J. DIFFERENTIAL GEOMETRY
13 (1978) 341-359

MANIFOLDS WITHOUT FOCAL POINTS

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0. Introduction

The behavior of geodesics in Riemannian manifolds without conjugate or focal points has been discussed by many geometers such as Morse, Hedlund, Green, Eberlein and others. It is known that the properties, e.g., the topological transitivity of geodesic flows on certain Riemannian manifolds, are connected closely to some instability property of geodesics. On a complete simply connected Riemannian manifold without conjugate points, L. Green proved an instability property of geodesics under the condition that the sectional curvature is bounded from below. His proof for the higher dimensional case was incomplete as was pointed out by Eberlein.

The purpose of this paper is to extend the theory of L. Green in [6], [7] and [8], reproducing the results there without the condition on curvature assumed by Green. Consequently our results turn out to be extensions of some fundamental notions and results for nonpositively curved manifolds to manifolds without focal points.

A complete Riemannian manifold M is said to have *no focal points* if no maximal geodesic σ of M has focal points along any perpendicular geodesic, where σ is considered as an imbedded one-dimensional submanifold of M. This property can be stated as follows: For any geodesic ray γ and any nontrivial Jacobi field along γ vanishing at $t = 0$, $(d/dt)\langle Y(t), Y(t)\rangle > 0$ for $t > 0$, where $\langle\ ,\ \rangle$ denotes the inner product with respect to the Riemannian metric of M, see [12].

In this paper we shall be concerned only with Riemannian manifolds without focal points. In addition manifolds are always assumed to be connected, complete and differentiable (of class C^∞). Geodesics are assumed to have unit speed unless otherwise stated.

In § 1 we introduce the basic results on the Jacobi equations by L. Green for later use.

In § 2 we prove

Theorem 1. *Let M be a complete Riemannian manifold without focal points. Let γ be a geodesic ray with $\gamma(0) = p \in M$. If $Y^x(t)$ denotes the Jacobi field along γ with $Y^x(0) = 0$, $(Y^x)'(0) = x$, where x is nonzero vector at p, then*

Communicated by E. Calabi, October, 1976.

$$\lim_{t\to\infty} |Y^x(t)| = \infty$$

uniformly on $\{x \in T_pM; |x| = 1\}$.

In § 3 we modify the technique by Kobayashi, and using it as a tool, we prove the following instability property of geodesic rays in § 4.

Theorem 2. *Let M be a complete simply connected Riemannian manifold without focal points. Then no two geodesic rays with a common origin can be asymptotic to each other.*

As we mentioned in the beginning we have the following theorem in § 5 using Theorem 2.

Theorem 3. *If* $M(f)$ *(defined in* § 5*) has no focal points, then the geodesic flow of* $M(f, G)$ *(defined in* § 5*) is topologically transitive.*

Next, the final goal of § 6 is to prove

Theorem 4. *Let M be a complete simply connected Riemannian manifold without focal points. Given a geodesic ray* γ *and a point* $p \in M$*, there exists a unique geodesic ray asymptotic to* γ *passing through p.*

Also from the proof of Theorem 4 we know that, if α and β are geodesic rays asymptotic to each other, then $d(\alpha(t), \beta(t))$ is nonincreasing in t, $t \geq 0$, where d denotes the distance function of M.

In § 7, we define a boundary of a complete simply connected Riemannian manifold M without focal points, and a topology of the closure $\bar{M}$ based on each point of M. The topology is independent of the points in the case where $\dim M = 2$. But the author does not know whether it is true in the higher dimensional case. For this purpose one would need to establish a sort of a uniformity condition. We consider this problem again in § 8.

This paper was written as a portion of author's dictoral thesis, submitted to the University of Pennsylvania. The author would like to express her hearty thanks to Professor F. Warner, her thesis advisor, for many valuable discussions and comments. The author would also like to thank Professors E. Calabi, L. Green and M. Goto with whom she had helpful conversations.

Lastly, L. Green has informed us of the results by R. Gulliver, which provide examples of a manifold without conjugate or focal points but with sectional curvatures of both signs (see Trans. Amer. Math. Soc. **210** (1975) 185–201).

1. Matrix Jacobi differential equation

In this section we shall introduce the basic results on Jacobi differential equations due to L. Green. Because his proofs are rather concise, we shall give the details here for later use.

Consider the $n \times n$ matrix differential equation in one variable

$$X''(t) + K(t)X(t) = 0 , \tag{J}$$

where $K(t)$ is a continuous $n \times n$ symmetric matrix valued function, and derivatives are taken componentwise.

For two solutions X, Y of (J) we define the Wronskian: $W(X, Y) = X^{*\prime}Y - X^*Y'$, where * denotes the transpose operation. From the definition $W(X, Y) = -W(Y, X)^*$.

(i) *For two solutions X and Y of* (J)

(1) $W(X, Y) =$ *a constant matrix*;

(2) *if* $X(c) = Y(c) = 0$ *or* $X'(c) = Y'(c) = 0$ *for some* $c \in \boldsymbol{R}$, *then* $W(X, Y) = 0$;

(3) *if* $W(X, X) = 0$ *and* $\det X \neq 0$, *then* $X'X^{-1}$ *is symmetric.*

Proof. (1) From $X''^* + X^*K = 0$ and $Y'' + KY = 0$ we have $(W(X, Y))' = X''^*Y - X^*Y'' = -X^*KY + X^*KY = 0$.

(2) and (3) follow from the definition. q.e.d.

From now on we shall assume the following condition (condition that (J) be free from conjugate points):

(C) If a nonzero solution X of (J) vanishes at some point, say c, then $\det X(t) \neq 0$ for $t \neq c$.

Under (C), a solution of (J) is uniquely determined by its values at any two distinct points.

Let A denote the solution of (J) with $A(0) = 0$, $A'(0) = I$, the identity matrix. By (C), $\det A(t) \neq 0$ for $t \neq 0$, and $A'(t)A^{-1}(t)$ is a symmetric matrix using (i) (2) and (3).

Next, we construct another solution $B(t)$ of (J), which is useful for later purposes, as a limit of solutions $B_c(t)$, $0 < c < \infty$. Let c be any positive constant; we denote by $B_c(t)$ the solution of (J) with $B_c(c) = 0$, $B_c'(c) = -A^{*-1}(c)$.

(ii) *In* $(0, \infty)$, $B_c(t)$ *can be written as*

$$(*) \qquad B_c(t) = A(t)\int_t^c A^{-1}(s)A^{-1}(s)^* ds ,$$

and satisfies $B_c(0) = I$, $W(A, B_c) = I$.

Proof. Let $D(t)$ denote the right side of $(*)$. Then setting

$$P(t) = \int_t^c A^{-1}(s)A^{-1}(s)^* ds$$

and using that $(A^{-1})' = -A^{-1}A'A^{-1}$, we obtain

$$D' = A'P - A(A^{-1}A^{-1*}) = A'P - A^{-1*} ,$$
$$D'' = A''P - A'A^{-1}A^{-1*} + A^{-1*}A'^*A^{-1*} .$$

On the other hand, since $W(A, A) = 0$, $A'A^{-1}$ is symmetric by (i) so $D'' = A''P$. Hence $D'' + KD = A''P + KAP = 0$, and D is a solution defined in $(0, \infty)$. Since $P(c) = 0$, we have $D(c) = 0$ and $D'(c) = -A^{-1*}(c)$. Hence $D = B_c$ on $(0, \infty)$.

Next, $W(A, B_c) = (A'^*D - A^*D')(c) = -A^*(c)D'(c) = I$, and by substituting $t = 0$ we have

$$B_c(0) = A'^*(0)B_c(0) - A^*(0)D'(0) = I . \qquad \text{q.e.d.}$$

Let $\mathscr{S}$ denote the totality of positive definite symmetric matrices in $GL(n, \boldsymbol{R})$. We define a partial ordering in $\mathscr{S}$: $S_1 \succ S_2$ if $S_1 - S_2 \in \mathscr{S}$. Then $S_1 \succ S_2$ if and only if $\langle x, S_1 x\rangle > \langle x, S_2 x\rangle$ for each $x \in \boldsymbol{R}^n - \{0\}$. For any S in $GL(n, \boldsymbol{R})$, clearly SS^*, $S^*S \in \mathscr{S}$.

