LINEARLY INDUCED VECTOR FIELDS AND R^2-ACTIONS ON SPHERES

GILDA DE LA ROCQUE PALIS

1. Introduction

We prove here results on the generic and structurally stable properties of linearly induced vector fields and R^2-actions on spheres. These actions are obtained from linear actions on R^n which are naturally extended to the standard sphere S^n via central projection. Similarly, one can use radial projection to get quite a large number of vector fields and R^2-actions on spheres which are structurally stable or at least Ω-stable.

In 1881 Poincaré [12] began the qualitative theory of polynomial vector fields on the plane R^2 looking at the central projection of their trajectories on the sphere S^2. This work appears in other texts [3], [6], [11], [13] always in a form similar to the original one. More recently Gonzalez [5] characterized the polynomial vector fields on R^2 which are structurally stable in a neighborhood of infinity. He also began the study of linearly induced vector fields on S^3.

In § 2 we consider linearly induced vector fields on the sphere S^n. Let $X(x) = Ax$ be a linear vector field on R^n. The central projection is the map which associates to each point $x = (x_1, \cdots, x_n)$ of R^n two points in S^n, $f(x) = (x_1, \cdots, x_n, 1)/\Delta x$ and $f_1(x) = -(x_1, \cdots, x_n, 1)/\Delta x$ where $\Delta x = (1 + x_1^2 + \cdots + x_n^2)^{1/2}$. The linearly induced vector fields $Df(X)$ and $Df_1(X)$ extend naturally to the whole S^n, and one gets a vector field called the Poincaré vector field $\pi(X)$. Let $\pi_\infty(X)$ be its restriction to the equator S^{n-1} which is an invariant set. The radial projection $\tau: R^n - 0 \to S^{n-1}$, $\tau(x) = x/|x|$, also induces a vector field $D\tau(X)$ on the sphere S^{n-1}.

Theorem 1. *Let $\pi(X)$, $X(x) = Ax$, be a Poincaré vector field on S^n. Then $\pi(X)$ is a Morse-Smale vector field if and only if the eigenvalues of A have distinct (except for pairs of conjugate complex eigenvalues) nonzero real parts.*

Let $\pi(\mathcal{X})$ be the set of Poincaré vector fields on S^n with the C^r-topology, $r \geq 1$, and $\Sigma \subset \pi(\mathcal{X})$ the subset of structurally stable ones. In Theorem 2 we prove that the Morse-Smale Poincaré vector fields on S^n form an open and dense set in $\pi(\mathcal{X})$ which coincides with Σ.

Similar results hold for linearly induced vector fields by radial projection, as shown in Theorems 3 and 4.

In the proof of Theorem 1 we exibit the phase diagrams of the Morse-Smale vector fields on S^n. It is not true in general that isomorphism between phase diagrams implies topological equivalence. An example in S^2 is given by Peixoto [10]. In our context we have

Theorem 5. *Two Morse-Smale Poincaré vector fields on S^n are topologically equivalent if and only if their phase diagrams are isomorphic.*

Thus we obtain a complete classification up to a topological equivalence of the Morse-Smale Poincaré vector fields on S^n.

The work of Camacho [1], [2], introduced the study of actions of the group $R^k \times Z^l$ on manifolds from the viewpoint of generic properties and structural stability. In §3 we study linearly induced R^2-actions on spheres. Let $\rho\colon R^2 \to$ Aut (R^{n+1}) be a linear action and ψ be the action $\psi\colon Gl(n + 1, R) \to \mathrm{Dif}\,(S^n)$, $\psi(Ax) = Ax/|Ax|$ where $A \in Gl(n + 1, R)$ and $x \in S^n$. An action $\bar\rho$ of R^2 on S^n where $\bar\rho = \psi \circ \rho$ is called a linearly induced action of R^2 on S^n. If the action ρ is generated by commuting linear vector fields X and Y in R^{n+1}, then $\bar\rho$ is generated by $D\tau(X)$ and $D\tau(Y)$. Similarly one can use the central projection.

Theorem 6. *Let $\bar\rho = \psi \circ \rho$ be a linearly induced action of R^2 on S^n where $X(x) = Ax$ and $Y(x) = Bx$ are generators of ρ. Then the compact orbits of $\bar\rho$ are hyperbolic if and only if for some basis of R^{n+1}, A and B are represented as*

(i) $A = \mathrm{diag}\,(\lambda_1, \cdots, \lambda_{n+1})$, $B = \mathrm{diag}\,(\bar\lambda_1, \cdots, \bar\lambda_{n+1})$ *and* $(\lambda_j - \lambda_i)(\bar\lambda_l - \bar\lambda_i) - (\bar\lambda_j - \bar\lambda_i)(\lambda_l - \lambda_i) \neq 0$ *for all distinct i, j, l,*

(ii) $A = \mathrm{diag}\,\left(\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}, \lambda_1, \cdots, \lambda_{n-1}\right)$, $B = \mathrm{diag}\,\left(\begin{pmatrix} \bar\alpha & -\bar\beta \\ \bar\beta & \bar\alpha \end{pmatrix}, \bar\lambda_1, \cdots, \bar\lambda_{n-1}\right)$,

β or $\bar\beta$ nonzero and

$$(\alpha - \lambda_i)\bar\beta - (\bar\alpha - \bar\lambda_i)\beta \neq 0\,,$$
$$(\alpha - \lambda_i)(\bar\lambda_j - \bar\lambda_i) - (\bar\alpha - \bar\lambda_i)(\lambda_j - \lambda_i) \neq 0\,,$$
$$(\lambda_j - \lambda_i)(\bar\lambda_l - \bar\lambda_i) - (\bar\lambda_j - \bar\lambda_i)(\lambda_l - \lambda_i) \neq 0$$

for all distinct i, j, l.

Let $\mathscr{A}(R^2, S^n)$ be the set of linearly induced actions of R^2 on S^n with the C^r $(r \geq 1)$ topology.

Corollary. *The set of linearly induced actions of R^2 on S^n having only hyperbolic compact orbits forms an open and nonempty set in $\mathscr{A}(R^2, S^n)$. This subset is not dense in $\mathscr{A}(R^2, S^n)$ for $n \geq 3$.*

The following theorems characterize the Ω-stability of these actions on S^n and the structural stability on S^3.

Theorem 7. *A linearly induced action of R^2 on S^n is Ω-stable in $\mathscr{A}(R^2, S^n)$ if and only if it has only hyperbolic compact orbits.*

Theorem 8. *A linearly induced action of R^2 on S^3 is structurally stable if and only if all its compact orbits are hyperbolic.*

Let $\chi^r(R^2, S^3)$ be the set of C^r, $r \geq 1$, actions of R^2 on S^3. From the C^r-topology on the space of C^r vector fields on S^3, one can define in a natural way

a C^r-topology on $\chi^r(R^2, S^3)$ using generators of the actions. Then we have the following theorem.

Theorem 9. *A linearly induced R^2-action on S^3 having only hyperbolic compact orbits is structurally stable in $\chi^r(R^2, S^3)$.*

It is an interesting open question if this last theorem is true for linearly induced R^2-actions on S^n when $n \geq 4$. Also one can pose similar questions for actions of R^k, $k \geq 3$.

The results which were announced in [7] and are presented here correspond to the author's doctoral thesis at the Institute forPure and Applied Mathematics at Rio de Janeiro under the guidance of M. Peixoto. The author expresses her gratitude to him, C. Camacho, J. Palis and to her colleagues at the Catholic University of Rio de Janeiro for their constant support.

2. Linearly induced vector fields on spheres

Consider in R^{n+1} the hyperplane $R^n = \{y \in R^{n+1} | y_{n+1} = 1\}$ and the sphere $S^n = \{y \in R^{n+1} | |y| = 1\}$. The central projection associates to each point $x = (x_1, \cdots, x_n)$ of R^n two points in S^n, $f(x) = (x_1, \cdots, x_n, 1)/\Delta x$ and $f_1(x) = -(x_1, \cdots, x_n, 1)/\Delta x$ where $\Delta x = (1 + x_1^2 + \cdots + x_n^2)^{1/2}$. The points at infinity in R^n are in a one-one correspondence with the points in the equator $S^{n-1} = \{y \in S^n | y_{n+1} = 0\}$. Let U_i, V_i, φ_i and ψ_i, $1 \leq i \leq n + 1$, be coordinate neighborhoods and maps in S^n:

$$U_i = \{y \in S^n | y_i > 0\}\,, \qquad i = 1, \cdots, n + 1\,,$$
$$V_i = \{y \in S^n | y_i < 0\}\,, \qquad i = 1, \cdots, n + 1\,,$$
$$\varphi_i \colon U_i \to R^n\,, \qquad \varphi_i(y) = (y_1, \cdots, \hat{y}_i, \cdots, y_{n+1})/y_i\,,$$
$$\psi_i \colon V_i \to R^n\,, \qquad \psi_i(y) = (y_1, \cdots, \hat{y}_i, \cdots, y_{n+1})/y_i\,,$$

where $\hat{y}_i$ means that the ith coordinate y_i is to be deleted.

Let $X(x) = Ax$ be a vector field in R^n where A is an $n \times n$ constant real matrix. By means of the central projection one gets the induced vector fields $Df(X)$ and $Df_1(X)$ in the north and south hemispheres of S^n. This field on $S^n - S^{n-1}$ extends naturally to a field on the whole S^n, which is denoted by $\pi(X)$ and called a Poincaré vector field. If $X(x) = Ax$ where $A = (a_{ij})$, $1 \leq i, j \leq n$, then the field $\pi(X)$ is given in U_i, V_i, $i = 1, \cdots, n$, by the following expressions:

$$(\cdots, (a_{rj})(\tilde{x}_i) - x_r(a_{ij})(\tilde{x}_i), \cdots,$$
$$(a_{l+1,j})(\tilde{x}_i) - x_l(a_{ij})(\tilde{x}_i), \cdots, -x_n(a_{ij})(\tilde{x}_i))\,,$$

where $\tilde{x}_i = (x_1, \cdots, 1, \cdots, x_{n-1})$, 1 being the ith coordinate, $1 \leq r < i, i \leq l < n$, and $(a_{ij})(\tilde{x}_i)$ is the multiplication of the row matrix $(a_{i1}, \cdots, a_{in})$ by the column matrix

$$\begin{bmatrix} x_1 \\ \vdots \\ 1 \\ x_i \\ \vdots \\ x_{n-1} \end{bmatrix}.$$

In U_{n+1} and V_{n+1}, $\pi(X)$ is given by $(\sum_{j=1}^{n} a_{1j}x_j, \sum_{j=1}^{n} a_{2j}x_j, \cdots, \sum_{j=1}^{n} a_{nj}x_j)$.

