J. DIFFERENTIAL GEOMETRY
13 (1978) 251-261

TOTALLY FOCAL EMBEDDINGS

SHEILA CARTER & ALAN WEST

1. Introduction

In this paper we investigate manifolds embedded in Euclidean space with the property that for any distance function either all its critical points are nondegenerate or all its critical points are degenerate. We will give a complete classification of such manifolds when the codimension of the embedding is one.

Let M be a connected smooth m-dimensional manifold without boundary. Let $f: M \to \boldsymbol{R}^n$ be a smooth proper embedding and let N be the corresponding normal bundle. Thus $N \subset M \times \boldsymbol{R}^n$ is the subset defined by $(p, v) \in N$ if and only if $f(p) + v$ lies on the normal plane to $f(M)$ at $f(p)$. The end-point map $\eta: N \to \boldsymbol{R}^n$ is then defined by $\eta(p, v) = f(p) + v$. The set of critical points of η will be denoted by Γ. Thus $\eta(\Gamma)$ is the set of focal points of the embedding. Now the distance function from $x \in \boldsymbol{R}^n$ is nondegenerate if and only if $x \notin \eta(\Gamma)$, (see Milnor [6, P. 36]), so the property which we investigate can be defined as

$$\Gamma = \eta^{-1} \circ \eta(\Gamma) .$$

Embeddings with this property will be said to be *totally focal.*

The round sphere and a flat plane, with arbitrary dimension and codimension, are examples of such embeddings. So also are products of such embeddings. This follows from the following two propositions.

Proposition 1.1. *Let $f: M \to \boldsymbol{R}^n$ be totally focal then, for any $d > 0$, $j \circ f: M \to \boldsymbol{R}^{n+d}$ is totally focal where $j: \boldsymbol{R}^n \to \boldsymbol{R}^n \times \boldsymbol{R}^d \equiv \boldsymbol{R}^{n+d}$ is given by $j(x) = (x, 0)$.*

The proof of this is trivial.

Proposition 1.2. *Let $f_i: M_i \to \boldsymbol{R}^{n_i}$ $(i = 1, 2)$ be embeddings. Then the product embedding*

$$f_1 \times f_2: M_1 \times M_2 \to \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2} \equiv \boldsymbol{R}^{n_1+n_2}$$

is totally focal if and only if both f_1 and f_2 are totally focal.

Proof. We write η_1, η_2 and η for the end-point maps of f_1, f_2 and $f_1 \times f_2$ respectively, with a similar notation for the normal bundles and critical point sets. Then it is easy to check that $N \equiv N_1 \times N_2$, $\eta \equiv \eta_1 \times \eta_2$ and thus $\Gamma \equiv (\Gamma_1 \times N_2) \cup (N_1 \times \Gamma_2)$. Hence

Received September 6, 1976, and, in revised form, November 21, 1977.

$$\eta^{-1}\circ\eta(\Gamma) \equiv [\eta_1^{-1}\circ\eta_1(\Gamma_1) \times N_2] \cup [N_1 \times \eta_2^{-1}\circ\eta_2(\Gamma_2)]\ .$$

The result follows immediately.

However although by the above theorem the flat torus $\boldsymbol{S}^1 \times \boldsymbol{S}^1 \to \boldsymbol{R}^4$ is totally focal, the torus does not have a totally focal embedding in $\boldsymbol{R}^3$. In fact the main theorem of this paper is the following classification theorem.

Classification theorem. *The only totally focal embeddings with codimension* 1 *are*

(i) *the round spheres* $f\colon \boldsymbol{S}^{n-1} \to \boldsymbol{R}^n$,

(ii) *the flat hyperplanes* $g\colon \boldsymbol{R}^{n-1} \to \boldsymbol{R}^n$,

(iii) *the round cylinders* $h \times 1\colon \boldsymbol{S}^{l-1} \times \boldsymbol{R}^k \to \boldsymbol{R}^l \times \boldsymbol{R}^k \equiv \boldsymbol{R}^n$, *where* $l + k = n$ *and* $h\colon \boldsymbol{S}^{l-1} \to \boldsymbol{R}^l$ *is a round sphere.*

The proof is the cumulation of a series of results which form the body of the paper. Results that apply to any codimension are collected together in § 2. Then in § 3 we specialize to codimension one and show that the manifold must be a convex hypersurface. Finally in § 4 we look at special cases which enable us to complete the classification theorem.

2. General results

We prove some general theorems about the end-point map $\eta\colon N \to \boldsymbol{R}^n$ associated with a fixed totally focal embedding $f\colon M \to \boldsymbol{R}^n$. For this we need the Morse index theorem [6]. For $x \in \boldsymbol{R}^n$ the distance function $L_x\colon M \to \boldsymbol{R}$ is defined by $L_x(p) = \|f(p) - x\|^2$. If $(p, v) \in N\backslash\Gamma$ and $\eta(p,v) = x$, then p is a nondegenerate critical point of L_x and we define the index of the normal (p, v) to be the index of L_x at p.

Theorem 2.1. *Let* $f\colon M \to \boldsymbol{R}^n$ *be a totally focal embedding. Then* $\boldsymbol{R}^n\backslash\eta(\Gamma)$ *is connected and contains* $f(M)$.

Proof. First observe that since η has no critical points on the zero section $M \times \{0\} \subset N$, the condition $\Gamma = \eta^{-1}\circ\eta(\Gamma)$ implies that $f(M) \cap \eta(\Gamma) = \emptyset$.

