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A TOPOLOGICAL GAUSS-BONNET THEOREM

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0. Introduction

The generalized Gauss-Bonnet theorem of Allendoerfer-Weil [1] and Chern [2] has played an important role in the development of the relationship between modern differential geometry and algebraic topology, providing in particular one of the primary stimuli for the theory of characteristic classes. There are now a number of proofs in the literature, from the quite sophisticated (deducing it as a special case of the Atiyah-Singer index theorem for example) to the relatively elementary and straightforward. (For a particularly elegant example of the latter see [7, Appendix C].) In general these previous proofs have a definite cohomological flavor and invoke explicit appeals to general vector bundle or principal bundle theory. In view of the above historical fact this is perhaps natural, and yet from another point of view it is somewhat anomalous. For the theorem states the equality of two quantities:

$$(2\pi)^{-k}\int_M K^{(n)}d\mu = \chi(M)\ .$$

Here M is any closed (= compact, without boundary), smooth (= C^∞) Riemannian manifold of even dimension $n = 2k$, $K^{(n)}$ is a certain "natural" real valued function on M (which in local coordinates is a somewhat complicated but quite explicit rational function of the components of the metric tensor and its partial derivatives of order two or less), μ is the Riemannian measure, and $\chi(M)$ is the Euler characteristic of M. There is nothing fundamentally "cohomological" on either side of this identity. True, one tends to think of $\chi(M)$ as the alternating sum of the betti numbers, but equally well and more geometrically it is the self intersection number of the diagonal in $M \times M$ or equivalently the algebraic number of zeros of a generic vector field. Indeed $\chi(M)$ is perhaps the most primitive topological invariant of M beyond the number of connected components; the fact that $\Sigma(-1)^k n_k$ (where n_k is the number of faces of dimension k in a cellular decomposition of a polyhedron P) is a combinatorial invariant $\chi(P)$ goes back two hundred years before the development of homology theory. And on the left we are *really* integrating a function with respect

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to a measure, not integrating an n-form over the fundamental cycle; for the theorem is equally valid when M is not orientable. This suggests that it should be possible to give an elementary "combinatorial" proof of the generalized Gauss-Bonnet theorem, using only the basic techniques of differential topology and in what follows we shall present such a proof. In the remainder of the introduction we outline the main ideas of the argument and at the same time introduce the notation we shall use in the body of the paper.

Let $\mathscr{M}_n$ denote the class of compact, smooth n-manifolds with boundary. A function F mapping $\mathscr{M}_n$ into a field K is called a differential invariant (for compact n-manifolds) if $F(M_1) = F(M_2)$ whenever M_1 and M_2 are diffeomorphic. Let $M_1, M_2 \in \mathscr{M}_n$ and let N be a union of components of ∂M_1. Given a smooth embedding $\psi: N \to \partial M_2$ we can form a manifold $M_1 +_\psi M_2 \in \mathscr{M}_n$ called the result of "gluing M_1 to M_2 along ψ". As a space this is the topological sum of M_1 and M_2 with $x \in N$ identified with $\psi(x)$. The differentiable structure is characterized up to diffeomorphism by the condition that M_1 and M_2 are smooth submanifolds (see [6, Theorem 1.4]). By varying ψ we get a class of manifolds which can be distinct differentiably and even topologically; $M_1 +_N M_2$ will denote an arbitrary element of this class. A differential invariant $F: \mathscr{M}_n \to K$ will be called *additive* if for all M_1, M_2, N as above it satisfies $F(M_1 +_N M_2) = F(M_1) + F(M_2)$. Now the Euler characteristic (thought of as defined on compact triangulable spaces for definiteness and having values in $\boldsymbol{Z} \subseteq \boldsymbol{Q}$) is well-known to satisfy $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$ whenever A and B are subspaces of a space X which can be triangulated so that A and B are subcomplexes (in fact [9] this characterizes χ up to a multiplicative constant). By restriction to $\mathscr{M}_n$ we get a differential invariant $\chi: \mathscr{M}_n \to \boldsymbol{Q}$ satisfying $\chi(M_1 +_N M_2) = \chi(M_1) + \chi(M_2) - \chi(N)$. Now if n is even then $\dim(N) = n - 1$ is odd, and it is well known that the Euler characteristic of a closed odd dimensional manifold is zero, so that in this case $\chi: \mathscr{M}_n \to \boldsymbol{Q}$ is additive. Of course if K is any field and $\gamma \in K$ then more generally $M \mapsto \chi(M)\gamma$ is an additive differential invariant $F_\gamma: \mathscr{M}_n \to K$, and γ can be recovered as $F_\gamma(D^n)$, where D^n denotes the n-disk, $\{x \in \boldsymbol{R}^n \mid \|x\| \leq 1\}$. The crucial topological fact for us is the following theorem which says that there are no other additive differential invariants when characteristic $(K) \neq 2$ (by contrast the number of boundary components modulo two *is* an additive differential invariant $\mathscr{M}_n \to \boldsymbol{Z}/2\boldsymbol{Z}$ not of the form F_γ).

0.1. Topological Gauss-Bonnet theorem. *If K is a field of characteristic not two, and $F: \mathscr{M}_n \to K$ is any additive differential invariant, then $F(M) = \chi(M)\gamma$ for all $M \in \mathscr{M}_n$ where $\gamma = F(D^N)$. Moreover if n is odd then $\gamma = 0$, so F is identically zero.*

The proof is constructive and basically combinatorial; it shows that when we adjoin a handle of index k to a manifold then the value of F changes by $(-1)^k\gamma$. It easily follows that for any handle body decomposition of $M \in \mathscr{M}_n$ (cf. § 2) $F(M) = (\sum_{k=0}^{n} (-1)^k \beta_k)\gamma$, where β_k is the number of handles of index k in the decomposition. In one sense the theorem then follows by just defining this alternating sum to be $\chi(M)$. (Of course identifying this alternating sum with the alternating sum of the betti numbers of M, the so called Morse equality, of necessity does require homological arguments.) The author is grateful to W. Neumann for pointing out that Theorem 0.1 is a simple corollary of results contained in Jänich' paper [3], and also follows easily from the characterization of the "cutting and pasting" groups given in [4]. In fact our proof of Proposition 2.4 is closely related to an argument used in the latter reference.