Let $0 < d < c$. Then

$$B_c(t) - B_d(t) = A(t)\int_d^c A^{-1}(s)A^{-1}(s)^* ds\ .$$

From $A^{-1}(s)A^{-1}(s)^* \in \mathscr{S}$, it follows that

$$\int_d^c A^{-1}(s)A^{-1}(s)^* ds \in \mathscr{S}\ .$$

Since $A(t)(B_c'(0) - B_d'(0))$ is the solution of (J) with the initial condition $(0, B_c'(0) - B_d'(0))$ at 0, we have

$$B_c(t) - B_d(t) = A(t)(B_c'(0) - B_d'(0))\ ,$$

$$B_c'(0) - B_d'(0) = \int_d^c A^{-1}(s)A^{-1}(s)^* ds \in \mathscr{S}\ .$$

A function F from an interval in $\boldsymbol{R}$ to $\mathscr{S}$ is called (strictly monotone) increasing (resp. decreasing) if $F(t_1) \succ F(t_2)$ (resp. $F(t_1) \prec F(t_2)$) whenever $t_1 > t_2$.

(iii) *$(1, \infty) \ni c \to B_c'(0) - B_1'(0) \in \mathscr{S}$ is increasing, and there exists $U \in \mathscr{S}$ with $U \succ B_c'(0) - B_1'(0)$ for each $c > 1$.*

Proof. From the observation above it is clear that $c \to B_c'(0) - B_1'(0)$ is increasing. For $c > 0$ we set

$$(\sharp) \qquad B_{-1}(t) = A(t)N_c + B_c(t)\ , \qquad N_c = -A^{-1}(-1)B_c(-1)\ .$$

Then B_{-1} is a solution of (J) with $B_{-1}(0)(= B_c(0)) = I$ and $B_{-1}(-1) = 0$, and is independent of the choice of c by (C). By (i), $W(B_{-1}, B_{-1}) = W(A, A) = W(B_c, B_c) = 0$. Also we know that $W(A, B_c) = I = -W(B_c, A)$. Hence

$$\begin{aligned} 0 &= W(B_{-1}, B_{-1}) \\ &= N_c^* W(A, A)N_c + W(B_c, B_c) + N_c^* W(A, B_c) + W(B_c, A)N_c\ , \end{aligned}$$

and so $N_c^* = N_c$, i.e., N_c is symmetric.

Differentiating ($\sharp$) at $t = 0$ we have

$$N_c = B_{-1}'(0) - B_c'(0)\ .$$

Substituting $t = c$ in ($\sharp$) gives $B_{-1}(c) = A(c)N_c$. Indeed, $N_t^{-1} = B_{-1}^{-1}(t)A(t)$ for $t > 0$ since c is arbitrary.

Consider $E(t) = B_{-1}^{-1}(t)A(t)$ for $t \geq 0$, then E is a symmetric matrix valued function, and $E(0) = 0$, $E'(0) = B_{-1}^{-1}(0)A'(0) = I$. For any unit vector x,

$\langle E(t)x, x\rangle$ has the derivative $\langle x, x\rangle = 1$ at $t = 0$. Hence $\langle E(t)x, x\rangle > 0$ for small $t > 0$. Since the unit sphere is compact, we can find $\varepsilon > 0$ such that $\langle E(t)x, x\rangle > 0$ for $t \in (0, \varepsilon)$ and for all unit vectors x. Therefore $E(t)$ is positive definite for $0 < t < \varepsilon$. Since N_t^{-1} is symmetric and nonsingular for all $t > 0$, we have $N_t^{-1} = E(t)$ is positive definite for all $t > 0$, and hence $N_t \in \mathscr{S}$ for $t > 0$.

Since $N_c = B'_{-1}(0) - B'_c(0) \succ 0$, it follows that

$$B'_{-1}(0) - B'_1(0) \succ B'_c(0) - B'_1(0) \qquad \text{for } c > 1 .$$

(iv) (1) *$\lim_{c\to\infty} (B'_c(0) - B'_1(0)) = Q$ exists.*

(2) *$\lim_{c\to\infty} B_c(t) = B(t)$ exists (uniformly on any compact set). $B(t)$ is a solution of* (J) *with the initial condition*

$$B(0) = I , \qquad B'(0) = Q + B'_1(0) .$$

Proof. (1) follows from (iii).

(2) By (1) $\lim_{c\to\infty} B'_c(0) = Q + B'_1(0)$. Let $B_\infty(t)$ be the solution of (J) with $B_\infty(0) = I$, $B'_\infty(0) = Q + B'_1(0)$. Then $[1, \infty] \ni c \to B_c$ is continuous and hence $\lim_{c\to\infty} B_c(t) = B_\infty(t)$ uniformly on any compact set.

2. Divergence theorem

We assume that M is a complete Riemannian manifold of dimension $n + 1$ without conjugate points. Let γ be a maximal geodesic in M, and let $e_1(t), \cdots, e_{n+1}(t)$ be a system of parallel orthonormal vector fields along γ with $e_{n+1} = \gamma'$. If $Y(t) = \sum_{i=1}^n y_i(t)e_i(t)$ is a perpendicular vector field on γ, we may identify Y with the curve $t \to (y_1(t), \cdots, y_n(t))$ in $\boldsymbol{R}^n$. The covariant derivative $Y'(t) = \sum_{i=1}^n y'_i(t)e_i(t)$ is identified with the curve $t \to (y'_1(t), \cdots, y'_n(t))$ in $\boldsymbol{R}^n$. Conversely, any curve $t \to (y_1(t), \cdots, y_n(t))$ in $\boldsymbol{R}^n$ defines a perpendicular vector field on γ. For each $t \in \boldsymbol{R}$ let $K(t) = (R_{ij}(t))$, where $R_{ij}(t) = \langle R(e_i(t), \gamma'(t))\gamma'(t), e_j(t)\rangle$, $1 \le i, j \le n$, and R denotes the curvature tensor of M. $K(t)$ is a symmetric matrix. Consider the $n \times n$ matrix Jacobi differential equation (J) in § 1. If $X(t)$ is a solution of (J), then for any $x \in \boldsymbol{R}^n$ the curve $t \to X(t)x$ corresponds to a perpendicular Jacobi field along γ. If $A(t)$ is the solution of (J) with $A(0) = 0$, $A'(0) = I$, then a perpendicular Jaboci field $Y(t)$ along γ with $Y(0) = 0$, $Y'(0) \neq 0$ is expressed by $Y(t) = A(t)Y'(0)$. Since M has no conjugate points, $A(t)$ is nonsingular for $t \neq 0$.

The following theorem was proved by L. Green under the condition that the sectional curvature of M is bounded below, see [6], [7] and [1, p. 168]. When the sectional curvature of M is nonpositive, the conclusion is known to be true.

Theorem 1. *Let M be a complete Riemannian manifold without focal points. Let p be a point in M, and γ a geodesic ray with $\gamma(0) = p$. If $Y^x(t)$ denotes the*

Jacobi field along γ *with* $Y^x(0) = 0$, $(Y^x)'(0) = x$, *where x is a nonzero vector at p, then*

$$\lim_{t\to\infty} |Y^x(t)| = \infty$$

uniformly on $\{x \in T_pM; |x| = 1\}$.

For $S \in GL(n, \boldsymbol{R})$, let $\|S\| = \operatorname{Max}_{|x|=1}|Sx|$. We denote by $\mathscr{S}$ the totality of positive definite symmetric matrices in $GL(n, \boldsymbol{R})$.

Lemma 1. *Let* $S \in GL(n, \boldsymbol{R})$ *and* $\sigma = \operatorname{Min}_{|x|=1}|Sx|$. *Then* σ^2 *is the minimum eigenvalue of* $S^*S \in \mathscr{S}$ *and* $\|S^{-1}S^{*-1}\| = 1/\sigma^2$.

