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INTEGRAL INVARIANTS OF CONVEX CONES

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Introduction

Let E be an $(n + 1)$-dimensional real vector space, E^* its dual space,[1] K a convex nondegenerate pointed cone in E, and K^* the dual cone in E^*. It is our main purpose to study geometric objects in K (and K^*) from the viewpoint of invariance under transformations of the general linear group $GL(n + 1, R)$, and of the unimodular group $SL(n + 1, R)$. So, this matter will occupy most of the present work. However, since there is a natural correspondence between flat bounded cross-sections of $(n + 1)$-dimensional convex cones and n-dimensional convex bodies, our first chapter will be somehow diverse from that main object.

More precisely, if B is a convex body with nonempty interior, relative to an n-dimensional affine space F, we can imbed F in E as a hyperplane not passing through the origin and define a convex nondegenerate pointed cone in E by

$$K(B) = \{\lambda X: X \in B, \lambda > 0\} .$$

Conversely, given E, E^*, K and K^* as above, for each nonzero $\mathscr{X} \in E^*$ we can define a hyperplane $P_{\mathscr{X}} \subset E$ by

$$P_{\mathscr{X}} = \{X: X \in E, \mathscr{X} \cdot X = 1\} .$$

$P_{\mathscr{X}} \cap K$ is a convex body with nonempty interior, relative to the n-dimensional affine space $P_{\mathscr{Y}}$ if and only if $\mathscr{X} \in \operatorname{Int}(K^*)$. This correspondence suggests a connection between geometrical properties of $(n + 1)$-dimensional convex cones and those of n-dimensional convex bodies.

In § 1 we study a class of real valued functionals on the set of convex bodies with nonempty interior, relative to an n-dimensional affine space F; these functionals will be invariant under the action of the group $AGL(n, R)$ of all affine transformations as acting on F, and one of them, which we shall call the mean square fractional volume, will play a fundamental role in the sections which follow.

The volume of truncated cones in K can be expressed in a natural way by

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means of a function defined on Int (K^*). The properties of this volume function $V(\mathscr{X})$ are discussed in § 2. In particular it is shown that $V(\mathscr{X})$ is analytic, convex, positively homogeneous of degree $-n-1$, and infinite at the boundary.

In § 3 we show that Int (K^*) and Int (K) are analytically diffeomorphic in a canonical way. This realizes, for each $c > 0$, the identification as abstract manifolds of the level hypersurface M_c^* of $V(\mathscr{X})$, in K^*, with its "dual" hypersurface $M_c \subset K$, which is characterized as being the envelope of the hyperplanes $P_{\mathscr{X}}$ as $\mathscr{X}$ ranges over M_c^*. We shall call M_c the constant volume envelope of hyperplanes relative to the cone K with volume equal to c. Some general properties of M_c are also discussed in this section.

For each $\mathscr{X}^c \in M_c^*$, let us call $X_c \in M_c$ the corresponding element under the above identification. We show in § 4 that the quadratic form $\mathscr{X}^c \cdot d^2 X_c = -d\mathscr{X}^c \cdot dX_c = d^2\mathscr{X}^c \cdot X_c$ is positive definite as $\mathscr{X}^c$ ranges over M_c^*, and hence it defines a natural Riemannian structure on M_c^* (or M_c). As hypersurfaces immersed in the (affine) spaces E and E^* respectively, M_c and M_c^* can also be equiped with the so called Berwald-Blaschke Riemannian metric, denoted by ds_B^2 and $ds_{B^*}^2$. In addition to relating these metrics, we also compare their volume elements to two more volume elements induced by the immersions of M_c and M_c^* in the vector spaces E and E^*. Furthermore, we extend the comparison to the Laplace-Beltrami operators, and prescribe a suitable condition for the manifolds (M_c, ds_B^2) and $(M_c^*, ds_{B^*}^2)$ to be affine hyperspheres.

The mean square fractional volume for a convex body, mentioned above, leads to the concept of the volume ratio function of a convex cone, defined on Int (K^*) (or Int (K)). All of the relations mentioned in the previous paragraph are expressed in terms of this function, which we shall denote by ρ_K. In § 5 we show that the asymptotic behaviour of $\rho_K(X)$, as X approaches infinity in the direction of any ray in the boundary of K, depends only on the local behavior of the boundary of K near that ray.

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1. Affine invariant functionals of convex bodies

The subject matter of this section originates historically with a problem by J. J. Sylvester, to which a first solution was given by M. W. Crofton. Our treatment has been inspired by the work of Blaschke in the plane (cf. [1]), who uses the generalization of a lemma by Carleman (cf. [2]):

Let F be an n-dimensional real vector space. If we choose a nonzero n-covector $\Delta \in \Lambda^n(F^*)$, then the (Δ)-volume of the parallelepiped

$$\left\{X: X = X_0 + \sum_i r_i(X_i - X_0), 0 < r_i < 1\right\}$$

will be denoted by

$$|[X_1 - X_0, \cdots, X_n - X_0]| = |\Delta(X_1 - X_0, \cdots, X_n - X_0)|,$$

and if $X: R^n \to F$ is an affine coordinate map, the volume element associated with X, relative to Δ, will be denoted by

$$\omega_X = \frac{1}{n!}[dX, dX, \cdots, dX],$$

where [ ] denotes a determinant.

Let $\mathfrak{B}$ be the set of convex bodies with nonempty interior relative to F. For $B \in \mathfrak{B}$, we denote by $V(B)$ the volume of B, and by $\langle X_0, X_1, \cdots, X_n \rangle_B$ the (absolute) fractional volume of the simplex with vertices $X_0, \cdots, X_n$, with respect to B:

$$\langle X_0, X_1, \cdots, X_n \rangle_B = \frac{1}{n!}|[X_1 - X_0, \cdots, X_n - X_0]|/V(B).$$

We define the following class of real valued functionals on $\mathfrak{B}$: if $f \in R^{[0,1]}$ is any integrable, real valued function defined on the closed unit interval, the functional $\mu_f: \mathfrak{B} \to R$ is defined to be the average value of $f(\langle X_0, \cdots, X_n \rangle_B)$ on B, i.e.,

$$\mu_f(B) = (V(B))^{-n-1} \int_{B^{n+1}} f(\langle X_0, \cdots, X_n \rangle_B) \omega_{X_0} \times \cdots \times \omega_{X_n}.$$

It is clear that the functional μ_f is invariant under the group $AGL(n, R)$ of all affine transformations as acting on F. This fact makes μ_f independent of the choice of the nonzero element $\Delta \in \Lambda^n(F^*)$.

We shall denote by $\mathfrak{F} \subset R^{[0,1]}$ the subset of those functions which are *strictly monotone increasing*. In the special case where f is defined by $f(t) = t^2$, we shall denote μ_f by μ and call it the *mean square fractional volume*. In the following sections we find some geometrical applications for this particular functional μ.

For the time being, we concentrate our attention to the subclass of functionals indexed by the set $\mathfrak{F}$ and state the following two propositions:

Proposition 1.1. *For any function $f \in \mathfrak{F}$, the functional μ_f has a greatest lower bound $\mu_f > f(0)$, which is attained only in the case where the boundary of B is an ellipsoid.*

Proposition 1.2. *For any function $f \in \mathfrak{F}$, the functional μ_f has a least upper bound $\bar{\mu}_f < f(1)$; when $n = 2$ this bound is attained only in the case where B is a nondegenerate simplex (i.e., a triangle).*

We shall preface the proof of the above propositions by some remarks and a couple of lemmas.

Let S be an affine line in F. We choose an affine coordinate system in F so as to have S as one of the axes, and call s the corresponding coordinate function. Let $S_0, S_1, \cdots, S_n$ be straight line segments in F parallel to S, m the midpoint of each S_i, and s_i the restrictions of the coordinate function s to the affine line containing S_i.

For any strictly monotone increasing function $g\colon R^+ \to R$ we define

$$J_g(X_0; S_1, \cdots, S_n) = \int_{S_1} \cdots \int_{S_n} g(\langle X_0, X_1, \cdots, X_n \rangle) ds_1 \times \cdots \times ds_n ,$$

where

$$\langle X_0, X_1, \cdots, X_n \rangle = \frac{1}{n!} |[X_1 - X_0, \cdots, X_n - X_0]|$$

denotes the (absolute) volume of the simplex with vertices $X_0, X_1, \cdots, X_n$.

We want to study the variations of the above integral when X_0 ranges over any affine line parallel to S. Obviously, if $m_1, m_2, \cdots, m_n$ are all contained in a hyperplane parallel to S (including the case where they may lie in an affine subspace of dimension strictly less than $n-1$), $J_g(X_0; S_1, \cdots, S_n)$ is constant with respect to X_0. In fact, in this case $\langle X_0, X_1, \cdots, X_n \rangle$ is in itself constant with respect to the variation of X_0, for any choice of the remaining X_i's. However, if $m_1, m_2, \cdots, m_n$ actually determine a hyperplane M transversal to S, the value of $J_g(X_0; S_1, \cdots, S_n)$ depends on the absolute value of the difference in s-coordinates between X_0 and its projection on M, parallel to S. More precisely, calling A_0 this projection and assuming that M is characterized as

$$M = \{X\colon X \in F, s(X) = 0\} ,$$

we have

Lemma 1.3. *Within the above conditions, $J_g(X_0; S_1, \cdots, S_n)$ is a strictly monotone increasing function of $|s(X_0)|$.*

Proof. First, we observe that by extending g to be an even function, i.e., by defining $g\colon R \to R$, $g_0(t) = g(|t|)$, we can write

$$\begin{aligned} &J_g(X_0; S_1, \cdots, S_n) \\ &\quad = \int_{S_1} \cdots \int_{S_n} g_0\Big(\frac{1}{n!}[X_1 - X_0, \cdots, X_n - X_0]\Big) \mathrm{ds}_1 \times \cdots \times ds_n . \end{aligned}$$

Next, calling A_1 the point on the affine line passing through A_0, parallel to S, such that $s(A_1) = 1$, and writing

$$X_0 = A_0 + s(X_0)(A_1 - A_0) ,$$
$$X_i = m_i + s(X_i)(A_1 - A_0) , \qquad i = 1, \cdots, n ,$$

we have

$$\frac{1}{n!}[X_1 - X_0, \cdots, X_n - X_0] = a_0 s(X_0) + \sum_i a_i s(X_i) ,$$

where

$$a_0 = \frac{1}{n!}[m_1 - A_1, m_2 - A_1, \cdots, m_n - A_1] ,$$

$$a_i = \frac{1}{n!}[m_1 - A_0, \cdots, m_{i-1} - A_0, A_1 - A_0, m_{i+1} - A_0, \cdots, m_n - A_0] ,$$
$$i = 1, \cdots, n .$$

The conditions prescribed assure that a_0 is always different from zero. Let us assume, for the time being, that all of the remaining a_i's are also nonzero. Then, denoting by $\overline{Z}_i$, $\underline{Z}_i$ the endpoints of each segment S_i, such that

$$b_i = s(\overline{Z}_i) > s(\underline{Z}_i) = -b_i ,$$

we define inductively the following set of functions: g_0 as above,

$$G_i\colon R \rightarrow R, \text{ by } G_i(x) = \int_0^x g_i(t)dt , \qquad i = 0, 1, \cdots, n - 1 ,$$

$$g_i\colon R \rightarrow R, \text{ by } g_i(x) = \frac{1}{a_i}\{G_{i-1}(x + a_i b_i) - G_{i-1}(x - a_i b_i)\} ,$$
$$i = 1, \cdots, n .$$

All of the functions g_i's have the two leading properties of g_0, namely, all are even, and, for each i, the restriction of g_i to R^+ is strictly monotone increasing. Therefore, since $J_g(X_0; S_1, \cdots, S_n) = g_n(a_0 s(X_0))$, the lemma would be proved. Finally, if for some $i \neq 0$, the corresponding $a_i = 0$, (let us observe, though, that not all of them can be zero), for that particular i we put, instead of the above,

$$g_i(x) = 2b_i g_{i-1}(x) ,$$

and proceed to the next inductive step. This concludes the proof of the lemma.