Now let R_n denote the class of compact smooth n-dimensional Riemannian manifolds with boundary; i.e., pairs (M, g) where $M \in \mathscr{M}_n$ and g is a smooth metric tensor for M. The metric g will be said to be *reflectable* if it is the restriction to M of a smooth metric tensor on DM, the double of M, with respect to which the canonical "reflection" automorphism of DM across ∂M is an isometry. A *natural scalar function* (for n-dimensional Riemannian manifolds) is a map F which associates to each $(M, g) \in R_n$ a smooth function $F_g\colon M \to \boldsymbol{R}$ such that if $\psi\colon M_1 \to M_2$ is an isometric embedding of (M_1, g_1) into (M_2, g_2) then $F_{g_1} = F_{g_2} \circ \psi$. Such functions of course abound; for example the scalar curvature, the length of the curvature tensor or of any of its covariant derivatives. Given such an F we associate to each $(M, g) \in R_n$ a real number $\bar{F}(M, g) = \int_M F_g d\mu_g$, where μ_g is the Riemannian measure on M defined by g. The natural scalar function F is called an *integral invariant* if whenever $M \in \mathscr{M}_n$ is without boundary $\bar{F}(M, g)$ has a value $\bar{F}(M)$ independent of g. It is then not hard to show (cf. § 4) that even if M has a nonempty boundary $\bar{F}(M, g)$ still has a value $\bar{F}(M)$ independent of g *provided* we consider only reflectable g, and in fact $\bar{F}(M) = \frac{1}{2}\bar{F}(DM)$. Moreover it follows from the naturality of $F(M, g)$ that $\bar{F}\colon \mathscr{M}_n \to \boldsymbol{R}$ is a differential invariant, and from the additivity of the integral it is not hard to see that $\bar{F}$ is even an additive differential invariant (cf. § 4) so by (0.1) we get

0.2. Abstract geometric Gauss-Bonnet theorem. *If F is an integral invariant for compact smooth n-dimensional Riemannian manifolds, then $\bar{F}(M) = \chi(M)\gamma$ for $M \in \mathscr{M}_n$ where $\gamma = \bar{F}(D^n) = \frac{1}{2}\bar{F}(S^n)$.*

Finally, in § 5, following in part the approach in [7, Appendix C] we define the natural scalar function $K^{(n)}$ and prove its integral invariance. Since this is a point where other proofs make an argument using the de Rham cohomology of TM or its frame bundle, we have taken some pains to give an elementary argument. Except for an application of the simplest form of Stokes theorem $\left(\text{if } \omega \text{ is an } (n-1)\text{-form on } \boldsymbol{R}^n \text{ with compact support, then } \int_{\boldsymbol{R}^n} d\omega = 0\right)$ the argument is in fact essentially formal.

Why this emphasis on an elementary proof? What after all is wrong with cohomology? Nothing of course, and the point is not to make the proof accessible to students at a lower level. Rather, with theorems which have played a role so central as Gauss-Bonnet it is author's feeling that it is important to understand their mathematical essence, and this can only be done by peeling away all the layers of elegant sophistication.

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1. Some differential topological constructions

In this section $F: \mathscr{M}_n \to K$ is an additive differential invariant and characteristic $(K) \neq 2$, as in the statement of the topological Gauss-Bonnet theorem. We shall investigate how F behaves with respect to several basic differential topological constructions.

1.1. Products of manifolds with boundaries. Let $M_1 \in \mathscr{M}_k$ and $M_2 \in \mathscr{M}_l$. Then there is a well-known way to put a differential structure on the topological product giving an element $M_1 \times M_2 \in \mathscr{M}_{k+l}$. If one of M_1 and M_2 has empty boundary, the process is elementary and the product is categorical (with respect to smooth maps). If $\partial M_1 \neq 0$ and $\partial M_2 \neq 0$, then this simple method of putting a differential structure on $M_1 \times M_2$ leads to "corners" along $\partial M_1 \times \partial M_2$. These can be removed by Milnor's method of "straightening angles", but the resulting product is no longer categorical. The basic facts that we shall need about the product are that it is associative and commutative up to diffeomorphism, and that

$$\partial(X \times Y) = \partial X \times Y +_{\partial X \times \partial Y} X \times \partial Y ,$$

where $\partial X \times \partial Y \subseteq \partial X \times Y$ is glued via the identity map to $\partial X \times \partial Y \subseteq X \times \partial Y$.

1.2. Interior deletion. Given $M \in \mathscr{M}_n$ let us denote is *interior*, $M - \partial M$, by $\mathring{M}$. Given M_1 and M_2 in $\mathscr{M}_n$ with M_2 smoothly embedded in $\mathring{M}_1$ we can remove $\mathring{M}_2$ from M_1 getting an element $M_1 - \mathring{M}_2$ of $\mathscr{M}_n$. Clearly $\partial(M_1 - \mathring{M}_2)$ is $\partial M_1 +_\emptyset \partial M_2$, the disjoint union of ∂M_1 and ∂M_2. If we glue M_2 to $M_1 - \mathring{M}_2$ along ∂M_2 (by the identity map) we of course get back M_1. Since F is additive we now see that it is also *subtractive*; i.e., $F(M_1 - \mathring{M}_2) = F(M_1) - F(M_2)$.