Proof. Let $T \in \mathscr{S}$ with eigenvalues $\alpha_1 \geq \cdots \geq \alpha_n > 0$. Then by a suitable orthogonal change of coordinates we may assume that $T = \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix}$. For $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\langle x, Tx \rangle = x^*Tx = \sum_{i=1}^n \alpha_i x_i^2$, and $\alpha_1\langle x, x\rangle = \alpha_1 \sum_{i=1}^n x_i^2 \geq \langle x, Tx\rangle \geq \alpha_n \sum_{i=1}^n x_i^2 = \alpha_n\langle x, x\rangle$. Therefore

$$\operatorname{Max}_{|x|=1} \langle x, Tx\rangle = \alpha_1 = \|T\| \quad \text{and} \quad \operatorname{Min}_{|x|=1} \langle x, Tx\rangle = \alpha_n .$$

On the other hand, putting $S^*S = T$, where $S \in GL(n, \boldsymbol{R})$, we have $\langle x, Tx\rangle = \langle x, S^*Sx\rangle = \langle Sx, Sx\rangle = |Sx|^2$. Since for

$$T = \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix}, \qquad T^{-1} = \begin{pmatrix} 1/\alpha_1 & & 0 \\ & \ddots & \\ 0 & & 1/\alpha_n \end{pmatrix} = S^{-1}S^{*-1} ,$$

the proof is completed.

Proof of Theorem 1. Let $Y(t)$ be a nontrivial Jacobi field with $Y(0) = 0$ along a geodesic ray γ. Note that any Jacobi field Z along γ can be decomposed as follows; $Z = (at + b)\gamma' + Z^\perp$, where a, b are real numbers, and $Z^\perp$ is a perpendicular Jacobi field along γ. Since $Y(0) = 0$, we have $Y(t) = at\gamma'(t) + Y^\perp(t)$ and $|Y(t)|^2 = a^2t^2 + |Y^\perp|^2$. Hence it suffices to prove the theorem in two cases separately.

Suppose Y is tangent to γ, i.e., $Y = at\gamma'$ for a determined by $Y'(0) = a\gamma'(0)$. Then $|Y(t)| = |a|\, t$, and therefore the conclusion is clear.

Next, we suppose Y is perpendicular to γ. Let $A(t)$ be the solution of (J) in § 1 with $A(0) = 0$, $A'(0) = I$. Then $Y(t) = A(t)Y'(0)$. We shall prove that $\lim_{t\to\infty} |A(t)x| = \infty$ for any $x \in \boldsymbol{R}^n - \{0\}$. Let $x \in \boldsymbol{R}^n - \{0\}$. By the property that M has no focal points, $|A(t)x|$ is strictly monotone increasing in t. Set $F(t) = A^*(t)A(t)$. Then it follows that $\langle F(t)x, x\rangle = |A(t)x|^2$ is strictly monotone increasing in t, and hence $F'(t) \in \mathscr{S}$ for all $t > 0$. Therefore $\langle F^{-1}(t)x, x\rangle$ is strictly monotone decreasing in t. In fact, $(F^{-1})' = -F^{-1}F'F^{-1}$ and so

$$\langle (F^{-1})'x, x\rangle = -\langle F^{-1}F'F^{-1}x, x\rangle = -\langle F'(F^{-1}x), F^{-1}x\rangle .$$

Since by § 1 (iii),

$$\int_1^\infty A^{-1}(t)A^{*-1}(t)dt$$

exists, so does

$$\int_1^\infty \langle A^{-1}(t)A^{*-1}(t)x, x\rangle dt .$$

This, together with the argument above, yields $\lim_{t\to\infty}\langle A^{-1}(t)A^{*-1}(t)x, x\rangle = 0$. By Lemma 1, $\|A^{-1}(t)A^{*-1}(t)\| = 1/\sigma(t)^2$, where $\sigma(t) = \mathrm{Min}_{|x|=1}|A(t)x|$. Therefore $\lim_{t\to\infty} 1/\sigma(t)^2 = 0$, and $\lim_{t\to\infty}\sigma(t) = \infty$. Thus $\lim_{t\to\infty}|A(t)x| = \infty$ uniformly on $\{x \in T_pM; |x| = 1\}$.

Corollary. *Let M be a complete Riemannian manifold without focal points. Let Y be a Jacobi field along a geodesic ray γ in M with $Y(0) \neq 0$. We denote by $Y_u, u > 0$, the Jacobi field along γ such that $Y_u(0) = Y(0)$ and $Y_u(u) = 0$. Then $|Y(t)|$ is bounded from above for $t \geq 0$ if and only if $Y = \lim_{u\to\infty} Y_u$.*

Proof. If Y is a limit of $\{Y_u\}$ as $u \to \infty$, $|Y(t)|$ is clearly bounded from above by $|Y(0)|$ for $t \geq 0$ since $|Y_u(t)|$ is strictly monotone decreasing in $t \in [0, u]$ for each $u > 0$.

Conversely, suppose that $|Y(t)|$ is bounded from above for $t \geq 0$. Let $X(t)$ be a limit Jacobi field along γ of $\{Y_u\}$, as $u \to \infty$. Then Y-X is a Jacobi field along γ with $(Y-X)(0) = 0$. If $Y \neq X$, $\lim_{t\to\infty}|Y-X| = \infty$ by the theorem. But $|Y - X| \leq |Y| + |X|$, and $|Y|, |X|$ are bounded above, a contradiction. Thus $Y = X$.

3. A Riemannian geometry on T_pM

Let M be a complete simply connected Riemannian manifold without focal points, and let $p \in M$. Using the technique of Kobayashi in [10], we shall modify the Euclidean metric of $T_p = T_pM$ slightly so that T_p is complete and $\exp = \exp_p\colon T_p \to M$ is distance-increasing.

For $w \in T_p$, let $\mathscr{F}_w\colon T_p \to T_w(T_p)$ be the canonical isomorphism. Let S_p denote the unit sphere with center at the origin. We define a map H by $H(w) = \inf\{|(d\exp)_w \mathscr{F}_w x|; x \in S_p\}$. By the continuity of eigenvalues, H is continuous in w. Since S_p is compact,

$$h(t) = \mathrm{Min}\,\{H(tv); v \in S_p\}$$

exists for $t \geq 0$. Since M has no focal points and $t \to (d\exp)_{tv}\mathscr{F}_{tv}(tx) = t(d\exp)_{tv}\mathscr{F}_{tv}(x) = Y^x(t)$ is a Jacobi field with $Y^x(0) = 0$ and $(Y^x)'(0) = x$, $th(t)$ is nondecreasing and $\lim_{t\to\infty} th(t) = \infty$, by Theorem 1. The following lemma is obvious.

Lemma 2. *Let g be a real-valued, nondecreasing function on* $[0, \infty)$ *such that* $g(0) = 0$, $g(t) > 0$ *for* $t > 0$ *and* $\lim_{t\to\infty} g(t) = \infty$. *Then we can find a nondecreasing function f, C^∞ on* $(0, \infty)$, C^1 *at* $t = 0$, *such that*

$$f(0) = 0\ , \qquad 0 < f(t) < g(t) \qquad \textit{for } t > 0,$$
$$\lim_{t\to\infty} f(t) = \infty\ .$$

Let $f(t)$ be the function obtained from Lemma 2 for our function $th(t)$.

To $tv \in T_p$ with $t \geq 0$, $v \in S_p$, we associate $(t, v) \in [0, \infty) \times S_p$ and call (t, v) the polar coordinate of tv. The Euclidean metric in T_p is given by

$$(dt)^2 + t^2(d\nu)^2\ ,$$

where $(d\nu)^2$ is the ordinary Riemannian metric of the unit sphere S_p. Using the function $f(t)$, we define a new metric $(d\sigma)^2$ in T_p in terms of the polar coordinate system by

$$(d\sigma)^2 = (dt)^2 + f(t)^2(d\nu)^2\ .$$

Let us denote by the same notation T_p the manifold T_p equipped with $(d\sigma)^2$. Then we can show that exp is distance-increasing as in Lemma 3 in [10].

T_p is complete, and in T_p, each geodesic sphere $S(t)$ of radius $t > 0$, centered at 0, has constant curvature $f(t)^{-1}$, which is nonincreasing in t, and approaches 0 as $t \to \infty$. Let d denote the distance function in T_p.

