J_m(E)_x \end{array}$$

is commutative, where the horizontal arrows are given by (15.25). Moreover, by [9, Proposition 5.4], we have

$$(15.44) \qquad [\phi_{m+1,x}(\xi), \phi_{m+1,x}(\eta)] = \phi_{m,x}([\xi, \eta]) ,$$

for all $\xi, \eta \in R_{k+m+1,a}$. By Lemma 15.3, the bundles N_{m+j}, for $j = 0, 1$, are stable under the covariant derivatives ∇; therefore by [9, Proposition 3.2],

$$(15.45) \qquad \psi_{m+j,x}(N_{m+j,a}) = N_{m+j,x} ,$$

for $j = 0, 1$ and $x \in U$. For $j = 0, 1$, let

$$\Phi_{m+j,x}: L_a/L_a^{m+j} \to L_x/L_x^{m+j}$$

be the isomorphism $\psi_{m+j,x} \times \phi_{m+j,x}$; from the commutativity of (15.43) and (15.44), we infer that

$$[\Phi_{m+1,x}(\xi), \Phi_{m+1,x}(\eta)] = \Phi_{m,x}([\xi, \eta]) ,$$

for $\xi, \eta \in L_a/L_a^{m+1}$, where the brackets are the ones induced by the brackets on L_x and L_a. Now take m to be equal to the integer $l_0 \geq l$ considered above. By the results of Guillemin and Sternberg [13], for $x \in U$ there exists an isomorphism $\Phi_x: L_a \to L_x$ of transitive Lie algebras such that $\Phi_x(L_a^m) = L_x^m$, for all $m \geq 0$, and such that the mapping $L_a/L_a^{l_0} \to L_x/L_x^{l_0}$ induced by Φ_x is equal to $\Phi_{l_0,x}$. If ψ_x is the restriction of Φ_x to $J_\infty(E)_x$, then

$$\psi_x(J_\infty(E)_a) \subset J_\infty(E)_x + L_x^{l_0} , \qquad \psi_x(N_{\infty,a}) \subset N_{\infty,x} + L_x^{l_0}$$

by the equality (15.45) with $m = l_0$ and $j = 0$. Hence the ideals $\psi_x(J_\infty(E)_a) + J_\infty(E)_x$ and $\psi_x(N_{\infty,a}) + N_{\infty,x}$ of L_x satisfy

$$J_\infty(E)_x \subset \psi_x(J_\infty(E)_a) + J_\infty(E)_x \subset J_\infty(E)_x + L_x^{l_0} ,$$
$$N_{\infty,x} \subset \psi_x(N_{\infty,a}) + N_{\infty,x} \subset N_{\infty,x} + L_x^{l_0} .$$

Since (15.41) holds for all $m \geq l_0$, by [10, Proposition 10.1] we know that $N_{\infty,x}$ is defined by a foliation in $(L_x, L_x^{l_0})$. As $J_\infty(E)_x$ is also defined by a foliation in $(L_x, L_x^{l_0})$, we see that

$$\psi_x(J_\infty(E)_a) \subset J_\infty(E)_x , \qquad \psi_x(N_{\infty,a}) \subset N_{\infty,x} .$$

The same argument applied to Φ_x^{-1} gives us the equalities

$$\psi_x(J_\infty(E)_a) = J_\infty(E)_x , \qquad \psi_x(N_{\infty,a}) = N_{\infty,x} ,$$

and thus

$$\psi_x(J_\infty^m(E)_a) = J_\infty^m(E)_x .$$

Let $\phi_x \colon R_{\infty,a} \to R_{\infty,x}$ be the isomorphism of transitive Lie algebras which makes the diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
J_\infty(E)_a & \xrightarrow{\psi_x} & J_\infty(E)_x \\
\downarrow & & \downarrow \\
L_a & \xrightarrow{\Phi_x} & L_x \\
\downarrow & & \downarrow \\
R_{\infty,a} & \xrightarrow{\phi_x} & R_{\infty,x} \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

commute. Since L_a and L_x are abelian extensions of $R_{\infty,a}$ and $R_{\infty,x}$, the diagram (15.42) commutes with $b = x$, $\phi = \phi_x$ and $\psi = \psi_x$. Moreover $\phi_x(R_a^m) = R_x^m$ for $m \geq k$, and the mapping $R_{l_0,a} \to R_{l_0,x}$ induced by ϕ_x is equal to $\phi_{l_0,x}$. Therefore $\phi_x(R_a^m) = R_x^m$ for $m \geq 0$, and by the results of Guillemin and Sternberg [13] there is an element of $Q_\infty(a, x)$ which induces the isomorphism ϕ_x. Thus the conclusions of the proposition hold for all $b \in U$; since X is connected, they also hold for all $b \in X$.