Now take $x \in \boldsymbol{R}^n\backslash\eta(\Gamma)$ and consider the distance function L_x. Since f is proper there exists an absolute minimum for L_x, that is, we can find $p \in M$ such that for all $q \in M$, $\|f(p) - x\| \leq \|f(q) - x\|$. This minimum is nondegenerate and has index zero. Thus, if $v = x - f(p)$, then $\eta(p, v) = x$,and (p, v) has index 0 and so the Morse index theorem tells us that

$$\{(p, tv)\colon t \in [0, 1]\} \cap \Gamma = \emptyset\ .$$

Since $\eta^{-1}\circ\eta(\Gamma) = \Gamma$, then the path γ where $\gamma(t) = \eta(p, tv)$ lies in $\boldsymbol{R}^n\backslash\eta(\Gamma)$ and joins x to a point in $f(M)$. As M is path-connected this is sufficient to prove the theorem.

Remark. We have actually proved that $\eta(\Gamma)$ lies in the *cut-locus* of the embedding, that is, in the closure of the set of points $x \in \boldsymbol{R}^n$ such that L_x does not have a unique nondegenerate absolute minimum. However we do not need this result.

A central fact in our investigation is that $\eta: N\backslash\Gamma \to \boldsymbol{R}^n\backslash\eta(\Gamma)$ is a covering. We know of course that η restricted to $N\backslash\Gamma$ is a local homeomorphism but to show it is a covering we require something more. In fact the following theorem suffices.

Theorem 2.2. *Let γ be a smooth path in $\boldsymbol{R}^n\backslash\eta(\Gamma)$ with initial point x_0, and let $(p_0, v_0) \in N$ be such that $\eta(p_0, v_0) = x_0$. Then γ can be lifted to a unique smooth path $\hat{\gamma}$ in $N\backslash\Gamma$ with initial point (p_0, v_0) such that $\eta \circ \hat{\gamma} = \gamma$.*

Proof. Let $I = [0, 1]$ be the unit interval so that $\gamma: I \to \boldsymbol{R}^n\backslash\eta(\Gamma)$. The technique is to show that the subinterval of I over which γ can be lifted is open and closed and thus must be I itself. It is easy to show that it is open since η is a local homeomorphism. Thus we have only to prove that it is closed. This amounts to assuming that $\hat{\gamma}$ is defined on $[0, 1) \subset I$ and showing that it can be extended uniquely to I.

Let $x_1 = \gamma(1) \in \boldsymbol{R}^n\backslash\eta(\Gamma)$, and suppose $\hat{\gamma}: [0, 1) \to N\backslash\Gamma$ with $\eta \circ \hat{\gamma} = \gamma$. Let l be the length of the curve γ. First we show that if for $s \in [0, 1)$ we write $\hat{\gamma}(s) = (p_s, v_s)$ then

$$\|v_0\| - l \leq \|v_s\| \leq \|v_0\| + l\ .$$

To do this we define $\tilde{f}$ and ν as maps $N \to \boldsymbol{R}^n$ by $\tilde{f}(p, v) = f(p)$, $\nu(p, v) = v$, and then we can write $\eta = \tilde{f} + \nu$. Hence $d\eta = d\tilde{f} + d\nu$. However by definition we have $\langle d\tilde{f}, \nu\rangle = 0$ and so

$$\langle d\eta, \nu\rangle = \langle \nu, d\nu\rangle = \tfrac{1}{2}d\langle \nu, \nu\rangle = \tfrac{1}{2}d\,\|\nu\|\rangle^2 = \|\nu\|\, d\,\|\nu\|\ .$$

This means that $|\|\nu\|\, d\,\|\nu\|| \leq \|d\eta\|\,\|\nu\|$ and therefore $|d\,\|\nu\|| < \|d\eta\|$.

Consequently, integrating along the path $\hat{\gamma}$, we obtain

$$|\|v_s\| - \|v_0\|| = \left|\int_0^s d\,\|\nu\|\right| \leq \int_0^s |d\,\|\nu\|| \leq \int_0^s \|d\eta\| = \int_0^s \|d\gamma\| < l\ .$$

This can be written $\|v_0\| - l \leq \|v_s\| \leq \|v_0\| + l$.

Now observe that since $\gamma(I)$ is compact we can find $k > 0$ such that for all $s \in I$, $\|\gamma(s)\| \leq k$. Hence, since $\gamma(s) = \eta \circ \hat{\gamma}(s) = f(p_s) + v_s$,

$$\|f(p_s)\| = \|\gamma(s) - v_s\| \leq k + l + \|v_0\| = K\,, \qquad \text{say}\ .$$

Now the ball $B_k = \{x \in R^n : \|x\| \leq K\}$ is compact and f is proper so $f^{-1}(B_k)$ is compact. We have now shown that if $s \in [0, 1)$ then $(p, v_s) \in (f^{-1}(B_k) \times B_k) \cap N$ which is a compact subset of N. Thus there exists

$$(p_1, v_1) \in \bigcap_{\varepsilon \in (0,1)} \overline{\{\hat{\gamma}(s) : s \in (\varepsilon, 1)\}}\ .$$

Further, by continuity of η, $\eta(p_1, v_1) \in \bigcap_{\varepsilon \in (0,1)} \overline{\{\gamma(s) : s \in (\varepsilon, 1)\}} = \gamma(1)$. Now since $\gamma(1) \notin \eta(\Gamma)$ then $(p_1, v_1) \notin \Gamma$, so η is a diffeomorphism on some neighborhood

V of (p_1, v_1) in $N\backslash\Gamma$. If $\eta(V) = U$ we can extend $\hat{\gamma}$ by defining $\hat{\gamma}(s) = \eta^{-1} \circ \gamma(s)$ for $\gamma(s) \in U$ and obtain the required lifting $\hat{\gamma}: I \to N\backslash\Gamma$.