Note that the equator S^{n-1} is an invariant set. Let $\pi_\infty(X)$ be the restriction of $\pi(X)$ to S^{n-1}. In particular if $A = (a_{ij})$ is such that $a_{in} = 0$ for $i \neq n$ and $a_{nj} = 0$ for $j \neq n$, then the $(n-1) \times (n-1)$ matrix $B = (b_{ij})$, where $b_{ij} = a_{ij}$ for $i \neq j$ and $b_{ii} = a_{ii} - a_{nn}$, is such that $\pi_\infty(X(x) = Ax) = \pi(Y(x) = Bx)$. For $X(x) = Ax$ a linear vector field in R^n, the radial projection $\tau: R^n - 0 \rightarrow S^{n-1}$, $\tau(x) = x/|x|$, also induces a vector field $D\tau(X)$ on the sphere S^{n-1}. If $\pi(X)$, $X(x) = Ax$, is a Poincaré vector field on S^n, then the linear vector field $Y(x) = Bx$ in R^{n+1}, $B = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$, is such that $\pi(X) = D\tau(Y)$. Also given a linearly induced vector field $D\tau(Y)$ one has $D\tau(Y) = \pi_\infty(Y)$.

A vector field X on a compact manifold M without boundary is called Morse-Smale if

(1) X has only a finite number of critical elements (singularities and closed orbits) all of which are hyperbolic,

(2) the stable and unstable manifolds of the critical elements of X have transversal intersections,

(3) the α and ω-limit sets of every trajectory of X are critical elements.

The phase diagram of a Morse-Smale vector field X is the set of critical elements of X partially ordered by the relation: $A \leq B$ if and only if $W^s(A) \cap W^u(B) \neq \emptyset$, where $W^s(A)$ and $W^u(B)$ are the stable and unstable manifolds of the critical elements A and B. We recall that the index of a hyperbolic critical element is the dimension of its stable manifold. Let $D(X)$ and $D(Y)$ be the phase diagrams of the Morse-Smale vector fields X and Y. A map $H: D(X) \rightarrow D(Y)$ is a diagram isomorphism if it is bijective, index and order-preserving. For more detailed discussion of the above definitions we refer to [8] and [14].

Theorem 1. *Let $\pi(X)$, $X(x) = Ax$, be a Poincaré vector field on S^n. Then $\pi(X)$ is a Morse-Smale vector field if and only if the eigenvalues of A have distinct (except for pairs of conjugate complex eigenvalues) nonzero real parts. In this case, the number of closed orbits is the number of pairs of complex conjugate eigenvalues of A, and there are $2k + 2$ singularities where k is the number of real eigenvalues of A.*

Proof. We may assume the operator A given, in standard coordinates, by a matrix in real canonical form. The theorem can be proved by induction on the

dimension n. The coordinate neighborhoods and maps (U_i, φ_i), (V_i, ψ_i) are the ones defined previously in this section.

For $n = 2$, $\pi_\infty(X)$ is very simple and the result can easily be checked by looking at the expressions of the vector field $\pi(X)$ in the coordinate neighborhoods considered. If $X(x) = Ax$ in R^2 is such that A is not diagonalizable, then $\pi(X)$ has a nonhyperbolic singularity and the phase portrait looks like Fig. A.
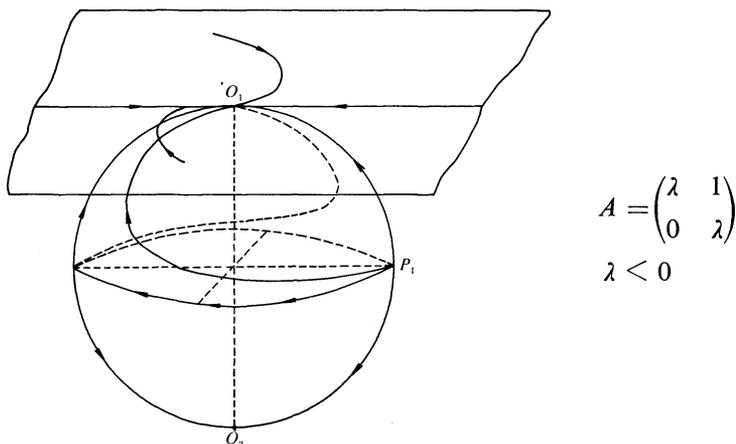

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$\lambda < 0$$

Fig. A

If A is diagonalizable, then it has one of the following two forms:

$$\text{(I)} \quad \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad \text{(II)} \quad \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}.$$

In the first case, $\pi(X)$ has only a finite number of singularities if and only if λ_1 and λ_2 are nonzero and distinct. Furthermore these are sufficient conditions for $\pi(X)$ to be a Morse-Smale vector field. In the second case, $\pi(X)$ has only a finite number of closed orbits if and only if α is nonzero. Again this suffices for the Poincaré vector field $\pi(X)$ to be Morse-Smale. The phase portrait and diagrams for these cases are given in Fig. B and Fig. C.

Assume the truth of the theorem for dimension less than n, and let us consider the case where the dimension is $n > 2$.

If $X(x) = Ax$ in R^n is such that A is not diagonalizable, then $\pi(X)$ has a nonhyperbolic critical element. In fact, if A has a real eigenvalue, from the expressions of the associated Poincaré vector field $\pi(X)$ we conclude that $\pi(X)$ has a singularity such that its linear part at this fixed point has zero as eigenvalue. In the case where A has a complex eigenvalue, there is a nonhyperbolic closed orbit. To see this it is useful to consider the previously mentioned relation between vector fields linearly induced by central and radial projection. Let Y be the linear vector field in R^{n+1} such that $D\tau(Y) = \pi(X)$, and Z_t the flow
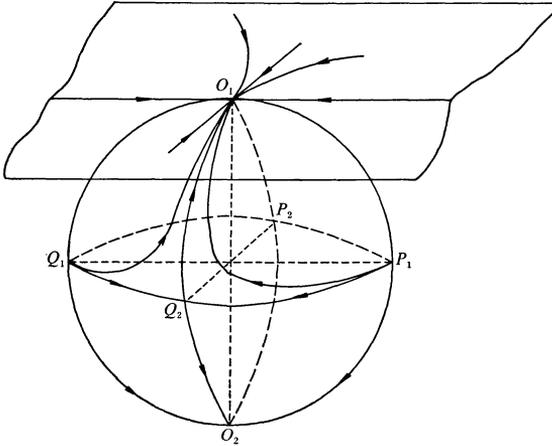
$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$\lambda_1 < \lambda_2 < 0$$

Fig. B
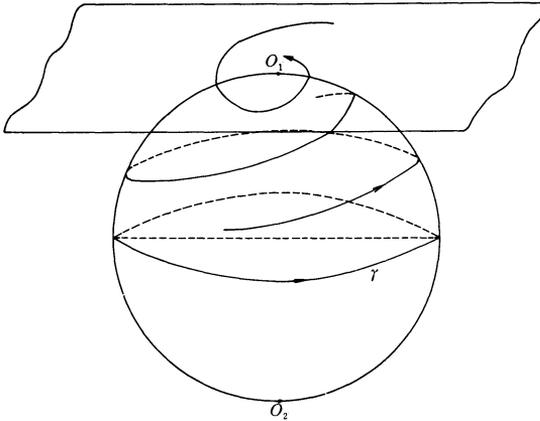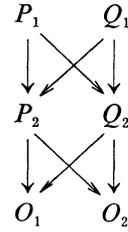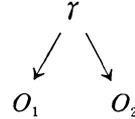
$$A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

$$\alpha < 0 , \qquad \beta > 0$$

Fig. C

determined by $D\tau(Y)$. One can check that $D_q Z_\tau$ has 1 as eigenvalue with multiplicity greater than one, where τ is the period of some closed orbit γ and q a point in γ. It then follows that γ is nonhyperbolic.

If A is diagonalizable we have to consider two possibilities:

(I) A has a real eigenvalue.

Let $A = \mathrm{diag}\left(\begin{pmatrix} \alpha_1 & -\beta_1 \\ \beta_1 & \alpha_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 & -\beta_2 \\ \beta_2 & \alpha_2 \end{pmatrix}, \cdots, \begin{pmatrix} \alpha_m & -\beta_m \\ \beta_m & \alpha_m \end{pmatrix}, \lambda_1, \cdots, \lambda_k \right)$, where $\beta_i \neq 0$ for $i = 1, \cdots, m$ and $k \geq 1$. The vector field $\pi(X)$ is given by the following expressions:

in $U_{2i-1}, V_{2i-1}, 1 \leq i \leq m$:

$$(x_1(\alpha_1 - \alpha_i + \beta_i x_{2i-1}) - \beta_1 x_2, x_1 \beta_1 + x_2(\alpha_1 - \alpha_i + \beta_i x_{2i-1})) ,$$

$$\cdots, x_{2i-3}\beta_{i-1} + x_{2i-2}(\alpha_{i-1} - \alpha_i + \beta_i x_{2i-1}), \beta_i(1 + x_{2i-1}^2),$$

$$x_{2i}(\alpha_{i+1} - \alpha_i + \beta_i x_{2i-1}) - \beta_{i+1}x_{2i+1}), \cdots,$$

$$x_{2m+j-1}(\lambda_j - \alpha_i + \beta_i x_{2i-1}), \cdots, x_n(\beta_i x_{2i-1} - \alpha_j)),$$

where $1 \leq j \leq k$;

in U_{2m+j}, V_{2m+j}, $1 \leq j \leq k$:

$$(x_1(\alpha_1 - \lambda_j) - \beta_1 x_2, \beta_1 x_1 + x_2(\alpha_1 - \lambda_j), \cdots, \beta_m x_{2m-1} + x_{2m}(\alpha_m - \lambda_j),$$

$$x_{2m+1}(\lambda_1 - \lambda_j), \cdots, x_{2m+j-1}(\lambda_{j-1} - \lambda_j), \cdots, -x_n\lambda_j);$$

in U_{n+1}, V_{n+1}:

$$(\alpha_1 x_1 - \beta_1 x_2, \cdots, \beta_m x_{2m-1} + \alpha_m x_{2m}, \lambda_1 x_{2m+1}, \cdots, \lambda_k x_n).$$

Notice that in the coordinate neighborhoods $U_1, V_1, \cdots, U_{2m}, V_{2m}$ the vector field $\pi(X)$ has no singularities.