1.3. General gluing. Let M_1, $M_2 \in \mathscr{M}_n$ and let N be a smoothly embedded compact submanifold of M_1 of dimension $n - 1$, and $\psi: N \to \partial M_2$ a smooth embedding. If $\partial N = \emptyset$ we are in the case of the introduction and we can form $M_1 +_\psi M_2$. Henceforth we will refer to this process as *simple gluing*. In case $\partial N \neq 0$, we still get a topological n-manifold with boundary, which we now denote by $M_1 \cup_\psi M_2$, by taking the disjoint union of M_1 and M_2 and identifying $x \in N$ with $\psi(x) \in \partial M_2$. The analogous attempt to impose a differential structure on $M_1 \cup_\psi M_2$ leads to corners along ∂N; however once again the process of straightening angles permits us to smooth these corners and get a differential structure on $M_1 \cup_\psi M_2$. We shall refer to this process as *general gluing*, and once again where convenient we shall use the alternative notation $M_1 \cup_N M_2$ when we wish to emphasize N rather than ψ. Let $\partial\psi: \partial N \to \partial M_2$ denote the restriction of ψ. Then we have the following easy but important formula relating interior deletion, general gluing, and simple gluing:

$$\partial(M_1 \cup_\psi M_2) = (\partial M_1 - N^\circ) +_{\partial\psi} (\partial M_2 - \psi(N)^\circ) .$$

There is also a simple and obvious relation we shall need relating general gluing and product, namely the latter distributes through the former:

$$(M_1 \cup_\psi M_2) \times Y = (M_1 \times Y) \cup_{\psi \times \mathrm{id}_Y} (M_2 \times Y) .$$

1.4. Doubling. The double of a manifold M in $\mathscr{M}_n$ is usually defined by

$$DM = M +_{\mathrm{id}} M ,$$

i.e., by (simply) gluing together two copies of M along ∂M using the identity map. The involution, which maps an x in one copy of M to the corresponding point in the other copy, will be denoted by ρ. Its fixed point set is of course $\mathrm{Fix}\,(\rho) = \partial M$. It is immediate from the additivity of F that $F(DM) = 2F(M)$.

There is another well-known method for constructing a manifold diffeomorphic to DM which will be important for us, namely taking the boundary of $M \times I$ (where $I = D^1 = [0, 1]$). Indeed

$$\begin{aligned}\partial(M \times I) &= (\partial M \times I) +_j (M \times \{0, 1\}) \\ &= (M \times \{0\}) +_{j_0} (\partial M \times I) +_{j_1} (M \times \{1\}) ,\end{aligned}$$

where j_1 is the obvious inclusion of $\partial M \times \{1\}$ into $\partial M \times I$. Now by the collar neighborhood theorem $M \times \{0\} +_{j_0} (\partial M \times I)$ is diffeomorphic to M and of course so is $M \times \{1\}$, and it follows easily that $\partial(M \times I) \approx DM$. We are now prepared to prove

1.5. Proposition. *Let M_1, $M_2 \in \mathscr{M}_n$, N be a smoothly embedded $(n - 1)$-dimensional submanifold of ∂M_1, and $\psi \colon N \to \partial M_2$ be a smooth embedding. Then*

$$D(M_1 \cup_\psi M_2) \approx (DM_1 - (N \times I)^\circ) +_{\partial\psi \times \mathrm{id}} (DM_2 - (\psi(N) \times I)^\circ) .$$

Proof. Recall the distributive law

$$(M_1 \cup_\psi M_2) \times I = (M_1 \times I) +_{\psi \times \mathrm{id}} (M_2 \times I) .$$

Then the conclusion is immediate from the facts noted earlier that $DM \approx \partial(M \times I)$ and $\partial(M_1 \cup_\varphi M_2) \approx (\partial M_1 - N^\circ) +_{\partial\varphi} (\partial M_2 - \varphi(N)^\circ)$. q.e.d.

If we use the additivity and subtractivity (cf. § 1.2) of F on the conclusion of Proposition 1.5 we get

$$F(D(M_1 \cup_\psi M_2)) = F(DM_1) + F(DM_2) - F(N \times I) - F(\psi(N) \times I) .$$

On the other hand recall that $F(DM) = 2F(M)$, and $F(\psi(N) \times I) = F(N \times I)$ since F is a differential invariant and ψ is a diffeomorphism. It follows that

$$2F(M_1 \cup_N M_2) = 2(F(M_1) + F(M_2) - F(N \times I)),$$

and since by assumption F has values in a field of characteristic different from two, finally we get

1.6. Extended additivity theorem. *If $F: \mathscr{M}_n \to K$ is an additive differential invariant, and K has characteristic $\neq 2$, then $F(M_1 \cup_N M_2) = F(M_1) + F(M_2) - F(N \times I)$.*

1.7. Remark. If $\partial N = \emptyset$, then $M_1 \cup_N M_2 = M_1 +_N M_2$ so $F(M_1 \cup_N M_2) = F(M_1) + F(M_2)$. This of course strongly suggests that when N is any closed $(n-1)$-manifold, then $F(N \times I) = 0$. In fact, this is an immediate consequence of the additivity of F and the fact that $(N \times I) +_\psi (N \times I) \approx N \times I$, where ψ is the obvious diffeomorphism of $N \times \{1\}$ with $N \times \{0\}$. This latter remark allows us to give a completely elementary proof of Theorem 0.1 for two-manifolds. For simplicity we consider only the orientable case. Suppose $\sum_g$ is an orientable surface of genus g. We can construct $\sum_g$ from S^2 by "adding g handles". Now $F(S^2) = F(D(D^2)) = 2F(D^2) = 2\gamma$, and each time we add a handle, we delete the interiors of two 2-disks, which by § 1.2 reduces the value of F by 2γ, and then (simply) glue on a cylinder $S^1 \times I$, which by the remark above does not change the value of F. Thus $F(\sum_g) = 2\gamma - g(\gamma 2) = (2-2g)\gamma$, and it is well-known that $\chi(\sum_g) = 2 - 2g$.