Theorem 2.3. *The map $\eta: N\backslash\Gamma \to \boldsymbol{R}^n\backslash\eta(\Gamma)$ is a covering.*

Proof. We would like to use a theorem such as Theorem 4.6, Chap. IV, Vol I of [4], or the theorem given in the appendix of [3]. However $N\backslash\Gamma$ is not complete in the obvious metric, and instead we modify the proof given in [3].

We show that for any $x_0 \in \boldsymbol{R}^n\backslash\eta(\Gamma)$ there is an open ball B centre x_0, radius ρ, with $B \cap \eta(\Gamma) = \emptyset$ such that for any $u_i \in \eta^{-1}(x_0)$ there is a corresponding neighborhood U_i of u_i in $N\backslash\Gamma$ for which $\eta: U_i \to B$ is one-one and onto, $U_i \cap U_j = \emptyset$ if $u_i \neq u_j \in \eta^{-1}(x_0)$, $\bigcup_i U_i = \eta^{-1}(B)$, and U_i is open. Since η is a local diffeomorphism on $N\backslash\Gamma$, this is enough to ensure that $\eta: U \to B$ is a diffeomorphism on any connected component U of $\eta^{-1}(B)$, and consequently $\eta: N\backslash\Gamma \to \boldsymbol{R}^n\backslash\eta(\Gamma)$ is a covering.

Choose any open ball B^*, centre x_0, radius ρ^* with $B^* \cap \eta(\Gamma) = \emptyset$, and take $\rho < \rho^*$ so that $B \subset B^*$.

Now from Theorem 2.2 if γ is the radial line joining x_0 to any x in B there is a unique path $\hat{\gamma}$ which covers γ and joins u_i to a point $\psi_i(x)$ in $\eta^{-1}(B)$. This defines a map $\psi_i: B \to \eta^{-1}(B)$, and we take its image to be U_i. Thus by definition $\eta \circ \psi_i$ and $\psi_i \circ \eta$ are identity maps, and so $\eta: U_i \to B$ is one-one and onto.

Also $U_i \cap U_j = \emptyset$ if $u_i \neq u_j \in \eta^{-1}(x_0)$ since if $u \in U_i \cap U_j$, $\eta(u) = x$, there would be two paths covering the radial line from x to x_0: one from u to u_i and another from u to u_j. This contradicts Theorem 2.2.

It is just as easy to see that $\bigcup_i U_i = \eta^{-1}(B)$ since if $u \in \eta^{-1}(B)$, $\eta(u) = x$, the radial line from x to x_0 lifts to a path from x to some $x_j \in \eta^{-1}(x_0)$. Reversing these paths we see that by definition $u \in U_j$.

Now there exists an open neighborhood V_i of u_i in $N\backslash\Gamma$ such that $\eta: V_i \to \eta(V_i)$ is a diffeomorphism, and without loss of generality we may suppose that $\eta(V_i) = B_i$ is an open ball, centre x_0, radius $\rho_i \leq \rho$. Clearly $V_i \subset U_i$ so U_i is a neighborhood of u_i. We define a map $\theta_i: V_i \to U_i$ by first projecting to B_i then dilating B_i to B and then lifting to U_i as in the diagram

$$\begin{tikzcd} V_i \arrow[r, "\theta_i"] \arrow[d, "\eta"'] & U_i \\ B_i \arrow[r] & B \arrow[u, "\psi_i"'] \end{tikzcd}$$

where the dilation $B_i \to B$ is given by $x \mapsto x_0 + (\rho/\rho_i)(x - x_0)$. We show that θ_i is the restriction of a diffeomorphism $\eta^{-1}(B^*) \to \eta^{-1}(B^*)$, and hence U_i is open and the theorem is proved.

To construct this diffeomorphism we take the radial vector field on B. In fact it is convenient to extend it to a vector field on B^* defined by

$$x \to (x, \mu(\|x - x_0\|)(x - x_0))$$

where $\mu\colon \boldsymbol{R} \to \boldsymbol{R}$ is given by $\mu(r) = 1$ if $r \leq \rho$ and $\mu(r) = (\rho^* - r)/(\rho^* - \rho)$ if $r \geq \rho$. We lift this vector field to $\eta^{-1}(B^*)$ and consider its maximal flow. It is easy to deduce from Theorem 2.2 that this is complete, in other words, it is a map $F\colon \eta^{-1}(B^*) \times \boldsymbol{R} \to \eta^{-1}(B^*)$. By the properties of flows if $F_t(w) = F(w, t)$ then for any $t \in \boldsymbol{R}$, $F_t\colon \eta^{-1}(B^*) \to \eta^{-1}(B^*)$ is a diffeomorphism. A straightforward calculation shows that θ_i is the restriction of F_t where $t = \log \rho/\rho_i$. This finishes the proof.

