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METRIC RIGIDITY OF HOLOMORPHIC MAPS TO KÄHLER MANIFOLDS

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It is an interesting general question what differential-geometric invariants of a smooth map from a differentiable manifold V to a Riemannian manifold M are needed to determine the map up to congruence, i.e., up to composition with an isometry of M, or up to local congruence. For hypersurfaces in $\boldsymbol{R}^n$, the first and second fundamental forms are sufficient, and for a generic hypersurface in $\boldsymbol{R}^n$, $n \geq 4$, the first fundamental form alone is enough (see [3, Vol. II, p. 45]). In higher codimensions, the first fundamental form can be insufficient for the rigidity even of a generic map.

In the complex-analytic analogue, where V is a complex manifold, M a Kähler manifold with real-analytic Kähler metric, and the maps under consideration are holomorphic, it will be true in arbitrary codimension that a generic holomorphic map from V to M is determined up to local congruence by its first fundamental form. Our interest in the question arose from considering the special case of holomorphic curves in the Siegel upper half-plane suggested by Griffiths in [2], but the result turns out to be general. The method is based on Calabi [1].

The main theorem is

Theorem. *A nondegenerate holomorphic map from a connected complex manifold V to a Kähler manifold M with real-analytic Kähler metric is determined up to local congruence in M by its first fundamental form.*

Several of the above terms require explanation. Two maps f, g from V to M are *locally congruent* if for every $z \in V$, there is a local isometry F of M from a neighborhood of $f(z)$ to a neighborhood of $g(z)$ such that $g = F \circ f$ on a neighborhood of z. If M is also connected, simply-connected, and complete, this is the same for analytic maps as being congruent, as local isometries extend (see [3, Vol. I, pp. 255–256]).

By the *first fundamental form* we mean the pullback of the metric.

The notion of nondegeneracy is more complicated. The proof of the theorem will associate to each M a covering $\{U_\alpha\}_{\alpha \in A}$ by open sets and on each U_α a finite-dimensional family $\mathscr{F}_\alpha$ of real-analytic hypersurfaces of U_α. A map $V \xrightarrow{f} M$ will be said to be *degenerate* if $f(V) \cap U_\alpha$ lies in a hyperfurface in

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$\mathscr{F}_\alpha$ for some $\alpha \in A$ for which $f(V) \cap U_\alpha \neq \emptyset$. All other maps are *nondegenerate*. Thus a generic holomorphic map from V to M is nondegenrate. In certain cases, we can dispense with the covering $\{U_\alpha\}$ and simply call a map degenerate if its image lies in a countable union of finite-dimensional families of global real-analytic hypersurfaces on M, namely, if either

(1) M is a projective algebraic variety with a Hodge metric or

(2) M is a bounded domain in $\boldsymbol{C}^n$ with the Bergmann metric or

(3) $H^2(M, \boldsymbol{R}) = H^1(M, \mathcal{O}) = 0$.

Before proceeding to the proof, we give some illustrative examples. Examples 1, 2, 3 are results of Calabi [1]; Example 4 is new and illustrates how complex the degeneracy condition can be. For simplicity, in these examples we treat only the case $V = \varDelta$, the unit disc.

Example 1. $M = \boldsymbol{C}^n$ with the Euclidian metric $\langle\ ,\ \rangle$. The theorem is true with no degeneracy condition.

Proof. Following Lawson [4, p. 149], we show that if $f = (f_1, \cdots, f_n)$, $g = (g_1, \cdots, g_n)$ are two holomorphic maps, $\varDelta \to \boldsymbol{C}^n$, and $\langle f, f\rangle \equiv \langle g, g\rangle$, then there exists a unitary transformation $U \in U(n)$ so that $g \equiv U \circ f$. This would suffice, for if we only knew $\langle f', f'\rangle = \langle g', g'\rangle$, where $f' = (f_1', \cdots, f_n')$, etc., then $g' \equiv U \circ f'$, and integrating, $g = Uf + b$, so f and g are congruent if their first fundamental forms agree.