We still suppose that the Lie equation R_k is formally transitive, and that E is associated to $\tilde{R}_k$. Using the method of proof of Lemma 15.2 and the formulas (15.24) and (15.32), the proofs of [10, Lemma 10.4 and Theorem 10.1] can be suitably modified in order to obtain the following generalizations of results of [10, § 10]:

Lemma 15.5. *Let E be a vector bundle associated to a formally transitive Lie equation $\tilde{R}_k$, and W a subspace of E_x, with $x \in X$, such that $R^0_{k,x} \cdot W \subset W$. Then there exists a sub-bundle F of E over a neighborhood U of x such that $F_x = W$ and*

$$\mathscr{L}(\xi)\mathscr{F} \subset \mathscr{F}\ , \qquad \textit{for all } \xi \in \tilde{\mathscr{R}}_{k|U}\ .$$

If F is a closed $R_{\infty,x}$-submodule of $J_\infty(E)_x$, with $x \in X$, we set $F^l = F \cap J^l_\infty(E)_x$; then $R^0_{\infty,x} \cdot F^0 \subset F^0$ and we can identify

$$\operatorname{gr} F = \bigoplus_{l=-1}^{\infty} F^l/F^{l+1}$$

with a T_x-submodule of $\operatorname{gr} J_\infty(E)_x$.

Theorem 15.1. *Assume that X is simply connected, and that R_k is formally transitive and formally integrable. Let E be a vector bundle associated to $\tilde{R}_k$, and F a closed $R_{\infty,x}$-submodule of $J_\infty(E)_x$, with $x \in X$.*

(i) *For all $m \geq 1$, there exists a unique differential equation $N_m \subset J_m(E)$ such that*

$$N_{m,x} = \pi_m F\ ,$$
$$\mathscr{L}(\xi)\mathscr{N}_m \subset \mathscr{N}_m\ , \qquad \textit{for all } \xi \in \tilde{\mathscr{R}}_{k+l}\ .$$

(ii) *For $m \geq 1$, we have*

$$R_{k+l} \cdot N_{m+1} \subset N_m\ .$$

(iii) *There is an integer $l \geq 1$ such that $H^{m-1,1}(T_x, \operatorname{gr} F) = 0$ for all $m \geq l$. If l is such an integer, N_l is formally integrable and N_{l+m} is the m-th prolongation of N_l. Moreover $\pi_m(N_l) = N_m$ for $m < l$, and*

$$N_{\infty,x} = F\ .$$

Corollary 15.1. *Assume that X is simply connected, and R_k is a formally transitive and formally integrable Lie equation. Let E be a vector bundle associated to $\tilde{R}_k$, and F a closed $R_{\infty,x}$-submodule of $J_\infty(E)_x$, with $x \in X$. There exist an integer $l \geq 0$ and a unique formally integrable differential equation $N_l \subset J_l(E)$ such that*

$$N_{\infty,x} = F\ ,$$

$$\mathscr{L}(\xi)\mathscr{N}_l \subset \mathscr{N}_l\ , \qquad \textit{for } \xi \in \tilde{\mathscr{R}}_{k+l}\ . \tag{15.46}$$

These last results imply that, if X is simply connected and $x \in X$, the mapping between the set of differential equations $N_l \subset J_l(E)$ in E satisfying (15.46) and the set of closed $R_{\infty,x}$-submodules of $J_\infty(E)_x$, which assigns to N_l the submodule $N_{\infty,x}$ of $J_\infty(E)_x$, is surjective; moreover, two such equations correspond to the same submodule of $J_\infty(E)_x$ if and only if one of these equations is a prolongation of the other.

16. Characteristic varieties of geometric modules

Let L be a linearly compact Lie algebra over the field K, and E a linearly compact L-module. Let $L^0 \subset L$, $E^0 \subset E$ be open subspaces satisfying $L^0 \cdot E^0 \subset E^0$. If $E^k = D_L^k E^0$ for $k \geq 1$, then according to Proposition 14.3,

$$(16.1) \qquad (\mathrm{gr}\, E)^* = \bigoplus_{k=-1}^{\infty} (E^k/E^{k+1})^* ,$$

where $E^{-1} = E$, is a finitely generated graded module over the symmetric algebra SV of $V = L/L^0$. If M is an SV-module, we denote by $\mathscr{I}_M$ the annihilator ideal of M; if K' is an extension field of K, we denote by $\mathscr{V}(M, K')$ the subvariety of $V^* \otimes K'$ defined by the ideal $\mathscr{I}_M$. We write

$$\mathscr{V}^0(L, E, K')_{L^0,E^0} = \mathscr{V}((\mathrm{gr}\, E)^*, K') .$$

The natural projection mapping $L \to V$ gives us a dual injective mapping $V^* \otimes K' \to L^* \otimes K'$, where L^* is the topological dual of L. We denote by $\mathscr{V}(L, E, K')_{L^0,E^0}$ the image of $\mathscr{V}^0(L, E, K')_{L^0,E^0}$ in $L^* \otimes K'$. If E is finite-dimensional, then $\mathscr{V}(L, E, K')_{L^0,E^0} = 0$.

The proof of the following lemma is left to the reader.