Of course $N\backslash\Gamma$ is not in general connected, and η will be a covering when restricted to any connected component of $N\backslash\Gamma$. Note that the index is a continuous function on $N\backslash\Gamma$ taking only integer values. So to each of the connected components of $N\backslash\Gamma$ there is associated a fixed index which is an integer k, $0 \leq k \leq m$.

Let U be a connected component of $N\backslash\Gamma$ with maximal index k (if M is compact $k = m$ but this need not be true in general). We may as well assume $k > 0$, otherwise $\Gamma = \emptyset$ and $f(M)$ must be just a hyperplane. Then since the zero section of N has index 0, U does not intersect this zero section.

Theorem 2.4. *Let U be a connected component of $N\backslash\Gamma$ with maximal index, then the closure of U in N is homeomorphic to $\partial U \times [1, \infty)$, and ∂U is connected.*

Proof. Observe that $\bar{U} \cap \Gamma = \partial U$. In fact, if $(p, v) \in U$ there exist a unique τ, $0 < \tau < 1$, such that $(p, \tau v) \in \Gamma$ and $(p, sv) \in U$ if and only if $\tau < s$. Further the map $\sigma\colon U \to \boldsymbol{R}$, defined by $\sigma(p, v) = \tau$, is continuous. Essentially this is because $1/\tau$ is defined as the smallest positive eigenvalue of a certain matrix (associated with the second fundamental form, see [6]), and this matrix varies continuously with $(p, v) \in N$. Also if $(p, v) \in U$ this minimum eigenvalue is never zero and so also varies continuously with (p, v). We deduce that σ is continuous, and a homeomorphism $\phi\colon \bar{U} \to \partial U \times [1, \infty)$ is given by $\phi(p, v) = ((p, \tau v), 1/\tau)$ where $\tau = \sigma(p, v)$.

Since U is connected, so is $\bar{U}$, and hence so is ∂U.

Theorem 2.5. *$\eta(\partial U) = \eta(\Gamma)$ and $\eta(\Gamma)$ is connected.*

Proof. Suppose $x \in \eta(\Gamma)$ and let p be a minimum for the distance function L_x. Define a path $\gamma\colon I \to \boldsymbol{R}^n$ by $\gamma(s) = (1 - s)f(p) + sx$. Since $\eta^{-1} \circ \eta(\Gamma) = \Gamma$ the Morse index theorem tells us that $\gamma(s) \in \boldsymbol{R}^n\backslash\eta(\Gamma)$ if $s \in [0, 1)$.

Take a point $(p_0, v_0) \in U$ with $\eta(p_0, v_0) = f(p)$; such a point exists since $\eta\colon U \to \boldsymbol{R}^n\backslash\eta(\Gamma)$ is a covering. Then we can lift $\gamma\colon [0, 1) \to \boldsymbol{R}^n\backslash\eta(\Gamma)$ to a path $\hat{\gamma}\colon [0, 1) \to U$. We then apply the method in Theorem 2.2 to show that there exists

$$(p_1, v_1) \in \bigcap_{\varepsilon \in (0,1)} \overline{\{\hat{\gamma}(s)\colon s \in (\varepsilon, 1)\}} \subset \bar{U}$$

such that $\eta(p_1, v_1) = \gamma(1) = x$. Since $x \in \eta(\Gamma)$, $(p_1, v_1) \in \Gamma \cap \bar{U} = \partial U$. Hence $\eta(\partial U) = \eta(\Gamma)$.

Since ∂U is connected it follows that $\eta(\Gamma)$ is connected.

3. Hypersurfaces

Now let us consider the case when the embedding f is of codimension 1. Thus $n = m + 1$, $f(M)$ separates $\boldsymbol{R}^n$, M must be orientable and the normal bundle is trivial.

The aim in this section is to prove the following:

Theorem 3.1. *Let $f: M \to \boldsymbol{R}^n$ be a totally focal embedding with codimension 1. Then $f(M)$ is a convex hypersurface, that is, the boundary of an open convex set in $\boldsymbol{R}^n$.*

The proof is given by Lemmas 3.2–3.4.

Let us change our notation somewhat and identify N with $M \times \boldsymbol{R}$. We can do this by choosing a field of unit normals $\boldsymbol{n}: M \to \boldsymbol{R}^n$ (we write $\boldsymbol{n}_p$ for the image of p). Then $(p, t) \in M \times \boldsymbol{R}$ is identified with $(p, t\boldsymbol{n}_p) \in N$ in the last section. We can thus write $\eta(p, t) = f(p) + t\boldsymbol{n}_p$, if necessary.

As in the previous section we let U be a connected component of $N \backslash \Gamma$ with maximal index $k > 0$. We may as well assume that if $(p, t) \in U$ then $t > 0$. We let W^+, W^- be the two connected components of $\boldsymbol{R}^n \backslash f(M)$ (labelled arbitrarily).

Lemma 3.2. *One of the two sets W^+ or W^- does not intersect $\eta(\Gamma)$. Further, supposing $W^- \cap \eta(\Gamma) = \emptyset$ then $U \cap \eta^{-1}(W^+)$ is connected.*

Proof. From Theorem 2.5, $\eta(\Gamma)$ is connected, and from Theorem 2.1, $\eta(\Gamma) \subset W^+ \cap W^-$. Hence, since $W^+ \cap W^- = \emptyset$, $\eta(\Gamma)$ is contained in W^+ or W^-. We assume $\eta(\Gamma) \subset W^+$.