Suppose $\pi(X)$ is a Morse-Smale vector field. Then $\pi(X)$ has only a finite number of singularities, in which case we have that $\lambda_i \neq \lambda_j$ for all $i \neq j$ and $\lambda_j \neq 0$ for every j. Let O_1, O_2 be the singularities on the neighborhoods U_{n+1}, V_{n+1}, and P_j, Q_j the ones on the neighborhoods U_{2m+j}, V_{2m+j}. The linear part of $\pi(X)$ at P_j and Q_j is given by

$$(*) \quad \operatorname{diag}\left(\begin{pmatrix} (\alpha_1 - \lambda_j) & -\beta_1 \\ \beta_1 & (\alpha_1 - \lambda_j) \end{pmatrix}, \cdots, \right.$$
$$\left. \begin{pmatrix} (\alpha_m - \lambda_j) & -\beta_m \\ \beta_m & (\alpha_m - \lambda_j) \end{pmatrix}, (\lambda_1 - \lambda_j), \cdots, -\lambda_j \right).$$

The hyperbolicity of the singularities implies that $\alpha_i \neq \lambda_j$ for all i, j; $\lambda_i \neq \lambda_j$ for all $i \neq j$; $\lambda_j \neq 0$ and $\alpha_j \neq 0$ for every j.

We now consider the closed orbits of $\pi(X)$. For that take the linear vector field $Y = Bx$ in R^{n+1}, $B = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$, where $D\tau(Y) = \pi(X)$ and the flow Z_t determined by $D\tau(Y)$. In this way we can see that the trajectories by $p_i = (0, \cdots, 1, \cdots, 0) \in S^n \subset R^{n+1}$, where 1 is the $(2i)$th coordinate, are closed orbits of period $\tau_i = 2\pi/\beta_i$, $i = 1, \cdots, m$. The fact that these are hyperbolic critical elements implies the following inequalities: $\alpha_i \neq \alpha_j$ for all $i \neq j$; $\lambda_j \neq \alpha_i$ for all i, j and $\alpha_i \neq 0$ for every i. This is because $D_{p_i} Z_{\tau_i}$ is given by

$$(**) \quad \operatorname{diag}\left(e^{(\alpha_1-\alpha_i)\tau_i} \begin{pmatrix} \cos\beta_1\tau_i & -\sin\beta_1\tau_i \\ \sin\beta_1\tau_i & \cos\beta_1\tau_i \end{pmatrix}, \cdots, 1, \cdots, e^{(\lambda_k-\alpha_i)\tau_i}, e^{-\alpha_i\tau_i} \right),$$

where 1 is the $(2i - 1)$th diagonal element. Therefore if $\pi(X)$ is a Morse-Smale vector field the eigenvalues of A have nonzero distinct real parts.

Next we prove that these are sufficient conditions for $\pi(X)$ to be Morse-

Smale. We now assume that A has eigenvalues with distinct nonzero real parts. Let us consider the linear vector field $W(x) = Cx$ in R^{n+1} where

$$C = \text{diag}\left(\begin{pmatrix} (\alpha_1 - \lambda_k) & -\beta_1 \\ \beta_1 & (\alpha_1 - \lambda_k) \end{pmatrix}, \cdots, (\lambda_{k-1} - \lambda_k)\right).$$

As previously mentioned we have that $\pi_\infty(X) = \pi(W)$. By induction hypothesis, $\pi_\infty(X)$, a vector field in the equator S^{n-1}, is a Morse-Smale vector field, and has m closed orbits and $2k$ singularities, and the α and ω-limit sets of its trajectories are critical elements. Observe that $\pi(X)$ in $S^n - S^{n-1}$ consists of two copies of the linear system $X(x) = Ax$. We conclude that outside equator S^{n-1} there are only two more singularities, and the geometrical behaviour of the trajectories can be completely analysed. It follows that $\pi(X)$ has m closed orbits and $2k + 2$ singularities, all hyperbolic as we can see in the expressions (*) and (**). Also the α and ω-limit sets of its trajectories are the critical elements.

It remains to show that the stable and unstable manifolds of the critical elements have transversal intersections. For simplicity let us call $u_1 < u_2 < \cdots < u_r < 0 < u_{r+1} < \cdots < u_{m+k}$ the ordered set of real parts of the eigenvalues of A, and $O_1, O_2, \Omega_1, \cdots, \Omega_{m+k}$ the critical elements of $\pi(X)$. Interpret the notation as follows: if $u_l = \lambda_i$ for some i, then Ω_l consists of two singularities, and if $u_l = \alpha_j$ for some j, then Ω_l is a closed orbit. If $u_i < u_j$ then $W^s(\Omega_j) \cap W^u(\Omega_i) \neq \emptyset$ in S^{n-1}. This can be verified by looking at the special cases for lower dimensions. For $\pi_\infty(X)$, a Morse-Smale vector field on S^{n-1}, by induction hypothesis we then have that $W^s(\Omega_j)$ is transversal to $W^u(\Omega_i)$ in S^{n-1}. Suppose $p \in W^s(\Omega_j) \cap W^u(\Omega_i)$, the invariant manifolds now on S^n and determined by $\pi(X)$. If $p \in S^{n-1}$ then $j > i$, for otherwise p would be a nonwandering point. Note that for $j > i$, the tangent space $T_p W^s(\Omega_j)$ or $T_p W^u(\Omega_i)$ contains the normal direction to S^{n-1} by p. This implies that the manifolds $W^s(\Omega_j)$ and $W^u(\Omega_i)$ are in general position. On the other hand if $p \notin S^{n-1}$ we must have $u_i > 0$ and $u_j < 0$. Also the phase portrait for $\pi(X)$ outside S^{n-1} is completely known. The transversality is then easily checked. The same procedure can be applied to show that if $p \in W^s(O_1) \cap W^u(\Omega_j)$, then $1 \leq j \leq r$ and the manifolds intersect transversally. We conclude that $\pi(X)$ is a Morse-Smale vector field, and this completes the proof. The phase diagram of $\pi(X)$ has the following configuration:

$$\Omega_1$$
$$\downarrow$$
$$\vdots$$
$$\Omega_r$$

$$O_1 \qquad\qquad O_2$$

$$\Omega_{r+1}$$
$$\downarrow$$
$$\vdots$$
$$\Omega_{m+k}$$
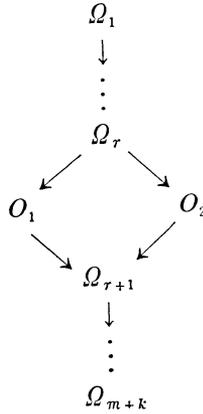
(II) A has only complex eigenvalues.
Let

$$A = \operatorname{diag}\left(\begin{pmatrix}\alpha_1 & -\beta_1 \\ \beta_1 & \alpha_1\end{pmatrix}, \cdots, \begin{pmatrix}\alpha_m & -\beta_m \\ \beta_m & \alpha_m\end{pmatrix}\right),$$

where $\beta_i \neq 0$ for every i. We can write the expressions of $\pi(X)$ in the coordinate neighborhoods $U_1, \cdots, U_{2m}, V_{2m}, U_{n+1}, V_{n+1}$ as it was done in case (I). From these expressions it is clear that $\pi(X)$ has only two singularities O_1 and O_2.

Suppose $\pi(X)$ is a Morse-Smale vector field. Then it has only a finite number of closed orbits so that $\alpha_i \neq 0$ for every i. Consider, as before, the linear vector field $Y = Bx$ in R^{n+1}, where $B = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ and $D\tau(Y) = \pi(X)$, and the flow Z_t determined by $D\tau(Y)$. In this way we can see that the trajectories by $p_i = (0, \cdots, 1, \cdots, 0) \in S^n \subset R^{n+1}$, where 1 is the $(2i)$th coordinate, are closed orbits of period $\tau_i = 2\pi/\beta_i$, $i = 1, \cdots, m$. Since $D_{p_i} Z_{\tau_i}$ is given by

(***) $\operatorname{diag}\left(1, e^{(\alpha_1-\alpha_i)\tau_i}\begin{pmatrix}\cos\beta_1\tau_i & -\operatorname{sen}\beta_1\tau_i \\ \operatorname{sen}\beta_1\tau_i & \cos\beta_1\tau_i\end{pmatrix}, \cdots, e^{-\alpha_i\tau_i}\right),$

the hyperbolicity of these closed orbits implies that $\alpha_i \neq 0$ for every i and $\alpha_i \neq \alpha_j$ for all $i \neq j$. Hence, if $\pi(X)$ is a Morse-Smale vector field, the eigenvalues of A have nonzero distinct real parts.

Next, to prove sufficiency, let us suppose that the eigenvalues of A are such that $\alpha_i \neq 0$ for every i and $\alpha_i \neq \alpha_j$ for all $i \neq j$. Assume $\alpha_1 < \alpha_2 < \cdots < \alpha_m$. The singularities O_1 and O_2 are hyperbolic, and from the expression (***) it follows that $\pi(X)$ has m hyperbolic closed orbits $\gamma_1, \cdots, \gamma_m$. We now examine the limit sets of the trajectories of $\pi(X)$. For trajectories in $S^n - S^{n-1}$ the limit sets are completely known since $\pi(X)$ on this region consists of two copies of $X(x) = Ax$. To investigate other trajectories, let p be a point in the equator S^{n-1}. In what follows, if M is a subspace of R^n, $\tilde{M}$ will mean the continuous exten-

sion of its central projection to the equator S^{n-1}. Let $D_1 = \{(x_1, x_2, \cdots, x_n) \in R^n \mid x_{n-1} = x_n = 0\}$. By induction hypothesis if $p \in \tilde{D}_1$, the limit sets of the trajectories through p are critical elements. If neither $p \in \tilde{D}_1$ nor $p \in \gamma_m$, we have to examine further. Consider the subspace $D_2 = \{(x_1, \cdots, x_n) \in R^n \mid x_1 = \cdots = x_{n-2} = 0\}$. Now let V be a neighborhood of γ_m, $V \subset W^s(\gamma_m)$. Observe that if $\alpha_m < 0$, V is a neighborhood in S^{n-1}. Next choose θ such that

　　(i)　if $W_1 = \{x \in R^n \mid \text{the slope of } x \text{ with } D_1 \text{ is } \leq \theta\}$, then $p \notin \tilde{W}_1$,
　　(ii)　if $W_2 = \{x \in R^n \mid \text{the slope of } x \text{ with } D_2 \text{ is } \leq \theta\}$, then $\tilde{W}_2 \cap S^{n-1} \subset V$.