Applying $\partial^{i+j}/\partial z^i \partial \bar{z}^j$ to the equation $\langle f, f\rangle \equiv \langle g, g\rangle$ yields

$$\langle f^{(i)}, f^{(j)}\rangle = \langle g^{(i)}, g^{(j)}\rangle \qquad \text{for all } i, j \geq 0 ,$$

superscripts denoting coordinatewise derivatives. This implies there exists a unitary transformation $U \in U(n)$ so

$$g^{(i)}(0) = U(f^{(i)}(0)) \qquad \text{for all } i \geq 0 .$$

Hence the equation

$$g(z) = U \circ (f(z))$$

holds identically in z as the power series agree.

Example 2. $M = \boldsymbol{P}_n$ with the Fubini-Study metric. The theorem is true with no degeneracy condition.

Proof. Choose homogeneous liftings $\tilde{f} = (f_0, \cdots, f_n)$, $\tilde{g} = (g_0, \cdots, g_n)$ of f and g such that $f_0, \cdots, f_n$ never all vanish simultaneously, nor do $g_0, \cdots, g_n$. Then $f^*\omega = g^*\omega$ becomes

$$\partial\bar{\partial} \log \langle \tilde{f}, \tilde{f}\rangle = \partial\bar{\partial} \log \langle \tilde{g}, \tilde{g}\rangle .$$

Thus

$$\partial\bar{\partial} \log \frac{\langle \tilde{f}, \tilde{f}\rangle}{\langle \tilde{g}, \tilde{g}\rangle} = 0 ,$$

so

$$\frac{\langle \tilde{f}, \tilde{f} \rangle}{\langle \tilde{g}, \tilde{g} \rangle} = |\alpha|^2$$

for some holomorphic function α. Replacing $\tilde{f} = (f_0, \cdots, f_n)$ by the equivalent map $(\alpha f_0, \alpha f_1, \cdots, \alpha f_n)$ and relabeling, we may assume $\alpha = 1$. Thus

$$\langle \tilde{f}, \tilde{f} \rangle = \langle \tilde{g}, \tilde{g} \rangle .$$

By the result used in Example 1, there is a $U \in U(n+1)$ so that $\tilde{g} = U(\tilde{f})$ holds identically, and hence g and f differ by an isometry of $\boldsymbol{P}_n$.

Example 3. $M = \Delta$ with the Poincaré metric. The theorem holds with no degeneracy condition.

Proof. We may assume $f(0) = g(0) = 0$ without loss of generality. The condition $f^*\omega = g^*\omega$ is

$$\partial\bar{\partial} \log (1 - |f|^2) = \partial\bar{\partial} \log (1 - |g|^2) ,$$

so

$$\partial\bar{\partial} \log \frac{1 - |f|^2}{1 - |g|^2} = 0 ,$$

and hence

$$\frac{1 - |f|^2}{1 - |g|^2} = |\alpha|^2$$

for some holomorphic function α. Writing a power series for the left-hand side in terms of z and $\bar{z}$, we see that the leading term after the initial 1 is divisible by $z\bar{z}$. The right-hand side looks like $(1 + a_k z^k + \cdots)(1 + \bar{a}_k \bar{z}^k + \cdots) = 1 + 2\,Re\,(a_k z^k) +$ higher order terms. The leading terms after 1 can thus never be equal, leading to a contradiction unless both sides are identically 1. So

$$\frac{1 - |f|^2}{1 - |g|^2} = 1 ,$$

or $|f|^2 = |g|^2$ which implies that $g = e^{i\theta} f$ for some constant θ. Hence f and g differ by an isometry of Δ.

Example 4. $M = \Delta \times \Delta$ with the product Poincaré metric. The theorem holds, but there is a nontrivial degeneracy condition.