2. Handles and handle-bodies

F still denotes an additive differential invariant $\mathscr{M}_n \to K$ (characteristic $(K) \neq 2$) and $\gamma = F(D^n)$. We will denote $D^k \times D^{n-k}$ by h_k^n which we call the handle of (dimension n and) index k. Of course $h_k^n \approx D^n$ so $F(h_k^n) = \gamma$. Included in ∂h_k^n is $\partial D^k \times D^{n-k} = S^{k-1} \times D^{n-k}$ so that given a smooth embedding φ of $S^{k-1} \times D^{n-k}$ into $M \in \mathscr{M}_n$ we can form $M \cup_\varphi h_k^n$, which we call M with a handle of index k attached. Note that by Theorem 1.6

2.1. Proposition. $F(M \cup_\varphi h_k^n) = F(M) + \gamma - f_{k-1}$.
where:

2.2. Definition. $f_k = F(S^k \times D^{n-k})$.

Note $F(S^0 \times D^n) = F(D^n +_\emptyset D^n) = 2\gamma$ and $F(S^n \times D^0) = F(S^n) = F(D(D^n)) = 2\gamma$. Thus

2.3. Lemma. $f_0 = f_n = 2\gamma$.

2.4. Proposition. $f_k = \gamma - (-1)^{k+1}\gamma$. *That is, for k even $f_k = 2\gamma$, and for k odd $f_k = 0$. Moreover, if n is odd then $\gamma = 0$, so that all the f_k are zero.*

Proof. $S^k = D(D^k) = D^k \cup_{S^{k-1}} D^k$ and hence by § 1.3

$$S^k \times D^{n-k} = (D^k \times D^{n-k}) \cup_{S^{k-1} \times D^{n-k}} D^k \times D^{n-k},$$

and therefore by Theorem 1.6 since $D^k \times D^{n-k} \approx D^n$

$$\begin{aligned} f_k &= F(S^k \times D^{n-k}) \\ &= F(D^n) + F(D^n) - F(S^{k-1} \times D^{n-k} \times I) = 2\gamma - f_{k-1} . \end{aligned}$$

That is, $f_k + f_{k-1} = 2\gamma$. Since $f_0 = 0$, from Lemma 2.3 it follows that $f_k = 2\gamma$ for k even and $f_k = 0$ for k odd, as required. Finally, if n is odd then by Lemma 2.3, $2\gamma = f_n = 0$ so $\gamma = 0$.

2.5. Theorem. $F(M \cup_\varphi h_k^n) = F(M) + (-1)^k\gamma$.

Proof. Immediate from Propositions 2.1 and 2.4.

2.6. Definition. Let $M \in \mathscr{M}_n$. A *Morse-Smale presentation* (or *handle-body decomposition*) of M is a sequence $(M_0, \cdots, M_r)$ in $\mathscr{M}_n$, and embeddings φ_j: $S^{k_j-1} \times D^{n-k_j} \to M_j$, $0 \leq j < r$ such that $M_0 = D^n$, $M_{j+1} = M_j \cup_{\varphi_j} h_{k_j}^n$, and $M_r = M$. The kth *type number* β_k of the presentation ($k = 0, 1, \cdots, n$) is the number of j in $\{0, 1, \cdots, r - 1\}$ such that $k_j = k$.

2.7. Theorem. *Let $M \in \mathscr{M}_n$ and let $\{\beta_0, \beta_1, \cdots, \beta_n\}$ be the type numbers of any Morse-Smale presentation of M. Then $F(M) = (\sum_{k=0}^n (-1)^k\beta_k)\gamma$.*

Proof. Immediate from Theorem 2.5.

3. Proof of the topological Gauss-Bonnet theorem

It is of course not immediately evident that an arbitrary $M \in \mathscr{M}_n$ admits any Morse-Smale presentation. However this is well-known and reasonably elementary. See for example [6, Theorem 2.5] and [8, § 12, Theorem]. Moreover, it is also well-known that if $(\beta_0, \cdots, \beta_n)$ are the type numbers of any Morse-Smale presentation of M, then $\sum_{k=0}^n (-1)^k\beta_k = \chi(M)$. This in fact is essentially equivalent to the "Morse equality" [6, Theorem 7]. These two facts together with Theorem 2.7 complete the proof of the topological Gauss-Bonnet theorem (Theorem 0.1). The fact that $\gamma = 0$ when n is odd is already contained in Proposition 2.4.