Lemma 16.1. *Let*

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of SV-modules. Then

$$\mathscr{I}_{M'} \cdot \mathscr{I}_{M''} \subset \mathscr{I}_M \subset \mathscr{I}_{M'} \cap \mathscr{I}_{M''}$$

and, if K' is an extension field of K,

$$\mathscr{V}(M, K') = \mathscr{V}(M', K') \cup \mathscr{V}(M'', K') .$$

Proposition 16.1. *Let $\phi\colon L \to L''$ be a continuous epimorphism of linearly compact Lie algebras, and E a geometric L''-module. If $L^0 \subset L$, $L''^0 \subset L''$ are open subspaces, and E^0 a fundamental subspace of E such that*

$$\phi(L^0) \subset L''^0 , \qquad L''^0 \cdot E^0 \subset E^0 ,$$

then E^0 *is a fundamental subspace of* $\phi^* E$ *satisfying* $L^0 \cdot E^0 \subset E^0$; *moreover, if* K' *is an extension field of* K, *we have*

$$(\phi^* \otimes \mathrm{id})(\mathscr{V}(L'', E, K')_{L''^0, E^0}) = \mathscr{V}(L, \phi^* E, K')_{L^0, E^0}\,, \tag{16.2}$$

where $\phi^* \otimes \mathrm{id}: L''^* \otimes K' \to L^* \otimes K'$ *is the injective mapping induced by* ϕ.

Proof. Since

$$L^0 \cdot E^0 \subset L''^0 \cdot E^0 \subset E^0\,,$$

the first part of the proposition is obvious. Let $V = L/L^0$ and $W = L''/L''^0$; the mapping ϕ induces a surjective mapping $\rho: V \to W$. Since the diagram

$$\begin{array}{ccc} L''^* \otimes K' & \xrightarrow{\phi^* \otimes \mathrm{id}} & L^* \otimes K' \\ \uparrow & & \uparrow \\ W^* \otimes K' & \xrightarrow{\rho^* \otimes \mathrm{id}} & V^* \otimes K' \end{array}$$

is commutative, to prove (16.2) it suffices to verify that

$$(\rho^* \otimes \mathrm{id})(\mathscr{V}^0(L'', E, K')_{L''^0, E^0}) = \mathscr{V}^0(L, E, K')_{L^0, E^0}\,. \tag{16.3}$$

We have

$$E^k = D^k_L E^0 = D^k_{L''} E^0\,, \qquad \text{for } k \geq 1\,.$$

Then (16.1) is a graded module over SV and SW; moreover, if $\rho: SV \to SW$ is the natural projection,

$$p \cdot a = \rho(p) \cdot a\,,$$

for all $p \in SV$, $a \in (\mathrm{gr}\, E)^*$. Thus if $\mathscr{I}'' \subset SW$ is the annihilator ideal of the SW-module $(\mathrm{gr}\, E)^*$, the annihilator ideal of the SV-module $(\mathrm{gr}\, E)^*$ is equal to $\rho^{-1}(\mathscr{I}'')$; now (16.3) is an immediate consequence of this fact.

The proof of the following proposition should be compared with that of [6, Theorem 1].

Proposition 16.2. *Let*

$$0 \longrightarrow E' \xrightarrow{\ \phi\ } E \xrightarrow{\ \psi\ } E'' \longrightarrow 0$$

be an exact sequence of linearly compact L-modules, whose mappings are continuous. Let $L^0 \subset L$, $E'^0 \subset E'$, $E^0 \subset E$, $E''^0 \subset E''$ *be open subspaces such that*

$$\begin{gathered} L^0 \cdot E'^0 \subset E'^0\,, \quad L^0 \cdot E^0 \subset E^0\,, \quad L^0 \cdot E''^0 \subset E''^0\,, \\ \phi(E'^0) = \phi(E') \cap E^0\,, \qquad \psi(E^0) \subset E''^0\,. \end{gathered} \tag{16.4}$$

Assume that E'' is a geometric L-module, and that E''^0 is a fundamental subspace of E''. If K' is an extension field of K, then

$$\mathscr{V}(L, E, K')_{L^0, E^0} = \mathscr{V}(L, E', K')_{L^0, E'^0} \cup \mathscr{V}(L, E'', K')_{L^0, E''^0} .$$

Proof. For $k > 0$, set

$$E'^k = D_L^k E'^0 , \quad E^k = D_L^k E^0 , \quad E''^k = D_L^k E''^0 ,$$
$$E'^{-k} = E' , \quad E^{-k} = E , \quad E''^{-k} = E'' .$$

Then

$$\phi(E'^k) = \phi(E') \cap E^k , \qquad \psi(E^k) \subset E''^k ;$$

if we write $F^k = \psi^{-1}(E''^k)$, we have

$$L \cdot F^{k+1} \subset F^k , \qquad L^0 \cdot F^k \subset F^k ,$$
$$E^k \subset F^k , \qquad F^{k+1} \subset F^k ,$$

and since E''^0 is a fundamental subspace,

$$\bigcap_k F^k = \phi(E') .$$

Let k be a fixed integer; since the open subspace $E^k + \phi(E')$ of E is a neighborhood of $\phi(E')$, by Proposition 14.1 (i) there exists an integer m such that

$$F^m \subset E^k + \phi(E') ,$$

and hence

$$\bigcap_{m=0}^{\infty} (E^k + F^m) = E^k + \phi(E') . \tag{16.5}$$

Consider the mapping

$$E''^k / E''^{k+1} \longrightarrow F^k/(E^k + F^{k+1})$$

sending the class of $e'' \in E''^k$ into the class of an element $e \in F^k$ satisfying $\psi(e) = e''$ and the exact sequence