Proof. Let $f = (f_1, f_2)$, $g = (g_1, g_2)$, and without loss of generality we may take $f_1(0) = f_2(0) = g_1(0) = g_2(0) = 0$. The condition $f^*\omega = g^*\omega$ is

$$\begin{aligned} &\partial\bar{\partial} \log (1 - |f_1|^2) + \partial\bar{\partial} \log (1 - |f_2|^2) \\ &\quad = \partial\bar{\partial} \log (1 - |g_1|^2) + \partial\bar{\partial} \log (1 - |g_2|^2) , \end{aligned}$$

so

$$\frac{(1-|f_1|^2)(1-|f_2|^2)}{(1-|g_1|^2)(1-|g_2|^2)}=|\alpha|^2$$

for some holomorphic function α. By the same power-series argument as in the previous example, we see $\alpha \equiv 1$. Thus

$$(1-|f_1|^2)(1-|f_2|^2)=(1-|g_1|^2)(1-|g_2|^2)\,,$$

which simplifies to

$$|f_1|^2+|f_2|^2+|g_1g_2|^2=|g_1|^2+|g_2|^2+|f_1f_2|^2\,.$$

By previous results, there thus exists $U \in U(3)$ such that

$$\begin{pmatrix} f_1 \\ f_2 \\ g_1g_2 \end{pmatrix} = U\begin{pmatrix} g_1 \\ g_2 \\ f_1f_2 \end{pmatrix}.$$

If $U=(a_{ij})$, we have

$$\begin{aligned} f_1 &= a_{11}g_1+a_{12}g_2+a_{13}f_1f_2\,, \\ f_2 &= a_{21}g_1+a_{22}g_2+a_{23}f_1f_2\,, \\ g_1g_2 &= a_{31}g_1+a_{32}g_2+a_{33}f_1f_2\,. \end{aligned}$$

Thus

$$\begin{aligned} a_{11}g_1+a_{12}g_2 &= f_1-a_{13}f_1f_2\,, \\ a_{21}g_1+a_{22}g_2 &= f_2-a_{23}f_1f_2\,. \end{aligned}$$

If the matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is nonsingular, by solving for g_1 and g_2 and substituting in the third equation we obtain a polynomial equation in f_1 and f_2 of degree ≤ 4. This equation may be seen to be trivial only in case U has one of the two forms

$$\begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{-i(\theta_1+\theta_2)} \end{pmatrix}, \qquad \begin{pmatrix} 0 & e^{i\theta_1} & 0 \\ e^{i\theta_2} & 0 & 0 \\ 0 & 0 & e^{-i(\theta_1+\theta_2)} \end{pmatrix},$$

which imply f and g differ by an isometry of $\Delta \times \Delta$. If the matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is singular, we obtain a nontrivial polynomial relation among f_1, f_2 of degree ≤ 2.

Conversely, if f_1, f_2 satisfy a relation arising in this way, there will be a $g = (g_1, g_2)$ so $g^*\omega = f^*\omega$ but not differing from f by an isometry of $\Delta \times \Delta$. We thus need the assumption that $f(\Delta)$ does not lie in a member of our finite-dimensional family of hypersurfaces to assure rigidity.

Our interest in this example arose as follows. We may equivalently view $\Delta \times \Delta$ as $\mathscr{H} \times \mathscr{H}$, where $\mathscr{H}$ is the upper half-plane. It lies inside the Siegel upper half-space $\mathscr{H}_2$ of genus 2 as diagonal matrices in $\mathscr{H}$, and the metric is the restriction of the invariant metric on $\mathscr{H}_2$ considered by Siegel. It is not hard to show, as a consequence of Example 4, that the theorem requires a nontrivial degeneracy condition in order to hold for $\mathscr{H}_2$ with this metric. Thus, while the only differential invariant of a generic holomorphic family of abelian varietries parametrized by a complex manifold will be the pullback of the invariant metric on $\mathscr{H}_g$ by virtue of the main theorem, this example shows that there do exist cases where this is not enough to determine the family up to isometries of $\mathscr{H}_g$. See [2] for a discussion of this question.