Let us write $U^+ = U \cap \eta^{-1}(W^+)$ and $U^- = U \cap \eta^{-1}(W^-)$.

We construct a continuous function $\alpha: \partial U \to \boldsymbol{R}$ such that if $(p, \tau) \in \partial U$ then $\alpha(p, \tau) > \tau$, and $(p, t) \in U^+$ for all $t, \tau < t < \alpha(p, \tau)$.

To show that this can be done we observe that if we define $\nu(p, \tau) = \tau$ and $\mu(p, \tau) = \inf\{t: (p, t) \in \bar{U}^-\}$ for all $(p, \tau) \in \partial U$, then $\nu(p, \tau) < \mu(p, \tau)$. Now $\mu: \partial U \to \boldsymbol{R}$ is a lower-semicontinuous function and $\nu: \partial U \to \boldsymbol{R}$ is continuous. Moreover ∂U is σ-compact, so there exists a continuous function $\alpha: \partial U \to \boldsymbol{R}$ such that $\nu(p, \tau) < \alpha(p, \tau) < \mu(p, \tau)$ for all $(p, \tau) \in \partial U$, (see also [7, Ex. 21B]). Then since $U^+ \cap U^- = \emptyset$, and $U^+ \cup U^- = U$, α has the required property.

Now since ∂U is connected (Theorem 2.4), so is the "collar" $C = \{(p, t): \tau < t < \alpha(p, \tau), (p, \tau) \in \partial U\} \subset U^+$. Also $\partial U \subset \bar{C}$. Further we showed in the same theorem that $\bar{U}$ is homeomorphic to $\partial U \times [1, \infty)$. Under the homeomorphism, $\partial U \cup C$ maps into an open set containing $\partial U \times \{1\}$ and hence $\partial U \cup C$ is an open set in $\bar{U}$ containing ∂U. So any component of U^+ whose closure intersects ∂U must intersect C, and this component must contain C since $C \subset U^+$ and is connected.

On the other hand, if U_0^+ is any connected component of U^+, then $\eta(U_0^+) = W^+ \backslash \eta(\Gamma)$ since $\eta: U \to \boldsymbol{R}^n \backslash \eta(\Gamma)$ is a covering,. Now consider the way in which a point $(p_1, v_1) \in \Gamma \cap \bar{U}$ was chosen in Theorem 2.5. The path γ defined there lies in W^+ if $s \in (0, 1]$, and so the same method shows that we can find (p_1, v_1)

$\in \Gamma \cap \overline{U_0^+}$, and hence $\overline{U_0^+}$ intersects ∂U. It follows that $C \subset U_0^+$. Hence there is only one connected component of U^+. This proves the lemma.

Lemma 3.3. *If $\mu \colon X \to \boldsymbol{R}^n \backslash \eta(\Gamma)$ is a covering where X is connected, then $\mu^{-1}(W^+ \backslash \eta(\Gamma))$ is connected.*

Proof. We use the notation of Lemma 3.2. Consider the following commutative diagram of the fundamental groups and the induced homomorphisms, where i and j are inclusions:

$$\begin{array}{ccc} \pi_1(U^+) & \xrightarrow{j_*} & \pi_1(U) \\ \Big\downarrow{\scriptstyle \eta_*} & & \Big\downarrow{\scriptstyle \eta_*} \\ \pi_1(W^+ \backslash \eta(\Gamma)) & \xrightarrow{i_*} & \pi_1(\boldsymbol{R}^n \backslash \eta(\Gamma)) \end{array}$$

Since $\eta \colon U \to \boldsymbol{R}^n \backslash \eta(\Gamma)$ is a covering, the fact that U^+ is connected (Lemma 3.2) is equivalent to saying that $\operatorname{im}(i_*)$ meets every coset of $\operatorname{im}(\eta_*)$ in $\pi_1(\boldsymbol{R}^n \backslash \eta(\Gamma))$, [5]. Also the fact that the collar $C \subset U^+$ is a strong deformation of U implies that j_* is onto. Then a simple algebraic argument shows that i_* must be onto. Applying the same argument in reverse to the corresponding diagram with η replaced by μ we obtain the required result.

Lemma 3.4. *W^+ is convex.*

Proof. Let V denote the connected component of $N \backslash \Gamma$ which contains the zero section. So V has index 0. Put $V^+ = V \cap \eta^{-1}(W^+)$, $V^- = V \cap \eta^{-1}(W^-)$. As before $\eta \colon V \to \boldsymbol{R}^n \backslash \eta(\Gamma)$ is a covering and so by Lemma 3.3, V^+ is connected. But $V^+ \cap \eta^{-1} \circ f(M) = \emptyset$, so in particular $V^+ \cap (M \times \{0\}) = \emptyset$. Hence either $V^+ \subset M \times (0, \infty)$ or $V^+ \subset M \times (-\infty, 0)$. Without loss of generality we suppose that $V^+ \subset M \times (0, \infty)$.