Let $q \in R^n$, $q \notin W_1 \cup W_2$, and let X_t be the flow in R^n determined by $X(x) = Ax$. It can be verified that there exists T such that $X_T(q) \in W_2$. Assume p is a point in the equator S^{n-1}, $p \notin \tilde{D}_1$ and $p \notin \gamma_m$. Consider a sequence (q_i) on the sphere S^n converging to p, such that $q_i \notin S^{n-1}$ and also does not belong to the central projection of $D_1 \cup D_2$. Since X_T is a diffeomorphism, one can see that $X_T(p) \in V$ which implies that the ω-limit set of the trajectory through p is γ_m. The same procedure can be applied to study the α-limit sets of the trajectories of $\pi(X)$. It follows that the critical elements $O_1, O_2, \gamma_1, \cdots, \gamma_m$ are the α and ω-limit sets of the trajectories of $\pi(X)$. As we proved before, these critical elements are all hyperbolic. Next we show that the invariant manifolds of these critical elements have transversal intersections. Assume $\alpha_1 < \alpha_2 < \cdots < \alpha_k < 0 < \alpha_{k+1} < \cdots < \alpha_m$. Let $p \in W^s(\gamma_i) \cap W^u(\gamma_j)$, $i \neq j$. We may suppose that neither $i = m$ nor $j = m$, in which cases the transversality is obvious. If p is in the equator S^{n-1} and $p \in \tilde{D}_1$, the transversality of $W^s(\gamma_i)$ and $W^u(\gamma_j)$ can be checked by induction hypothesis and by looking at a special four-dimensional case. To see this consider the Poincaré vector fields: $\pi(Y)$ in $S^{n-2} \subset S^n$ where $Y(x) = Bx$,

$$B = \text{diag}\left(\begin{pmatrix} \alpha_1 & -\beta_1 \\ \beta^1 & \alpha_1 \end{pmatrix}, \cdots, \begin{pmatrix} \alpha_{m-1} & -\beta_{m-1} \\ \beta_{m-1} & \alpha_{m-1} \end{pmatrix} \right),$$

and $\pi(Z)$ in $S^4 \subset S^n$ where $Z(x) = Cx$,

$$C = \text{diag}\left(\begin{pmatrix} \alpha_j & -\beta_j \\ \beta_1 & \alpha_j \end{pmatrix}, \begin{pmatrix} \alpha_m & -\beta_m \\ \beta_m & \alpha_m \end{pmatrix} \right).$$

Now, if $p \in S^{n-1}$ and $p \notin \tilde{D}_1$ then $\gamma_i = \gamma_m$, and since $\dim W^s(\gamma_i) = n$ the manifolds $W^s(\gamma_i)$ and $W^u(\gamma_m)$ are in general position. On the other hand if $p \notin S^{n-1}$, we must have $\alpha_i > 0$, $\alpha_j < 0$, the phase portrait of $\pi(X)$ in $S^n - S^{n-1}$ is known, and the transversality is easily checked. Similarly we can verify that the invariant manifolds of singulalities (O_1 and O_2) and closed orbits intersect transversaly. As a conclusion $\pi(X)$ is a Morse-Smale vector field, and this finishes the proof of this last case. The phase diagram for $\pi(X)$ is given below:

$$\gamma_1$$
$$\downarrow$$
$$\vdots$$
$$\gamma_k$$

$$O_1 \qquad\qquad O_2$$

$$\gamma_{k+1}$$
$$\downarrow$$
$$\vdots$$
$$\gamma_m$$
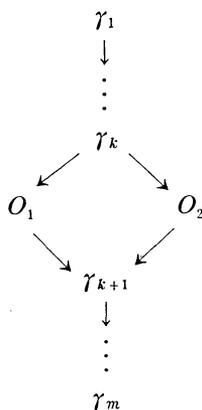
Let $\mathscr{X}$ be the set of linear vector fields in R^n, $\chi = \{X(x) = Ax\}$, identified with the Euclidean space R^{n^2}, space of the entries of the matrices A. Let $\pi(\mathscr{X})$ be the set of Poincaré vector fields on S^n with the topology which arises from the metric $\|\pi(X) - \pi(Y)\| = |X - Y|$. This coincides with the topology induced by the C^r-topology on the space $\mathscr{X}(S^n)$ of C^r vector fields on S^n, $r \geq 1$. A vector field $X \in \mathscr{X}(S^n)$ is structurally stable if there exists a neighborhood $V(X)$ such that if $Y \in V(X)$ then Y is topologically equivalent to X, that is, there is a homeomorphism of M taking trajectories of X onto trajectories of Y.

Theorem 2. *The Morse-Smale Poincaré vector fields on S^n form an open and dense set in $\pi(\mathscr{X})$ which coincides with the subset $\Sigma \subset \pi(\mathscr{X})$ of the structurally stable ones.*

Proof. From Theorem 1, density is obvious and openess follows from continuity of eigenvalues with linear operators.

Next we show that a Poincaré vector field $\pi(X)$ on S^n is structurally stable if and only if it is Morse-Smale.

To prove necessity let us examine a Poincaré vector field $\pi(X)$ which is not Morse-Smale. If $\pi(X)$ has an infinite number of critical elements, it can not be structurally stable as a consequence of the density mentioned above. On the other hand if $\pi(X)$, $X(x) = Ax$, has a finite number of critical elements then the operator A is not diagonalizable. Furthermore if $(x - \lambda_1)^{p_1}$, $(x - \lambda_2)^{p_2}$, $\cdots$ are the elementary divisors of A, $p_1, p_2 \geq 2$, then $\lambda_i \neq 0$ for every i and $\lambda_i \neq \lambda_j$ for all pairs i, j. Also if $[(x - z_1)(x - \bar{z}_1)]^{p_1}$, $[(x - z_2)(x - \bar{z}_2)]^{p_2}$, $\cdots$ are elementary divisors of A, $p_1, p_2 \geq 2$, $z_i = \alpha_i + i\beta_i$, then $\alpha_i \neq 0$ for every i and there is a pair i, j such that $\alpha_i = \alpha_j$ with β_j/β_i irrational. We now perturb X so that all the real eigenvalues become distinct or the ratios β_j/β_i above become rational. In both cases, the perturbed vector field has a larger number of critical elements which proves that it is not structurally stable.

Sufficiency follows directly from a theorem of Palis-Smale [9]. This finishes the proof.

For linearly induced vector fields on S^n by radial projection, we have the following similar results.

Theorem 3.   *Let $D\tau(X)$ be a linearly induced vector field on S^n by redial projection, where $X(x) = Ax$ is a linear vector field in R^{n+1}. Then $D\tau(X)$ is a Morse-Smale vector field if and only if the eigenvalues of A have distinct (except for pairs of conjugate complex eigenvalues) real parts.*

Let $D\tau(\mathcal{X})$ be the space of linearly induced vector fields on S^n by radial projection with the topology induced by the C^r topology on $\mathcal{X}(S^n)$, $r \geq 1$.

Theorem 4.   *The Morse-Smale linearly induced vector fields on S^n by radial projection form an open and dense set in $D\tau(\mathcal{X})$, which coincides with the subset $\Sigma \subset D\tau(\mathcal{X})$ of the structurally stable ones.*

We conclude this section with the following statement.

Theorem 5.   *Two Morse-Smale Poincaré vector fields on S^n are topologically equivalent if and only if their phase diagrams are isomorphic.*

Proof.   Let $\pi(X)$, $X(x) = Ax$, and $\pi(Y)$, $Y(x) = Bx$, be two Poincaré Morse-Smale vector fields on S^n. If they are equivalent it is obvious that they have isomorphic phase diagrams. To prove the converse let us suppose that they have isomorphic phase diagrams. Let $\alpha_p \pm i\beta_p$ where $p = 1, \cdots, m$, and λ_l where $l = 1, \cdots, k$ be the eigenvalues of A, and $\bar{\alpha}_p \pm i\bar{\beta}_p$ where $p = 1, \cdots, m$, and $\bar{\lambda}_l$ where $l = 1, \cdots, k$ the eigenvalues of B. In the following, interpret the notation as in the proof of Theorem 1. Let us call $u_1 < \cdots < u_{m+k}$ and $\bar{u}_1 < \bar{u}_2 < \cdots < \bar{u}_{m+k}$ the ordered set of the real parts of the eigenvalues of A and B respectively. Consider the vector fields $\pi(X_1)$, $X_1(x) = A_1 x$, $A_1 = \mathrm{diag}\,((u_1), \cdots, (u_{m+k}))$ and $\pi(Y_1)$, $Y_1(x) = B_1 x$, $B_1 = \mathrm{diag}\,((\bar{u}_1), \cdots, (\bar{u}_{m+k}))$. Then $\pi(X)$ is topologically equivalent to $\pi(X_1)$, $\pi(Y)$ is topologically equivalent to $\pi(Y_1)$, and the two vector fields $\pi(X_1)$ and $\pi(Y_1)$ have isomorphic phase diagrams. Observe that if $u_1 < \cdots < u_r < 0 < \cdots < u_{m+k}$ and $\bar{u}_1 < \cdots < \bar{u}_s < 0 < \cdots < \bar{u}_{m+k}$ with $r \neq s$ then $k \neq 0$. In particular if $r < s$, then $\bar{u}_{r+1}$ is a real eigenvalue of B_1. The vector field $\pi(Y_1)$ on S^n coincides with the vector field linearly induced on S^n by $Y_2(x) = B_2 x$, $B_2 = \mathrm{diag}\,((\bar{u}_1 - \bar{u}_{r+1}), \cdots, (-\bar{u}_{r+1}))$ through the coordinate maps φ_j^{-1} and ψ_j^{-1} for some j, defined previously in this section. Moreover, $\pi(Y_1)$ coincides with $D\tau(Y_3)$, $Y_3(x) = B_3 x$ being a linear vector field in R^{n+1}, where if $B_2 = (b_{il})$, then

$$B_3 = \begin{bmatrix} b_{11} & \cdots & b_{1,j-1} & 0 & b_{1j} & \cdots & b_{1n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ b_{j-1,1} & \cdots & b_{j-1,j-1} & 0 & b_{j-1,j} & \cdots & b_{j-1,n} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ b_{j1} & \cdots & b_{j,j-1} & 0 & b_{jj} & \cdots & b_{jn} \\ \vdots & & \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{n,j-1} & 0 & b_{nj} & \cdots & b_{nn} \end{bmatrix},$$