Example 5. An example similar to the foregoing was suggested by the referee, to whom the author wishes to express his thanks. Consider $M = \boldsymbol{P}_1 \times \boldsymbol{P}_2$ with the product of the Fubini-Study metrics

$$\omega = i\partial\bar{\partial} \log [(1 + |x|^2)(1 + |y_1|^2 + |y_2|^2)] .$$

We may embed $\boldsymbol{C}$ (or $\boldsymbol{P}_1$) by

$$f(z) = (x; y_1, y_2) = (z; z^2, z^4) ,$$
$$g(z) = (x; y_1, y_2) = (z^3; z, z^2) .$$

In $H^2(M, Z) \simeq \boldsymbol{Z}^2$, the cohomology classes are

$$[f(\boldsymbol{P}_1)] = (1, 4) , \qquad [g(\boldsymbol{P}_1)] = (3, 2) .$$

Thus there can be no biholomorphic map F of M to itself so that $g = F \circ f$ on topological grounds. They are isometric since

$$f^*\omega = g^*\omega = i\partial\bar{\partial} \log (1 + |z|^2 + |z|^4 + |z|^6 + |z|^8 + |z|^{10}) .$$

Example 6. Let $M = G(2, 4)$, the Grassmannian of lines in $\boldsymbol{P}_3$, embedded as a quadric Q in $\boldsymbol{P}_5$ by Plücker coordinates. We take the metric ω induced from the Fubini-Study metric on $\boldsymbol{P}_5$. If $\Delta \xrightarrow{f} Q$, $\Delta \xrightarrow{g} Q$ are two holomorphic maps with $f^*\omega = g^*\omega$, then by the theorem for profective spaces, there is an isometry U of $\boldsymbol{P}_5$ such that $f = U \circ g$. If $U(Q) = Q$, then U is an isometry of $G(2, 4)$, and they all arise in this way. If not, then $f(\Delta) \subset Q \cap U(Q)$, a hypersurface of $G(2, 4)$. Thus $f^*\omega = g^*\omega$ implies that f and g differ by an isometry of $G(2, 4)$, unless $f(\Delta)$ lies in the finite-dimensional family of hypersurfaces $Q \cap U(Q)$, $U \in U(n + 1)$, $U(Q) \neq Q$. If $f(\Delta)$ does lie in $Q \cap U(Q)$, $U \in U(n + 1)$, $U(Q) \neq Q$, then as long as f lies in no proper algebraic subvariety of $Q \cap U(Q)$, $g = U \circ f$ is not congruent to f by an isometry of $G(2, 4)$, but $g^*\omega = f^*\omega$.

For a general projective algebraic variety $M \hookrightarrow \boldsymbol{P}_N$ with the metric induced from the Fubini-Study metric by the embedding, we have that a holomorphic map $V \xrightarrow{f} M$ is determined up to congruence by an isometry of M unless $f(V) \subset M \cap U(M)$, $U \in U(N + 1)$, $U(M) \neq M$.

With this as motivation, we proceed to the proof of the main theorem. This is surprisingly elementary, once we employ an ingenious idea due to Calabi [1], the diastasis function.

If ω is the (1, 1)-form representing a Kähler metric on M, locally there is a real valued function Φ such that $\omega = i\partial\bar{\partial}\Phi$. This can be done globally if $H^2(M, \boldsymbol{R}) = H^1(M, \mathcal{O}) = 0$, or if ω is the Bergmann metric on M. If the metric is real-analytic, then Φ has a power series in $z_1, \cdots, z_n, \bar{z}_1, \cdots, \bar{z}_n$,

$$\Phi(z, \bar{z}) = \sum b_{I,J} z^I \bar{z}^J$$

written here in multi-index notation. We can then define

$$\Phi(z, \bar{w}) = \sum b_{I,J} z^I \bar{w}^J$$

as a function of two variables. The *diastasis function* $D(z, w)$ is defined by

$$D(z, w) = \Phi(z, \bar{z}) + \Phi(w, \bar{w}) - \Phi(z, \bar{w}) - \Phi(w, \bar{z}) ,$$

which is symmetric, real-valued, and $D(z, z) = 0$ for all z. Although there is some ambiguity in the choice of Φ, this drops out when we define D, which depends only on the metric.

Two properties of the diastasis we will need are:

(1) if $\rho(z, w)$ is the distance function on M for the given metric, then for w near z, $D(z, w) = \rho(z, w)^2 + 0(\rho(z, w)^4)$,

(2) if $M_1 \subset M_2$ and if the metric on M_1 comes from the metric on M_2 by restriction, then the same is true of their diastases.