Now since $\eta(\Gamma) \subset W^+$ and $\partial V \subset \Gamma$, $\partial V^- \cap \partial V = \emptyset$. Hence $\partial V \subset \partial V^+ \subset M \times [0, \infty)$. Thus, since we certainly cannot have $V \subset M \times [0, \infty)$, we must have $M \times (-\infty, 0) \subset V^-$. In fact, this means that $M \times (-\infty, 0)$ is a connected component of V^- since the zero section does not meet V^-. Note also that $\eta^{-1} \circ f(M) \subset M \times [0, \infty)$. This shows that $M \times (-\infty, 0]$ is a connected component in V of $\eta^{-1}(W^- \cup f(M))$, and so $\eta \colon M \times (-\infty, 0] \to W^- \cup f(M)$ is a covering. But $f(M)$ is covered just once (i.e., by the zero section), and so this covering is in fact a homeomorphism.

Now take any $p \in M$. Consider $x \in \eta(p, t)$ where $t < 0$. We will show that L_x has a unique absolute minimum at p. Observe that $(p, t) \in V^-$ and so $x \in W^-$. Suppose that L_x has an absolute minimum at p' so that $x = \eta(p', t')$ for some t'. Clearly no point $\eta(p', s)$ with s between 0 and t' can belong to $f(M)$ so $\eta(p', s) \in W^-$, and hence $(p', s) \in V^-$, for all s between 0 and t'. But for s sufficiently small this implies $s < 0$. Hence $t' < 0$. Since η is a homeomorphism on $M \times (-\infty, 0]$, $x = \eta(p, t) = \eta(p', t')$ implies $p = p'$. Thus p is the unique absolute minimum of L_x. This means that $f(M)$ lies outside every open ball with boundary touching $f(M)$ at $f(p)$ and centre $\eta(p, t)$ for some $t < 0$. We deduce that, for any $p \in M$, $f(M)$ lies on one side of the tangent hyperplane at $f(p)$. Hence $f(M)$ is the boundary of a convex set which must be W^+.

This completes the proof of Theorem 3.1. The classification theorem is now reduced to classifying totally focal convex hypersurfaces.

The convex hypersurfaces, i.e., the boundaries of convex sets in finite dimensional Euclidean spaces, have been classified [1]. There are two basic types; these are given by embeddings (i) $f_1: \boldsymbol{S}^{n-1} \to \boldsymbol{R}^n$ and (ii) $f_2: \boldsymbol{R}^{n-1} \to \boldsymbol{R}^n$ where $f_2(\boldsymbol{R}^{n-1})$ does not contain any (complete) straight line. The others are given by product embeddings; either $f_1 \times 1: \boldsymbol{S}^{n-1} \times \boldsymbol{R}^d \to \boldsymbol{R}^n \times \boldsymbol{R}^d$ or $f_2 \times 1: \boldsymbol{R}^{n-1} \times \boldsymbol{R}^d \to \boldsymbol{R}^n \times \boldsymbol{R}^d$, or just the inclusion $\boldsymbol{R}^{n-1} \to \boldsymbol{R}^n$ as a flat hyperplane. Of course these are only classified up to Euclidean transformations of $\boldsymbol{R}^n$.

By Proposition 1.2 this reduces our problem to considering the totally focal convex hypersurfaces $f_1: \boldsymbol{S}^{n-1} \to \boldsymbol{R}^n$ and $f_2: \boldsymbol{R}^{n-1} \to \boldsymbol{R}^n$.

4. Special cases

Theorem 4.1. *Let $f: \boldsymbol{S}^{n-1} \to \boldsymbol{R}^n$ be a totally focal embedding, then $\eta(\Gamma)$ consists of a single point $x_0 \in \boldsymbol{R}^n$.*

Proof. We put $M = \boldsymbol{S}^{n-1}$ and use the notation of § 3. If $p \in M$, then the normal line $\{\eta(p, t): t \in \boldsymbol{R}\}$ through p must intersect $f(M)$ in precisely two points, since $f(M) = \partial W^+$ and W^+ is convex. This means that $\eta^{-1} \circ f(M)$ consists of the zero section and some other cross-section in $M \times (0, \infty)$. This in turn implies that $\eta^{-1}(W^+)$ is homeomorphic to $M \times (0, 1)$, and $\eta^{-1}(W^-)$ is homeomorphic to $M \times [(-\infty, 0) \cup (1, \infty)]$.

Now since M is compact the distance function L_x for any $x \in W^-$ must have a critical point of index 0 and a critical point of index $(n-1)$. But $\eta^{-1}(W^-)$ only has two connected components and $\eta^{-1}(W^-) \subset N \backslash \Gamma$, so we can say that one of these, V^-, has index 0 and the other must have index $(n-1)$. Since every connected component of $N \backslash \Gamma$ intersects $\eta^{-1}(W^-)$, this means that $N \backslash \Gamma$ has just two components, one with index 0, and the other with index $(n-1)$.

The Morse index theorem then enables us to deduce that each point $p \in M$ is an umbilic and the result is then well-known if $n > 2$ [4, Vol. II]. For the case $n = 2$ we need to use more of the information available and in fact the method we use will prove the result in general anyway.