$$\begin{aligned} 0 \longrightarrow (F^{k+1} \cap E^k)/E^{k+1} \longrightarrow E^k/E^{k+1} \\ \xrightarrow{\psi} E''^k/E''^{k+1} \longrightarrow F^k/(E^k + F^{k+1}) \longrightarrow 0 . \end{aligned} \tag{16.6}$$

whose mapping ψ is induced by $\psi : E^k \to E''^k$. For $l \geq 0$, we set

$$g_k^{(l)} = (E^{k+1} + F^{k+l+1} \cap E^k)/E^{k+1} ,$$
$$h_k^{(l)} = (E^{k+1} + F^{k+l+1})/(E^{k+1} + F^{k+l+2}) ;$$

the sequence

$$0 \to g_k^{(l+1)} \to g_k^{(l)} \to h_k^{(l)} \to h_{k-1}^{(l+1)} \to 0 , \tag{16.7}$$

whose mappings are induced by inclusions, is easily seen to be exact. The vector space $V = L/L^0$ considered as an abelian Lie algebra has natural representations on the graded vector spaces

$$g^{(l)} = \bigoplus_k g_k^{(l)}, \qquad h^{(l)} = \bigoplus_k h_k^{(l)} .$$

If we set

$$h^{(l)}(-1) = \bigoplus_k h^{(l)}(-1)_k ,$$

where $h^{(l)}(-1)_k = h_{k-1}^{(l)}$, we obtain from (16.6) and (16.7) the exact sequences of V-modules

$$0 \longrightarrow g^{(0)} \longrightarrow \operatorname{gr} E \xrightarrow{\psi} \operatorname{gr} E'' \longrightarrow h^{(0)}(-1) \longrightarrow 0 , \tag{16.8}$$

$$0 \to g^{(l+1)} \to g^{(l)} \to h^{(l)} \to h^{(l+1)}(-1) \to 0 , \tag{16.9}$$

for $l \geq 0$. The mapping ϕ induces injective mappings of V-modules

$$\phi\colon \operatorname{gr} E' \to g^{(l)} , \qquad \text{for } l \geq 0 , \tag{16.10}$$

$$\phi\colon \operatorname{gr} E' \to \operatorname{gr} E \tag{16.11}$$

such that the diagrams

$$\begin{array}{ccc} \operatorname{gr} E' & \xrightarrow{\phi} & g^{(l+m)} \\ & \searrow \quad \swarrow & \\ & g^{(l)} & \end{array} \tag{16.12}$$

$$\begin{array}{ccc} \operatorname{gr} E' & \xrightarrow{\phi} & \operatorname{gr} E \\ & {\scriptstyle\phi}\searrow \quad \swarrow & \\ & g^{(l)} & \end{array}$$

commute, for all $l, m \geq 0$.

We next verify certain properties of these V-modules.

Lemma 16.2. *Let $l \geq 0$; the following assertions are equivalent:*

(i) *the mapping* (16.10) *is an isomorphism of V-modules;*

(ii) *for all $m \geq 0$, the mapping*

$$\phi\colon \operatorname{gr} E' \to g^{(l+m)}$$

is an isomorphism of V-modules;

(iii) *for all* $m \geq 0$, *the mapping*

$$g^{(l+m)} \to g^{(l)}$$

is an isomorphism of V-modules;

(iv) *for all* k, *we have*

$$F^{k+l+1} \cap E^k \subset E^{k+1} + \phi(E') \; ; \tag{16.13}$$

(v) *for all* k, *we have*

$$F^{k+l} \subset E^k + \phi(E') \; ; \tag{16.14}$$

(vi) $h^{(l)} = 0$;

(vii) *for all* $m \geq 0$, *we have* $h^{(l+m)} = 0$.

Proof. (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) follow from the commutativity of (16.12). Next, we remark that (iii) is equivalent to

$$F^{k+l+1} \cap E^k \subset E^{k+1} + F^{k+l+m+1} \, ,$$

for all k and m, and hence also by (16.5) to

$$F^{k+l+1} \cap E^k \subset \bigcap_{m=0}^{\infty} (E^{k+1} + F^{k+l+m+1}) = E^{k+1} + \phi(E') \, ,$$

for all k, that is to (iv). We now show that (iv) implies (16.14) for all $k \geq 0$ by induction on k. First, the inclusion (16.13) with $k = -1$ is the same as (16.14) with $k = 0$; if (16.14) holds for some $k \geq 0$, we deduce from (iv) that

$$F^{k+l+1} \subset F^{k+l+1} \cap E^k + \phi(E') \subset E^{k+1} + \phi(E') \, .$$

Clearly, (v) implies that the mapping (16.10) is surjective and hence that it is an isomorphism, showing that the assertions (i)-(v) are equivalent. Now $h^{(l)} = 0$ if and only if

$$F^{k+l+1} \subset E^{k+1} + F^{k+l+2} \, ,$$

for all k; by (16.5), this last condition is equivalent to

$$F^{k+l+1} \subset \bigcap_{m=1}^{\infty} (E^{k+1} + F^{k+l+m}) = E^{k+1} + \phi(E') \, ,$$

for all k, that is to (v). The equivalence of (vi) and (vii) follows from the exactness of the sequence (16.7) or the equivalence of (v) and (vi), concluding the proof of the lemma.