With the same notation as above, let S_0 be another straight line segment also parallel to S, and m_0 its midpoint. Let us define now

$$I_g(S_0, \cdots, S_n) = \int_{S_0} \cdots \int_{S_n} g(\langle X_0, \cdots, X_n \rangle) ds_0 \times \cdots \times ds_n$$
$$= \int_{S_0} J_g(X_0; S_1, \cdots, S_n) ds_0 .$$

We shall study the variation of the above integral in the case where the segments $S_1, S_2, \cdots, S_n$ are kept fixed, while S_0 is translated parallel to itself along the affine line in which it is contained, i.e., the relative rigid displacements of S_0, parallel to S, with respect to the remaining S_i's.

Lemma 1.4. *If the points $m_1, m_2, \cdots, m_n$ determine an affine hyperplane M, transversal to S, then $I_g(S_0, S_1, \cdots, S_n)$ is a strictly monotone increasing function of $|s(m_0)|$.*

Proof. We can assume, without loss of generality, that $s(m_0) \geq 0$. Then, with $h > 0$, we call ${}_hS_0$ the segment obtained by translating S_0 parallel to itself along the affine line in which it is contained, h units in the positive direction. Let $\bar{Z}_0$, $\underline{Z}_0$ be the endpoints of S_0, such that $s(\bar{Z}_0) > s(\underline{Z}_0)$, and ${}_h\bar{Z}_0$, ${}_h\underline{Z}_0$ be the corresponding endpoints of ${}_hS_0$. If for each point X_0 in the segment $\overline{\bar{Z}_0, {}_h\bar{Z}_0}$, we call X_0' the point in the segment $\overline{\underline{Z}_0, {}_h\underline{Z}_0}$ such that $s(X_0) - s(X_0') = s(\bar{Z}_0) - s(\underline{Z}_0)$, then obviously

$$|s(X_0)| \geq |s(X_0')| ,$$

equality holding only for $X_0 = \bar{Z}_0$ and for the special case where $s(m_0) = 0$. This together with Lemma 1.3 imply that

$$I_g({}_hS_0, S_1, \cdots, S_n) - I_g(S_0, S_1, \cdots, S_n)$$
$$= \int_{s(\bar{Z}_0)}^{s({}_h\bar{Z}_0)} J_g(X_0; S_1, \cdots, S_n) ds_0 - \int_{s(\underline{Z}_0)}^{s({}_h\underline{Z}_0)} J_g(X_0; S_1, \cdots, S_n) ds_0 > 0 ,$$

which concludes the proof of the lemma.

Proof of Proposition 1.1. Let S be an affine line in F, and H be any affine hyperplane *transversal* to S. We assume that an affine coordinate system is defined in F so as to have S as the last axis, the remaining axes being contained in H. More precisely, let X_0 be the point where S meets H, X_n be another point in S different from X_0, and $X_1, \cdots, X_n$ be points in H such that $X_1 - X_0, \cdots, X_n - X_0$ are linearly independent. Then we define an affine coordinate map X: $R^n \rightarrow F$ by

$$X(u_1, \cdots, u_{n-1}, s) = X_0 + s(X_n - X_0) + \sum_i u_i(X_i - X_0) .$$

The volume element associated with X is given by

$$\omega_X = [X_1 - X_0, \cdots, X_n - X_0] du_1 \wedge \cdots \wedge du_{n-1} \wedge ds ,$$

and we can without loss of generality normalize the constant factor in this equation, and also write

$$\omega_X = dH \wedge ds$$

with

$$dH = du_1 \wedge \cdots \wedge du_{n-1} .$$

Let $B \in \mathfrak{B}$ be a convex body. If p_H denotes the projection on H parallel to S, and $\underline{g}, \bar{g}: p_H(B) \to R$ are suitable continuous functions, then we can characterized B as

$$B = \{(Y, s): Y \in p_H(B), \underline{g}(Y) < s < \bar{g}(Y)\} .$$

If we symmetrize B in the sense of Steiner's respect to H, in the direction of S, we obtain a convex body $B' \in \mathfrak{B}$, which is g ven by

$$B' = \{(Y, s): Y \in p_H(B), \tfrac{1}{2}(\underline{g} - \bar{g})(Y) < s < \tfrac{1}{2}(\bar{g} - \underline{g})(Y)\};$$

more descriptively, each affine line parallel to S, intersecting B nontrivially, determines a segment. Each of these segments is imagined to slide in such a way that its center falls on H. The transformation $B \mapsto B'$ is obviously volume-preserving, and we can assume without loss of generallity that $V(B) = V(B') = 1$. The affine invariant functional μ_f, as applied to B and B', can thus be written

$$\mu_f(B) = \int_{B^{n+1}} f(\langle X_0, \cdots, X_n \rangle)\omega_{X_0} \times \cdots \times \omega_{X_n} ,$$

$$\mu_f(B') = \int_{B'^{n+1}} f(\langle X_0, \cdots, X_n \rangle)\omega_{X_0} \times \cdots \times \omega_{X_n} .$$

By using Fubini's theorem, we can also write

$$\mu_f(B) = \int_{p_H(B)^{n+1}} dH_0 \times \cdots \times dH_n \int_{[S]} f(X_0, \cdots, X_n \rangle)ds_0 \times \cdots \times ds_n ,$$

$$\mu_f(B') = \int_{p_H(B')^{n+1}} dH_0 \times \cdots \times dH_n$$

$$\cdot \int_{[S']} f(\langle X_0, \times \cdots, \times X_n \rangle)ds_0 \times \cdots \times ds_n ,$$

where

$$[S] = \prod_{i=0}^{n} \{(Y_i, s): g(Y_i) < s < \bar{g}(Y_i)\} ,$$

$$[S'] = \prod_{i=0}^{n} \{(Y_i, s): \tfrac{1}{2}(\underline{g} - \bar{g})(Y_i) < s < \tfrac{1}{2}(\bar{g} - \underline{g})(Y_i)\} .$$

Hence as a consequence of Lemma 1.4 we get that

$$\mu_f(B) \geq \mu_f(B') .$$

The equality sign can hold only if B is affinely symmetric in the direction of S, i.e., if the set

$$\{(Y, s): Y \in p_H(B), s = \tfrac{1}{2}(\bar{g}(Y) + \underline{g}(Y))\}$$

is contained in an affine hyperplane of F. Therefore $\mu_f(B)$ can be minimal only if for every direction the centers of the corresponding set of parallel chords of B lie on the same affine hyperplane, i.e., only if B is bounded by an ellipsoid.

In order to prove that the minimum is actually achieved, we use a suitable sequence $\{S_j\}$ of affine lines, and symmetrize successively $B = B_0$ with respect to suitable affine hyperplanes H_j so as to get a sequence $\{B_j\} \subset \mathfrak{B}$ of convex bodies converging to an *ellipsoidal* convex body B_∞. Thus for a given $\varepsilon > 0$ there exists a j_0 such that for every $j > j_0$

$$B_\infty \subset (1 + \varepsilon)B_j ,$$

which implies that for every $j > j_0$

$$\mu_f(B_\infty) < \int_{((1+\varepsilon)B_j)^{n+1}} f(\langle X_0, \cdots, X_n \rangle)\omega_{X_0} \times \cdots \times \omega_{X_n} ,$$

where in the last integral we extend trivially, if necessary, the domain of definition of f by putting, for example, $f(t) = f(1)$ for $t > 1$. But observing that the right-hand side of the above inequality can be written as

$$\mu_f(B_j) + \sum_{p=1}^{n+1} \binom{n+1}{p} \int_{[B_{j,p}(\varepsilon)]} f(\langle X_0, \cdots, X_n \rangle)\omega_{X_0} \times \cdots \times \omega_{X_n} ,$$

where

$$[B_{j,p}(\varepsilon)] = ((1 + \varepsilon)B_j - B_j)^p \times (B_j)^{n-p} ,$$

we find that for every $j > j_0$

$$\mu_f(B_\infty) < \mu_f(B_j) + f(1) \sum_{p=1}^{n+1} \binom{n+1}{p} \left(\sum_{h=1}^{n} \binom{n}{h} \varepsilon^h \right)^p .$$

This inequality, together with the fact that the sequence $\mu_f(B_j)$ is monotone descreasing, concludes the proof of the proposition.

As for proposition 1.2, the first part of the statement is fairly obvious. A proof of the second part can be found in [1, § 25], while it is still an open question to prove that the proposition is true with the restriction $n = 2$ removed.

In § 4, as an application of the differential geometric properties of the constant volume envelopes, we shall compute the actual values of the mean square fractional volume in two distinguished cases, namely, the minimum value (achieved, as we have just proved, only by the interior of an ellipsoid), and in the case of a nondegenerate simplex, which is conjectured to represent the maximum.

2. Convex cones: The volume function

Let E be an $(n + 1)$-dimensional real vector space, and E^* its dual space, Unless specified otherwise, the elements of E will be denoted by capital letters $X, Y, Z, \cdots$, etc., and those of E^* by script capital letters $\mathscr{X}, \mathscr{Y}, \cdots$, etc. The scalar product describing the duality between E and E^* will be denoted by a dot. Thus for $\mathscr{X} \in E^*$ and $X \in E$ the map $(\mathscr{X}, X) \to \mathscr{X} \cdot X$ is a bilinear function from $E^* \times E$ to the field of real numbers.

A subset $K \subset E$ is called a convex cone if

(1) $X \in E$, $r > 0$ imply $rX \in K$,

(2) $X, Y \in K$ imply $X + Y \in K$.

The set defined by

$$K^* = \{\mathscr{X} : \mathscr{X} \in E^*, \mathscr{X} \cdot X \geq 0 \text{ for every } X \in K\}$$

is a closed convex cone in E^*, called the *dual cone of* K. In using the qualification "closed", for future reference we assume that E and E^* are provided with their natural topologies as real vector spaces.

The convex cone K is said to be *pointed* if

(3) Int (K^*) is nonempty (equivalently, K contains no affine line),

and to be *nondegenerate* if

(4) Int (K) is nonempty.