Proposition 1. *Let α, β, γ be distinct rays in T_p starting from* 0, *parametrized by their arc-lengths. Then*:

(1) $\lim_{t\to\infty} d(\alpha(t), \beta(t)) = \infty$,

(2) $\lim_{t\to\infty} d(\alpha(t), \beta) = \infty$,

(3) *if* $\sphericalangle(\alpha'(0), \beta'(0)) = \sphericalangle(\alpha'(0), \gamma'(0))$, *we have* $d(\alpha(t), \beta) = d(\alpha(t), \gamma)$.

Proof. (1) Suppose it were not true. Then we can find a sequence $\{t_j\}$ with $\lim t_j = \infty$, and a constant $c > 0$ such that $d(\alpha(t_j), \beta(t_j)) < c$ for $j = 1, 2, \cdots$. For $v, w \in S(t)$, let $\widehat{vw}$ denote the distance between v and w on $S(t)$. Then $\widehat{\alpha(t)\beta(t)} = (f(t)/f(1))\cdot\widehat{\alpha(1)\beta(1)}$. Since $\lim_{t\to\infty} f(t) = \infty$ we can find a number $N > 0$ such that

$$\widehat{\alpha(t)\beta(t)} \geq c \qquad \text{for } t \geq N\ .$$

Pick t_j with $t_j > N + c$. Let $\lambda\colon [0, 1] \to T_p$ be a shortest geodesic segment connecting $\alpha(t_j)$ and $\beta(t_j)$. Since length$(\lambda) = d(\alpha(t_j), \beta(t_j)) < c$, λ lies outside of $S(N)$. Let $\tilde{\lambda}$ be the projection of λ into $S(N)$, i.e.,

$$\lambda(s) = (t(s), v(s)) \to \tilde{\lambda}(s) = (N, v(s))\ .$$

Since $t(s) > N$,

$$\int_0^1 \left|\frac{d\lambda}{ds}\right| ds = \int_0^1 \left\{\left|\frac{dt}{ds}\right|^2 + f(t(s))^2 \left|\frac{dv}{ds}\right|^2\right\}^{1/2} ds$$
$$\geq \int_0^1 f(t(s)) \left|\frac{dv}{ds}\right| ds$$
$$\geq \int_0^1 f(N) \left|\frac{dv}{ds}\right| ds = \text{length}\,(\tilde{\lambda}) \,.$$

On the other hand, length $(\tilde{\lambda}) \geq \widehat{\alpha(N)\beta(N)} \geq c$, which is a contradiction.

(2) Otherwise, we can find a number $c > 0$ and a sequence $\{t_j\}$ with $\lim t_j = \infty$, such that

$$d(\alpha(t_j), \beta) < c \qquad \text{for } j = 1, 2, \cdots \,.$$

We can find s_j with $d(\alpha(t_j), \beta(s_j)) = d(\alpha(t_j), \beta) < c$. By the triangle inequality, we have $|t_j - s_j| < c$. Then

$$d(\alpha(t_j), \beta(t_j)) \leq d(\alpha(t_j), \beta(s_j)) + d(\beta(s_j), \beta(t_j)) < 2c \,,$$

for all t_j, which is contrary to (1).

(3) Let $\mathcal{T}$ be the unit sphere in T_0T_p about 0. Let us put $\alpha'(0) = x$, $\beta'(0) = y$, and $\gamma'(0) = z$. Then $x, y, z \in \mathcal{T}$, and $\measuredangle(x, y) = \measuredangle(x, z)$ by the assumption. Hence we can find an isometry (orthogonal transformation) μ of $\mathcal{T}$ such that $\mu x = x$ and $\mu y = z$.

On the other hand, for an isometry μ of $\mathcal{T}$, the map $\tilde{\mu}$ defined by

$$\tilde{\mu}(\exp tv) = \exp t\mu(v) \qquad t \geq 0 \,,$$

for $v \in \mathcal{T}$, is an isometry of T_p by the definition of the metric $(d\sigma)^2$ in T_p, where exp denotes the exponential map: $T_0T_p \to T_p$. Hence

$$d(\alpha(t), \beta) = d(\tilde{\mu}\alpha(t), \tilde{\mu}\beta) = d(\alpha(t), \gamma) \,.$$

4. Geodesic instability

Two geodesic rays α, β are said to be *asymptotic* if there exists a number c, $0 < c < \infty$, such that $d(\alpha(t), \beta(t)) \leq c$ for all $t \geq 0$. In complete simply connected Riemannian manifolds of nonpositive curvature, any two distinct geodesic rays starting from a point cannot be asymptotic to each other. In manifolds without conjugate points L. Green obtained the same conclusion under the additional condition that the sectional curvature is bounded from below (see [6], [7] and also [1, p. 168]).

In this section we shall prove the following theorem.

Theorem 2. *Let M be a complete simply connected Riemannian manifold without focal points. Then any two distinct geodesic rays starting from any $p \in M$ cannot be asymptotic to each other.*

Proof. Let, α, β be distinct geodesic rays starting from p. We introduce the metric $(d\sigma)^2$ in T_pM as in § 3. Note that $\exp_p$ preserves the distance of radial direction. Let, $\tilde{\alpha}, \tilde{\beta}$ be rays in T_pM such that $\exp_p \alpha(t) = \alpha(t)$ and $\exp_p \tilde{\beta}(t) = \beta(t)$ for $t \geq 0$. We denote by $\tilde{d}$ the distance function in T_pM equipped with $(d\sigma)^2$. Because $\exp_p$ is distance-increasing, we have $\tilde{d}(\tilde{\alpha}(t), \tilde{\beta}(t)) \leq d(\alpha(t), \beta(t))$, by Proposition 1(1), $\lim_{t\to\infty} \tilde{d}(\tilde{\alpha}(t), \tilde{\beta}(t)) = \infty$. Therefore we have $\lim_{t\to\infty} d(\alpha(t), \beta(t)) = \infty$.

Proposition 2. *Let M be a complete simply connected Riemannian manifold without conjugate points. Then M has no focal points if and only if for every maximal geodesic γ of M and every $p \in M$ with $p \notin \gamma$, there exists only one geodesic from p to γ, which is perpendicular to γ.*

This proposition was proved by L. Green for dim $M = 2$, and by P. Eberlein for arbitrary dimension. We shall outline the proof. Suppose that the unique perpendicular property holds in M. By a first variation argument, the unique geodesic α from p to $q \in \gamma$, the nearest point to p, is perpendicular to γ. Suppose that $\alpha(0) = q$ and $\alpha(l) = p$ where $l = d(p, q)$. If there were a focal point $\alpha(a)$ of γ along α for some $a > 0$, then $d(\alpha(t), r) < t$ for any $t > a$. Therefore for any $t > a$ there exist two perpendiculars from $\alpha(t)$ to γ. Conversely, let M have no focal points, and γ, p as above. We assume that γ has unit speed and $\gamma(0)$ is a point on γ nearest to p. Let $g(s) = d^2(p, \gamma(s))$. Then $g'(s) = 2\langle dr(\partial/\partial t), dr(\partial/\partial s)\rangle (1, s)$, where $r\colon [0, 1] \times \boldsymbol{R} \to M$ is given by $r(t, s) = \exp_p tv(s)$ and $v(s) = \exp_p^{-1}\gamma(s)$. Hence the unique geodesic from p to $\gamma(s)$ is perpendicular to γ if and only if $g'(s) = 0$. By assumption, $g'(0) = 0$. Also, since M has no focal points, it is shown that $g''(s) > 0$. Therefore $g'(s) \neq 0$ if $s \neq 0$. Thus there is exactly one perpendicular from p to γ.