We now return to the proof of the proposition. For $l \geq 0$, we write

$$M_k^{(l)} = g_k^{(l)*}\,, \qquad N_k^{(l)} = h_k^{(l)*}\,,$$
$$M^{(l)} = \bigoplus_k M_k^{(l)}\,, \quad N^{(l)} = \bigoplus_k N_k^{(l)}\,, \quad N^{(l)}(-1) = \bigoplus_k N_k^{(l)}(-1)\,,$$

where $N_k^{(l)}(-1) = N_{k-1}^{(l)}$. The abelian Lie algebra V has natural representations on the graded vector spaces $M^{(l)}$, $N^{(l)}$, $N^{(l)}(-1)$; we may consider these vector spaces as graded modules over the symmetric algebra SV of V. The sequences (16.8) and (16.9) give the exact sequences of SV-modules

$$\text{(16.15)} \quad 0 \longrightarrow N^{(0)}(-1) \longrightarrow (\text{gr } E'')^* \xrightarrow{\psi^*} (\text{gr } E)^* \longrightarrow M^{(0)} \longrightarrow 0\,,$$

$$\text{(16.16)} \qquad 0 \to N^{(l+1)}(-1) \to N^{(l)} \to M^{(l)} \to M^{(l+1)} \to 0\,,$$

and (16.10) and (16.11) the epimorphisms of SV-modules

$$\text{(16.17)} \qquad \begin{aligned} &\phi^* \colon M^{(l)} \to (\text{gr } E')^*\,, \\ &\phi^* \colon (\text{gr } E)^* \to (\text{gr } E')^*\,, \end{aligned}$$

for $l \geq 0$. Let $Q^{(l)}$ be the kernel of the epimorphism of SV-modules $(\text{gr } E)^* \to M^{(l)}$. We obtain the ascending chain of SV-submodules of $(\text{gr } E)^*$

$$Q^{(0)} \subset Q^{(1)} \subset \cdots \subset Q^{(l)} \subset Q^{(l+1)} \subset \cdots \subset (\text{gr } E)^*\;;$$

by Proposition 14.3, $(\text{gr } E)^*$ is finitely generated and so this chain stabilizes. Hence there exists an integer $l_0 \geq 0$ such that $Q^{(l_0)} = Q^{(l_0+m)}$ for all $m \geq 0$. The mappings

$$M^{(l_0)} \to M^{(l_0+m)}$$

are isomorphisms for all $m \geq 0$ and thus assertion (iii) of Lemma 16.2 holds with $l = l_0$. Therefore $h^{(l_0+m)} = 0$ and $N^{(l_0+m)} = 0$ for all $m \geq 0$, and (16.17) is an isomorphism for all $l \geq l_0$. By Lemma 16.1, since

$$\mathscr{V}(N^{(l)}(-1), K') = \mathscr{V}(N^{(l)}, K')\,,$$

from the exact sequence (16.15) we deduce

$$\text{(16.18)} \qquad \mathscr{V}((\text{gr } E)^*, K') \subset \mathscr{V}(M^{(0)}, K') \cup \mathscr{V}((\text{gr } E'')^*, K')\,,$$

$$\text{(16.19)} \quad \mathscr{V}(N^{(0)}, K') \subset \mathscr{V}((\text{gr } E'')^*, K') \subset \mathscr{V}((\text{gr } E)^*, K') \cup \mathscr{V}(N^{(0)}, K')\,,$$

and from (16.16) that

$$\text{(16.20)} \qquad \mathscr{V}(M^{(l)}, K') \subset \mathscr{V}(M^{(l+1)}, K') \cup \mathscr{V}(N^{(l)}, K')\,,$$

$$\text{(16.21)} \quad \mathscr{V}(N^{(l+1)}, K') \subset \mathscr{V}(N^{(l)}, K') \subset \mathscr{V}(N^{(l+1)}, K') \cup \mathscr{V}(M^{(l)}, K')\,,$$

for all $l \geq 0$. We obtain by induction on l

$$\mathscr{V}((\mathrm{gr}\, E)^*, K') \subset \mathscr{V}(M^{(l)}, K') \cup \mathscr{V}((\mathrm{gr}\, E'')^*, K') , \tag{16.22}$$

for all $l \geq 0$; indeed, the inclusion (16.22) for $l = 0$ is (16.18) and, if (16.22) holds for $l \geq 0$, then by (16.20), (16.21) and (16.19) we have

$$\begin{aligned}\mathscr{V}((\mathrm{gr}\, E)^*, K') &\subset \mathscr{V}(M^{(l+1)}, K') \cup \mathscr{V}(N^{(l)}, K') \cup \mathscr{V}((\mathrm{gr}\, E'')^*, K') \\ &\subset \mathscr{V}(M^{(l+1)}, K') \cup \mathscr{V}(N^{(0)}, K') \cup \mathscr{V}((\mathrm{gr}\, E'')^*, K') \\ &\subset \mathscr{V}(M^{(l+1)}, K') \cup \mathscr{V}((\mathrm{gr}\, E'')^*, K') .\end{aligned}$$