4. Proof of the abstract geometric Gauss-Bonnet theorem

In this section F will denote some integral invariant for compact n-manifolds. Then by assumption $\bar{F}(M, g) = \int_M F_g d\mu_g$ depends only on M and not on g when $\partial M = \emptyset$, and we denote its value in this case by $\bar{F}(M)$. It is clear from the fact that F is a natural scalar function that $\bar{F}(M)$ depends only on the diffeomorphism type of M. Since $\bar{F}(M)$ is defined to be $\frac{1}{2}\bar{F}(DM)$ when $\partial M \neq \emptyset$, and the diffeomorphism type of M determines that of DM, it follows that $\bar{F}$: $\mathscr{M}_n \to \boldsymbol{R}$ is a differential invariant and we now will show that it is additive. Recall that a smooth Riemannian metric g for $M \in \mathscr{M}_n$ is said to be *reflectable* when it is the restriction to M of a smooth Riemannian metric λ on DM for which ρ is an isometry (i.e., $\rho^*\lambda = \lambda$), where ρ: $DM \approx DM$ is the canonical involution. (Since the fixed point set of a Riemannian isometry is totally geodesic, this implies ∂M is totally geodesic with respect to g, conversely it is not difficult to see that if a smooth Riemannian metric g on M has ∂M as a totally geodesic submanifold, then the result of reflecting this metric across ∂M is a metric on DM which is C^2 across ∂M, but *not* necessarily smoother.) It is trivial that reflectable metrics always exist. For if λ_1 is any smooth metric on DM and $\lambda = \frac{1}{2}(\lambda_1 + \rho^*\lambda_1)$, then $\rho^*\lambda = \lambda$ since $\rho^2 = \text{id}$. Now put $M' = \rho(M)$ so that (since $DM = M \cup M'$, and $\partial M = M \cap M'$ has measure zero)

$$\bar{F}(DM) = \int_{DM} F_\lambda d\mu_\lambda = \int_M F_g d\mu_g + \int_{M'} F_{g'} d\mu_{g'} ,$$

where g and g' are respectively the restrictions of λ to M and M'. Since ρ maps (M, g) isometrically onto (M', g'), $\mu_{g'} \circ \rho = \mu_g$, and since F is a natural scalar function we have $F_{g'} \circ \rho = F_g$. Hence $\int_{M'} F_{g'} d\mu_{g'} = \int_M F_g d\mu_g$ and so

$$\bar{F}(M, g) = \int_M F_g d\mu_g = \tfrac{1}{2}\bar{F}(DM) = \bar{F}(M) ,$$

provided g is reflectable. It is of course clear that a metric g for M which is a product metric on some collar neighborhood $U \approx \partial M \times I$ of ∂M is reflectable.

Now suppose $M = M_1 +_N M_2$ and let g be a product Riemannian metric on U, the union of a tubular neighborhood of N and a collar neighborhood of ∂M. By a classical extension theorem after restricting g to a slightly smaller neighborhood of $N \cup \partial M$ it can be extended to as mooth metric on M. By the preceding remark g is a reflectable metric for M, and its restriction g_i to M_i is a reflectable metric for M_i, and hence

$$\begin{aligned} \bar{F}(M_1 +_N M_2) &= \int_M F_g d\mu_g = \int_{M'} F_g d\mu_g + \int_{M_2} F_g d\mu_g \\ &= \int_{M_1} F_{g_1} d\mu_{g_1} + \int_{M_2} F_{g_2} d\mu_{g_2} = \bar{F}(M_1) + \bar{F}(M_2) . \end{aligned}$$

This proves the additivity of $\bar{F}$ and completes the proof of the abstract geometric Gauss-Bonnet theorem (Theorem 0.2).

5. The classical generalized Gauss-Bonnet theorem

The abstract geometric Gauss-Bonnet theorem of the preceding section only gains content with the demonstration that nontrivial integral invariants exists. In this section we will give an elementary, almost formal argument to show that the classical Pfaffian expression in the components of the curvature tensor is, as first noted by S. S. Chern, an integral invariant.

We shall work locally, in a coordinate neighborhood $\mathcal{O}$ of a closed manifold M of dimension $n = 2m$. An n-triple $E = (E_1, \cdots, E_n)$ of smooth vector fields is called a framing of $\mathcal{O}$ if $E(x) = (E_1(x), \cdots, E_n(x))$ is linearly independent and hence a basis for TM_x for all $x \in \mathcal{O}$. In this case we denote by $\theta =$

$(\theta_1, \cdots, \theta_n)$ the n-tuple of one-forms in $\mathcal{O}$ such that $\theta(x) = (\theta_1(x), \cdots, \theta_n(x))$ is the dual basis to $E(x)$. We note that E defines a unique Riemannian metric in $\mathcal{O}$ with respect to which it is orthonormal. Moreover any metric g in $\mathcal{O}$ is defined in this way; merely take E to be defined by orthonormalizing $(\partial/\partial x_1, \cdots, \partial/\partial x_n)$ with respect to g using the Gram-Schmidt process. Given a framing E of $\mathcal{O}$ and a smooth map $T: \mathcal{O} \to GL(n)$ we get another framing $E' = TE$ of $\mathcal{O}$, where $E'_j(x) = \sum_{i=1}^n T_{ij}(x)E_i(x)$, and clearly every framing of $\mathcal{O}$ arises in this way for a unique such map T. Of course E' and E define the same Riemannian metric in $\mathcal{O}$ if and only if they are orthogonally related, i.e., T has its image in the orthogonal group $\boldsymbol{O}(n)$. We note that $\theta_1 \wedge \cdots \wedge \theta_n$ is a nonvanishing n-form in $\mathcal{O}$, so any smooth n-form λ in $\mathcal{O}$ can be written uniquely as $f\theta_1 \wedge \cdots \wedge \theta_n$ where f is a smooth function in $\mathcal{O}$. Since $\theta'_1 \wedge \cdots \wedge \theta'_n = \det(T)\theta_1 \wedge \cdots \wedge \theta_n$ we easily get the following general principle for defining natural scalar functions on n-dimensional Riemannian manifolds.