Using perpendiculars we can also restate the divergence property of intersecting geodesics as follows:

Theorem 2′. *Let, α, β be distinct geodesic rays starting from a point $p \in M$. Let $\gamma_t\colon [0, 1] \to M$ be the unique geodesic from $\alpha(t) = \gamma_t(0)$ to β, perpendicular to β. Then the length of γ_t is strictly monotone increasing in t and approaches ∞ as $t \to \infty$.*

Proof. For a fixed $u > 0$, we define a curve $\varphi_u\colon [0, 1] \to T_pM$ by $\varphi_u(s) = (\exp_p^{-1}\gamma_u(s))/|\exp_p^{-1}\gamma_u(s)|$, and a C^∞-variation $f_u\colon [0, \infty) \times [0, 1] \to M$ by

$$f_u(t, s) = \exp_p t(d(p, \gamma_u(s))/u)\varphi_u(s) .$$

For each fixed s, the variational vector field $Y_u(t, s) = df_u(\partial/\partial s)$ is a Jacobi field along the geodesic ray $t \to f_u(t, s)$, and $Y_u(0, s) = 0$. Since M has no focal points, $|Y_u(t, s)|$ is strictly monotone increasing in t. For $u_1 < u_2$, we put $\lambda(s) = f_{u_2}(u_1, s)$. Then $\lambda(0) = \alpha(u_1)$, and $\lambda(1)$ lies on β. Hence

$$\begin{aligned} \text{length}(\gamma_{u_1}) \leq \text{length}(\lambda) &= \int_0^1 |Y_{u_2}(u_1, s)|\, ds \\ &< \int_0^1 |Y_{u_2}(u_2, s)|\, ds = \text{length}(\gamma_{u_2}) . \end{aligned}$$

Therefore the length of γ_t is strictly monotone increasing in t. The rest of the proof follows from Proposition 1(2) and the fact that $\exp_p$ is distance-increasing after introducing the metric $(d\sigma)^2$ in T_pM as before.

5. Topological transitivity of certain geodesic flow

In this section, we shall give an application of our Theorem 2.

We specialize the manifold M to the unit open disk D in $\boldsymbol{R}^n$ endowed with a metric

$$(*) \qquad (ds)^2 = 4(f(x))^2 \sum_{i=1}^n dx_i dx_i / (1 - \sum_{i=1}^n x_i x_i)^2 ,$$

where f is a differentiable function in $D = M$ such that there exist constants a, b with $0 < a \leq f(x) \leq b$. Let $M(f)$ denote the Riemannian manifold thus obtained.

Let G be an isometry group of $M(f)$ with properties:

(1) G is properly discontinuous, has no fixed points, and ceases to be properly discontinuous at every point of the boundary ∂D of D;

(2) Each element of G leaves invariant the hyperbolic metric, the metric given by $(*)$ with $f(x) = 1$.

We denote by $M(f, G)$ the Riemannian manifold $M(f)/G$. Let us define the geodesic flow of $M(f, G)$. We denote by $SM = SM(f, G)$ the unit tangent bundle over $M(f, G)$. For $v \in S_pM = \{w \in T_pM; |w| = 1\}$, let $\gamma_v(t)$ denote the geodesic parametrized by arc-length with $\gamma_v(0) = p$ and $\gamma_v'(0) = v$. For a real number t, we define a map T_t by

$$T_t(v) = \gamma_v'(t) \in S_{\gamma_v(t)}M .$$

The one-parameter group T_t of transformations of SM is called the *geodesic flow* of $M(f, G)$ (or in SM). If there is $v \in SM$ such that the orbit $\{T_t(v); t \in (-\infty, \infty)\}$ is dense in SM, the geodesic flow is said to be *topologically transitive*.

The spaces $M(f, G)$ and their geodesic flows have been studied by M. Morse, G. Hedlund, W. R. Utz, and L. Green. Here we shall only give a remark that from the proof of Theorem 4 in [7] (cf. [1, p. 168]) and our Theorem 2, follows

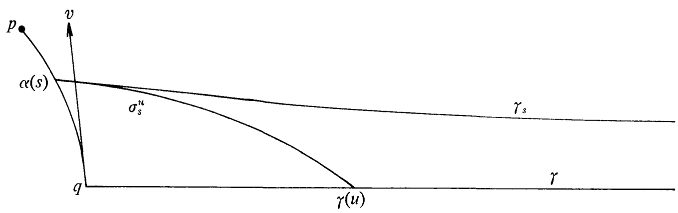
Theorem 3. *If $M(f)$ has no focal points, then the geodesic flow of $M(f, G)$ is topologically transitive.*

6. The existence of asymptote

Let M be a complete simply connected Riemannian manifold without focal points. Let $p \in M$ and let γ be a geodesic ray in M with $\gamma(0) = q \in M$. Let $v = \exp_q^{-1} p$ and let $\alpha(s) = \exp_q sv$. Our purpose is now to construct a geodesic asymptotic to γ passing through $\alpha(s)$ for each s in $(0, 1]$.

For $s \in [0, 1]$ and $u \geq 0$ we join $\alpha(s)$ and $\gamma(u)$ by the unique geodesic σ_s^u. σ_s^u are parametrized proportionally to arc-lengths, ranged in $[0, u]$. For a fixed u, we have a two-dimensional surface $\{\sigma_s^u(t); 0 \leq t \leq u, 0 \leq s \leq 1\}$. We put $f_u(t, s) = \sigma_s^u(t)$. Then $Y_u(t, s) = df_u(\partial/\partial s)(t, s)$ are Jacobi fields along σ_s^u vanishing at $t = u$. Since M has no focal points, $|Y_u(t, s)|$ is strictly monotone decreasing in $t \in [0, u]$. Therefore, for each $t \in (0, u)$,

$$d(q, \alpha(s_0)) = \int_0^{s_0} |Y_u(0, s)|\, ds > \int_0^{s_0} |Y_u(t, s)|\, ds \geq d(\sigma_{s_0}^u(t), \gamma(t)) .$$



Next, we fix $t > 0$ arbitrarily. We carry over the situation above to the limit as $u \to \infty$, where $u > t$. By the result of L. Green in § 1, there is a limit Jacobi field Y with $Y(0) = v$ of $\{Y_u(t, 0); u > t\}$ as $u \to \infty$. The differentiable variation of γ corresponding to Y would consist of limit geodesics γ_s of sequences of geodesics $\{\sigma_s^u; u > t\}$. Actually we shall prove the existence of differentiable variation of γ corresponding to Y. Let us start from some preparations.

Let M be a complete C^∞-Riemannian manifold and let $\pi: TM \to M$ be the projection. Let $w \in TM$. The kernel of $(d\pi)_w: T_w(TM) \to T_{\pi w}M$ is called the *vertical subspace* of $T_w(TM)$. *A connection map* $K: T(TM) \to TM$ is defined geometrically as follows: For $\xi \in T_w(TM)$, let $X: (-\delta, \delta) \to TM$ be a C^∞-curve with initial velocity ξ, and let $\alpha = \pi \cdot X$. Define $K(\xi) = X'(0)$, where $X'(0)$ denotes the covariant derivative of X along α evaluated at $t = 0$. Clearly, $K_w: T_w(TM) \to T_{\pi w}M$ is a linear map. The kernel of K_w is called the *horizontal subspace* of $T_w(TM)$. More detailed description of K is in [9, pp. 43–46].

We define an inner product $\langle\langle\, ,\, \rangle\rangle$ on TM by letting

$$\langle\langle \xi, \eta \rangle\rangle_w = \langle d\pi\xi, d\pi\eta \rangle_{\pi w} + \langle K\xi, K\eta \rangle_{\pi w}$$

for $\xi, \eta \in T_w(TM)$. Then the horizontal and vertical subspaces of $T_w(TM)$ are orthogonal to each other.

If $w \in TM$ and $\xi \in T_w(TM)$ are given, the Jacobi field Y along the maximal geodesic γ_w, defined by $\gamma_w(t) = \exp_{\pi w} tw$, with initial condition $Y(0) = d\pi\, \xi$ and $Y'(0) = K\xi$ (by interchanging the order of differentiation) is determined naturally, and vice versa. Therefore there is a linear isomorphism between $T_w(TM)$ and the set of all Jacobi fields along γ_w for any $w \in TM$.

Now we retain the notation in § 1. If we put $b = \|B'_{-1}(0) - B'_1(0)|$, then

$$\|B'_u(0) - B'_1(0)\| \leq b \qquad \text{for } u > 1 ,$$

by § 1 (iii), where $\|B\|$ denotes the operator norm (maximum eigenvalue of a positive definite symmetric matrix B).