Let K be a compact subset of M such that the diastasis is defined everywhere on $K \times K$. An N-tuple $(p_1, \cdots, p_N) \in K^N$ will be said to have property P if the map $K \to \boldsymbol{R}^N$, $p \to (D(p, p_1), \cdots, D(p, p_N))$ is injective.

Lemma 1. *For N sufficiently large, a generic point of the image of $K^N \to \boldsymbol{R}^{\binom{N}{2}}$, $(p_1, \cdots, p_N) \to (D(p_i, p_j))$, $1 \leq i < j \leq N$, has the property that every preimage has property P. Here "generic" means the complement of a lower-dimensional real-analytic subvariety.*

Proof. Let $E_{p,q} = \{s \in K \mid D(p, s) = D(q, s)\}$, and $E^N_{p,q}$ denote its N-fold Cartesian product. Then an N-tuple has property $P \leftrightarrow$ it lies in $K^N - \bigcup_{p,q \in K} E^N_{p,q}$. Let $n = \dim_C M$. As $D \equiv \rho^2$ up to $0(\rho^4)$, the dimension of the image of $K^N \to \boldsymbol{R}^{\binom{N}{2}}$ is at least $2nN - \dim 0(2n)$, which is what we get for the Euclidean metric on $\boldsymbol{R}^{2n}$. The image of $\bigcup_{p,q \in K} E^N_{p,q}$ has real dimension $\leq N(2n - 1) + 4n = 2Nn + 4n - N$. So for $N > 4n + \dim 0(2n)$, the points in the image of K^N having a preimage for which property P fails are contained in a real-analytic hypersurface of the image.

Lemma 2. *Given compact sets K_1, K_2 in M such that the diastasis is defined on $K_1 \times K_1$ and $K_2 \times K_2$, for N sufficiently large, for a generic $(p_1, \cdots, p_N) \in K_1^N$ and any $(q_1, \cdots, q_N) \in K_2^N$ such that $D(p_i, p_j) = D(q_i, q_j)$ for all $1 \leq i < j \leq N$, then there exists a unique local isometry F of M defined on K_1 so $q_i = F(p_i)$ for all $i = 1, \cdots, N$. Here generic means outside a lower-dimensional real-analytic subvariety of K_1^N.*

Proof. Consider the maps $K_1^N \xrightarrow{F_1} \boldsymbol{R}^{\binom{N}{2}}$, $K_2^N \xrightarrow{F_2} \boldsymbol{R}^{\binom{N}{2}}$ defined as in Lemma 1. If a generic point of the image of $F_1(K_1^N)$ does not lie in $F_2(K_2^N)$ for N sufficiently large, there is nothing to prove. If it does, then we may choose $(p, \cdots, p_N)$ so its image satisfies the conclusion of Lemma 1 for both maps.

Let $S(p_1, \cdots, p_N) = \{(q_1, \cdots, q_N) \in K_2^N \mid D(p_i, p_j) = D(q_i, q_j)$ for all $1 \leq i < j \leq N\}$. For any $p \in K_1$, there is at most one $q \in K_2$ so that $D(p, p_i) = D(q, q_i)$ for all $i = 1, \cdots, N$. Therefore, if we take $S(p_1, \cdots, p_N, p_{N+1})$, the projection $K_2^{N+1} \to K_2^N$ induces an inclusion in $S(p_1, \cdots, p_N)$. Now by compactness, $S(p_1, \cdots, p_N)$ contains at most a finite number of irreducible components besides the one containing the images of $(p_1, \cdots, p_N)$ under local isometries of M. We can cut away all extraneous matter by the following procedure. Assume $(q_1, \cdots, q_N) \in S(p_1, \cdots, p_N)$ but there is no local isometry F as in the statement of the lemma. If we can find $p_{N+1} \in K_1$ so that there is no $q_{N+1} \in K_2$ with $D(q_{N+1}, q_i) = D(p_{N+1}, p_i)$ for all $i = 1, \cdots, N$, then there is no element of $S(p_1, \cdots, p_{N+1})$ with $q_1, \cdots, q_N$ as its first N entries. Otherwise, for all $p_{N+1} \in K_1$ there exists a unique $q_{N+1} \in K_2$ with $D(q_{N+1}, q_i) = D(p_{N+1}, p_i)$, $i = 1, \cdots, N$ (uniqueness is by Lemma 1). Define $K_1 \xrightarrow{F} K_2$ by letting $F(p_{N+1})$ equal this unique q_{N+1}. If $D(F(p_{N+1}), F(p_{N+2})) = D(p_{N+1}, p_{N+2})$ for all $p_{N+1}, p_{N+2} \in K_1$, then F is a local isometry. If not, then by adjoining such a p_{N+1}, p_{N+2}, we get no element of $S(p_1, \cdots, p_{N+2})$ with first N entries $q_1, \cdots, q_N$. So by suitably increasing N, we eventually reduce $S(p_1, \cdots, p_N)$ to $\{$images of $(p_1, \cdots, p_N)$ under isometries of $M\} \cap K_2\}$. Given such a $(p_1, \cdots, p_N)$, the isometry is unique as for all p, $D(F(p), q_i) = D(p, p_i)$, $i + 1, \cdots, N$; so by Lemma 1, $F(p)$ is uniquely determined.