In what follows, those sets K satisfying conditions (1) through (4) will be called, for brevity, *convex cones.*

Given a convex cone K, it is easy to see that Int (K^*) is characterized by

$$\text{Int}\,(K^*) = \{\mathscr{X} : \mathscr{X} \in E^*, \mathscr{X} \cdot X > 0 \text{ for every } X \in \bar{K} - \{0\}\} .$$

For each nonzero $\mathscr{X} \in E^*$ we define a hyperplane $P_{\mathscr{X}} \subset E$ by

$$P_{\mathscr{X}} = \{X : X \in E, \mathscr{X} \cdot X = 1\} .$$

$P_{\mathscr{X}}$ has a nonempty intersection with Int (K) if and only if $\mathscr{X} \in$ Int (K^*). Moreover, $P_{\mathscr{X}} \cap K$ is a convex body with nonempty interior relative to $P_{\mathscr{X}}$. Correspondingly, the set defined by

$$K_{\mathscr{X}} = \{X: X \in K, \mathscr{X} \cdot X < 1\}$$

is a convex body with nonempty interior in E. We shall call the sets of the form $K_{\mathscr{X}}$, for $\mathscr{X} \in \text{Int}(K^*)$, *truncated cones.*

The invariants of E under the action of $SL(n+1, R)$ are generated by a nonzero exterior $(n+1)$-form $D \in \Lambda^{n+1}(E^*)$, which we choose once and for all; its absolute value represents an invariant measure under translations. In the sequel we will abbreviate $D(A_0, A_1, \cdots, A_n)$ by denoting its value by $[A_0, A_1, \cdots, A_n]$. We also consider the corresponding dual $(n+1)$-fom D^* in $\Lambda^{n+1}(E)$, and similarly $D^*(\mathscr{A}_0, \cdots, \mathscr{A}_n)$ is denoted by $[\mathscr{A}_0, \cdots, \mathscr{A}_n]$ and defined by the equation

$$[\mathscr{A}_0, \mathscr{A}_1, \cdots, \mathscr{A}_n] \cdot [A_0, A_1, \cdots, A_n] = \det(\mathscr{A}_i \cdot A_j) .$$

The choice of D induces in a natural way geometrically natural volume elements on certain immersed hypersurfaces in E. Let M be an n-dimensional abstract differentiable manifold, together with a differentiable immersion $Y: M \to E$, then by writing the differential of the map in terms of local parameters $(t_1, \cdots, t_n)$ as

$$dY = \sum_i \partial_i Y \otimes dt_i$$

we consider the n-form defined by

$$[Y, (dY)^n] = [Y, dY, \cdots, dY] = n!\, [Y, \partial_1 Y, \cdots, \partial_n Y] dt_1 \wedge \cdots \wedge dt_n .$$

The immersion is said to be *radially transversal* if $[Y, (dY)^n]$ is uniformly nonzero on M; in this case the n-form

$$\frac{1}{n!}[Y, (dY)^n]$$

will be called the *prismatic volume element* of the immersion.

Let U be a bounded connected open subset of R^n. An immersion $Y: U \to E$ is said to be *cross-sectional to the convex cone* K if it satisfies the following conditions:

(i) Y is of class C^1,

(ii) each ray in $\bar{K}$ meets $Y(U)$ in exactly one point,

(iii) Y is radially transversal.

Given a convex cone K, the existence of such Y is obvious; for instance $Y(U)$ could be a properly chosen relative open subset of an affine hyperplane, or a suitable subdomain of a euclidean hypersphere.

Let $\mathscr{X} \in \operatorname{Int}(K^*)$. If $Y: U \to E$ is cross-sectional to K, then so is

$$\bar{Y} = \frac{Y}{\mathscr{X} \cdot Y}.$$

The latter map immerses U into the hyperplane $P_{\mathscr{X}} = \{X: \mathscr{X} \cdot X = 1\}$. Furthermore, we have

$$[\bar{Y}, (d\bar{Y})^n] = \frac{[Y, (dY)^n]}{(\mathscr{X} \cdot Y)^{n+1}},$$

from which we deduce that if $Z: V \to E$ is another such immersion, $\mathscr{X} \in \operatorname{Int}(K^*)$ and $\bar{Z} = \dfrac{Z}{\mathscr{X} \cdot Z}$, then

$$\int_{P_{\mathscr{X}} \cap K} [\bar{Y}, (d\bar{Y})^n] = \int_{P_{\mathscr{X}} \cap K} [\bar{Z}, (d\bar{Z})^n],$$

where we have denoted, by definition,

$$\int_{P_{\mathscr{X}} \cap K} [\bar{Y}, (d\bar{Y})^n] = \int_{Y^{-1}(K)} [\bar{Y}, (d\bar{Y})^n],$$

and similarly for the right-hand side. From now on we shall omit the domain of integration in the cases where there is no possibility of confusion. In every instance integration shall be interpreted in the sense just defined.

The volume of the truncated cone $K_{\mathscr{X}}$, for any $\mathscr{X} \in \operatorname{Int}(K^*)$, can therefore be written as

$$V(\mathscr{X}) = \frac{1}{(n+1)!} \int [\bar{Y}, (d\bar{Y})^n] = \frac{1}{(n+1);} \int \frac{[Y, (dY)^n]}{(\mathscr{X} \cdot Y)^{n+1}}. \tag{2.1}$$

The map $V: \operatorname{Int}(K^*) \to R$ thus introduced is a real analytic function, positively homogeneous of degree $-n-1$, and will be called the *volume function* associated with the convex cone K.

We can get a second integral representation for $V(\mathscr{X})$ as follows:

Let $\mathscr{X} \in \operatorname{Int}(K^*)$. Then taking A to be any element in E such that $\mathscr{X} \cdot A \neq 0$, writing

$$\bar{A} = \frac{A}{\mathscr{X} \cdot A}, \qquad \bar{Y} = \frac{Y}{\mathscr{X} \cdot Y},$$

and using the fact that Y is radially transversal, we have

$$[\bar{A}, (d\bar{Y})^n] = [\bar{Y}, (d\bar{Y})^n] ,$$

so that

$$(2.2)\quad V(\mathscr{X}) = \frac{1}{(n+1)!}\int [\bar{A}, (d\bar{Y})^n] = \frac{1}{(n+1)!}\,\frac{1}{\mathscr{X}\cdot A}\int [A, (d\bar{Y})^n] .$$

Finally, it is also possible to represent the volume function by means of an $(n-1)$-dimensional integral:

Since the $(n-1)$-form $[\bar{A}, \bar{Y}; (d\bar{Y})^{n-1}]$ satisfies

$$d([\bar{A}, \bar{Y}, (d\bar{Y})^{n-1}]) = [\bar{A}, (d\bar{Y})^n] ,$$

from Stoke's theorem it follows that

$$(2.3)\qquad V(\mathscr{X}) = \frac{1}{(n+1)!}\int_{P_{\mathscr{X}}\cap\partial K} [\bar{A}, \bar{Y}, (d\bar{Y})^{n-1}] .$$

During the course of the present work we will need to use the volume function only as given by expressions (2.1) and (2.2), leaving (2.3) for possible future use.

Proposition 2.1. *The volume function $V(\mathscr{X})$ tends $+\infty$ when $\mathscr{X}$ approaches any point on the boundary of K^*.*

Proof. Let $\{\mathscr{X}_j\}$ be a sequence of points in Int (K^*) converging to the point $\mathscr{X}_0$ on the boundary of K^*. We have $\mathscr{X}_0 \cdot Y(m) \geq 0$ for every point m on the compact set $Y^{-1}(\bar{K})$. So, if $\mathscr{X}_0 = 0$, the result follows at once from (2.1). Let us assume next that $\mathscr{X}_0 \neq 0$. Then there exists a point $A = Y(m_0)$ on the boundary of K such that $\mathscr{X}_0 \cdot A = 0$, while $\mathscr{X}_0 \cdot Y(m) > 0$ for every $Y(m) \in \mathrm{Int}(K)$. Thus using this same A, and substituting $\mathscr{X}_j$ for $\mathscr{X}$ in the right-hand side member of (2.2), we prove the proposition since the integral is uniformly bounded away from zero in terms of j for every $\mathscr{X}_j$ in a suitable neighborhood of $\mathscr{X}_0$, and $(\mathscr{X}_j \cdot A) \to 0$ as $\mathscr{X}_j \to \mathscr{X}_0$.

From now on we will need to use the following extensions of the scalar product between E^* and E to vector-valued and tensor valued products.

Let M be an abstract differentiable manifold, and $\mathscr{X}: M \to E^*$ a differentiable immersion. Write the differential of the map in terms of local coordinates $(t_1, \cdots, t_h)$ as

$$d\mathscr{X} = \sum_i \partial_i\mathscr{X} \otimes dt_i ,$$

and put, for $X \in E$,

$$d\mathscr{X}\cdot X = \sum_i (\partial_i\mathscr{X}\cdot X)dt_i , \qquad d\mathscr{X}\cdot(X\otimes X) = (d\mathscr{X}\cdot X)\otimes X ,$$

$$(d\mathscr{X} \otimes d\mathscr{X})\cdot(X \otimes X) = \sum_{i,j} (\partial_i \mathscr{X} \cdot X)(\partial_j \mathscr{X} \cdot X) dt_i \otimes dt_j \,,$$

$$d^2\mathscr{X} \cdot X = \sum_{i,j} (\partial_{ij}\mathscr{X} \cdot X) dt_i \otimes dt_j \,.$$

It is easy to check that these products are well defined. We go on to prove an important differential property of the volume function.

Proposition 2.2. *The volume function is a strongly convex function at each point* $\mathscr{X} \in \operatorname{Int}(K^*)$ *in the sense that the quadratic form* d^2V *is definite everywhere.*

Proof. Using one of the above definitions, from (2.1) we get that $d^2V(\mathscr{X})$ is given in any *affine* coordinate system by

$$d^2V(\mathscr{X}) = \frac{(n+2)}{n!}(d\mathscr{X} \otimes d\mathscr{X})\cdot \int \frac{Y \otimes Y}{(\mathscr{X} \cdot Y)^{n+3}}[Y, (dY)^n] \,.$$

It is fairly obvious to observe that this quadratic form is definite at each $\mathscr{X} \in \operatorname{Int}(K^*)$.

Note. Since $\operatorname{Int}(K^*)$ is a convex set, Proposition 2.2 implies, in particular, that V is a strictly convex function. This latter property can also be obtained independently from the same expression (2.1) by putting $t = \mathscr{X} \cdot Y$ and using the fact that the function $t \mapsto \dfrac{1}{t^{n+1}}$ is strictly convex on the interval $(0, \infty)$.

3. Constant volume envelopes: General properties

As we pointed out in the previous section, for any $\mathscr{X} \in \operatorname{Int}(K^*)$, the affine hyperplane $P_{\mathscr{X}} = \{X : \mathscr{X} \cdot X = 1\}$ intersects K in a convex body with nonempty interior relative to $P_{\mathscr{X}}$. There is a point in this hyperplane which should be of particular geometrical interest in connection with our convex cone K, namely the barycenter of the convex body $B_{\mathscr{X}} = P_{\mathscr{X}} \cap K$.

If for each $\mathscr{X} \in \operatorname{Int}(K^*)$ we define

$$Z(\mathscr{X}) = \frac{1}{(n+1)!}\int \frac{Y}{(\mathscr{X} \cdot Y)^{n+2}}[Y, (dY)^n] \,,$$

the barycenter of $B_{\mathscr{X}}$ is given by

$$X(\mathscr{X}) = \frac{Z(\mathscr{X})}{V(\mathscr{X})} \,.$$

Lemma 3.1. *The map* $\mathscr{X} \to X(\mathscr{X})$ *is an analytic diffeomorphism from* $\operatorname{Int}(K^*)$ *onto* $\operatorname{Int}(K)$. *For each* $\mathscr{X} \in \operatorname{Int}(K^*)$, *the hyperplane* $P_{\mathscr{X}}$ *minimizes the volume which it cuts out of* K, *among all hyperplanes passing through* $X(\mathscr{X})$.