Also $D\tau(Y_3)$ is topologically equivalent to $D\tau(Y_4)$ where $Y_4(x) = B_4 x$, $B_4 =$

$\begin{pmatrix} B_2 & 0 \\ 0 & 0 \end{pmatrix}$, which coincides with $\pi(Y_2)$. We conclude that $\pi(Y_1)$ is topologically equivalent to $\pi(Y_2)$. Consider $B_5 = \mathrm{diag}\,((v_1), (v_2), \cdots, (v_{m+k}))$ where $v_1 < v_2 < \cdots < v_r < 0 < \cdots < v_{m+k}$ are the ordered real parts of the eigenvalues of B_2. Then $\pi(Y_2)$ is topologically equivalent to $\pi(Y_5)$, $Y_5(x) = B_5 x$. Denote by $\bar{\bar{\alpha}}_p \pm i\bar{\bar{\beta}}_p$ and $\bar{\bar{\lambda}}_l$ the eigenvalues of B_5. Observe that we can suppose $\bar{\bar{\beta}}_p$ and β_p with the same sign for each p. Now let $\gamma\colon [0, 1] \to \{\pi(\mathscr{X})\}$ be the continuous function defined by $\gamma(s) = \pi(Z_s)$, $Z_s(x) = C_s x$ where

$$C_s = \mathrm{diag}\,((sv_1 + (1 - s)u_1), \cdots, (sv_{m+k} + (1 - s)u_{m+k}))\,.$$

Here interpret the notation as follows: if u_i is the real part of a complex eigenvalue $\alpha_p \pm i\beta_p$, then

$$(sv_i + (1 - s)u_i) = \begin{pmatrix} s\bar{\bar{\alpha}}_p + (1 - s)\alpha_p & -[s\bar{\bar{\beta}}_p + (1 - s)\beta_p] \\ s\bar{\bar{\beta}}_p + (1 - s)\beta_p & s\bar{\bar{\alpha}}_p + (1 - s)\alpha_p \end{pmatrix}\,.$$

Therefore $\gamma(0) = \pi(X_1)$, $\gamma(1) = \pi(Y_5)$ and for each $s \in (0, 1)$, $\gamma(s) = \pi(z_s)$ is a Morse-Smale vector field. By compactness of $\gamma[0, 1]$ and the fact that Morse-Smale vector fields are structurally stable we conclude that $\pi(X_1)$ and $\pi(Y_5)$ are topologically equivalent. It follows that $\pi(X)$ and $\pi(Y)$ are topologically equivalent, and hence the proof is complete.

3. Linearly induced actions of R^2 on spheres

Let $\varphi\colon G \to \mathrm{Dif}\,(M)$ be a C^r, $r \geq 1$, action of a Lie group G on a differentiable manifold M. The orbit $\mathcal{O}_x(\varphi)$ of φ through $x \in M$ and the isotropy group $G_x(\varphi)$ of φ on $x \in M$ are the sets $\mathcal{O}_x(\varphi) = \{\varphi_g(x) \mid g \in G\}$, $G_x(\varphi) = \{g \in G \mid \varphi_g(x) = x\}$. An action $\varphi\colon G \to \mathrm{Dif}\,(M)$ is said to be transitive if for $x \in M$ one has $\mathcal{O}_x(\varphi) = M$. Given two actions φ and ψ of a group G on M, they are topologically equivalent if there is a homeomorphism of M taking orbits of φ onto orbits of ψ.

Definition. Let $\rho\colon G \to \mathrm{Aut}\,(R^n)$ be a linear action of a group G on R^n. Suppose that G is isomorphic to R or Z. Then the action ρ is hyperbolic if for every $g \in G$, $g \neq 0$, the eigenvalues of $\rho(g)$ have absolute value different from one. In the case where G is isomorphic to R^2 or $R \times Z$, the action ρ is said to be hyperbolic if the following hold:

 (i)   There exists a ρ-invariant splitting $R^n = \oplus\, E_i$ such that ρ is transitive on the connected components of each $E_i - \{0\}$.

 (ii) Let G_i be the isotropy group of some nonzero element of E_i. By (i) G_i is isomorphic to R or Z. Let $\rho_i\colon G_i \to \mathrm{Aut}\,(\oplus_{j \neq i} E_j)$ be $\rho|_{G_i}$ acting on $\oplus_{j \neq i} E_j$. Then for each i, ρ_i is hyperbolic.

 Moreover, if $G = Z^2$ we say that the action is not hyperbolic.

Definition.   Let $\varphi\colon G \to \text{Dif}\,(M)$ be a C^r, $r \geq 1$, action of G on M, and γ a compact orbit of φ. Then γ is hyperbolic if

(i)   there exists an invariant continuous splitting of the tangent bundle of M restricted to $\gamma\colon T_\gamma(M) = E + \Gamma$, where Γ is the subbundle of tangent vectors to γ,

(ii)   the induced linear action $\rho|_{G_x} \to \text{Aut}\,(E_x)$ is hyperbolic, where $x \in \gamma$ and $\rho(g) = D\varphi_g|_{E_x}$.

In particular, a fixed point $x \in M$ of φ is said to be hyperbolic if the induced linear action $D\varphi_g(x)\colon G \to \text{Aut}\,(T_x(M))$ is hyperbolic.

Definition.   Let $\varphi\colon G \to \text{Dif}\,(M)$ be an action of G on M. A point $x \in M$ is said to be nonwandering if for an increasing sequence K_n of compact neighborhoods of zero covering G and for every neighborhood V of x and $n_0 \in Z^+$, there exists $g \notin K_{n_0}$ such that $\varphi_g(V) \cap V \neq \emptyset$. The set of nonwandering points of φ is denoted by $\Omega(\varphi)$. An orbit $\mathcal{O}_x(\varphi)$ through x is a singular orbit if $G_x(\varphi) \neq 0$. Observe that if $G_x(\varphi) \neq 0$ then $\mathcal{O}_x(\varphi) \subset \Omega(\varphi)$.

One can give similar definitions for $R^k \times Z^l$ actions [1].

Let $\rho\colon R^2 \to \text{Aut}\,(R^n)$ be a linear action. Then ρ can be written as $\rho(s, t) = e^{sA + tB}$ where A and B are $n \times n$ commutative matrices. The linear vector fields $X(x) = Ax$ and $Y(x) = Bx$ are generators of the action. We may assume the operator A given in standard coordinates by a matrix in the real canonical form.

Proposition.   *Let $\rho\colon R^2 \to \text{Aut}\,(R^n)$ be a linear action of R^2 on R^n where $X(x) = Ax$ and $Y(x) = Bx$ are generators of ρ. Then ρ is hyperbolic if and only if there exists a basis of R^n where*

$$A = \text{diag}\left(\begin{pmatrix} \alpha_1 & -\beta_1 \\ \beta_1 & \alpha_1 \end{pmatrix}, \ldots, \begin{pmatrix} \alpha_m & -\beta_m \\ \beta_m & \alpha_m \end{pmatrix}, \lambda_1, \ldots, \lambda_k \right),$$

$$B = \text{diag}\left(\begin{pmatrix} \overline{\alpha}_1 & -\overline{\beta}_1 \\ \overline{\beta}_1 & \overline{\alpha}_1 \end{pmatrix}, \ldots, \begin{pmatrix} \overline{\alpha}_m & -\overline{\beta}_m \\ \overline{\beta}_m & \overline{\alpha}_m \end{pmatrix}, \overline{\lambda}_1, \ldots, \overline{\lambda}_k \right),$$

and $\alpha_i \overline{\beta}_i - \beta_i \overline{\alpha}_i \neq 0$, $\lambda_i \overline{\lambda}_j - \lambda_j \overline{\lambda}_i \neq 0$, $\alpha_i \overline{\alpha}_j - \alpha_j \overline{\alpha}_i \neq 0$, $\alpha_i \overline{\lambda}_j - \alpha_j \overline{\lambda}_i \neq 0$ for all distinct i, j. In this case all orbits are homeomorphic to R^2 except for one fixed point, $2k$ orbits homeomorphic to R, and m orbits homeomorphic to $R \times S^1$. The nonwandering set $\Omega(\rho)$ consists of the singular orbits.

Proof.   If the action $\rho\colon R^2 \to \text{Aut}\,(R^n)$, $\rho(s, t) = e^{sA + tB}$, is such that either the operator A or B is not diagonalizable, then there exists a ρ-invariant and transitive splitting of R^n. This implies that the action is not hyperbolic. On the other hand, if both A and B are diagonalizable, we can suppose the action ρ expressed in standard coordinates in the following form: $\rho(s, t) = e^{sA + tB}$ where

$$A = \text{diag}\left(\begin{pmatrix} \alpha_1 & -\beta_1 \\ \beta_1 & \alpha_1 \end{pmatrix}, \ldots, \begin{pmatrix} \alpha_m & -\beta_m \\ \beta_m & \alpha_m \end{pmatrix}, \lambda_1, \ldots, \lambda_k \right),$$

$$B = \text{diag}\left(\begin{pmatrix} \overline{\alpha}_1 & -\overline{\beta}_1 \\ \overline{\beta}_1 & \overline{\alpha}_1 \end{pmatrix}, \ldots, \begin{pmatrix} \overline{\alpha}_m & -\overline{\beta}_m \\ \overline{\beta}_m & \overline{\alpha}_m \end{pmatrix}, \overline{\lambda}_1, \ldots, \overline{\lambda}_k \right).$$

In this case, the necessary and sufficient conditions for the action ρ to be hyperbolic follow easily from the definition of hyperbolicity, as well as the orbit structure. We proceed now to examine the nonwandering set $\Omega(\rho)$, ρ being a hyperbolic linear action. This is done by induction on the dimension n. For simplicity we will consider next a special case. Let us suppose that the action $\rho\colon R^2 \to \mathrm{Aut}\,(R^n)$, $\rho(s,t) = e^{sA+tB}$, where

$$A = \mathrm{diag}\left(\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}, \lambda_1, \cdots, \lambda_k\right),$$

$$B = \mathrm{diag}\left(\begin{pmatrix} \bar{\alpha} & -\bar{\beta} \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \bar{\lambda}_1, \cdots, \bar{\lambda}_k\right),$$

and either β or $\bar{\beta}$ is nonzero, is hyperbolic. Then $\alpha\bar{\beta} - \bar{\alpha}\beta \neq 0$, $\alpha\bar{\lambda}_i - \bar{\alpha}\lambda_i \neq 0$ for every i and $\lambda_i\bar{\lambda}_j - \bar{\lambda}_i\lambda_j \neq 0$ for all distinct i, j. The orbit set of ρ consists of a fixed point (origin), $2k$ orbits homeomorphic to R, one orbit homeomorphic to $R \times S^1$ and all the others homeomorphic to R^2. We will prove, by induction on k, that the nonwandering set $\Omega(\rho)$ of ρ consists of the orbits nonhomeomorphic to R^2. Therefore we have to show that if $x = (x_1, \cdots, x_n)$ is such that x_m and x_l are nonzero, and either l or m greater than 2, then x is a wandering point of ρ. For $k = 1$, the inequalities which hold for hyperbolicity permit to check the result without difficulty. By induction hypothesis, let us suppose that the conclusion holds for a linear action of R^2 on R^{k+2}, $k \leq n - 3$. Now we want to prove for $k = n - 2$. Set $R^n = E_{12} \oplus \cdots \oplus E_n$ where E_{12} is two-dimensional. If $x = (x_1, \cdots, x_{n-1}, 0)$ is a wandering point for the induced action $\rho_n\colon R^2 \to \mathrm{Aut}\,(E_{12} \oplus \cdots \oplus E_{n-1})$, then it is a wandering point for ρ. To see this it is enough to observe that the projection in $E_{12} \oplus \cdots \oplus E_j$, $2 < j < n$, of the orbit through $(x_1, \cdots, x_n)$ is the orbit through $(x_1, \cdots, x_j)$. Now, if $x = (x_1, \cdots, x_n)$, $x_n \neq 0$, we have the following three cases to consider.