We have already shown that $\eta: V^- \to W^-$ is a homeomorphism, so the connected component of $N \backslash \Gamma$ with index 0 must cover $\boldsymbol{R}^n \backslash \eta(\Gamma)$ just once. Hence for every $x \in \boldsymbol{R}^n \backslash \eta(\Gamma)$ the distance function L_x has just one minimum and at least one maximum, but no other critical points. But in this case the Morse inequalities show that L_x must have just one maximum also. Now take any $p \in M$, and then there exists a unique $t_0 > 0$ with $(p, t_0) \in \Gamma$. Let $x_0 = \eta(p, t_0)$, and $z = f(p) - x_0$. Then p is the unique minimum of L_x if $x = x_0 + sz$, $s > 0$. So

$$f(M) \subset \{x: \|x - x_0 - sz\| \geq (1 - s)\|z\|, s > 0\},$$

and it is the unique maximum of L_x if $x = x_0 + sz$, $s < 0$. So $f(M) \subset \{x: \|x - x_0 - sz\| \leq (1 - s)\|z\|, s < 0\}$. We deduce that $f(M) = \{x: \|x - x_0\| = \|z\|\}$, and then $\eta(\Gamma) = \{x_0\}$.

Note that we have in fact shown that embedding is taut [2]. Notice also that the conclusion $\eta(\Gamma)$ is z single point just says that $f(M)$ is a "round" sphere.

Theorem 4.2. *There does not exist a totally focal embedding $f: \boldsymbol{R}^{n-1} \to \boldsymbol{R}^n$ such that $f(\boldsymbol{R}^{n-1})$ does not contain a straight line.*

Proof. Suppose otherwise. We put $M = \boldsymbol{R}^{n-1}$ and use the notation of § 3 so that W^+ is convex but $f(M) = \partial W^+$ does not contain a line. This means that $\overline{W}^+$ contains at least one half-line but no lines.

We require the notion of the characteristic cone [1] C of $\overline{W}^+$. This is defined as follows: we take any point $x \in \overline{W}^+$ and defines $y \in C$ if and only if $x + ty \in \overline{W}^+$ for all $t \geq 0$. It is known that C does not depend on $x \in \overline{W}^+$ and is a closed convex cone. In our case C is nonempty and pointed. This means that there exists at least one supporting hyperplane for C which intersects only at the origin.

We define another cone D associated with C, by putting $z \in D$ if and only if $\langle y, z\rangle > 0$ for all $y \in C$, $y \neq 0$. Clearly D is an open convex cone and, since C is pointed, it is nonempty.

Note that the closure of D is defined by $z \in \bar{D}$ if and only if $\langle y, z\rangle \geq 0$, for all $y \in C$. To see this observe that if $\langle y, z\rangle \geq 0$ and $z' \in D$ so that $\langle y, z'\rangle > 0$ for all $y \in C$, $y \neq 0$, then $(1 - t)z + tz' \in D$ for all t, $0 < t \leq 1$.

We claim that $C \cap D \neq \emptyset$. Suppose not, then, by the Hahn-Banach theorem, since D is open we can separate C and D by a hyperplane which must go through the origin. More precisely we can find $w \neq 0$ (the normal to the hyperplane) such that $\langle w, y\rangle \geq 0$ far all $y \in C$ and $\langle w, z\rangle < 0$ for all $z \in D$. But this means that $w \in D$, $w \neq 0$ and $\langle w, z\rangle < 0$ for all $z \in D$, which is impossible.

The next step is to show that there exists a point $p_0 \in M$ such that $\eta^{-1} \circ f(p_0)$ is a single point. In other words only one normal line passes through $f(p_0)$. Choose $z \in C \cap D$ with $\|z\| = 1$. The idea is to show that $z \in D$ implies that there is some point $p_0 \in M$ with $\boldsymbol{n}_{p_0} = z$, and $z \in C$ implies that for any $q \in M$, $f(q) + z \in \overline{W}^+$. From this we deduce that no other normal line passes through $f(p_0)$.

Consider the height function $\phi: M \to \boldsymbol{R}$ given by $\phi(p) = \langle z, f(p)\rangle$. Suppose $x_0 \in W^+$ and consider the hyperplane $H = \{x: \langle z, x\rangle = \langle z, x_0\rangle\}$. The intersection $H \cap W^+$ is convex, relatively open in H, and contains no half-lines since $z \in D$. Hence $H \cap \overline{W}^+$ is homeomorphic to a closed $(n - 1)$-ball in H. This means that the boundary of $H \cap \overline{W}^+$ in H, which is the level curve $\{p \in M: \phi(p) = \langle z, x_0\rangle\}$, is homeomorphic to an $(n - 2)$-sphere in $M \equiv \boldsymbol{R}^{n-1}$. Hence ϕ must have a maximum or minimum point $p_0 \in M$. In fact $\boldsymbol{n}_{p_0} = z$, and ϕ has an absolute minimum at p_0.

We now claim that there is no other normal through p_0; in other words $\eta(q, t) = f(p_0)$ implies $t = 0$ and thus $q = p_0$. Suppose that $\eta(q, t) = f(q) +$

$t\boldsymbol{n}_q = f(p_0)$. Then $0 \leq \phi(q) - \phi(p_0) < \langle z, f(q) - f(p_0)\rangle = -t\langle z, \boldsymbol{n}_q\rangle$. Further, equality would imply that the tangent planes at p_0 and q were identical and so $z = \boldsymbol{n}_q$. In particular this means $\langle z, \boldsymbol{n}_q\rangle \neq 0$. But $z \in C$ and $f(q) + C \subset \overline{W}^+$ and the tangent hyperplane at $f(q)$ supports $\overline{W}^+$ so $\langle \boldsymbol{n}_q, z\rangle > 0$. Hence $t \leq 0$. But also $f(p_0) \in \overline{W}^+$ so $0 \leq \langle \boldsymbol{n}_q, f(p_0) - f(q)\rangle = t$. Hence $t = 0$ as required.