We fix $u > 1$, and decompose $\alpha'(s)$ and $Y_u(t, s)$ into the tangential parts and the orthogonal parts to the geodesic σ_s^u, respectively:

$$\alpha'(s) = c \cdot (\sigma_s^u)'(0) + \alpha'(s)^\perp ,$$
$$Y_u(t, s) = c \cdot \left(1 - \frac{t}{u}\right) \cdot (\sigma_s^u)'(t) + Y_u^\perp(t, s) ,$$

where c is a constant. Since $(Y_u^\perp)'(0, s) = B_u'(0)\alpha'(s)^\perp$, we have

$$\begin{aligned} &| Y_u'(0, s) - Y_1'(0, s)|^2 \\ &\quad = |(B_u'(0) - B_1'(0))\alpha'(s)^\perp|^2 + \left|\left(1 - \frac{1}{u}\right)(\alpha'(s) - \alpha'(s)^\perp)\right|^2 \\ &\quad \leq b^2 |\alpha'(s)|^2 + |\alpha'(s)|^2 = (b^2 + 1) |v|^2 . \end{aligned}$$

Putting

$$a = \operatorname{Max} \{| Y_1'(0, s)| ; s \in [0, 1]\}/|v| = \|B_1'(0)\|$$

we have

$$\begin{aligned} | Y_u'(0, s) &\leq | Y_1'(0, s)| + | Y_u'(0, s) - Y_1'(0, s)| \\ &\leq (a + (b^2 + 1)^{1/2}) |v| . \end{aligned}$$

Let $w^u(s) = (\sigma_s^u)'(0)$. As we observed before, to the Jacobi field $Y_u(t, s)$ there corresponds a vector $\xi^u(s)$ in $T_{w^u(s)}(TM)$ uniquely, and $[0, 1] \ni s \to w^u(s)$ is a curve in TM, starting at v. Since $\xi^u(s)$ is the tangent vector of the curve $w^u(s)$, the length of the curve $w^u(s)$, $s \in [s_1, s_2]$, is given by

$$\begin{aligned} \int_{s_2}^{s_1} \langle\!\langle \xi^u(s), \xi^u(s) \rangle\!\rangle^{1/2} ds &= \int_{s_1}^{s_2} (|d\pi\xi^u(s)|^2 + |K\xi^u(s)|^2)^{1/2} ds \\ &= \int_{s_1}^{s_2} (| Y_u(0, s)|^2 + | Y_u'(0, s)|^2)^{1/2} ds \\ &\leq (s_2 - s_1)(1 + (a + (b^2 + 1)^{1/2})^2)^{1/2} |v| . \end{aligned}$$

Therefore, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|s_1 - s_2| < \delta$ implies distance $(w^u(s_1), w^u(s_2)) < \varepsilon$, for all $u > 1$. Namely, $\{w^u ; u > 1\}$ is a family of equicontinuous maps. Next, because

$$|w^u(s)| = d(\alpha(s), \gamma(u))/u \leq (u + d(p, q))/u < 1 + d(p, q) ,$$

w^u are considered as maps from $[0, 1]$ to the compact set $\{z \in TM ; \pi(z) \in \{\alpha(s); s \in [0, 1]\}$ and $|z| \leq 1 + d(p, q)\}$. By virtue of Ascoli's theorem, for each sequence $\{u_j\}$ in $(1, \infty)$, there is a subsequence $\{v_j\}$ so that $\{w^{v_j}\}$ converges to a continuous map w uniformly on $[0, 1]$.

We recall that $w^u(s)' = (\nabla/ds)w^u = Y_u'(0, s)$. Since $(Y_u'(0, s))^\perp = B_u'(0)(\alpha'(s))^\perp$, $|(\alpha'(s))^\perp| \leq |v|$, and

$$\lim_{u_1, u_2 \to \infty} \|B_{u_1}'(0) - B_{u_2}'(0)\| = 0$$

by § 1 (iv), $(Y_u'(0, s))^\perp$ converges uniformly on $[0, 1]$.

Nrxt, let $1 < u_1 < u_2 < \cdots$ be a sequence with $\lim_{j \to \infty} u_j = \infty$, and let $\{v_j\}$ be a subsequence of $\{u_j\}$ such that

$$\lim_{j \to \infty} w^{v_j} = w \qquad \text{uniformly} .$$

Then the tangential part of $Y_{v_j}'(0, s)$ to $w^{v_j}(s)$ converges uniformly. Therefore $\{w^{v_j}(s)'\}$ converges uniformly. The uniform convergences of $\{w^{v_j}\}$ and $\{w^{v_j\prime}\}$ implies easily that $\lim_{j \to \infty} w^{v_j}(s)' = w(s)'$, and in particular $w(s)$ is of class C^1.

Let $f(t, s) = \gamma_s(t) = \exp_{\alpha(s)} tw(s)$, and $Y(t, s) = df(\partial/\partial s)(t, s)$. The function $f(t, s)$ is of class at least C^1 in s and C^∞ in t. Since

$$1 - \frac{1}{u} d(q, \alpha(s)) \leq \frac{1}{u} d(\alpha(s), \gamma(u)) = |w^u(s)| \leq 1 + \frac{1}{u} d(q, \alpha(s)) ,$$

we have $|w(s)| = 1$, and we see that γ_s are parametrized by arc-lengths.

Now we are ready to prove the following theorem:

Theorem 4. *Let M be a complete simply connected Riemannian manifold without focal points. Given a geodesic ray γ and a point $p \in M$, there exists the unique geodesic ray asymptotic to γ passing through p.*

Proof. Let us fix $t > 0$. Then $Y_{v_j}(t, s)$ converges uniformly to $Y(t, s)$. Also $|Y_u(t, s)| \leq |Y_u(0, s)| = |\alpha'(s)| = |v|$. Hence by Lebesque convergence theorem we have

$$\int_0^{s_0} |Y(t, s)| \, ds = \lim_{j = \infty} \int_0^{s_0} |Y_{v_j}(t, s)| ds .$$

for each $s_0 \in [0, 1]$. Therefore

$$\begin{aligned} d(q, \alpha(s_0)) &= \int_0^{s_0} |Y(0, s)| \, ds = \lim_{j \to \infty} \int_0^{s_0} |Y_{v_j}(0, s)| \, ds \\ &\geq \lim_{j \to \infty} \int_0^{s_0} |Y_{v_j}(t, s)| \, ds = \int_0^{s_0} |Y(t, s)| \, ds \\ &\geq d(\gamma(t), \gamma_{s_0}(t)) . \end{aligned}$$

Thus γ_s is asymptotic to γ for all $s \in [0, 1]$.

The uniqueness follows from Theorem 2.

Remark. By the uniqueness of $w(s)$, we have

$$\lim_{u\to\infty} w^u(s) = w(s) , \qquad \lim_{u\to\infty} w^u(s)' = w(s)'$$

uniformly in s.

From Theorem 4 and its proof we easily have

Corollary 1. *Let $\alpha(t)$ and $\beta(t)$, $t \in [0, \infty)$, be geodesic rays asymptotic to each other. Then $d(\alpha(t), \beta(t))$ is non increasing in t.*

Corollary 2. *Let p be a point in M, and α a geodesic ray. For $t \in [0, \infty)$, let $\gamma_{p\alpha(t)}$ denote the geodesic ray starting from p and passing through $\gamma(t)$. If $\gamma_{p\alpha(\infty)}$ denotes the unique geodesic ray from p, asymptotic to α, then*

$$\lim_{t\to\infty} \gamma'_{p\alpha(t)}(0) = \gamma'_{p\alpha(\infty)}(0) .$$

7. A boundary of M and a topology on $\bar{M}$

Let M be a complete simply connected Riemannian manifold without focal points. A *point at infinity* is defined as an asymptote class of (oriented) maximal geodesics of M. Let $M(\infty)$ denote the totality of points at infinity of M, and let $\bar{M}$ be the set theoretic union of M and $M(\infty)$. We shall call $\bar{M}$ the *closure* of M.