Proof. Since the volume function $V(\mathscr{X})$ is analytic and uniformly positive on Int (K^*), it is obvious that both of the above maps $\mathscr{X} \mapsto Z(\mathscr{X})$ and $\mathscr{X} \mapsto X(\mathscr{X})$ are analytic functions. We also observe that both are positively homogeneous: the first of degree $-n-2$, the second of degree -1. Hence either of them maps each ray in Int (K^*) *bijectively* onto some ray in Int (K). From the expression of $Z(\mathscr{X})$ above we get

$$-dZ(\mathscr{X}) = \frac{(n+2)}{(n+1)!} d\mathscr{X} \cdot \int \frac{Y \otimes Y}{(\mathscr{X} \cdot Y)^{n+3}} [Y, (dY)^n] .$$

Thus the differential map of Z is definite, and it follows from the inverse function theorem that Z is locally invertible. Since Int (K^*) is a convex set, the mean value theorem implies that Z is injective. This and the above mentioned homogeneity property imply that X is injective. To prove that X is onto, we let A be a given point in Int (K), and consider the affine hyperplane in E^* defined by

$$P_A = \{\mathscr{X} : \mathscr{X} \cdot A = 1\} .$$

By Proposition 2.2, the restriction of $V(\mathscr{X})$ to the convex set $P_A \cap$ Int (K^*) is a strictly convex function, and by Proposition 2.1 it tends to infinity when $\mathscr{X}$ approaches the boundary of this set. Hence it has a unique minimum at an interior point. Let us call this point $\mathscr{X}_A$. Then by the Lagrange multipliers' theorem it follows that

$$A = \frac{Z(\mathscr{X}_A)}{V(\mathscr{X})} = X(\mathscr{X}_A) .$$

This concludes the proof of the first part in the statement of the lemma. The second part follows by observing, in the above proof of surjectivity, that as $\mathscr{X}$ ranges over $P_A \cap$ Int (K^*) the set $P_{\mathscr{X}} = \{X : \mathscr{X} \cdot X = 1\}$ describes the family of all hyperplanes passing through A and cutting out of K a truncated cone $K_{\mathscr{X}}$, and that the unique minimum volume is achieved precisely at the interior point $\mathscr{X}_A$ such that $X(\mathscr{X}_A) = A$.

Before definitg what we shall call the constant volume envelopes of the convex cone K, we will make a few remarks about the level hypersurfaces of the volume function.

As we stated in the previous section, the volume function satisfies a homogeneity relation of degree $-n = 1$, that is, for any real number $r > 0$ and any $\mathscr{X} \in$ Int (K^*),

$$V(r\mathscr{X}) = \frac{1}{n^{n+1}} V(\tilde{\mathscr{X}}) .$$

Hence by the implicit function theorem, for any fixed real number $c > 0$ the set

$$M_c^* = \{\mathscr{X}: \mathscr{X} \in \operatorname{Int}(K^*),\ V(\mathscr{X}) = c\}$$

is a radially transversal, analytic hypersurface in E^*. From Proposition 2.2 it follows that M_c^* is a (strictly) convex hypersurface, in the classical sense of being the boundary of a convex set in E^*. Moreover, it will follow from the implicit function theorem and Proposition 2.2 that M_c^* is a strongly convex hypersurface, in the following sense:

Let M be an n-dimensional differentiable manifold, and let $\tilde{\mathscr{X}}: M \to E^*$ be a differentiable immersion of class C^2 from M into the (affine) space E^*. We consider the quadratic form $[d^2\tilde{\mathscr{X}}, (d\tilde{\mathscr{X}})^n]$, which in a local coordinate system $(t_1, \cdots, t_n)$ is defined by

$$[d^2\tilde{\mathscr{X}}, (d\tilde{\mathscr{X}})^n] = \sum_{i,j} [\partial_{ij}\tilde{\mathscr{X}}, (d\tilde{\mathscr{X}})^n] dt_i \otimes dt_j ;$$

this definition is invariant both under changes of coordinates and under the action of the unimodular affine group $ASL(n+1, R)$. Its geometrical interpretation, that represents an affinely invariant analog of the second fundamental form, is of importance. In particular, we will say that $\tilde{\mathscr{X}}$ (or $\tilde{\mathscr{X}}(M)$) is *strongly convex* at a given point $m \in M$ if $[d^2\tilde{\mathscr{X}}, (d\tilde{\mathscr{X}})]|_m$ is definite. In this case, there exists a neighborhood U of m such that $\tilde{\mathscr{X}}(U - \{m\})$ is interior to one of the halfspaces determined by the tangent hyperplane to $\tilde{\mathscr{X}}(M)$ at $\tilde{\mathscr{X}}(m)$, in other words, $\tilde{\mathscr{X}}(m)$ is an "elliptic" point, borrowing the term from classical theory of surfaces in euclidean 3-space.

Let $f: E^* \to R$ be a differentiable function of class C^2, and let us assume f to be strongly convex at a given, noncritical point $\mathscr{X}_0 \in E^*$. Then it follows from the implicit function theorem that the level hypersurface

$$\{\mathscr{X}: f(\mathscr{X}) = f(\mathscr{X}_0)\}$$

is strongly convex at $\mathscr{X}_0$. Thus, in particular, for any fixed $c > 0$, the level hypersurface of the volume function $M_c^*$is a strongly convex hypersurface, by Proposition 2.2.

The properties of M_c^* which we have discussed insofar are enough to ensure the existence of its "dual" hypersurface, a concept which we now define.

Let M be an n-dimensional differentiable manifold, and $\tilde{\mathscr{X}}: M \to E^*$ a radially transversal, differentiable immersion of class C^k, $k \geq 1$. For each $m \in M$, we call $\tilde{X}(m)$ the point in E defined by the conditions

$$\tilde{\mathscr{X}}(m) \cdot \tilde{X}(m) = 1 , \qquad d\tilde{\mathscr{X}}(m) \cdot \tilde{X}(m) = 0.$$

The existence of $\tilde{X}(m)$ is guaranteed by the radial transversality of $\tilde{\mathscr{X}}$. It is also obvious that $\tilde{X}(m) \neq 0$ and that as m ranges over M the map $\tilde{X}: M \to E$ is of class C^{k-1}; in particular, the map $\tilde{X}$ could be reduced to a constant: this is the case when $\tilde{\mathscr{X}}(M)$ is contained in an affine hyperplane. We can avoid this trivial case by prescribing strong convexity. Actually, something weaker is needed.

Lemma 3.2. *Let $\tilde{\mathscr{X}}$ be of class C^k, $k > 2$, and let us assume that the bilinear form $[d^2\tilde{\mathscr{X}}, (d\tilde{\mathscr{X}})^n]$ is nondegenerate at some point and hence on a nonempty open subset $U \subset M$. Then $\tilde{X}|_U : U \to E$ is a radially transversal, differentiably immersed hypersurface in E, and the bilinear form $[d^2\tilde{X}, (d\tilde{X})^n]$ is uniformly nondegenerate on U. In particular, if $\tilde{\mathscr{X}}$ is strongly convex, then so is $\tilde{X}$ on the same open subset of M.*

Proof. We observe first that the conditions

$$\tilde{\mathscr{X}} \cdot \tilde{X} = 1 , \qquad d\tilde{\mathscr{X}} \cdot \tilde{X} = 0$$

imply that $\tilde{\mathscr{X}} \cdot d\tilde{X} = 0$ and $d^2\tilde{\mathscr{X}} \cdot \tilde{X} = -d\tilde{\mathscr{X}} \cdot d\tilde{X} = \tilde{\mathscr{X}} \cdot d^2\tilde{X}$. The latter bilinear form is invariant under changes of coordinates and, unlike $[d^2\tilde{\mathscr{X}}, (d\tilde{\mathscr{X}})^n]$, it is also invariant under the action of the general linear group $GL(n+1, R)$. However, in the present case we can relate one to the other. In fact, since $\tilde{\mathscr{X}}$ is radially transversal we can write the identiffication

$$[\_, (d\tilde{\mathscr{X}})^n] \equiv [\tilde{\mathscr{X}}, (d\tilde{\mathscr{X}})^n]\tilde{X} ,$$

which implies that

$$[d^2\tilde{\mathscr{X}}, (d\tilde{\mathscr{X}})^n] = [\tilde{\mathscr{X}}, (d\tilde{\mathscr{X}})^n](d^2\mathscr{X} \cdot \tilde{X}) . \tag{3.1}$$

Hence $d\tilde{\mathscr{X}} \cdot \tilde{X}$ is nondegenerate on the same set as $[d^2\mathscr{X}, (d\tilde{\mathscr{X}})^n]$. In a local coordinate system $(t_1, \cdots, t_n)$ let us write

$$d^2\tilde{\mathscr{X}} \cdot \tilde{X} = \sum_{i,j} g_{ij} dt_i \otimes dt_j ,$$

and define a set of vector fields

$$\{Z_i = Z_i(t_1, \cdots, t_n) : i = 1, \cdots, n\}$$

by requiring that at each point the set

$$\{\tilde{X}, Z_1, \cdots, Z_n\}$$

be the linear basis of E, dual to the basis of E^*,

$$\{\tilde{\mathscr{X}}, \partial_1\tilde{\mathscr{X}}, \cdots, \partial_n\tilde{\mathscr{X}}\} .$$

Therefore we can write the equations

$$\partial_i\tilde{X} = -\sum_j g_{ij} Z_j ,$$

$$[\tilde{X}, \partial_1\tilde{X}, \cdots, \partial_n\tilde{X}] = (-1)^n \det(g_{ij})[\tilde{X},, Z_1, \cdots, Z_n]$$

$$(3.2) \qquad = (-1)^n \frac{\det(g_{ij})}{[\tilde{\mathscr{X}}, \partial_1\tilde{\mathscr{X}}, \cdots, \partial_n\tilde{\mathscr{X}}]} .$$

This last implies that $\tilde{X}|_U : U \to E$ is rdially transversal. Finally, by considering the relation dual to (3.1)

$$(3.3) \qquad [d^2\tilde{X}, (d\tilde{X})^n] = [\tilde{X}, (d\tilde{X})^n](\tilde{\mathscr{X}} \cdot d^2\tilde{X}) ,$$

we get that the bilinear form $[d^2\tilde{X}, (d\tilde{X})^n]$ is nondegenerate on U. The last part in the statement of the lemma is now obvious.

If the conditions prescribed in the preceding lemma are satisfied, we shall call $\tilde{X}(U)$ the *dual hypersurface* of $\tilde{\mathscr{X}}(U)$. We observe that, for each $m \in U$, the hyperplane tangent to $\tilde{X}(U)$ at $\tilde{X}(m)$ coincides with $P_{\tilde{\mathscr{X}}(m)} = \{X : \tilde{\mathscr{X}}(m) \cdot X = 1\}$, hence $\tilde{X}(U)$ could also be described as the envelope of the set of hyperplanes $P_{\tilde{\mathscr{X}}(m)}$ for m ranging over U.

We shall denote by M_c the dual hypersurface of M_c^*, that is, the envelope of hyperplanes $P_{\mathscr{X}}$ as $\mathscr{X}$ ranges over the level hypersurface of the volume function M_c^*. M_c will be called the *constant volume envelope* of hyperplanes relative to the convex cone K, corresponding to the positive real number c.