For $l = l_0$, the inclusion (16.22) becomes

$$\mathscr{V}((\mathrm{gr}\, E)^*, K') \subset \mathscr{V}((\mathrm{gr}\, E')^*, K') \cup \mathscr{V}((\mathrm{gr}\, E'')^*, K') . \tag{16.23}$$

On the other hand, since $(\mathrm{gr}\, E')^*$ is a quotient of $(\mathrm{gr}\, E)^*$, we have

$$\mathscr{V}((\mathrm{gr}\, E')^*, K') \subset \mathscr{V}((\mathrm{gr}\, E)^*, K') , \tag{16.24}$$

and, since $M^{(l)}$ is a quotient of $(\mathrm{gr}\, E)^*$,

$$\mathscr{V}(M^{(l)}, K') \subset \mathscr{V}((\mathrm{gr}\, E)^*, K') . \tag{16.25}$$

We obtain by induction on l

$$\mathscr{V}((\mathrm{gr}\, E'')^*, K') \subset \mathscr{V}((\mathrm{gr}\, E)^*, K') \cup \mathscr{V}(N^{(l)}, K') ; \tag{16.26}$$

indeed, the inclusion (16.26) for $l = 0$ is given by (16.19) and, if (16.26) holds for $l \geq 0$, then by (16.21) and (16.25) we have

$$\begin{aligned}\mathscr{V}((\mathrm{gr}\, E'')^*, K') &\subset \mathscr{V}((\mathrm{gr}\, E)^*, K') \cup \mathscr{V}(N^{(l+1)}, K') \cup \mathscr{V}(M^{(l)}, K') \\ &\subset \mathscr{V}((\mathrm{gr}\, E)^*, K') \cup \mathscr{V}(N^{(l+1)}, K') .\end{aligned}$$

For $l = l_0$, the inclusion (16.26) becomes

$$\mathscr{V}((\mathrm{gr}\, E'')^*, K') \subset \mathscr{V}((\mathrm{gr}\, E)^*, K') . \tag{16.27}$$

From (16.23), (16.24) and (16.27), we obtain the equality

$$\mathscr{V}((\mathrm{gr}\, E)^*, K') = \mathscr{V}((\mathrm{gr}\, E')^*, K') \cup \mathscr{V}((\mathrm{gr}\, E'')^*, K') ,$$

which is the desired result.

We now deduce from Propositions 16.1 and 16.2 the following generalization of Theorem 1 of the appendix of [27] and of [28, Theorem 9.1]:

Theorem 16.1. *Let E be a geometric L-module; let L^0, L'^0 be open subspaces of L and E^0, E'^0 fundamental subspaces of E such that*

$$L^0 \cdot E^0 \subset E^0 , \qquad L'^0 \cdot E'^0 \subset E'^0 .$$

If K' is an extension field of K, then

$$\mathscr{V}(L, E, K')_{L^0, E^0} = \mathscr{V}(L, E, K')_{L'^0, E'^0} .$$

Proof. First assume that $E^0 = E'^0$ and $L^0 \subset L'^0$; then apply Proposition 16.1 with $L = L''$, $L'^0 = L''^0$ and ϕ the identity mapping of L to deduce the result in this case. Next, if $E^0 \subset E'^0$ and $L^0 = L'^0$, by Proposition 16.2, with $E = E''$ and ψ the identity mapping of E, we see that the theorem holds under these assumptions. If $L^0 \subset L'^0$ and $E^0 \subset E'^0$, we derive the equality

$$\mathscr{V}(L, E, K')_{L^0, E^0} = \mathscr{V}(L, E, K')_{L^0, E'^0} = \mathscr{V}(L, E, K')_{L'^0, E'^0}$$

from the previous cases. The general case now follows from the above; indeed, we have

$$\mathscr{V}(L, E, K')_{L^0, E^0} = \mathscr{V}(L, E, K')_{L^0 \cap L'^0, E^0 \cap E'^0} = \mathscr{V}(L, E, K')_{L'^0, E'^0} .$$

We therefore write

$$\mathscr{V}(L, E, K') = \mathscr{V}(L, E, K')_{L^0, E^0}$$

and call this subset of $L^* \otimes K'$ the *characteristic variety* of the geometric L-module E over the extension field K' of K. If L is a transitive Lie algebra, which we consider as an L-module via the adjoint representation of L, we write $\mathscr{V}(L, K') = \mathscr{V}(L, L, K')$.

We have the following generalization of Theorem 2 of the appendix of [27] and Proposition 9.3 of [28]:

Theorem 16.2. *A geometric L-module E is finite-dimensional if and only if $\mathscr{V}(L, E, K') = 0$ for all extension fields K' of K.*

Proposition 16.1 can now be reformulated as

Proposition 16.3. *Let $\phi: L \to L''$ be a continuous epimorphism of linearly compact Lie algebras, and E a geometric L''-module. If K' is an extension field of K, then*

$$\mathscr{V}(L, \phi^* E, K') = (\phi^* \otimes \mathrm{id})(\mathscr{V}(L'', E, K')) ,$$

where $(\phi^* \otimes \mathrm{id}): L''^* \otimes K' \to L^* \otimes K'$ is the injective mapping induced by ϕ.