5.1. Proposition. *Let σ be a function which assigns to each orthonormal framing E of an open set $\mathcal{O}$ of an n-dimensional Riemannian manifold (M, g) an n-form σ^E in $\mathcal{O}$, and suppose that whenever E and E' are two orthonormal framings of the same open set $\mathcal{O}$ with $E' = TE$, then $\sigma^{E'} = \det(T)\sigma^E$. Then there is a uniquely determined natural scalar function F for n-dimensional Riemannian manifolds such that for any orthonormal framing E of an open set $\mathcal{O}$ of (M, g), $\sigma^E = F_g\theta_1 \wedge \cdots \wedge \theta_n$.*

We now seek a local criterion for deciding when such a natural scalar function is an integral invariant.

5.2. Lemma. *Let F be a natural scalar function for compact n-dimensional Riemannian manifolds. Suppose that whenever (M, g_0) and (M, g_1) are two closed Riemannian manifolds with the same underlying manifold, and g_0 and g_1 agree outside some compact subset of a coordinate neighborhood $\mathcal{O}$ of M, then $\bar{F}_{g_0} = \bar{F}_{g_1}$. Then F is an integral invariant.*

Proof. Let M be any closed manifold of dimension n, and let $\varphi_1, \cdots, \varphi_k$ be a smooth partition of unity for M subordinate to a covering by coordinate neighborhood $\mathcal{O}_1, \cdots, \mathcal{O}_k$. Given two metrics for M, call them g_0 and g_k and let $s = (g_k - g_0)$. By the convexity of Riemannian metrics, if f is any smooth function or M with $0 \leq f \leq 1$ everywhere, then $g_0 + fs$ is also a smooth metric. In particular taking $f_j = \varphi_1 + \cdots + \varphi_j$, (so $f_k = 1$), $g_j = g_0 + f_j s$ is a smooth metric for M. Moreover g_{j+1} agrees with g_j outside the support of φ_j which is a compact subset of $\mathcal{O}_j$, and so $\bar{F}_{g_{j+1}} = \bar{F}_g$. It follows that $\bar{F}_{g_k} = \bar{F}_{g_0}$.

5.3. Proposition. *With the notation of Proposition* 5.1 *suppose that given a smooth one-parameter family $E(t)$, $0 \leq t \leq 1$, of framings of $\mathcal{O}$, the corresponding family $\sigma^{E(t)}$ of n-forms in $\mathcal{O}$ is smooth in t, and moreover $(d/dt)(\sigma^{E(t)}) = d(\lambda(t))$, where $\lambda(t)$ is an $(n-1)$-form in $\mathcal{O}$ vanishing on any open set where $E(t)$ is independent of t. Then the natural scalar function F of the conclusion of Proposition* 5.1 *is in fact an integral invariant.*

Proof. Let (M, g_0) and (M, g_1) be closed Riemannian manifolds, and suppose

g_0 agrees with g_1 outside some coordinate neighborhood $\mathcal{O}$. By Lemma 5.2 it will suffice to show that $\bar{F}_{g_1} = \bar{F}_{g_0}$. Let $g_t = g_0 + t(g_1 - g_0)$. Then we will show that $\bar{F}_{g_t}$ is independent of t. Since g_t (and hence F_{g_t} and μ_{g_t}) agree outside $\mathcal{O}$ and $\bar{F}_{g_t} = \int_M F_{g_t} d\mu_{g_t}$ it will suffice to show $\int_{\mathcal{O}} F_{g_t} d\mu_{g_t}$ is independent of t. In fact, we shall show that it is a differentiable function of t with derivative zero. Let $E(t)$ be the framing of $\mathcal{O}$ obtained by orthonormalizing $(\partial/\partial x_1, \cdots, \partial/\partial x_n)$ with respect to $g(t)$. Let γ be a cube in $\mathcal{O}$ (with respect to the coordinates x) including in its interior the set where g_0 and g_1 differ. The restriction of the coordinate map to γ is a singular n-cube C in M, and $E(t)$ is constant in a neighborhood of ∂C. Since by assumption $(d/dt)(\sigma^{E(t)}) = d(\lambda(t))$ where $\lambda(t)$ vanishes on ∂C, by Stokes theorem we have

$$\frac{d}{dt}\int_C \sigma^{E(t)} = \int_C d(\lambda(t)) = \int_{\partial C} \lambda(t) = 0 \, .$$

On the other hand, by definition of the Riemannian measure μ_{g_t}:

$$\frac{d}{dt}\int_{\mathcal{O}} F_{g_t} d\mu_{g_t} = \frac{d}{dt}\int_{\gamma} F_{g_t} d\mu_{g_t} = \frac{d}{dt}\int_C F_{g_t}\theta_1(t) \wedge \cdots \wedge \theta_n(t)$$
$$= \frac{d}{dt}\int_C \sigma^{E(t)} = 0 \, . \quad \text{q.e.d.}$$

Now let ∇ denote the covariant differentiation with respect to a Riemannian metric g in $\mathcal{O}$ defined by a framing E. For a vector Y based in $\mathcal{O}$ we can write

$$\nabla_Y E_i = \sum_{j=1}^{n} \omega_{ij}(Y) E_j \, ,$$

where $\omega = \omega_{ij}$ is an $n \times n$ matrix of one-forms in $\mathcal{O}$ (called the *connection forms* associated to E) defined by these equations. Since

$$0 = Y\delta_{ij} = Yg(E_i, E_j) = g(\nabla_Y E_i, E_j) + g(E_i, \nabla_Y E_j) \, ,$$

it follows easily that the matrix ω is skew symmetric, and hence so also is the matrix $\Omega = \Omega_{ij}$ of *curvature two forms* associated to E, defined by:

$$\Omega = d\omega - \omega \wedge \omega \, ,$$

i.e., $\Omega_{ij} = d\omega_{ij} - \sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj}$. Let $T\colon \mathcal{O} \to \boldsymbol{O}(n)$ be smooth and $E' = TE$, so since E' is orthogonally related to E it defines the same metric and hence the same covariant derivative ∇. Then an easy calculation shows that the matrix Ω' of curvature two-forms associated to E' is related to Ω by $\Omega' = T\Omega T^{-1}$, i.e., $\Omega'_{ij}(x) = \sum_{k,l=1}^{n} T_{ik}(x) T_{jl}(x) \Omega_{kl}(x)$.