Theorem 3.3. *The constant volume envelope M_c is a real analytic convex and strongly convex hypersurface in E asymptotic to the boundary of K at infinity, and is canonically diffeomorphic to M_c^*: let $\mathscr{X}^c \mapsto X_c$ denote the natural correspondence $M_c^* \to M_c$. As $\mathscr{X}^c$ ranges over M_c^*, the corresponding point X_c is the center of gravity of the n-dimensional convex body $B_{\mathscr{X}^c} = P_{\mathscr{X}^c} \cap K$. The hyperplane $P_{\mathscr{X}^c}$ minimizes the volume which it cuts out of K, among all hyperplanes through X_c.*

Proof. Let X: Int $(K^*) \to$ Int (K) be the analytic diffeomorphism of Lemma 3.1. As $\mathscr{X}^c$ ranges over M_c^*, the restriction of X to M_c^* sends $\mathscr{X}^c \mapsto X(\mathscr{X}^c)$, and this ranges over the analytic hypersurface $X(M_c^*) \subset$ Int (K). From the identity $V(\mathscr{X}^c) \equiv c$ we get that $d\mathscr{X}^c \cdot X(\mathscr{X}^c) = 0$, and it is also obvious that $\mathscr{X}^c \cdot X(\mathscr{X}^c) = 1$. Therefore the hypersurface

$$X(M_c^*) = M_c$$

is the constant volume envelope of the convex cone K corresponding to the positive real number c. From now on we shall call $X(\mathscr{X}^c) = X_c$. Thus the map $\mathscr{X}^c \mapsto X_c$ is a canonical diffeomorphism from M_c^* onto M_c. Most of the statement of the theorem is now an immediate consequence of Lemmas 3.1 and 3.2. We show next that M_c is convex in the classical sense: let A_c be a fixed point on M_c, and $\mathscr{A}^c$ the corresponding point on M_c^*. As $\mathscr{X}^c$ ranges over M_c^* the point $\dfrac{\mathscr{X}^c}{\mathscr{X}^c \cdot A_c}$ ranges over $P_{A_c} \cap$ Int (K^*) and, as in the proof of Lemma 3.1, the volume function restricted to this set attaches a unique minimum precisely at $\mathscr{A}^c$, i.e.,

$$V\left(\frac{\mathscr{X}^c}{\mathscr{X}^c \cdot A_c}\right) \geq V(\mathscr{X}^c) = c ,$$

the equality sign holding only for $\mathscr{X}^c = \mathscr{A}^c$. Hence, since V is positively homogeneous of degree $-n-1$, and $V(\mathscr{X}^c) = c$ for every $\mathscr{X}^c \in M_c^*$, we get that

$$\mathscr{X}^c \cdot A_c \geq 1 , \qquad \text{for every } \mathscr{X}^c \in M_c^* ,$$

with equality holding only for $\mathscr{X}^c = \mathscr{A}^c$. Therefore M_c is contained in the convex set

$$\tilde{M}_c = \{X : \mathscr{X}^c \cdot X \geq 1, \text{ for every } \mathscr{X}^c \in M_c^*\} ,$$

and a standard continuity argument shows that M_c is precisely the boundary of $\tilde{M}_c$.

Finally, we prove that M_c is asymptotic to the boundary of K in the sense that there exists no affine ray totally contained in Int $(K) - \tilde{M}_c$. In fact, let us assume that there is a ray $\overrightarrow{AB} = \{A + rB : r \geq 0\} \subset \text{Int}\,(K) - \tilde{M}_c$, and consider the cone Q defined by

$$Q = \{A + r(X - A) : r < 0, X \in M_c\} .$$

It is obvious that Int $(Q) \cap$ Int $(K) = \emptyset$. Hence, if H is an affine hyperplane transversal to $\overrightarrow{AB}$ passing through A and such that $H \cap \text{Int}\,(Q) = \emptyset$, we can choose n points $A_1, \cdots, A_n \in H \cap \text{Int}\,(K)$ such that $[A_1 - A, \cdots, A_n - A, B] > 0$. On the other hand, if for each $r \geq 0$ we call T_r the hyperplane tangent to M_c at the point where the ray $\overrightarrow{0, A + rB}$ meets M_c and K_r, the corresponding truncated cone, it follows that T_r meets $\overrightarrow{AB}$ at a point $A + sB$ such that $s > r$ (otherwise K_r would be unbounded). Thus K_r contains the simplex with vertices $A_1, A_2, \cdots, A_n, A, A + rB$. But, for $r \to \infty$, we have also $[A_1 - A, \cdots, A_n - A, rB] \to \infty$, and therefore vol $(K_r) \to \infty$ where vol (K_r) is the volume of K_r. This is a contradiction, since vol $(K_r) = c$ for every r. Hence the theorem is proved.

Remark. The property stating that X_c is the barycenter of $B_{\mathscr{X}^c}$ for each $\mathscr{X}^c$ characterizes the pairing of dual hypersurfaces M_c^*, M_c. In fact, it is easy to see that if $\tilde{\mathscr{X}}, \tilde{X}$ are dual immersions of a manifold M into Int (K^*) and Int (K) respectively, and $\tilde{X}(m)$ is the barycenter of $P_{\tilde{\mathscr{X}}(m)} \cap K$ for each $m \in M$, then there exists a constant $c > 0$ such that $\tilde{\mathscr{X}}(M) \subset M_c^*$ and $\tilde{X}(M) \subset M_c$.

4. Constant volume envelopes: Riemannian metrics, volume elements and normal vectors

With notatation as in the previous section, we identify M_c and M_c^* as abstract analytic manifolds under the canonical diffeomorphism of Theorem 3.3, and consider the bilinear form

$$d^2\mathscr{X}^c \cdot X_c = -d\mathscr{X}^c \cdot dX_c = \mathscr{X}^c \cdot d^2X_c ,$$

invariant under the action of the general linear group $GL(n + 1, R)$. Since M_c^*, and hence M_c, are strongly convex hypersurfaces, it follows from (3.1) that $d^2\mathscr{X}^c \cdot X_c$ is definite. In the present case we can find a suitable integral representation for this quadratic form. In fact, we saw in Theorem 3.3 that

$$X_c = \frac{1}{c}\frac{(n+2)}{(n+1)!}\int \frac{Y}{(\mathscr{X}^c \cdot Y)^{n+2}}[Y, (dY)^n] .$$

Hence the relation between the differentials of the position vectors is given by

$$-dX_c = \frac{1}{c}\frac{(n+2)}{(n+1)!}d\mathscr{X}^c \cdot \int \frac{Y \otimes Y}{(\mathscr{X}^c \cdot Y)^{n+3}}[Y, (dY)^n] ,$$

and we get for the quadratic form

$$-d\mathscr{X}^c \cdot dX_c = \frac{1}{c}\frac{(n+2)}{(n+1)!}(d\mathscr{X}^c \otimes d\mathscr{X}^c)\int \frac{Y \otimes Y}{(\mathscr{X}^c \cdot Y)^{n+3}}[Y, (dY)^n] ,$$

from which it follows that $-d\mathscr{X}^c \cdot dX_c$ is positive-definite everywhere. Thus by denoting it by^2 ds we have defined in a natural way a Riemannian structure on M_c^* or M_c.

We shall denote by σ the volume element corresponding to the above Riemannian metric. Also by considering the radially transversal immersions of M_c^* and M_c into the vector spaces E^* and E respectively, we have two more (prismatic) volume elements which are invariant under the action of the unimodular group $SL(n + 1, R)$:

$$\tau_c^* = \frac{1}{n!}|[\mathscr{X}^c, (d\mathscr{X}^c)^n]| , \qquad \tau_c = \frac{1}{n!}|[X_c, (dX_c)^n]| .$$

We are now going to compare these three volume elements: first we observe that, since the volume function is homogeneous of degree $-n - 1$, and the canonical diffeomorphism of Lemma 3.1 is homogeneous of degree -1, we have the relations

$$\mathscr{X}^c = c^{-1/(n+1)}\mathscr{X}^1 , \qquad X_c = c^{1/(n+1)}X_1 .$$

Hence there exists a real analytic function f, independent of c, such that

$$\tau_c = c^2 f(\hat{\mathscr{X}}^c)\tau_c^* ,$$

where $\hat{\mathscr{X}}^c$ is any point in the same ray as $\mathscr{X}^c$.In other words, the map f can be extended to a real analytic function, denoted also by f: Int $(K^*) \to R$, homogeneous of degree 0. More precisely:

With notation as in Lemma 3.2, let the bilinear form $d^2\tilde{\mathscr{X}} \cdot \tilde{X}$ be nondegenerate on the open subset $U \subset M$, and let $\tilde{\sigma}$ denote the corresponding pseudo-Riemannian volume element, the other two volume elements being given by

$$\tau = \frac{1}{n!}|[\tilde{X}, (d\tilde{X})]| , \qquad \tau^* = \frac{1}{n!}|[\tilde{\mathscr{X}}, (d\tilde{\mathscr{X}})^n]| .$$

From (3.2) it follows that these volume elements are related by

$$\tilde{\sigma} = \sqrt{\tau^*\tau} . \tag{4.1}$$

Furthermore, since the map from $\tilde{\mathscr{X}}(U)$ onto $\tilde{X}(U)$ defined by $\tilde{\mathscr{X}}(m) \mapsto \tilde{X}(m)$ is a local diffeomorphism of class C^{k-1}, there exists a positive valued function $\rho: \tilde{\mathscr{X}}(U) \to R^+$, also of class C^{k-1} such that for each $m \in U$

$$|[\tilde{X}, (d\tilde{X})^n]||_m = \rho(\tilde{\mathscr{X}}(m))|[\tilde{\mathscr{X}}, (d\tilde{\mathscr{X}})]||_m . \tag{4.2}$$

ρ is uniquely determined up to a multiplicative positive constant. We shall call it the *volume ratio function* of the pairing of dual immersed hypersurfaces $\tilde{\mathscr{X}}(U)$ and $\tilde{X}(U)$.

By using (3.2) again we get

$$|\det(g_{ij})| = \rho(\tilde{\mathscr{X}}(m))[\tilde{\mathscr{X}}, \partial_1\tilde{\mathscr{X}}, \cdots, \partial_n\mathscr{X}]^2 . \tag{4.3}$$

We denote by ρ_K the volume ratio function corresponding to the dual hypersurfaces M_c^*, M_c.

Theorem 4.1. *The volume elements τ_c, τ_c^*, σ are related to one another by the equations*

$$\tau_c = \rho_K(\mathscr{X}^c)\tau_c^* , \qquad \sigma = \sqrt{\tau_c^*\tau_c} ,$$

where

$$\rho_K(\mathscr{X}^c) = a_n c^2 \mu(B_{\mathscr{x}} c) , \tag{4.4}$$

$B_{\mathscr{x}}c$ being the n-dimensional convex body $P_{\mathscr{x}}c \cap K$, μ the mean square fractional volume defined in § 1, *and a_n an absolute constant equal to* $(n+1)!\,(n+2)^n$.

The following lemma will be useful in the proof of the theorem, as well as in providing an alternative expression for the calculation of the mean square fractional volume.

Lemma 4.2. *Let Y be a cross-sectional immersion to the convex cone K. By* *denoting* $\bar{Y} = \dfrac{Y}{\mathscr{X} \cdot Y}$, $\omega_{\bar{Y}} = \dfrac{1}{(n+1)!}[\bar{Y}, (d\bar{Y})^n]$, *and letting*

$$X = X(\mathscr{X}) = \frac{1}{V(\mathscr{X})} \int \bar{Y} \omega_{\bar{Y}}$$

to be the barycenter of the n-dimensional convex body $P_{\mathscr{X}} \cap K$, *we have the identity*

$$\int [\bar{Y}_0, \cdots, \bar{Y}_n]^2 \omega_{\bar{Y}_0} \times \cdots \times \omega_{\bar{Y}_n}$$
$$= (n+1) V(\mathscr{X}) \int [X, \bar{Y}_1, \cdots, \bar{Y}_n]^2 \omega_{\bar{Y}_1} \times \cdots \times \omega_{\bar{Y}_n} .$$