Let $\alpha:(-\infty, \infty) \to M$ be a geodesic. Let $\alpha(\infty)$ be the asymptote class of α and let $\alpha(-\infty)$ be that of the reversed geodesic: $t \to \alpha(-t)$. The resulting map $\alpha: [-\infty, \infty] \to \bar{M}$ is called the *asymptotic extension* of α.

Now we would like to construct a suitable topology of $\bar{M}$ which makes $\bar{M}$ homeomorphic to the closed unit disk in $\boldsymbol{R}^n$, $n = \dim M$. Let $D_p = \{v \in T_pM;$ $|v| < 1\}$, $p \in M$.

We pick a point $p \in M$ arbitrarily and fix it. We choose a homeomorphism r: $[0, 1] \to [0, \infty]$, $r(0) = 0$, and define a map ψ_p: $\bar{D}_p \to \bar{M}$ by $\psi_p(v) = \exp_p(r(|v|)/|v|)v$ for $v \neq 0$ and $\psi_p(0) = p$, where $\exp_p(\infty v)$ denotes the asymptote class containing the geodesic $t \to \exp_p tv$. By Theorem 4 in § 6, ψ_p is one-to-one and onto. Hence we can define a topology of $\bar{M}$ from that of $\bar{D}_p$ via the map ψ_p. Obviously, the relative topology of M in $\bar{M}$ coincides with the original one, and any asymptotic extension of a maximal geodesic passing through p is continuous.

Next we compare the topologies thus obtained based on different points.

Proposition 3. *Let p, q be distinct points in M. Let $\Phi = \psi_q^{-1}\cdot\psi_p$: $\bar{D}_p \to \bar{D}_q$. Then*

(1) *Φ is one to one and onto,*

(2) *the restriction of Φ into D_p is a homeomorphism,*

(3) $\lim_{t\to 1} \Phi(tv) = \Phi(v)$ *for $v \in S_p$.*

(3) *implies that an asymptotic extension of a maximal geodesic is continuous with respect to any topology defined above.*

Proof. (1) and (2) are obvious. (3) follows from Corollary 2 of Theorem 4. q.e.d.

That Φ is a homeomorphism is known for M with nonpositive curvature. Here we shall give a proof for the two-dimensional case.

Theorem 5. *Let M be a complete simply connected Riemannian manifold of two dimensions without focal points. Then the topology of $\bar{M}$ constructed above is independent of the choice of p.*

The author does not know when M satisfies the conclusion of Theorem 5 for higher dimensional case. (We can construct an example which shows that the conclusion of Theorem 5 does not hold for higher dimensional case even if Φ satisfies (1), (2) and (3) in Proposition 3.)

Proof. Let x_0 and x_1 be distinct points of S_p, and let $\alpha: [0, 1] \to S_p$ be one of the arcs of S_p from $x_0 = \alpha(0)$ to $x_1 = \alpha(1)$. Let $\beta: [0, 1] \to S_q$ be the arc of S_q from $\Phi(x_0) = \beta(0)$ to $\Phi(x_1) = \beta(1)$ passing through $\Phi(\alpha(\frac{1}{2}))$. We suppose that α and β are parametrized proportionally to arc-lengths. Let γ be the curve in $\bar{D}_p$ given by

$$\gamma(t) = \begin{cases} (1 - 2t)x_0 , & 0 \leq t \leq \frac{1}{2} , \\ (2t - 1) , & \frac{1}{2} \leq t \leq 1 . \end{cases}$$

By (3), $\delta(t) = \Phi(\gamma(t))$ is a Jordan curve in $\bar{D}_q$ with $\delta(0) = \Phi(x_0)$ and $\delta(1) = \Phi(x_1)$. Using Jordan's curve theorem we see that δ divides $\bar{D}_q$ into two connected components C and C'; $\bar{D}_q = \delta \cup C \cup C'$ (disjoint union). Since $\beta^0 = \{\beta(s); 0 < s < 1\}$ is connected and does not intersect with δ, β^0 is contained in one of the connected components, say C. Then C is bounded by the curves β and δ. Let β' be the complementary arc to β in S_q. Then C' is bounded by the curves β' and δ.

We put $A = \{t\alpha(s);\ t \in (0, 1],\ s \in (0, 1)\}$. Then $\Phi(A)$ is connected and $\Phi(A) \ni \Phi(\alpha(\frac{1}{2}))$. Hence $\Phi(A) \subset C$. Since $C \cap S_q = \beta^0$, we can find a function f; $[0, 1] \to [0, 1]$ such that $\Phi(\alpha(s)) = \beta(f(s))$.

Let $0 \leq s_1 < s_2 \leq 1$. In a similar way as above, we have $f(s_2) \in (f(s_1), 1]$, that is, f is monotone increasing.

Let $\alpha': [0, 1] \to S_p$ be the arc of S_p complementary to α, with $\alpha'(0) = x_0$ and $\alpha'(1) = x_1$. Put $A' = \{t\alpha'(s); t \in (0, 1], s \in (0, 1)\}$. Then $\bar{D}_q = \Phi(A) \cup \Phi(A') \cup \delta$ (disjoint union). Since $\Phi(A')$ is connected, we have $\Phi(A') \subset C'$, which implies that $\Phi(A) = C$, so that Φ induces a surjective map from α to β. Thus f is surjective, and is a homeomorphism. Therefore $\Phi|_{S_p}$ is a homeomorphism.

Next we shall prove the compatibility of the homeomorphism $\Phi|_{D_p}$ and the homeomorphism from S_p onto S_q obtained above. Let $y_i \in S_p$, $t_i \in [0, 1]$ and $\lim_{i\to\infty} t_i y_i = y \in \alpha$. Then $\lim_{i\to\infty} |\Phi(t_i y_i)| = 1$ and $\lim_{i\to\infty} \Phi(y_i) = \Phi(y)$. Suppose that $\Phi(y) = \beta(u)$, $0 < u < 1$. Let ε be a positive number such that $0 < u - \varepsilon$, $u + \varepsilon < 1$. We put $f^{-1}(u - \varepsilon) = s_1$ and $f^{-1}(u + \varepsilon) = s_2$, and set $B = \{t\alpha(s); 0 \leq t \leq 1, s_1 \leq s \leq s_2\}$. Then $\Phi(B_\varepsilon)$ is a closed set bounded by the curves $\{t\Phi(\alpha(s_1)); t \in [0, 1]\}$, $\{t\Phi(\alpha(s_2)); t \in [0, 1]\}$ and $\{\beta(s); s \in [u - \varepsilon, u + \varepsilon]\}$. Moreover $\Phi(B_\varepsilon)$ contains almost all $\Phi(t_i y_i)$.

Suppose that some subsequence of $\{\Phi(t_i y_i)\}$ converges to $z \in \bar{D}_q$. Since $\lim_{i\to\infty} |\Phi(t_i y_i)| = 1$, we have $z \in S_q$. Hence $z \in \Phi(B_\varepsilon) \cap S_q = \{\beta(s); s \in [u - \varepsilon, u + \varepsilon]\}$ for any sufficiently small ε. Therefore $z = \beta(u) = \Phi(y)$.

8. Continuation

Let M be a complete Riemannian manifold of dimension $n + 1$ without focal points. Here we recall the metric of $T(SM)$ defined in § 6. Let $\{T_t\}$ be the geodesic flow of M. $\{T_t\}$ is said to be *of Anosov type* if it satisfies the condition:

For each $v \in SM$

$$T_v(SM) = X_s^*(v) + X_u^*(v) + \boldsymbol{R}V(v) ,$$

where V is the flow vector field, and there exist positive numbers a, b, c such that

(i) for any $\xi \in X_s^*(v)$

$$\|dT_t\xi\| \leq ae^{-ct} \|\xi\| \text{ for } t \geq 0, \geq be^{-ct} \|\xi\| \text{ for } t \leq 0 ,$$

(ii) for any $\eta \in X_u^*(v)$

$$\|dT_t\eta\| \leq ae^{ct} \|\eta\| \text{ for } t \leq 0, \geq be^{ct} \|\eta\| \text{ for } t \geq 0 ,$$

(iii) $\dim X_s^*(v) = \dim X_u^*(v) = n$.