(i) There exist i and j, either i or j greater than 2, $i, j = 1, \cdots, n - 1$, both x_i and x_j being nonzero. Here the projection $(x_1, \cdots, x_{n-1}, 0)$ being a wandering point for an induced linear action of R^2 on R^{n-1} implies that x is a wandering point of ρ.

(ii) There exists $j \neq n$, $j \neq 1$, $j \neq 2$ such that $x_i = 0$ for $i \neq j$, $i = 1, \cdots, n - 1$. In this case, x is a wandering point for an induced linear action of R^2 on $E_j \oplus E_n$. Then it is a wandering point for ρ

(iii) The point x is such that $x_i = 0$ for $i > 2$, $i = 2, \cdots, n - 1$. x is a wandering point for ρ since it is a wandering point for an induced linear action of R^2 on $E_{12} \oplus E_n$. Thus the proof for this special case is finished. The proof for the other cases being similar to this one will be omitted. Therefore the proposition holds.

The conditions for hyperbolicity of linear actions of R^2 on R^n given on the previous proposition, equivalent to the ones found in [1], will be useful in the study of linearly induced actions on spheres.

Let $\mathscr{A}(R^2, R^n)$ be the set of linear actions of R^2 on R^n with the topology induced by the following metric. Let $\rho_1(s, t) = e^{sA_1+tB_1}$, $\rho_2(s, t) = e^{sA_2+tB_2}$ be two linear actions of R^2 on R^n, and put $X_i(x) = A_i(x)$ and $Y_i(x) = B_i(x)$. Define $\|\rho_2 - \rho_1\| = \max\{|X_1 - X_2|, |Y_1 - Y_2|\}$. As an immediate consequence of the previous proposition one has the following corollary.

Corollary. *The hyperbolic linear actions of R^2 on R^n form an open and dense set in $\mathscr{A}(R^2, R^n)$.*

Let $\rho\colon R^2 \to \mathrm{Aut}\,(R^{n+1})$ be a linear action, and ψ be the action $\psi\colon Gl(n+1, R) \to \mathrm{Dif}\,(S^n)$, $\psi(Ax) = Ax/|Ax|$ where $A \in Gl(n+1, R)$ and $x \in S^n$. An action $\bar\rho$ of R^2 on S^n where $\bar\rho = \psi \circ f$ is called a linearly induced action of R^2 on S^n. If the action ρ is generated by commuting linear vector fields X and Y in R^{n+1}, then $\bar\rho$ is generated by $D\tau(X)$ and $D\tau(Y)$. Note that if $x \in S^n$ then $\mathscr{O}_x(\bar\rho) = \tau(\mathscr{O}_x(\rho))$.

Theorem 6. *Let $\bar\rho = \psi \circ \rho$ be a linearly induced action of R^2 on S^n where $X(x) = Ax$ and $Y(x) = Bx$ are generators of ρ. Then the compact orbits of $\bar\rho$ are hyperbolic if and only if for some basis of R^{n+1}, the operators A and B are represented as $A = \mathrm{diag}(\lambda_1, \cdots, \lambda_{n+1})$, $B = \mathrm{diag}(\bar\lambda_1, \cdots, \bar\lambda_{n+1})$ and $(\lambda_j - \lambda_i)(\bar\lambda_l - \bar\lambda_i) - (\bar\lambda_j - \bar\lambda_i)(\lambda_l - \lambda_i) \neq 0$ for all distinct i, j, l or*

$$A = \mathrm{diag}\left(\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}, \lambda_1, \cdots, \lambda_{n-1}\right),$$

$$B = \mathrm{diag}\left(\begin{pmatrix} \bar\alpha & -\bar\beta \\ \bar\beta & \bar\alpha \end{pmatrix}, \bar\lambda_1, \cdots, \bar\lambda_{n-1}\right),$$

where β or $\bar\beta$ is nonzero and

$$(\alpha - \lambda_i)\bar\beta - (\bar\alpha - \bar\lambda_i)\beta \neq 0 ,$$
$$(\alpha - \lambda_i)(\bar\lambda_j - \bar\lambda_i) - (\bar\alpha - \bar\lambda_i)(\lambda_j - \lambda_i) \neq 0 ,$$
$$(\lambda_j - \lambda_i)(\bar\lambda_l - \bar\lambda_i) - (\bar\lambda_j - \bar\lambda_i)(\lambda_l - \lambda_i) \neq 0$$

for all distinct i, j, l.

Proof. Observe that if $\varphi\colon R^2 \to \mathrm{Dif}\,(S^n)$ is an action with fixed point y then in some local coordinates the derivative is the linear action $D\varphi_{(s,t)}(y) = e^{s\bar A+t\bar B}$ where

$$\bar A = \left(\frac{\partial}{\partial x^j}\left(\frac{d\varphi^i_{(s,0)}(x)}{ds}\bigg|_{s=0}\right)(y)\right) ,$$

$$\bar B = \left(\frac{\partial}{\partial x^j}\left(\frac{d\varphi^i_{(0,t)}(x)}{ds}\bigg|_{s=0}\right)(y)\right) .$$

As in the case of linear actions we can restrict the study to the cases where the operator A is given in standard coordinates by a matrix in the real canonical form. We will show first that if A is not diagonalizable, the action $\bar\rho$ will present

a nonhyperbolic compact orbit. Let us suppose the elementary divisors of A to be

$$(x - \lambda_1)^{p_{11}}, (x - \lambda_1)^{p_{12}}, \cdots, (x - \lambda_2)^{p_{21}}, (x - \lambda_2)^{p_{22}}, \cdots,$$
$$[(x - z_1)(x - \bar{z}_1)]^{q_{11}}, [(x - z_1)(x - \bar{z}_1)]^{q_{12}}, \cdots,$$
$$[(x - z_2)(x - \bar{z}_2)]^{q_{21}}, [(x - z_2)(x - \bar{z}_2)]^{q_{22}}, \cdots,$$
$$(x - u_1), \cdots, [(x - w_1)(x - \bar{w}_1)], \cdots,$$

where λ_i, u_i are reals, z_i, ω_i are complex numbers, and $p_{i1} \geq p_{i2} \geq \cdots \geq 2$, $q_{i1} \geq q_{i2} \geq \cdots \geq 2$, $p_{i1} \geq p_{i+1,1}$, $q_{i1} \geq q_{i+1,1}$. We have four possibilities to examine.

(i) $p_{11} > p_{12}$. In this case the action $\bar{\rho}$ presents a fixed point p with $D\bar{\rho}_g(p)$ nonhyperbolic. Therefore $\bar{\rho}$ has a nonhyperbolic fixed point.

(ii) $p_{11} = p_{12} = \cdots = p_{1m}$. Here there exists a sphere S of dimension $m - 1$ consisting of singularities of $D\tau(X)$, and this sphere S is an invariant set for $D\tau(Y)$. Then either $D\tau(Y)$ has on S a singularity p which will be a nonhyperbolic fixed point of $\bar{\rho}$, or the action $\bar{\rho}$ will present a nonhyperbolic orbit homeomorphic to S^1 on S.

(iii) $p_{11} = 0$, $q_{11} > q_{12}$. In this case the action $\bar{\rho}$ has a nonhyperbolic orbit homeomorphic to S^1.

(iv) $p_{11} = 0$, $q_{11} = q_{12} = \cdots = q_{1m}$. Here there is a sphere S of dimension $2m - 1$ consisting of closed orbits of $D\tau(X)$, and this sphere is an invariant set for $D\tau(Y)$, Then either the action $\bar{\rho}$ will have a nonhyperbolic orbit homeomorphic to S^1 or a nonhyperbolic orbit homeomorphic to T^2.

Now if both operators A and B are diagonalizable, then we can suppose the action ρ expressed in standard coordinates in three different ways, each of which we will consider separatedly.