Proof of Lemma 4.2. From

$$[\bar{Y}_0, \cdots, \bar{Y}_n] = [X, \bar{Y}_1, \cdots, \bar{Y}_n] + \sum_{i=1}^{n} (-1)^i [X, \bar{Y}_0, \cdots, \bar{Y}_{i-1}, \bar{Y}_{i+1}, \cdots, \bar{Y}_n]$$

we get

$$\int [\bar{Y}_0, \cdots, \bar{Y}_n]^2 \omega^{(n+1)} = \sum_{i=0}^{n} \int [X, \bar{Y}_0, \cdots, \hat{\bar{Y}}_i, \cdots, \bar{Y}_n]^2 \omega^{(n+1)}$$
$$+ \sum_{i \neq j} \int [X, \bar{Y}_0, \cdots, \hat{\bar{Y}}_i, \cdots, \bar{Y}_n][X, \bar{Y}_0, \cdots, Y_j, \cdots, \bar{Y}_n] \omega^{(n+1)} ,$$

where $\omega^{(n+1)} = \omega_{\bar{Y}_0} \times \cdots \times \omega_{\bar{Y}_n}$, and the "hat" $\wedge$ over a certain variable $\bar{Y}_i$ indicates, as usual, that that variable is omitted. But by using Fubini's theorem it follows that

$$\int [X, \bar{Y}_0, \cdots, \hat{\bar{Y}}_i, \cdots, \bar{Y}_n]^2 \omega^{(n+1)} = V(\mathscr{X}) \int [X, \bar{Y}_1, \cdots, \bar{Y}_n]^2 \omega_{\bar{Y}_1} \times \cdots \times \omega_{\bar{Y}_n} ,$$

and, for $i \neq j$,

$$\int [X, \bar{Y}_0, \cdots, \hat{\bar{Y}}_i, \cdots, \bar{Y}_n][X, \bar{Y}_0, \cdots, \hat{\bar{Y}}_j, \cdots, \bar{Y}_n] \omega^{(n+1)}$$
$$= \int [X, \bar{Y}_0, \cdots, \hat{\bar{Y}}_i, \cdots, \bar{Y}_n] \omega_{\bar{Y}_0} \times \cdots \times \hat{\omega}_{\bar{Y}_i}$$
$$\times \cdots \times \omega_{\bar{Y}_n} \int [X, \bar{Y}_0, \cdots, \hat{\bar{Y}}_j, \cdots, \bar{Y}_n] \omega_{\bar{Y}_i}$$
$$= \int [X, \bar{Y}_0, \cdots, \hat{\bar{Y}}_i, \cdots, \bar{Y}_n] \omega_{\bar{Y}_0} \times \cdots \times \hat{\omega}_{\bar{Y}_i}$$
$$\times \cdots \times \omega_{\bar{Y}_n} V(\mathscr{X})[X, \cdots, X, \cdots, \bar{Y}_j, \cdots, \bar{Y}_n] = 0 .$$

Hence the lemma is proved.

Proof of Theorem 4.1. The first two equations hold in general for any pair of dual hypersurfaces, as we can see from (4.1) and (4.2). To prove (4.4) we let as before $\mathscr{X}^c$ be the position vector on M_c^*, and Y a cross-sectional immersion to the convex cone K. By denoting

$$\bar{Y} = \frac{Y}{\mathscr{X}^c \cdot Y}, \qquad \omega_{\bar{Y}} = \frac{1}{(n+1)!}[\bar{Y}, (d\bar{Y})^n],$$

we can write for the Riemannian structure on M_c^*

$$ds^2 = -d\mathscr{X}^c \cdot dX_c = \frac{(n+2)}{c} d\mathscr{X}^c \otimes d\mathscr{X}^c) \int (\bar{Y} \otimes \bar{Y})\omega_{\bar{Y}},$$

and if $(t_1, \cdots, t_n)$ is a local coordinate system, the components of this tensor are

$$g_{ij} = \frac{(n+2)}{c} \int (\partial_i \mathscr{X}^c \cdot \bar{Y})(\partial_j \mathscr{X}^c \cdot \bar{Y})\omega_{\bar{Y}}.$$

By putting

$$g = \det(g_{ij}) = \frac{1}{n!} \mathscr{E}^{i_1, \cdots, i_n} \mathscr{E}^{j_1, \cdots, j_n} g_{i_1 j_1} \cdots g_{i_n j_n},$$

and observing that

$$\begin{aligned} &\mathscr{E}^{i_1 \cdots i_n} \mathscr{E}^{j_1 \cdots j_n} (\partial_{i_1} \mathscr{X}^c \cdot \bar{Y}_1)(\partial_{j_1} \mathscr{X}^c \cdot \bar{Y}_1) \cdots (\partial_{i_n} \mathscr{X}^c \cdot \bar{Y}_n)(\partial_{j_n} \mathscr{X}^c \cdot \bar{Y}_n) \\ &= [\mathscr{X}^c, \partial_1 \mathscr{X}^c, \cdots, \partial_n \mathscr{X}^c]^2 [X_c, \bar{Y}_1, \cdots, \bar{Y}_n]^2, \end{aligned}$$

we get, by using also Lemma 4.2,

$$g = \frac{(n+2)^n}{(n+1)!} \frac{1}{c^{n+1}} [\mathscr{X}^c, \partial_1 \mathscr{X}^c, \cdots, \partial_n \mathscr{X}^c]^2 \int [\bar{Y}_0, \cdots, \bar{Y}_n]^2 \omega^{(n+1)}.$$

Hence, if we replace $\tilde{\mathscr{X}}$ for $\mathscr{X}^c$ in (4.3) and compare with the above, we get

$$\rho_K(\mathscr{X}^c) = \frac{(n+2)^n}{(n+1)!} \frac{1}{c^{n+1}} \int [\bar{Y}_0, \cdots, \bar{Y}_n]^2 \omega^{(n+1)},$$

and since the mean square fractional volume of the n-dimensional convex body $B_{\mathfrak{x}} c = P_{\mathfrak{x}} c \cap K$ is given by

$$\mu(B_{\mathfrak{x}} c) = \frac{1}{(n+1)!^2} \frac{1}{c^{n+3}} \int [\bar{Y}_0, \cdots, \bar{Y}_n]^2 \omega^{(n+1)},$$

the proof of the theorem is concluded.

As an application of Theorem 4.1 we shall compute the mean square fractional volume in two particular cases, namely, the minimum possible value which according to Proposition 1.1 is attained only when the n-dimensional convex body is bounded by an ellipsoid, and for the conjectured maximum which should occur only when the convex body is a nondegenerate n-simplex. We preface those computations by observing, also from the above theorem, that the value of the volume ratio function $\rho_K(\mathscr{X}^c)$ is not constant, in general, as $\mathscr{X}^c$ ranges over M_c^*, since the n-dimensional convex bodies $B_{\mathscr{X}}c = P_{\mathscr{X}}c \cap K$, although projectively equivalent, do not carry necessarily the same affine shape. However, if K happens to be a homogeneous cone, all the convex bodies $B_{\mathscr{X}}c$ are affine equivalent and hence ρ_K is constant on M_c^*.

It is equally easy to observe that the computation of μ will be independent of the choice of any particular positive value for the constant c, since for each $\mathscr{X} \in \mathrm{Int}\,(K^*)$ the n-dimensional convex bodies $B_{r\mathscr{X}} = P_{r\mathscr{X}} \cap K$, parametrized by $r > 0$, are all affinely equivalent. In other words, the function

$$\mathscr{X} \to \mu(B_{\mathscr{X}}) = \frac{1}{(n+1)!^2} \frac{1}{V(\mathscr{X})^{n+3}} \int [\bar{Y}_0, \cdots, \bar{Y}_n]^2 \omega^{(n+1)}$$

is homogeneous of degree zero, as defined on $\mathrm{Int}\,(K^*)$.

Thus we consider first the quadric cone

$$Q = \{(x_0, \cdots, x_n) : x_0 \geq \sqrt{x_1^2 + \cdots + x_n^2}\} .$$

It is well known that this is a selfdual homogeneous convex cone. In this case it is convenient to take $c = \dfrac{1}{n+1} \dfrac{\pi^{n/2}}{\Gamma(n/2+1)}$, that is, $\dfrac{1}{n+1}$ times the volume of the n-dimensional euclidean unit ball. Then a parametrization for M_c^* is given by the function $\mathscr{X}^c\colon R^n \to \mathrm{Int}\,(Q)$ defined by the equation

$$\mathscr{X}^c(u_1, \cdots, u_n) = \left(\sqrt{1 + \sum_i u_i^2}, u_1, \cdots, u\right),$$

and, correspondingly, X_c can be written

$$X_c(u_1, \cdots, u_n) = \left(\sqrt{1 + \sum_i u_i^2}, -u_1, \cdots, -u_n\right).$$

A straightforward computation shows that $\rho_Q(\mathscr{X}^c) = 1$. Hence by using (4.4) we get

$$\mu(B_{\mathscr{X}}c) = \frac{(n+1)^2\{\Gamma(n/2+1)\}^2}{(n+1)!\,(n+2)^n\pi^n} .$$

Next, we consider the simplicial cone, that is, the cone subtended by a non-degenerate n-simplex:

$$S = (R^+)^n = \{(x_0, \cdots, x_n): x_i \geq 0\} .$$

This is also a selfdual homogeneous convex cone, and we put now $c = \frac{1}{(n+1)!}$, the volume of the standard $(n+1)$-simplex. We can parametrize M_c^* be the function $\mathscr{X}^c: (R^+)^n \to \operatorname{Int}(S)$:

$$\mathscr{X}^c(u_1, \cdots, u_n) = \left(\frac{1}{u_1 \cdots u_n}, u_1, \cdots, u_n\right) .$$

Hence X_c is given by

$$X_c(u_1, \cdots, u_n) = \frac{1}{n+1}(u_1 \cdots u_n, u_1^{-1}, \cdots, u_n^{-1}) ,$$

and again a straightforward computation shows that $\rho_S(\mathscr{X}^c) = (n+1)^{-(n+1)}$. Therefoore we get

$$\mu(B_{\mathscr{X}}c) = \frac{(n+1)!}{(n+1)^{n+1}(n+2)^n} .$$

Retaking the notation preceding the statement of Theorem 4.1, let $\tilde{\mathscr{X}}(U)$ and $\tilde{X}(U)$ be hypersurfaces dually immersed into the vector spaces E^* and E respectively. We are now going to compare the pseudo-Riemannian metric $ds^2 = -d\tilde{\mathscr{X}} \cdot d\tilde{X}$ with the Berwald-Blaschke pseudo-Riemannian metrics, which are defined on $\tilde{\mathscr{X}}(U)$ and $\tilde{X}(U)$, as hypersurfaces immersed into the (affine) spaces E^*, E. This will enable us to relate the corresponding Laplace-Beltrami operators and, consequently, to furnish a suitable condition for those manifolds to be affine hyperspheres.

First, a brief definition: following Blaschke (see [1, § 65]) we put, for a local coordinate system $(t_1, \cdots, t_n)$,

$$\Lambda_{ij} = [\partial_{ij}\tilde{X}, \partial_1\tilde{X}, \cdots, \partial_n\tilde{X}] , \quad \Lambda = \det(\Lambda_{ij}) , \quad G_{ij} = \frac{\Lambda_{ij}}{|\Lambda|^{1/(n+2)}} .$$

The bilinear form

$$ds_{\tilde{X}}^2 = \sum G_{ij} dt_i \otimes dt_j ,$$

which is invariant under the unimodular affine group $ASL(n+1, R)$, is the Berwald-Blaschke pseudo-Riemannian metric on the hypersurface $\tilde{X}(U)$.

Similarly, we denote by Λ^*_{ij}, Λ^*, and $ds^2_{\tilde{\mathscr{X}}}$ the corresponding objects associated with hypersurface $\tilde{\mathscr{X}}(U)$.

A straightforward computation and the use of (3.3), (4.2) and (4.3) provide the relation

$$ds^2_{\tilde{X}} = \rho(\tilde{\mathscr{X}})^{1/(n+2)} ds^2 . \tag{4.5}$$

Similarly, by using (3.1) and (4.3) we get

$$ds^2_{\tilde{\mathscr{X}}} = \rho(\tilde{\mathscr{X}})^{-1/(n+2)} ds^2 . \tag{4.6}$$

We now compute the Laplace-Beltrami operator corresponding to the metric $ds^2 = -d\tilde{\mathscr{X}} \cdot d\tilde{X} = \sum g_{ij} dt_i \otimes dt_j$, as applied to the position vectors $\tilde{\mathscr{X}}$ and $\tilde{X}$. As usual, let us denote $|\det(g_{ij})|$ by g, g^{ij} the contravariant components of the metric by g^{ij}, and the star operator by $*$.