In what follows X denotes $\frac{1}{2}n(n-1)$ indeterminates $X_{ij} (1 \leq i < j \leq n)$, and we define $X_{ii} = 0$ and $X_{ji} = -X_{ij}$ in the polynomial ring $\boldsymbol{R}[X]$, so we may regard X as a skew $n \times n$ matrix of elements of $\boldsymbol{R}[X]$. Similarly for Y and Z. If A is any $n \times n$ real matrix, then AX will denote the metrix $(AX)_{ij} = \sum_{k=1}^{n} A_{ik}X_{kj}$ of elements of $\boldsymbol{R}[X]$, etc. Thus for $T \in \boldsymbol{O}(n)$, TXT^{-1} denotes the matrix

$$(TXT^{-1})_{ij} = \sum_{k,l=1}^{n} T_{ik}T_{jl}X_{kl}$$

of elements of $\boldsymbol{R}[X]$, which is clearly skew. Now if $P(X)$ is in $\boldsymbol{R}[X]$, and $S = S_{ij}$ is any skew $n \times n$ matrix of elements from a commutative algebra $\mathscr{A}$ over $\boldsymbol{R}$, then we can substitute S_{ij} for X_{ij} in P to obtain $P(S) \in \mathscr{A}$. In particular for $T \in \boldsymbol{O}(n)$ we have $P(TXT^{-1}) \in \boldsymbol{R}[X]$.

5.4. Definition. $P(X) \in [X]$ is said to be *orthogonally covariant* if for all $T \in \boldsymbol{O}(n)$

$$P(TXT^{-1}) = \det(T)P(X) .$$

Now differential forms of even degree in an open set $\mathcal{O}$ of an n-manifold M also form a commutative algebra. Thus, if Ω is the matrix of curvature forms associated to a framing E of $\mathcal{O}$, then $P(\Omega)$ is a form in this algebra. In particular, if $P(X)$ is homogeneous of degree $m = n/2$, then $P(\Omega)$ is a form of degree n. If Ω' is the matrix of curvature forms associated to $E' = TE$ where $T\colon \mathcal{O} \to \boldsymbol{O}(n)$ is smooth, then as we noted above $\Omega' = T\Omega T^{-1}$, hence if $P(X)$ is orthogonally covariant then $P(\Omega') = \det(T)P(\Omega)$, so by Proposition 5.1 we see

5.5. Proposition. *If $P(X) \in \boldsymbol{R}[X]$ is orthogonally covariant and homogeneous of degree $m = n/2$, then there is a unique natural scalar function F for n-dimensional Riemannian manifolds such that for any orthonormal framing E of an open set $\mathcal{O}$ of (M, g), if Ω is the associated matrix of curvature two-forms in $\mathcal{O}$ then*

$$P(\Omega) = F_g\theta_1 \wedge \cdots \wedge \theta_n .$$

The remarkable and surprising fact is that, as was first shown by A. Weil, such a natural scalar function F is automatically an integral invariant. We give an elementary formal proof below.

Given $P(X) \in \boldsymbol{R}[X]$ we define $\nabla P(X)$ to be the matrix of elements of $\boldsymbol{R}[X]$ (the gradient of P) defined by

$$(\nabla P(X))_{ij} = \partial P(X)/\partial X_{ij} , \qquad 1 \le i < j \le n .$$

And we define $\Delta P(X; Y)$, the formal directional derivative of P in the direction Y, by

$$\Delta P(X; Y) = [(P(X + TY) - (X))/T]_{T=0} ,$$

so that clearly

$$\Delta P(X;\, Y) = \text{trace}\,(\nabla P(X)Y) = \sum_{i,j} \nabla P(X)_{ij} Y_{ij}\ .$$

5.6. Lemma. *If $P(X) \in \boldsymbol{R}[X]$ is orthogonally covariant, and A is any $n \times n$ skew real matrix, then*

$$\Delta P(X; [A, X]) = 0\ .$$

Proof. $\exp(tA)$ is a one-parameter group of orthogonal matrices, and $\exp(tA)^{-1} = \exp(-tA)$, so $P(\exp(tA)X \exp(-tA)) = \det(\exp tA)P(X) = P(X)$. Differentiating this with respect to t at $t = 0$ gives the result. q.e.d.

Next let

$$\Delta^2 P(X;\, Y, Z) = \left[\frac{1}{T}(\Delta P(X + TY;\, Z) - \nabla P(X;\, Z))\right]_{T=0}$$
$$= \sum_{i,j,k,l} \frac{\partial^2 P}{\partial X_{ij} \partial X_{kl}} Y_{ij} Z_{kl}\ ,$$

and note that $\Delta^2 P(X;\, Y, Z) = \Delta^2 P(X;\, Z, Y)$.