(1) $\bar{\rho}: R^2 \to \text{Dif}(S^n)$, $\bar{\rho} = \psi \circ \rho$, $\rho(s, t) = e^{sA + tB}$, $A = \text{diag}(\lambda_1, \cdots, \lambda_{n+1})$, $B = \text{diag}(\bar{\lambda}_1, \cdots, \bar{\lambda}_{n+1})$. The vector fields $X(x) = Ax$ and $Y(x) = Bx$ are generators of ρ; $D\tau(X) = \pi_\infty(X)$ and $D\tau(Y) = \pi_\infty(Y)$ are generators of $\bar{\rho}$. The $2(n + 1)$ points $p_i = (0, \cdots, 1, \cdots)$ and $q_i = (0, \cdots, -1, \cdots)$, where all but the ith coordinate are zero, are fixed points of the action $\bar{\rho}$. In the local coordinate systems (U_i, φ_i), (V_i, ψ_i) defined in the beginning of § 2, the derivative $D\bar{\rho}_g(p_i) = D\bar{\rho}_g(q_i) = e^{s\bar{A} + t\bar{B}}$ is such that

$$\bar{A} = \text{diag}(\lambda_1 - \lambda_i, \cdots, \lambda_{i-1} - \lambda_i, \lambda_{i+1} - \lambda_i, \cdots, \lambda_{n+1} - \lambda_i),$$
$$\bar{B} = \text{diag}(\bar{\lambda}_1 - \bar{\lambda}_i, \cdots, \bar{\lambda}_{i-1} - \bar{\lambda}_i, \bar{\lambda}_{i+1} - \bar{\lambda}_i, \cdots, \bar{\lambda}_{n+1} - \bar{\lambda}_i).$$

These fixed points are hyperbolic if and only if

$$(\lambda_j - \lambda_i)(\bar{\lambda}_k - \bar{\lambda}_i) - (\bar{\lambda}_j - \bar{\lambda}_i)(\lambda_k - \lambda_i) \neq 0$$

for all distinct i, j, k. Observe that if the above inequalities hold for points

$x \in S^n$ such that $x \neq p_i$ and $x \neq q_i$, then the isotropy group $G_x(\bar{\rho}) = \{0\}$ or R. It follows that the action $\bar{\rho}$ has these $2(n + 1)$ hyperbolic fixed points as its only compact orbits. Therefore all compact orbits are hyperbolic if and only if the previous inequalities hold. In this case the action $\bar{\rho}$ has all orbits homeomorphic to R^2 except for $2(n + 1)$ fixed points and $2n(n + 1)$ orbits homeomorphic to R. Also the nonwandering set $\Omega(\bar{\rho})$ is formed by the singular orbits which form $\frac{1}{2}n(n + 1)$ spheres of dimension one. To see this it is enough to use induction on the dimension n and observe that outside the equator $S^{n-1} = \{x \in S^n \,|\, x_{n+1} = 0\}$ the action $\bar{\rho}$ consists of two copies of the linear hyperbolic action φ where $\varphi(s, t) = e^{sA_1 + tB_1}$, $A_1 = \mathrm{diag}\,(\lambda_1 - \lambda_{n+1}, \cdots, \lambda_n - \lambda_{n+1})$ and $B_1 = \mathrm{diag}\,(\bar{\lambda}_1 - \bar{\lambda}_{n+1}, \cdots, \bar{\lambda}_n - \bar{\lambda}_{n+1})$.

In the case where $n = 2$, the orbit structure is shown in the figure below:
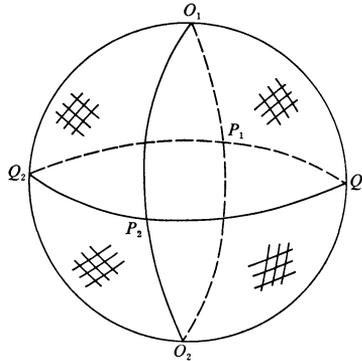
(2) $\bar{\rho} : R^2 \to \mathrm{Dif}\,(S^n)$, $\bar{\rho} = \psi \circ \rho$, $\rho : R^2 \to \mathrm{Aut}\,(R^{n+1})$, $\rho(s, t) = e^{sA + tB}$, where

$$A = \mathrm{diag}\left(\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}, \lambda_1, \cdots, \lambda_k\right),$$

$$B = \mathrm{diag}\left(\begin{pmatrix} \bar{\alpha} & -\bar{\beta} \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \bar{\lambda}_1, \cdots, \bar{\lambda}_k\right),$$

and β or $\bar{\beta}$ is nonzero. As before, the vector fields $X(x) = Ax$ and $Y(x) = Bx$ are generators of ρ; $D\tau(X) = \pi_\infty(X)$ and $D\tau(Y) = \pi_\infty(Y)$ are generators of $\bar{\rho}$. The $2k$ points $p_i = (0, \cdots, 1, \cdots)$ and $q_i = (0, \cdots, -1, \cdots)$, where all but the ith coordinate are zero, $i = 3, \cdots, k + 2$, are fixed points of the action $\bar{\rho}$. Evaluating $D\bar{\rho}_g(p_i) = D\bar{\rho}_g(q_i)$ we can see that these are hyperbolic fixed points if and only if

$$(\alpha - \lambda_i)\bar{\beta} - (\bar{\alpha} - \bar{\lambda}_i)\beta \neq 0 \,,$$

(*) $(\alpha - \lambda_i)(\bar{\lambda}_j - \bar{\lambda}_i) - (\bar{\alpha} - \bar{\lambda}_i)(\lambda_j - \lambda_i) \neq 0$ for all distinct i, j, l ,

$$(\lambda_j - \lambda_i)(\bar{\lambda}_l - \bar{\lambda}_i) - (\bar{\lambda}_j - \bar{\lambda}_i)(\lambda_l - \lambda_i) \neq 0 \,.$$

Now let us suppose that the above inequalities hold. For points $x \in S^n$, $x \neq p_i$, $x \neq q_i$, and $x \neq (x_1, x_2, 0, \cdots, 0)$, the isotropy group $G_x(\bar\rho) = \{0\}$, Z or R. It follows that the only compact orbits are the hyperbolic fixed points p_i, q_i and one orbit γ homeomorphic to S^1. This orbit γ is the intersection of the plane x_1, x_2 with the sphere S^n. It is easy to verify that a linear action $\varphi: R \times Z \to \mathrm{Aut}\,(R^n)$, $n \geq 2$, given by $\varphi(s, n) = e^{sA}B^n$, where $A = \mathrm{diag}\,(\lambda_1, \cdots, \lambda_n)$, $B = \mathrm{diag}\,(u_1, \cdots, u_n)$, $\lambda_i \neq 0$ and $u_i > 0$ for every i, is hyperbolic if and only if $\lambda_i \lg u_j - \lambda_j \lg u_i \neq 0$ for all distinct i, j. Using this fact and calculating the linear action $D\bar\rho_g|_{E_p}: G_p(\bar\rho) \to \mathrm{Aut}\,(E_p)$, where $p \in \gamma$ and $G_p(\bar\rho) = R \times Z$, we can conclude that the orbit γ is hyperbolic. Hence all compact orbits are hyperbolic if and only if the above inequalities (*) are true. In this case the action $\bar\rho$ presents $2k$ fixed points, $2k(k-1)$ orbits homeomorphic to R, one homeomorphic to S^1, $2k$ homeomorphic to $R \times S^1$ and all the others homeomorphic to R^2. The nonwandering set $\Omega(\bar\rho)$ consists of the singular orbits which form a complex of $\frac{1}{2}k(k-1)$ spheres of dimension one and k spheres of dimension two. As previously, to verify this we use induction on the dimension n and note that outside the equator S^{n-1} the action $\bar\rho$ consists of two copies of the linear hyperbolic action φ where

$$\varphi(s, t) = e^{sA_1 + tB_1},$$

$$A_1 = \mathrm{diag}\left(\begin{pmatrix} \alpha - \lambda_k & -\beta \\ \beta & \alpha - \lambda_k \end{pmatrix}, \lambda_1 - \lambda_k, \cdots, \lambda_{k-1} - \lambda_k\right),$$

$$B_1 = \mathrm{diag}\left(\begin{pmatrix} \bar\alpha - \bar\lambda_k & -\bar\beta \\ \bar\beta & \bar\alpha - \bar\lambda_k \end{pmatrix}, \bar\lambda_1 - \bar\lambda_k, \cdots, \bar\lambda_{k-1} - \bar\lambda_k\right).$$

In the particular case where $n = 2$, $k = 1$, the orbit structure is shown in the figure below:
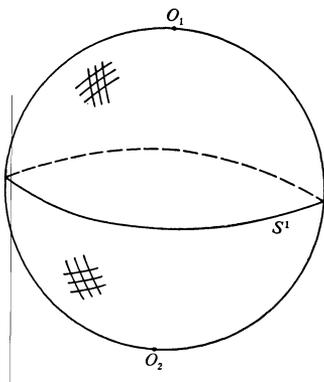
$$(3) \quad \bar\rho: R^2 \to \mathrm{Dif}\,(S^n), \quad \bar\rho = \psi \circ \rho, \quad \rho(s, t) = e^{sA + tB},$$

$$A = \text{diag}\left(\begin{pmatrix} \alpha_1 & -\beta_1 \\ \beta_1 & \alpha_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 & -\beta_2 \\ \beta_2 & \alpha_2 \end{pmatrix}, \; \cdots \right),$$

$$B = \text{diag}\left(\begin{pmatrix} \bar{\alpha}_1 & -\bar{\beta}_1 \\ \bar{\beta}_1 & \bar{\alpha}_1 \end{pmatrix}, \begin{pmatrix} \bar{\alpha}_2 & -\bar{\beta}_2 \\ \bar{\beta}_2 & \bar{\alpha}_2 \end{pmatrix}, \; \cdots \right),$$

and β_i or $\bar{\beta}_i$ is nonzero for $i = 1, 2$.

Now we can show that there is a nonhyperbolic orbit homeomorphic to S^1. Let us restrict ourselves to the case where $n = 3$, the generalization to $n > 3$ being similar. Suppose $\bar{\beta}_1 \neq 0$. Then the intersection of the plane $x_1 x_2$ with S^3 an orbit of $\bar{\rho}$ homeomorphic to S^1. Let $p \in \gamma$. By computing the linear action $D\bar{\rho}_g|_E : G_p(\bar{\rho}) \to \text{Aut}(E_p)$ which is given by $\varphi(s, n) = e^{sA}B^n$ where

$$A = \begin{pmatrix} [(\alpha_2 - \alpha_1)\bar{\beta}_1 - (\bar{\alpha}_2 - \bar{\alpha}_1)\bar{\beta}_1]/\bar{\beta}_1 & -(\beta_2\bar{\beta}_1 - \bar{\beta}_2\beta_1)/\bar{\beta}_1 \\ (\beta_2\bar{\beta}_1 - \bar{\beta}_2\beta_1)/\bar{\beta}_1 & [(\alpha_2 - \alpha_1)\bar{\beta}_1 - (\bar{\alpha}_2 - \bar{\alpha}_1)\beta_1]/\bar{\beta}_1 \end{pmatrix},$$

$$B = e^{2\pi(\bar{\alpha}_2\bar{\alpha}_1)/\bar{\beta}_1} \begin{pmatrix} \cos(2\pi\bar{\beta}_2/\bar{\beta}_1) & -\sin(2\pi\bar{\beta}_2/\bar{\beta}_1) \\ \sin(2\pi\bar{\beta}_2/\bar{\beta}_1) & \cos(2\pi\bar{\beta}_2/\bar{\beta}_1) \end{pmatrix},$$

we can verify that this is not a hyperbolic action. It follows that the orbit γ is nonhyperbolic. By putting all these results together the theorem is proved.