From the differential of the position vector

$$d\tilde{X} = \sum \partial_h \tilde{X} \otimes dt_h$$

we get

$$*d\tilde{X} = \sqrt{g} \sum_{h,k} (-1)^{k-1} g^{hk} \partial_h \tilde{X} \otimes dt_1 \wedge \cdots \wedge d\hat{t}_k \wedge \cdots \wedge dt_n .$$

Hence, if as in the proof of Lemma 3.2 we define the set of vector fields $\{Z_i = Z_i(t_1, \cdots, t_n) : i = 1, \cdots, n\}$ in such a way that the linear basis $\{\tilde{X}, Z_1, \cdots, Z_p\}$ of E be dual at each point to the basis $\{\tilde{\mathscr{X}}, \partial_1\tilde{\mathscr{X}}, \cdots, \partial_n\tilde{\mathscr{X}}\}$ of E^*, we can write

$$*d\tilde{X} = \sqrt{g} \sum_{k} (-1)^{k-1} Z_k \otimes dt_1 \wedge \cdots \wedge d\hat{t}_k \wedge \cdots \wedge dt_n .$$

Then by making use of the identification

$$[\tilde{\mathscr{X}}, \partial_1\tilde{\mathscr{X}}, \cdots, \partial_{k-1}\tilde{\mathscr{X}}, \_, \partial_{k+1}\tilde{\mathscr{X}}, \cdots, \partial_n\tilde{\mathscr{X}}] \equiv [\tilde{\mathscr{X}}, \partial_1\tilde{\mathscr{X}}, \cdots, \partial_n\tilde{\mathscr{X}}] Z_k$$

and (4.3), we get

$$*d\tilde{X} = \frac{\{\rho(\tilde{\mathscr{X}})\}^{1/2}}{(n-1)!} [\tilde{\mathscr{X}}, \_, (d\tilde{\mathscr{X}})^{n-1}] ,$$

from which it follows that

$$d*d\tilde{X} = \frac{1}{(n-1)!} \left\{ \frac{1}{2} \frac{d\rho(\tilde{\mathscr{X}})}{\rho(\tilde{\mathscr{X}})^{1/2}} \wedge [\tilde{\mathscr{X}}, \_, (d\tilde{\mathscr{X}})^{n-1}] - \rho(\tilde{\mathscr{X}})^{1/2} [\_, (d\tilde{\mathscr{X}})^n] \right\} .$$

The identification

$$[\_\_, (d\tilde{\mathscr{X}})^n] \equiv [\tilde{\mathscr{X}}, (d\tilde{\mathscr{X}})^n] \otimes \tilde{X}$$

and (4.3) furnish

$$[\_\_, (d\tilde{\mathscr{X}})^n] = n!\, \rho(\tilde{\mathscr{X}})^{-1/2}\tilde{\sigma} \otimes \tilde{X} .$$

Therefore we finally write

$$\Delta\tilde{X} = -*d*d\tilde{X} = n\tilde{X} - \tfrac{1}{2}*\{d\log\rho(\tilde{\mathscr{X}}) \wedge *d\tilde{X}\} . \tag{4.7}$$

An analogous calculation shows that

$$\Delta\tilde{\mathscr{X}} = n\tilde{\mathscr{X}} + \tfrac{1}{2}*\{d\log\rho(\tilde{\mathscr{X}}) \wedge *d\tilde{\mathscr{X}}\} . \tag{4.8}$$

Proposition 4.3. *The following conditions are equivalent:*

(1) *The volume ratio function ρ is constant on $\tilde{\mathscr{X}}(U)$.*

(2) *The manifold $(\tilde{X}(U), ds^2)$ satisfies the equation $\Delta\tilde{X} = n\tilde{X}$.*

(3) *The manifold $(\tilde{\mathscr{X}}(U), ds^2)$ satisfies the equation $\Delta\tilde{\mathscr{X}} = n\tilde{\mathscr{X}}$.*

If any of the above conditions holds, then both $(\tilde{X}(U), ds^2_{\tilde{X}})$ and $(\tilde{\mathscr{X}}(U), ds^2_{\tilde{\mathscr{X}}})$ are affine hyperspheres contained in E and E^* respectively.

Proof. The equivalences follow at once from (4.7) and (4.8). These same relations, together with (4.5) and (4.6), provide also easily the last part of the statement.

We close this section by observing that the above proposition shows, in particular, that if K is a homogeneous convex cone then M_c and M_c^* are both affine hyperspheres, also homogeneous.

5. The volume ratio function: symptotic behavior near the boundary

In the previous section we introduced the volume ratio function ρ_K of a convex cone K as defined on the level hypersurfaces M_c^*. In particular, we saw that if K is a homogeneous cone, then ρ_K is constant on each M_c^*. The question arises whether the converse holds. Namely, if the condition $\rho_K =$ const. on each M_c^* would imply that the convex cone K be homogeneous. We can assert this to be the case when the boundary of K is of class C^2; the proof will be a consequence of Proposition 1.1 and the study of the asymptotic behavior of ρ_K near the boundary of K^*. This motivates the current section.

It is clear that ρ_K can be extended in a natural way so that it is defined on the whole Int (K^*). Moreover, $\rho_K(\mathscr{X})$ can be written in terms of integrals on the n-dimensional convex body $P_{\mathscr{X}} \cap K$. Hence it will be possible to link the asymptotic behavior of $\rho_K(\mathscr{X})$ with the local behavior of the boundary of K.

However, it is more convenient to study, rather than ρ_K itself, the function μ_K defined by $\mu_K(\mathscr{X}) = \mu(P_{\mathscr{X}} \cap K)$, for the reason that the latter is homogeneous of degree zero and absolutely bounded. (4.4) can then be applied to relate one to the other, restricted to the hypersurfaces M_c^*.

More generally, we shall consider a wide class of functions: with notation as in § 1, for each $f \in \mathfrak{F}$ we define $\mu_{K,f}$: Int $(K^*) \to R$ by

$$\mu_{K,f}(\mathscr{X}) = \mu_f(P_{\mathscr{X}} \cap K) .$$

The function $\mu_{K,f}$ is homogeneous of degree zero and absolutely bounded. We observe that, by using the inverse of the natural diffeomorphism defined at the beginning of § 3 (Lemma 3.1), we can think of it as defined on Int (K). Furthermore, by its homogeneity property we can also assume it to be defined on the quotient space Int $(K)/R^+$, and this can be identified in an obvious way with a cross-sectional space of the form $P_{\mathscr{X}} \cap$ Int (K), for some $\mathscr{X} \in$ Int (K^*). This setting will be particularly suitable when we prescribe the condition that the convex boundary of K be locally smooth; we will discuss this particular situation later in this section. For the time being, without assuming any smoothness, we shall prescribe instead a strong contact condition on the boundary.

For technical reasons we assume, throughout this section, that the vector space E is provided with a Euclidean metric. We use the notation XY to denote directed distance from X to Y; it will be clear however from the context that the results we are about to present are independent of the choice of the Euclidean structure.

If K is a convex cone, for each $\mathscr{A} \in \partial K^* - \{0\}$ we denote

$$\mathscr{A}^* = \{X: \mathscr{X} \cdot X = 0, X \in \partial K\} .$$

Let $\mathscr{X} \in$ Int (K^*). A subset $U \subset P_{\mathscr{X}} \cap \partial K$ is called a *capped neighborhood* of $\mathscr{A}^* \cap P_{\mathscr{X}}$ if:

(i) $\mathscr{A}^* \cap P_{\mathscr{X}}$ is interior to U relative to $P_{\mathscr{X}} \cap \partial K$,

(ii) there exist a point $A \in \mathscr{A}^* \cap P_{\mathscr{X}}$ and a hyperplane $\pi = \{X: \mathscr{D} \cdot X = 0\}$ such that π separates A from ∂U where ∂ indicates now the boundary relative to $P_{\mathscr{X}} \cap \partial K$,

(iii) the closure $\overline{\pi \cap Q(A)}$ is contained in Int (K), where $Q(A)$ denotes the cone (in $P_{\mathscr{X}}$) defined by

$$Q(A) = \{A + r(X - A): X \in P_{\mathscr{X}} \cap \partial K - U, r > 0\} .$$

We are now in a position to state the first result of this section.

Theorem 5.1. *Let K_1, K_2 be convex cones with nonempty* Int $(K_1^*) \cap$ Int (K_2^*) *such that for some $\mathscr{X} \in$* Int $(K_1^*) \cap$ Int (K_2^*) *and $\mathscr{A} \in \partial K_1^* \cap \partial K_2^*$, the set $\partial K_1 \cap \partial K^2 \cap P_{\mathscr{X}}$ is a capped neighborhood of $\mathscr{A}^* \cap P_{\mathscr{X}}$. Then the absolute value of the difference between the corresponding functions $\mu_{K_1,f}$ and $\mu_{K_2,f}$ is arbitrarily small on suitable neighborhoods of $\mathscr{A}$.*

Proof. Let U be a capped neighborhood common to both K_1 and K_2, with notation as in the definition for A, π, and $Q(A)$. For each ray $\overrightarrow{AB}$ in $Q(A)$ we denote by V the point at which $\overrightarrow{AB}$ meets π, and by Y, Z the corresponding points, other than A, on the boundaries ∂K_1, ∂K_2, respectively. We can assume, without loss of generality, that the absolute value of the cross-ratio $|(A, Y; V, Z)| = \left|\frac{AV}{AZ} \div \frac{YV}{YZ}\right| < 1$, uniformly. Let $\{\mathscr{X}_h\}$ be a sequence of points in $\operatorname{Int}(K_1^*) \cap \operatorname{Int}(K_2^*)$ converging to $\mathscr{A}$. For each h we denote by π_h the hyperplane $\pi_h = \{X\colon \mathscr{X}_h \cdot X = 0\}$, and let P_h be the hyperplane parallel to π_h, passing through $\frac{1}{2}A$.

If $Q^-(A)$ denotes the cone in $P_{\mathscr{X}}$, opposite to $Q(A)$, then we are going to show that given $\varepsilon > 0$ there exists an h_0 such that the absolute value of the cross-ratio

$$|(A, V; X, Y)| = \left|\frac{AX}{AY} \div \frac{VX}{VY}\right| < \varepsilon$$

for every $h > h_0$ and (uniformly) for every $X \in P_h \cap Q^-(A)$. In fact, we observe first that the limit hyperplane $\pi_{\mathscr{A}} = \{X\colon \mathscr{A} \cdot X = 0\}$ supports the cone $Q(A)$. Moreover, since it also supports both K_1 and K_2, and by condition (iii) in the definition of a capped neighborhood, the closure $\overline{\pi \cap Q(A)}$ of a cross-section of $Q(A)$ is contained in $\operatorname{Int}(K_1) \cap \operatorname{Int}(K_2)$, it follows that no ray $\overrightarrow{AB}$ in $Q(A)$ can be contained in $\pi_{\mathscr{A}}$. In other words, $\pi_{\mathscr{A}} \cap Q(A) = \{A\}$. Hence, if we consider now the level set

$$Q^-(A, \varepsilon) = \{X\colon X \in \overline{Q^-(A)}, |(A, V; X, Y)| = \varepsilon\},$$

to each $X \in Q^-(A, \varepsilon)$ we can associate a pair $(N(X), h(X))$, where $N(X)$ is a neighborhood of X, and $h(X)$ is an integer such that, for every $h > h(X)$, π_h separates $N(X)$ from A. Since $Q^-(A, \varepsilon)$ is compact, from the covering $\{N(X)\colon X \in Q^-(A, \varepsilon)\}$ we can extract a finite number of neighborhoods $\{N(X_1), N(X_2), \cdots, N(X_k)\}$ also furnishing a covering for $Q^-(A, \varepsilon)$. Therefore defining $h_0 = \max\{h(X_1), \cdots, h(X_k)\}$ we have that for every $h > h_0$, π_h separates $Q^-(A, \varepsilon)$ from A, thus proving the above inequality.