A more general definition is given in [4].

We denote by γ_v the geodesic starting at $\pi(v)$ in the direction of v, where $\pi: SM \to M$ is the projection. We set

$$X_s(v) = \{\xi \in T_v(SM); \langle\langle \xi, V(v)\rangle\rangle = 0, \xi_u \to \xi \text{ as } u \to \infty\} ,$$
$$X_u(v) = \{\xi \in T_v(SM); \langle\langle \xi, V(v)\rangle\rangle = 0, \xi_u \to \xi \text{ as } u \to -\infty\}$$

where ξ_u, $u > 0$, denotes the unique vector corresponding to the Jacobi field Y_u along the geodesic γ_v such that $Y_u(u) = 0$, $Y_u(0) = d\pi\, \xi$. It is known that $X_s(v)$, $X_u(v)$ are vector subspaces of $T_v(SM)$ and $\dim X_s(v) = \dim X_u(v) = n$, see [4].

If $\xi \in X_s^*(v)$, the corresponding Jacobi field Y along γ_v is bounded for $t \geq 0$ by the definition. Therefore $\xi \in X_s(v)$ by Corollary of Theorem 1. Thus $X_s(v) = X_s^*(v)$, and $X_u(v) = X_u^*(v)$ for each $v \in SM$. Also $X_s(v) \cap X_u(v) = 0$. Consequently, when the geodesic flow is of Anosov type, it follows that M admits no nontrivial perpendicular Jacobi field Y along a maximal geodesic such that $|Y(t)|$ is bounded above for all $t \in \boldsymbol{R}$.

From now on we further suppose that M is simply connected and that the geodesic flow of M is of Anosov type.

Lemma 1. *Let p, $q \in M$. Given $\varepsilon > 0$, there exists a number $t_0 > 0$ such that*

$$d(\alpha(t), \beta(t)) < \varepsilon \qquad \text{for } t > t_0 ,$$

where, α, β are geodesic rays starting from p, q (resp.) asymptotic to each other.

Proof. We may assume that $p \neq q$. Join p and q by the geodesic γ. We construct a differentiable 2-dimensional surface $f(t, s)$ consisting of geodesic rays asymptotic to β passing through each point of γ as in § 6. Let $Y(t, s)$ be the associated Jacobi fields. Then using the notation in § 1

$$\begin{aligned} |Y(t, s)| &\leq ae^{-ct}(|Y(0, s)|^2 + |Y'(0, s)|^2)^{1/2} \\ &\leq ae^{-ct}d(p, q)(2 + (|||Q||| + |||B_1'(0)|||)^2)^{1/2}, \end{aligned}$$

where $|||\cdot|||$ denotes the operator norm, and a, c are constants. Since

$$d(\alpha(t), \beta(t)) = \int_0^1 |Y(t, s)|\, ds ,$$

the proof is completed.

From Lemma 1 we immediately have

Lemma 2. *Let $p, q \in M$. We introduce the metric $(d\sigma)^2$, defined in § 3, in T_qM. If $\Phi\colon D_p \to D_q$ is the map defined in § 7, then it furthermore satisfies that for any $\varepsilon > 0$, there exists a number T, $0 < T < 1$, such that*

$$|\Phi(tv) - t\Phi(v)| < \varepsilon \qquad \text{for } t,\ T < t \leq 1$$

for every $v \in S_p$, where $|\cdot - \cdot|$ denotes the distance with respect to $(d\sigma)^2$.

To prove that the topology defined in § 7 is independent of base points we shall prepare the following:

Proposition 4. *Let D_i, $i = 1, 2$, be the unit disks in $\boldsymbol{R}^{n+1}$ with boundaries S_i. Let Φ be a map from D_1 to D_2. Suppose that Φ satisfies properties,*

(1) *Φ is one to one and onto,*

(2) *$\Phi|_{D_1 - S_1}$ is a homeomorphism,*

(3) *for any $\varepsilon > 0$, there is a number T, $0 < T < 1$, such that*

$$|\Phi(tv) - t\Phi(v)| < \varepsilon \qquad \text{for } t > T \text{ and every } v \in S_1$$

where $|\cdot - \cdot|$ denotes the Euclidean distance.

Then Φ is a homeomorphism.

Proof. First we shall prove that Φ is continuous on S_1. Let $\{v_i\}$ be a sequence in S_1 converging to $v_0 \in S_1$. For simplicity we put $w_j = \Phi(v_j)$, $j = 0, 1, 2, \cdots$. Given $\varepsilon > 0$, by (3) there exists a number $T > 0$ such that

$$|\Phi(tv_i) - tw_i| < \frac{1}{3}\varepsilon \qquad \text{for } t > T .$$

Since Φ is a homeomorphism on $D_1 - S_1$ by (2), $\{\Phi(tv_i)\}$ converges for $0 \leq t < 1$. Therefore there is an integer $N > 0$ such that

$$|\Phi(tv_i) - \Phi(tv_k)| < \frac{1}{3}\varepsilon \qquad \text{for } i, k \geq N .$$

Hence for $t > T$ for $i, k \geq N$

$$t|w_i - w_k| \leq |\Phi(tv_i) - tw_i| + |\Phi(tv_i) - \Phi(tv_k)| + |\Phi(tv_k) - tw_k| < \varepsilon .$$

Thus $\{w_i\}$ converges to, say, $w \in S_2$. But for $i > N$ and for $t > T$

$$\begin{aligned} t|w - w_0| \leq |\Phi(tv_0) - tw_0| &+ |\Phi(tv_0) - \Phi(tv_i)| \\ &+ |\Phi(tv_i) - tw_i| + t|w_i - w| < 2\varepsilon . \end{aligned}$$

Since ε is arbitrary, $w = w_0$.

Another application of (3) gives that Φ is a homeomorphism.

Remark. In Proposition 4 we treated the Euclidean metric. However the proposition holds for the metric $(d\sigma)^2$ in § 3 as well.

Theorem 6. *Let M be a complete simply connected Riemannian manifold of dimension $n + 1$, without focal points. Suppose the geodesic flow of M is of Anosov type. Then the topology of $\overline{M}$ constructed in § 7 is independent of the choice of base points.*

Added in proof. The problem considered in §§ 7 and 8 has been completely solved by the author without further assumption; see her forthcoming paper to appear in this journal.

References

[1] P. Eberlein, *Geodesic flow in certain manifolds without conjugate points*, Trans. Amer. Math. Soc. **167** (1972) 151–170.

[2] ——, *Geodesic flows on negatively curved manifolds*. I, Ann. of Math. **95** (1972) 492–510.

[3] ——, *Geodesic flows on negatively curved manifolds*. II, Trans. Amer. Math. Soc. **178** (1973) 57–82.

[4] ——, *When is a geodesic flow of Anosov type*? I, J. Differential Geometry **8** (1973) 437–463.

[5] P. Eberlein & B. O'Neill, *Visibility manifolds*, Pacific J. Math. **46** (1973) 45–109.

[6] L. W. Green, *Surfaces without conjugate points*, Trans. Amer. Math. Soc. **76** (1954) 529–546.

[7] ——, *Geodesic instability*, Proc. Amer. Math. Soc. **7** (1956) 438–448.

[8] ——, *A theorem of E. Hopf*, Michigan Math. J. **5** (1958) 31–34.

[9] D. Gromoll, W. Klingenberg & W. Meyer, *Riemansche Geometric im Grossen*, Lecture Notes in Math. Vol. 55, Springer, Berlin, 1968.

[10] S. Kobayashi, *Riemannian manifolds without conjugate points*, Ann. Mat. Pura Appl. **53** (1961) 149–155.

[11] M. Morse & G. Hedlund, *Manifolds without conjugate points*, Trans. Amer. Math. Soc. **51** (1942) 362–386.

[12] J. J. O'Sullivan, *Manifolds without conjugate points*, Math. Ann. **210** (1974) 295–311.

[13] W. R. Utz, *Almost periodic geodesics on manifolds of hyperbolic type*, Duke Math. J. **18** (1951) 147–164.

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