Thus $\eta^{-1} \circ f(p_0)$ is a single point. This means that $\eta\colon N\backslash\Gamma \to \boldsymbol{R}^n\backslash\eta(\Gamma)$ is a homeomorphism and so $N\backslash\Gamma$ is connected. Hence $\Gamma = \emptyset$ and $f(\boldsymbol{R}^{n-1})$ is a flat hyperplane. This contradicts the hypothesis that $f(\boldsymbol{R}^{n-1})$ does not contain a straight line.

We have now completed the proof of the classification theorem stated in the introduction, § 1, of this paper, apart from the trivial observation that the inclusion $\boldsymbol{R}^{n-1} \subset \boldsymbol{R}^n$ as a flat hyperplane is totally focal.

5. Taut and totally focal embeddings

The classification of totally focal embeddings with higher codimension requires further study. One might conjecture that they must be products of the embeddings given above. However the Möbius band has a totally focal embedding in $\boldsymbol{R}^4$. Such an embedding can be obtained by taking the embedding of the real projective plane as a Veronese surface lying on the sphere $\boldsymbol{S}^4 \subset \boldsymbol{R}^5$ and using a stereographic projection from a point in the image of the projective plane. This embedding is taut (see [2]) and is an example of a more general result given in Theorem 5.2 below. For this theorem we require the following general lemma.

Lemma 5.1. *Let $f\colon M \to \boldsymbol{R}^n$ be an immersion and let $x \in \boldsymbol{R}^n$, $p \in U \subset M$ where U is a neighborhood of p. Suppose that p is the only critical point of the distance function $L_x\colon M \to \boldsymbol{R}$ which lies in U, and that it is nondegenerate with index k. Then there exists a neighborhood E of x in $\boldsymbol{R}^n$ such that if $y \in E$, then L_y has a nondegenerate critical point $q \in U$ which has index k.*

This lemma was proved in [2]. It is a weak version of Lemma 3.1 in [2]. In fact this Lemma 3.1 as it stands is incorrect although the given proof is valid for Lemma 5.1 above, which was all that was needed in the theorems of [2]. The stronger version of this lemma in which q is *unique* can be proved provided that U is *compact*.

Theorem 5.2. *Let $f\colon M \to \boldsymbol{R}^n$ be a taut embedding of a connected manifold where $H_r(M; \boldsymbol{Z}) = \boldsymbol{Z}$ for some $r > 0$ and $H_i(M; \boldsymbol{Z}) = 0$ if $i \neq 0, r$. Then f is totally focal.*

Proof. Observe that, by hypothesis, any nondegenerate distance function L_y, $y \in \boldsymbol{R}^n$, has precisely two critical points, one with index 0 and one with index r.

Now suppose that f is not totally focal so that there exists $x \in \boldsymbol{R}^n$ such that L_x has both a degenerate critical point p and a nondegenerate critical point q of index k, say.

Let U, V be disjoint open sets in M with $p \in U$, $q \in V$. Applying Lemma 5.1 we can find an open set $E \subset \boldsymbol{R}^n$ with $x \in E$ such that if $y \in E$, then L_y has a nondegenerate critical point of index k in V. We can find $y \in E$, lying on the line segment joining $f(p)$ to x. Then p is a nondegenerate critical point of L_y with index l, say. Again applying Lemma 5.1 we find an open set $E' \subset \boldsymbol{R}^n$ with $y \in E'$ such that if $z \in E'$, then L_z has a nondegenerate critical point in U with index l. We can choose $z \in E \cap E'$ so that L_z is nondegenerate. Then L_z has only two critical points which must be of index 0 and r. Hence $k = r$, $l = 0$ or $k = 0$, $l = r$.

Using a similar argument, but choosing $y \in E$, lying on the extension of the line joining $f(p)$ to x, so that p is a nondegenerate critical point of L_y with index $l' > l$, we can find $z \in \boldsymbol{R}^n$ such that L_z is nondegenerate and has a critical point in V with index k and another in U with index $l' > l$. We deduce $l' = 0$, $k = r$ or $l' = r$ and $k = 0$ which contradicts the above.

Hence no such point x exists and thus the embedding is totally focal.

Note that this result only gives more information about totally focal embeddings if $2r < \dim M < n - 1$ since otherwise we could apply Theorem 3.10 of [2] to deduce that f is essentially one of the embeddings already described in the classification theorem.

Note also that it makes sense to consider totally focal immersions but again some details need further investigation. However one would conjecture that these must be just covering projections combined with totally focal embeddings.

References

[1] H. Busemann, *Convex surfaces,* Interscience, New York, 1958.

[2] S. Carter & A. West, *Tight and taut immersions,* Proc. London Math. Soc. **25** (1972) 701–720.

[3] ——, *A characterisation of certain embedded spheres,* J. London Math. Soc. **7** (1973) 441–452.

[4] S. Kobayashi & K. Nomizu, *Foundations of differential geometry*, Vols. I and II, Interscience, New York, 1963 and 1969.

[5] W. S. Massey, *Algebraic topology: an introduction,* Harcourt, Brace and World, New York, 1967.

[6] J. Milnor, *Morse theory,* Annals of Math. Studies, No. 51, Princeton University Press, Princeton, 1963.

[7] S. Willard, *General topology,* Addison-Wesley, Reading, Massachusetts, 1971.

UNIVERSITY OF LEEDS, ENGLAND