5.7. Lemma. *If $P(X) \in \boldsymbol{R}[X]$ is orthogonally covariant, then*

$$\Delta^2 P(X; [Y, X], Z) = \text{trace}\ ([Y, \nabla P(X)]Z\ .$$

Proof. From Lemma 5.6 it follows that

$$\Delta P(X + tZ; [Y, X + tZ]) = 0\ .$$

If we "differentiate" this (i.e., subtract $\Delta P(X; [Y, X]) = 0$, divide by t and set $t = 0$) we get, using the linearity of $\Delta P(X;\, Y)$ in Y,

$$\Delta^2 P(X; Z, [Y, X]) + \Delta P(X; [Y, Z]) = 0\ .$$

Now

$$\begin{aligned} \Delta P(X; [Y, Z]) &= \text{trace}\,(\nabla P(X)(YZ - ZY)) \\ &= \text{trace}\,(\nabla P(X)YZ - \nabla P(X)ZY) \\ &= \text{trace}\,(\nabla P(X)YZ - Y\nabla P(X)Z) \\ &= \text{trace}\,([\nabla P(X),\, Y]Z)\ , \end{aligned}$$

and using the symmetry of $\Delta^2 P(X;\, Y, Z)$ in Y, Z we get the desired result.

q.e.d.

Now suppose $E(t)$ is a smooth one-parameter family of framings of an open set $\mathcal{O}$ of an n-manifold M with associated matrix of connection one-forms $\omega(t)$ and curvature two-forms $\Omega(t)$. From their definition it is clear that ω and Ω are smooth in t, and we put $\dot\omega(t) = (\partial/\partial t)\omega(t)$ and $\dot\Omega(t) = (\partial/\partial t)\Omega(t)$. Since $\Omega = d\omega - \omega \wedge \omega$, it follows that $\dot\Omega = d\dot\omega - \dot\omega \wedge \omega - \omega \wedge \dot\omega$. Now for $P(X) \in \boldsymbol{R}[X]$

$$
\begin{aligned}
(\partial/\partial t)P(\Omega(t)) &= \Delta P(\Omega(t);\dot{\Omega}(t))\\
&= \Delta P(\Omega(t); d\dot{\omega} - \dot{\omega}\wedge\omega - \omega\wedge\dot{\omega})\\
&= \operatorname{trace}(\nabla P(\Omega)d\dot{\omega} - \nabla P(\Omega)\dot{\omega}\omega - \nabla P(\Omega)\omega\dot{\omega})
\end{aligned}
$$

Since $\nabla P(\Omega)\omega$ and $\dot{\omega}$ are forms of odd degree,

$$\operatorname{trace}(\nabla P(\Omega)\dot{\omega}\omega) = -\operatorname{trace}(\omega\nabla P(\Omega)\dot{\omega}) ,$$

hence we have

$$(\partial/\partial t)P(\Omega(t)) = \operatorname{trace}(\nabla P(\Omega)d\dot{\omega} + [\omega, \nabla P(\Omega)]\dot{\omega}) .$$

On the other hand, we compute easily that $d(\Delta P(\Omega;\dot{\omega})) = \Delta^2 P(\Omega; d\Omega, \dot{\omega}) + \Delta P(\Omega; d\dot{\omega})$, and since the definition $\Omega = d\omega - \omega\wedge\omega$ implies $d\Omega = -d\omega\wedge\omega - \omega\wedge d\omega = \omega\wedge\Omega - \Omega\wedge\omega = [\omega, \Omega]$, we have

$$d(\Delta P(\Omega;\dot{\omega})) = \Delta^2 P(\Omega; [\omega, \Omega], \dot{\omega}) + \Delta P(\Omega; d\dot{\omega}) .$$

Thus, if we assume that $P(X)$ is orthogonally covariant, from Lemma 5.7 it follows that

$$d(\Delta P(\Omega;\dot{\omega})) = \operatorname{trace}([\omega, \nabla P(X)]\dot{\omega}) + \operatorname{trace}(\nabla P(\Omega)d\dot{\omega}) ,$$

so comparing above we finally have

$$(\partial/\partial t)P(\Omega(t)) = d(\Delta P(\Omega;\dot{\omega})) .$$

Note that on any open set where $E(t)$ is constant, $\dot{\omega} = 0$ so $\Delta P(\Omega;\dot{\omega}) = 0$.

Now from Proposition 5.3, follows

5.8. Theorem. *Let $P(X) \in \boldsymbol{R}[X]$ be orthogonally covariant and homogeneous of degree $m = n/2$. Let F be the natural scalar function defined for n-dimensional Riemannian manifolds by the condition that if E is an orthonormal framing of an open set $\mathcal{O}$ and Ω is the curvature matrix, then $P(\Omega) = F_g\theta_1\wedge\cdots\wedge\theta_n$ (see Proposition* 5.5). *Then F is an integral invariant.*

The final question of course is whether when $n = 2m$ there do in fact exist orthogonally covariant elements of $\boldsymbol{R}[X]$ homogeneous of degree m. This is answered by the following classical theorem of pure algebra (cf. [5, p. 372], [7, p. 309]).

5.9. Theorem. *If $n = 2m$, then there exists up to sign a unique polynomial $Pf(X)$ (the pfaffian) in $\boldsymbol{R}[X]$ such that $Pf(X)^2 = \det(X)$. Moreover Pf has integer coefficients, is orthogonally covariant, and is homogeneous of degree m.*

The sign of Pf is chosen so that $Pf(\operatorname{diag}(S, \cdots, S)) = 1$ where $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. In particular for $n = 2$, $Pf(X) = X_{12}$ and for $n = 4$, $Pf(X) = X_{12}X_{34} - X_{13}X_{24}$. With this choice let us call $K^{(n)}$ the corresponding integral invariant. Then a simple computation of $K^{(n)}(S^n)$, using the induced metric from the standard embedding in $\boldsymbol{R}^{n+1}$, gives $2(2\pi)^{n/2}$. It then follows that for any closed Riemannian manifold M of dimension n we have $\int_M K_g^n d\mu_g = (2\pi)^{n/2}\chi(M)$, which is the generalized Gauss-Bonnet theorem of Allendoerfer-Chern-Weil [1], [2].

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