We may now prove the theorem. Let $V \xrightarrow{f} M$, $V \xrightarrow{g} M$ be two holomorphic maps with $f^*\omega = g^*\omega$. Then $D(f(p), f(q)) = D(g(p), g(q))$ for all $p, q \in V$, whenever D is defined. We may assume, by shrinking V, that $f(V) \subset K_1$, $g(V)$

$\subset K_2$, where K_1 and K_2 are compact and D is defined on $K_1 \times K_1$ and $K_2 \times K_2$. If we can pick $z_1, \cdots, z_N \in V$, and N as in Lemma 2, so that $f(z_1), \cdots, f(z_N)$ is generic in the sense of both Lemmas 1 and 2, then as $D(f(z_i), f(z_j)) = D(g(z_i), g(z_j))$ for all $1 \leq i < j \leq N$, by Lemma 2 there is a unique local isometry F of M so $g(z_i) = F(f(z_i))$ for all $i = 1, \cdots, N$. If for an open set of $z_{N+1} \in V, f(z_1), \cdots, f(z_{N+1})$ is generic in the sense of Lemma 2, then $g(z_{N+1}) = F(f(z_{N+1}))$ on an open subset of V and we are done. The only source of trouble is if for $m = N$ or $N+1$, we have $R(f(z_1), \cdots, f(z_m)) = 0$ for $z_1, \cdots, z_m \in V$, where R is a real-analytic function on K_1 containing the nongeneric (in the sense of Lemma 2) m-tuples of K_1. Now fixing $z_1, \cdots, z_{m-1}$, we get a relation $R(a_1, \cdots, a_{m-1}, f(z)) = 0$, which either gives a hypersurface containing $f(V)$ or else we obtain a real-analytic relation $R_1(f(z_1), \cdots, f(z_{m-1})) = 0$ for all $z_1, \cdots, z_{m-1} \in V$, which says that $(f(z_1), \cdots, f(z_{m-1}))$ is an $(m-1)$-tuple making R vanish identically in last variable. In the latter case, by fixing $z_1, \cdots, z_{m-2}$, we either get a hypersurface containing $f(V)$ or a relation $R_2(f(z_1), \cdots, f(z_{m-2})) = 0$ for all $z_1, \cdots, z_{m-2} \in V$. Eventually this will lead to $R_{m-1}(f(z_1)) = 0$ for all $z_1 \in V$, hence a hypersurface containing $f(V)$, if we do not get one beforehand. All the hypersurfaces which come up this way, starting from R (which depended only on M and the metric and not on f), belong to a finite-dimensional family, since V is finite-dimensional and hence so are the possible values of $f(z_1), \cdots, f(z_{m-1})$ which we fix in the intermediate stages. This completes the proof.

In case the diastasis D is globally defined, we can exhaust M by compact sets $K_1 \subset K_2 \subset \cdots$, and the degeneracy conditions become a countable union of real-analytic hypersurfaces.

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