The main result of this section is

Theorem 16.3. *Let*

$$0 \longrightarrow E' \xrightarrow{\phi} E \xrightarrow{\psi} E'' \longrightarrow 0$$

be an exact sequence of geometric L-modules, whose mappings are continuous. If K' is an extension field of K, then

$$\mathscr{V}(L, E, K') = \mathscr{V}(L, E', K') \cup \mathscr{V}(L, E'', K') .$$

Proof. Let L^0 be an open subspace of L, and $E^0 \subset E$, $E''^0 \subset E''$ fundamental subspaces such that $L^0 \cdot E^0 \subset E^0$ and $L^0 \cdot E''^0 \subset E''^0$. Replace E^0 by $E^0 \cap \psi^{-1}(E''^0)$, and let E'^0 be the subspace of E' such that $\phi(E'^0) = \phi(E') \cap E^0$; we thus have constructed fundamental subspaces $E'^0 \subset E'$, $E^0 \subset E$, $E''^0 \subset E''$ such that (16.4) holds. We then apply Proposition 16.2 to obtain the result.

We now deduce the following generalization of [28, Proposition 9.2]:

Corollary 16.1. *Let $\phi: L \to L''$ be an epimorphism of transitive Lie algebras, and $I \subset L$, $I'' \subset L''$ be closed ideals of L and L'' such that $\phi(I) = I''$. Let I' be the closed ideal of L which is the kernel of $\phi: I \to I''$. Then, if K' is an extension field of K,*

$$\mathscr{V}(L, I, K') = \mathscr{V}(L, I', K') \cup (\phi^* \otimes \mathrm{id})(\mathscr{V}(L'', I'', K')) ,$$

where $\phi^* \otimes \mathrm{id}: L''^* \otimes K' \to L^* \otimes K'$ is the injective mapping induced by ϕ.

Proof. The sequence

$$0 \longrightarrow I' \longrightarrow I \xrightarrow{\phi} \phi^* I'' \longrightarrow 0$$

of geometric L-modules is exact, and so Theorem 16.3 together with Proposition 16.3 gives us the desired equality.

The following result is a special case of the above corollary or of [28, Proposition 9.2]:

Corollary 16.2. *Let $\phi: L \to L''$ be an epimorphism of transitive Lie algebras, and let J be the kernel of ϕ. If K' is an extension field of K,*

$$\mathscr{V}(L, K') = \mathscr{V}(L, J, K') \cup (\phi^* \otimes \mathrm{id})(\mathscr{V}(L'', K')) ,$$

where $\phi^* \otimes \mathrm{id}: L''^* \otimes K' \to L^* \otimes K'$ is the mapping induced by ϕ.

Definition 16.1. We say that a geometric L-module E is elliptic if $\mathscr{V}(L, E, K) = 0$. A closed ideal of a transitive Lie algebra L is elliptic if it is elliptic considered as an L-module.

Corollary 16.3. (i) *Let $\phi: L \to L''$ be an epimorphism of transitive Lie algebras, and $I \subset L$, $I'' \subset L''$ be closed ideals of L and L'' such that $\phi(I) = I''$. Let I' be the kernel of $\phi: I \to I''$. Then I is elliptic if and only if I' and I'' are elliptic.*

(ii) *Let I be a closed ideal of a transitive Lie algebra L; then L is elliptic if and only if I and L/I are elliptic.*

Let E be a vector bundle over X, and $N_l \subset J_l(E)$ be a formally integrable differential equation. Let $x \in X$, and M_x be the graded ST_x-module (15.36). Let $\mathscr{I}_{M_x} \subset ST_x$ be the annihilator ideal of M_x; if K' is $\boldsymbol{R}$ or $\boldsymbol{C}$, the variety $\mathscr{V}(M_x, K')$ of $T_x^* \otimes K'$ defined by the ideal $\mathscr{I}_{M_x}$ is called the *characteristic variety* over K' of the differential equation N_l at $x \in X$ and is denoted by $\mathscr{V}_x(N_l, K')$.

The following description of the characteristic variety of N_l is a consequence of [3, Proposition 6.3]:

Proposition 16.4. *If $K' = \boldsymbol{R}$ or $\boldsymbol{C}$, the characteristic variety $\mathscr{V}_x(N_l, K')$ is equal to the set of $\alpha \in T_x^* \otimes K'$ satisfying*

$$(\alpha^l \otimes (E_x \otimes K')) \cap (h_{l,x} \otimes K') = 0 ,$$

where the intersection is taken in $(S^l T_x^* \otimes E_x) \otimes K'$.

Definition 16.2. We say that N_l is elliptic if $\mathscr{V}_x(N_l, \boldsymbol{R}) = 0$ for all $x \in X$.