If $T \mapsto T'$ denotes the perspectivity centered at the origin 0 of E, mapping $P_{\mathscr{X}}$ onto P_h, the invariance of the cross-ratio under projective transformations implies that

$$|(A', V'; X', Y')| < \varepsilon$$

for every $h > h_0$ and for every $X' = X \in P_h \cap Q^-(A)$. Also, since $|(A, Y; V, Z)| < 1$, we can assume to have chosen h_0 so that $|Y'Z'| < |Y'V'|$ uniformly. On the other hand, since $|V'X'|$ is uniformly bounded and $|A'X'| \geq \frac{1}{2}|OA|$, we have that $\left|\frac{V'X'}{A'X'}\right|$ is also uniformly bounded, say by a constant k. Therefore we get that

$$\left|\frac{Z'Y'}{A'Y'}\right| < \left|\frac{V'Y'}{A'Y'}\right| < k\varepsilon, \qquad \text{uniformly} .$$

Now if for each $r > -1$ we denote by $B_h(r)$ the n-dimensional convex body in P_h defined by

$$B_h(r) = \{T' + r(T' - A') \colon T' \in P_h \cap K_1\} ,$$

then the last inequality implies that

$$B_h(-k\varepsilon) \subset P_h \cap K_2 \subset B_h(k\varepsilon) , \qquad \text{for every } h > h_0 .$$

Hence, by the continuity of the functional μ_f as defined on n-dimeniional convex bodies, the theorem is proved.

Remark. Let V_1, V_2 denote the volume functions of K_0 and K_2 respectively. The above inequality of sets implies, in particular, that

$$(1 - k\varepsilon)^n V_1(\mathscr{X}_h) < V_2(\mathscr{X}_h) < (1 + k\varepsilon)^n V_1(\mathscr{X}_h) ,$$

for every $h > h_0$. By homogeneity the same relation holds for any $\hat{\mathscr{X}}_h$ in the same ray as $\mathscr{X}_h$. Hence, if the sequence $\{\hat{\mathscr{X}}_h\}$ is taken in such a way that one of the sequences $\{V_1(\hat{\mathscr{X}}_h)\}$ or $\{V_2(\hat{\mathscr{X}}_h)\}$ is bounded, then so is the other and the absolute value of the difference $|V_1(\hat{\mathscr{X}}_h) - V_2(\hat{\mathscr{X}}_h)|$ converges to zero. Therefore, since the value at $\hat{\mathscr{X}}_h$ of each volume ratio function ρ_{K_i} can be written

$$\rho_{K_i}(\hat{\mathscr{X}}_h) = a_n V_i(\hat{\mathscr{X}}_h)^2 \mu_{K_i}(\mathscr{X}_h) , \qquad i = 1, 2 ,$$

it follows that the corresponding difference

$$\rho_{K_1}(\hat{\mathscr{X}}_h) - \rho_{K_2}(\hat{\mathscr{X}}_h)$$

also converges to zero.

We are now going to prescribe suitable local conditions on the boundary of a convex cone K, so that the behavior $\mu_{K,f}$ near the boundary is similar to that of the corresponding function of a quadric cone. The latter, being obviously constant, it also takes the minimum possible value, as a consequence of Proposition 1.1.

Let K be a convex cone. We assume now that for some $\mathscr{X} \in \operatorname{Int}(K^*)$, $P_{\mathscr{X}} \cap \partial K$ has third order contact with an ellipsoid L in $P_{\mathscr{X}}$ at a given point A, and consider the geometry the so called Kleinian model of Lobachevsky's hyperbolic geometry, defined by the projective transformations of $P_{\mathscr{X}}$ leaving the ellipsoid L invariant. It will be handy to make use of some of the objects from this particular geometry:

Given a straight line M through A, intersecting also the ellipsoid L at a second point B, the set of points which are at constant distance from M in the sense of hyperbolic geometry is an $(n-1)$-ellipsoid contained in L, and having strictly second order tangential contact with L at A and B only. We shall call it a *hyperbolic tube* with endpoints A, B.

We introduce here a non-standard topology on the union of $P_{\mathfrak{x}} \cap \operatorname{Int}(K)$ with $\{A\}$: In the interior $P_{\mathfrak{x}} \cap \operatorname{Int}(K)$ we define it as usual; for the boundary point A we define it by means of convergent sequences as follows: given a sequence of points $\{X_i\}$ in $P_{\mathfrak{x}} \cap \operatorname{Int}(K)$, we say that $\{X_i\}$ is *h-convergent* to A if it is eventually in the interior of some hyperbolic tube with A as one of its endpoints, and converges to A in the usual sense.

Denoting by $m_{Q,f}$ the (constant) value of the function $\mu_{Q,f}$ for any quadric cone, we have

Lemma 5.2. *With notation as above, if $\{X_i\}$ is h-convergent to A, then $\{\mu_{K,f}(X_i)\}$ converges to $m_{Q,f}$.*

Proof. For each $X \in P_{\mathfrak{x}} \cap \operatorname{Int}(K)$ let Y be the point, other than A, where the straight line $\overline{AX}$ meets the boundary $P_{\mathfrak{x}} \cap \partial K$, Z the corresponding point on the ellipsoid L, and $(A, Y; X, Z)$ the cross-ratio of these four points. Let $\{X_i\}$ be a sequence in $P_{\mathfrak{x}} \cap \operatorname{Int}(K)$ which is h-convergent to A, H a hyperbolic tube with A as one of its endpoints and eventually containing $\{X_i\}$, and $\tilde{H}$ the region bounded by H. For each i let P_i be the hyperplane tangent to the constant volume envelope passing through X_i, S_i the halfspace determined by P_i and containing A, and P_{iA} the hyperplane parallel to P_i passing through A. If 0 denotes again the origin of E, for each $T \in P_{\mathfrak{x}}$ let T' be the point where the ray $\overrightarrow{0T}$ meets P_i, and T'' the corresponding point on P_{iA}. We observethat, in a deleted neighborhood of A, H is interior to K. This, together with the facts that L has third order contact with, and H is tangent to, $P_{\mathfrak{x}} \cap \partial K$ at A, imply that given $\varepsilon > 0$ there exists an i_0 such that, for every $i > i_0$ the inequality

$$|(A, Y; X, Z)| < \varepsilon$$

holds uniformly for every $X \in \partial(\tilde{H} \cap S_i)$ where ∂ indicates now the boundary relative to $P_{\mathfrak{x}}$. The invariance of the cross-ratio under projective transformations implies that

$$|(A'', Y''; X'', Z'')| < \varepsilon$$

for every $i > i_0$ and uniformly for every $X \in \partial(\tilde{H} \cap S_i)$. Simultaneously, the contact conditions also imply that there exists a positive constant k depending on A, L and H such that

$$\left|\frac{Y''X''}{A''X''}\right| < k$$

for every $i > i_0$ and uniformly for every $X \in \partial(\tilde{H} \cap S_i)$. Hence

$$\left|\frac{Y''Z''}{A''Z''}\right| < k\varepsilon , \qquad \text{uniformly} .$$

A continuity argument, completely analogous to that in the last part of the proof in Theorem 5.1, concludes this lemma's proof.

Remark. We can rephrase the outcome of the proof in the previous lemma as follows:

With K, $\mathscr{X}$, A, and L as before, if H is a hyperbolic tube in L with A as one of its endpoints, and $\tilde{H}$ is the region bounded by H, then given $\varepsilon > 0$ there exists a ball $B(A)$ centered at A such that for every $V \in B(A) \cap \tilde{H}$ if P_V denotes the hyperplane tangent to the constant volume envelope passing through V, and P_{VA} the hyperplane parallel to P_V passing through A, then every ray emanating from A, contained in P_{VA}, and meeting the boundary of the cone K at Y'', also meets the boundary of the cone subtended by L at a point Z'', and the inequality

$$\left|\frac{Y''Z''}{A''Z''}\right| < k\varepsilon$$

holds uniformly with k being a positive constant depending on A, L and H.

We are now going to prescribe a suitable condition on the boundary of K, which will enable us to remove the restriction of h-convergency and replace it by convergency in the usual sense.

Theorem 5.3. *Let K be a convex cone, $\mathscr{X} \in \operatorname{Int}(K^*)$, and U be a relative open subset of $P_{\mathscr{X}} \cap \partial K$. If $P_{\mathscr{X}} \cap \partial K$ is smooth of class C^2 and strongly convex in U, then $\mu_{K,f}(X)$ approaches $m_{Q,f}$ uniformly as X approaches any $A \in U$.*

Proof. Since $P_{\mathscr{X}} \cap \partial K$ is smooth and strongly convex in U, we can associate to each $A \in U$ an ellipsoid $L(A)$ having third order contact with $P_{\mathscr{X}} \cap \partial K$ at A, and a hyperbolic tube $H(A)$. Furthermore, we can choose the families $\{L(A)\}$ and $\{H(A)\}$ to be *continuous*. The cross-ratio $|(A, Y; X, Z)|$, as defined in the preceding lemma, is therefore continuous in both arguments X and A. Hence given $\varepsilon > 0$ the family of $(n-1)$-hypersurfaces defined by

$$S(A;\varepsilon) = \{X : |(A, Y; X, Z)| = \varepsilon\}$$

is also continuous. It follows that there exists a *continuous* family of balls $\{B(A)\}$, centered at each $A \in U$, such that if $\tilde{H}(A)$ denotes the region bounded by the hyperbolic tube $H(A)$, then for every $V \in B(A) \cap \tilde{H}(A)$ the conditions stated in the remark preceding this theorem are satisfied. A standard continuity argument now concludes the proof.

Remark. It is clear how to apply the above theorem (or the lemma) to the volume ratio function ρ_K as defined on M_c^* (or M_c). To each $X \in P_{\mathscr{X}} \cap \operatorname{Int}(K)$ we make it correspond to the point X_c at which the ray $\overrightarrow{OX}$ meets M_c. As X

approaches $A \in \partial K$, X_c converges asymptotically to ∂K in the direction of the ray $\overrightarrow{OA}$ and, correspondingly, if the conditions of either Lemma 5.2 or Theorem 5.3 are satisfied, then $\rho_K(\mathscr{X}^c) = a_n c^2 \mu_K(\mathscr{X}^c)$ converges to $a_n c^2 m_Q$.

We can now prove the particular case of the conjecture stated at the beginning of this section. In fact, if for some $\mathscr{X} \in \operatorname{Int}(K^*)$, $P_{\mathscr{X}} \cap \partial K$ is of class C^2, then there exists an open subset $U \subset P_{\mathscr{X}} \cap \partial K$ such that U is strongly convex. By Theorem 5.3 and the above remark $\rho_K(\mathscr{X}^c)$ converges to $a_n c^2 m_Q$ as X_c converges asymptotically to ∂K in the direction of any ray $\overrightarrow{OA}$, $A \in U$. Hence assuming $\rho_K(\mathscr{X}^c)$ to be constant we have precisely $\rho_K(\mathscr{X}^c) = a_n c^2 \mu_K(\mathscr{X}^c)$ for every $X_c \in M_c$. By Proposition 1.1 all of the n-dimensional convex bodies $P_{\mathscr{X}} c \cap K$, with a quadric cone K which is well known to be homogeneous, should then be ellipsoidal and, therefore, in an appropriate affine coordinate system.

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