Now let $R_k \subset J_k(T)$ be a formally transitive and formally integrable Lie equation, and let P_k be a formally integrable finite form of R_k, whose m-th prolongation we denote by P_{k+m}. Assume that the vector bundle E is associated to $\tilde{R}_k$, and let $N_l \subset J_l(E)$ be a formally integrable differential equation such that

$$R_{k+l} \cdot N_{l+1} \subset N_l .$$

From Lemma 15.4 and Proposition 16.4, we obtain

Lemma 16.3. *If X is connected and $a, b \in X$, there exists $F \in P_k$, with source $F = a$, target $F = b$, such that, if $K' = \boldsymbol{R}$ or $\boldsymbol{C}$, the image of $\mathscr{V}_b(N_l, K')$ under the isomorphism*

$$(\nu^* \circ \pi_1 F \circ \nu^{*-1}) \otimes \mathrm{id}\colon T_b^* \otimes K' \to T_a^* \otimes K'$$

is equal to $\mathscr{V}_a(N_l, K')$.

Theorem 16.4. (i) *If $x \in X$ and $K' = \boldsymbol{R}$ or $\boldsymbol{C}$, then the image of $\mathscr{V}_x(N_l, K')$ under the injective mapping*

$$\pi_0^* \circ \nu^{*-1} \otimes \mathrm{id}\colon T_x^* \otimes K' \to R_{\infty,x}^* \otimes K'$$

is equal to $\mathscr{V}(R_{\infty,x}, N_{\infty,x}, K')$.

(ii) *If X is connected and $a, b \in X$, there exists $\phi \in Q_\infty(a, b)$ such that $\phi(R_{\infty,a}) = R_{\infty,b}$ and the image of $\mathscr{V}(R_{\infty,b}, N_{\infty,b}, K')$ under the isomorphism $\phi^* \otimes \mathrm{id}\colon R_{\infty,b}^* \otimes K' \to R_{\infty,a}^* \otimes K'$ is equal to $\mathscr{V}(R_{\infty,a}, N_{\infty,a}, K')$, with $K' = \boldsymbol{R}$ or $\boldsymbol{C}$.*

(iii) *If X is connected and $x \in X$, then N_l is elliptic if and only if $N_{\infty,x}$ is an elliptic $R_{\infty,x}$-module.*

Proof. (i) Let $x \in X$; we identify T_x with the quotient $R_{\infty,x}/R^0_{\infty,x}$ via the exact sequence (15.39), and we consider the graded T_x-modules $\mathrm{gr}\, N_{\infty,x}$ and $\bigoplus_{m=0}^\infty h_{m,x}$, as defined in § 15. From the isomorphism (15.40) of graded ST_x-modules, we obtain

$$\mathscr{V}(R_{\infty,x}, N_{\infty,x}, K') = (\pi_0^* \circ \nu^{*-1} \otimes \mathrm{id})\left(\mathscr{V}\left(\bigoplus_{m=0}^{\infty} h^*_{m,x}, K'\right)\right) .$$

By Lemma 16.1, we have

$$\mathscr{V}_x(N_l, K') = \mathscr{V}\left(\bigoplus_{m=l}^{\infty} h^*_{m,x}, K'\right) = \mathscr{V}\left(\bigoplus_{m=0}^{\infty} h^*_{m,x}, K'\right) ,$$

and from these equalities we obtain the desired equality among characteristic varieties.

(ii) If $a, b \in X$, let F be an element of P_k, with source $F = a$, target $F = b$, satisfying the condition of Lemma 16.3. Since P_k is formally integrable, we can choose $\phi \in Q_\infty(a)$ such that $\pi_m\phi \in P_m$, for all $m \geq k$, and $\pi_k\phi = F$. Then $\phi(R_{\infty,a}) = R_{\infty,b}$ and the diagram

$$\begin{array}{ccc} R^*_{\infty,b} & \xrightarrow{\phi^*} & R^*_{\infty,a} \\ \uparrow{\scriptstyle \pi_0^*} & & \uparrow{\scriptstyle \pi_0^*} \\ J_0(T)^*_b & \xrightarrow{\pi_1 F} & J_0(T)^*_a \end{array}$$

is commutative; the result is now an immediate consequence of the property of F.

(iii) This assertion follows directly from the definitions, (i) and (ii).

Assertion (ii) of Theorem 16.4 can also be derived from Proposition 15.1. In fact, let $\phi \in Q_\infty(a, b)$, and $\psi: J_\infty(E)_a \to J_\infty(E)_b$ be an isomorphism satisfying the conditions of Proposition 15.1. Then ψ induces an isomorphism

$$(\mathrm{gr}\ \psi)^*: (\mathrm{gr}\ N_{\infty,a})^* \to (\mathrm{gr}\ N_{\infty,b})^*$$

such that

$$(\mathrm{gr}\ \psi)^*(p \cdot u) = (\nu^{-1} \circ \pi_1\phi \circ \nu)(p) \cdot (\mathrm{gr}\ \psi)^*(u) ,$$

for all $p \in ST_a$, $u \in (\mathrm{gr}\ N_{\infty,a})^*$. From this identity, we deduce that the image of $\mathscr{V}^0_b(R_{\infty,b}, N_{\infty,b}, K')$ under the isomorphism

$$(\nu^* \circ \pi_1\phi \circ \nu^{*-1}) \otimes \mathrm{id}: T^*_b \otimes K' \to T^*_a \otimes K'$$

is equal to $\mathscr{V}^0_a(R_{\infty,a}, N_{\infty,a}, K')$, and hence that (ii) holds.

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