Let $\mathscr{A}(R^2, S^n)$ be the set of linearly induced actions of R^2 on S^n with the topology which arises from the following metric. If $\bar{\rho}_1$ is generated by $D\tau(X_1)$ and $D\tau(Y_1)$, and $\bar{\rho}_2$ is generated by $D\tau(X_2)$ and $D\tau(Y_2)$, define $\|\bar{\rho}_2 - \bar{\rho}_1\| = \max\{|D\tau(X_1) - D\tau(X_2)|, |D\tau(Y_1) - D\tau(Y_2)|\}$. As a consequence of the previous theorem we have the following corollary.

Corollary. *The set of linearly induced actions of R^2 on S^n having only hyperbolic compact orbits forms an open and nonempty set in $\mathscr{A}(R^2, S^n)$. This subset is not dense in $\mathscr{A}(R^2, S^n)$ for $n \geq 3$.*

We now discuss the important concepts of structural stability and Ω-stability for linearly induced R^2-actions on S^n.

Definition. A linearly induced action $\bar{\rho}_1 \in \mathscr{A}(R^2, S^n)$ is structurally stable if there is a neighborhood $V(\bar{\rho}_1)$ such that if $\bar{\rho}_2 \in V(\bar{\rho}_1)$ then $\bar{\rho}_2$ is topologically equivalent to $\bar{\rho}_1$.

Definition. A linearly induced action $\bar{\rho}_1 \in \mathscr{A}(R^2, S^n)$ is Ω-stable if there exists a neighborhood $V(\bar{\rho}_1)$ such that if $\bar{\rho}_2 \in V(\bar{\rho}_1)$ then there is a homeomorphism $h : \Omega(\bar{\rho}_1) \to \Omega(\bar{\rho}_2)$ taking orbits of $\bar{\rho}_1$ onto orbits of $\bar{\rho}_2$.

Theorem 7. *A linearly induced action of R^2 on S^n is Ω-stable in $\mathscr{A}(R^2, S^n)$ if and only if it has only hyperbolic compact orbits.*

Proof. Sufficiency follows easily from Theorem 6. To prove nesessity let us suppose $\bar{\rho}$ is not an action with all compact orbits hyperbolic. We have to examine several possibilities. In each case we want to show that $\bar{\rho}$ is not Ω-stable.

(I) $\bar{\rho}_1 = \psi \circ \rho$, $\rho(s, t) = e^{sA + tB}$ where both A and B are diagonalizable, and the action ρ has at most one pair of complex conjugate eigenvalues such

as in cases (1) and (2) in the proof of Theorem 6. Since it does not satisfy the inequalities of Theorem 6, $\bar{\rho}_1$ has an infinite number of singular orbits. In this case every neighborhood $V(\bar{\rho}_1)$ contains an action $\bar{\rho}_2$ with a finite number of singular orbits, namely, $\bar{\rho}_2$ satisfying the inequalities of Theorem 6.

(II) $\bar{\rho}_1 = \psi \circ \rho$, $\rho(s, t) = e^{sA + tB}$ where both A and B are diagonalizable but the action ρ has more than one pair of complex conjugate eigenvalues such as in case (3) in Theorem 6. Here $\bar{\rho}_1$ presents at least two orbits homeomorphic to S^1, and every neighborhood $V(\bar{\rho}_1)$ contains two actions $\bar{\rho}_2$ and $\bar{\rho}_3$ with the following orbit structures. Besides the two orbits homeomorphic to S^1, $\bar{\rho}_2$ has all orbits homeomorphic to $R \times S^1$ and $\bar{\rho}_3$ to R^2. It follows that $\bar{\rho}_1$ is not Ω-stable because it can be approximated by actions $\bar{\rho}_2$ and $\bar{\rho}_3$ with an infinite and only a finite number of singular orbits respectively.

(III) $\bar{\rho} = \psi \circ \rho$, $\rho(s, t) = e^{sA + tB}$ where either A or B is not diagonalizable. Let us suppose that A is not diagonalizable. We can apply the same procedure as in the proof of Theorem 6. If A has $(x - \lambda)^q$, λ real, $q > 1$, for elementary divisor, then either $\bar{\rho}$ has a finite number of singular orbits and can be approximated by an action with a larger number of singular orbits or $\bar{\rho}$ presents an infinite number of singular orbits and can be approximated by an action with only a finite number of singular orbits. On the other hand if A has $[(x - z)(x - \bar{z})]^q$, z complex, $q > 1$, for elementary divisor, then we can perturb $\bar{\rho}$ in order to get an action with more than one pair of complex conjugate eigenvalues. By what we have seen, an action of this kind is not Ω-stable implying that $\bar{\rho}$ is not Ω-stable either.

As an immediate corollary of case (II) before we have

Corollary. *The structurally stable linearly induced actions of R^2 on S^n do not form a dense set in $\mathscr{A}(R^2, S^n)$ for $n > 3$,*

However in $\mathscr{A}(R^2, S^3)$ the structurally stable actions form an open and dense subset which is characterized in the next theorem.

Theorem 8. *A linearly induced action of R^2 on S^3 is structurally stable if and only if all its compact orbits are hyperbolic.*

Proof. Necessity follows directly from the proof of the previous theorem. To prove sufficiency let us suppose $\bar{\rho}: R^2 \to \text{Dif}(S^3)$ is a linearly induced action with all its compact orbits hyperbolic. Then by Theorem 6 we only have to examine the two cases below.

(I) $\bar{\rho} = \psi \circ \rho$, $\rho(s, t) = e^{sA + tB}$ where $A = \text{diag}(\lambda_1, \cdots, \lambda_4)$, $B = \text{diag}(\bar{\lambda}_1, \cdots, \bar{\lambda}_4)$ and $(\lambda_j - \lambda_i)(\bar{\lambda}_l - \bar{\lambda}_i) - (\bar{\lambda}_j - \bar{\lambda}_i)(\lambda_l - \lambda_i) \neq 0$ for all distinct i, j, l. This action $\bar{\rho}$ is generated by the vector fields $D\tau(X)$, $X(x) = Ax$ and $D\tau(Y)$, $Y(x) = Bx$. Consider S^3 with the Riemannian metric induced by the usual metric on R^4. It is easy to show that $D\tau(X)$ and $D\tau(Y)$ are gradient vector fields. Thus the action $\bar{\rho}$ is a gradient action, meaning that is generated by commutative gradient vector fields. From a theorem in [2] $\bar{\rho}$ is a structurally stable action.

(II) $\bar{\rho} = \psi \circ \rho$, $\rho(s, t) = e^{sA + tB}$ where

$$A = \text{diag}\left(\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}, \lambda_1, \lambda_2\right), \qquad B = \text{diag}\left(\begin{pmatrix} \bar{\alpha} & -\bar{\beta} \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \bar{\lambda}_1, \bar{\lambda}_2\right),$$

$$(\alpha - \lambda_i)\bar{\beta} - (\bar{\alpha} - \bar{\lambda}_i)\beta \neq 0,$$

$$(\alpha - \lambda_i)(\bar{\lambda}_j - \bar{\lambda}_i) - (\bar{\alpha} - \bar{\lambda}_i)(\lambda_j - \lambda_i) \neq 0 \qquad \text{for all } i, j.$$

First we can find a real k such that the operator $B + kA = \text{diag}\left(\begin{pmatrix} \bar{\bar{\alpha}} & -\bar{\bar{\beta}} \\ \bar{\bar{\beta}} & \bar{\bar{\alpha}} \end{pmatrix},\right.$ $\left. \bar{\bar{\lambda}}_1, \bar{\bar{\lambda}}_2\right)$ has the following properties: $(\bar{\bar{\alpha}} - \bar{\bar{\lambda}}_1)$ and $(\bar{\bar{\alpha}} - \bar{\bar{\lambda}}_2)$ have the same sign, and both $(\bar{\bar{\lambda}}_1 - \bar{\bar{\lambda}}_2)$ and $\bar{\bar{\beta}}$ are nonzero. Since the vector fields $(B + kA)(x)$ and $B(x)$ are generators of the action ρ we proceed with the additional hypothesis that A has the above properties, meaning that $(\alpha - \lambda_1)$ and $(\alpha - \lambda_2)$ are of the same sign, and both $(\lambda_1 - \lambda_2)$ and β are nonzero. Consider the following vector fields in R^3:

$$X_1(x) = A_1 x \qquad \text{where } A_1 = \text{diag}\left(\begin{pmatrix} \alpha - \lambda_2 & -\beta \\ \beta & \alpha - \lambda_2 \end{pmatrix}, (\lambda_1 - \lambda_2)\right),$$

and

$$Y_1(x) = B_1 x \qquad \text{where } B_1 = \text{diag}\left(\begin{pmatrix} \bar{\alpha} - \bar{\lambda}_2 & -\bar{\beta} \\ \bar{\beta} & \bar{\alpha} - \bar{\lambda}_2 \end{pmatrix}, (\bar{\lambda}_1 - \bar{\lambda}_2)\right),$$

The Poincaré vector fields $\pi(X_1)$ and $\pi(Y_1)$ are also generators of the action $\bar{\rho}$, and we can verify that $\pi(X_1)$ is a Morse-Smale vector field and $e^{sA_1 + tB_1}$ is a hyperbolic linear R^2-action. We know from a corollary of Theorem 6 that the linearly induced actions of R^2 on S^n having only hyperbolic compact orbits form an open set in $\mathscr{A}(R^2, S^n)$. Hence it follows that there exists a neighborhood $V(\bar{\rho})$ in $\mathscr{A}(R^2, S^3)$ such that if $\bar{\rho}^* \in V(\bar{\rho})$ then $\bar{\rho}^*$ has all its compact orbits hyperbolic. Besides, $V(\bar{\rho})$ can be chosen such that $\bar{\rho}^* = \psi \circ \rho^*$ where $\rho^*(s, t) = e^{sA^* + tB^*}$,

$$A^* = \text{diag}\left(\begin{pmatrix} \alpha^* & -\beta^* \\ \beta^* & \alpha^* \end{pmatrix}, \lambda_1^*, \lambda_2^*\right), \qquad B^* = \text{diag}\left(\begin{pmatrix} \bar{\alpha}^* & -\bar{\beta}^* \\ \bar{\beta}^* & \bar{\alpha}^* \end{pmatrix}, \bar{\lambda}_1^*, \bar{\lambda}_2^*\right),$$

$(\alpha^* - \lambda_1^*)$ and $(\alpha^* - \lambda_2^*)$ have the same sign as $(\alpha - \lambda_1)$, $(\lambda_1^* - \lambda_2^*)$ has the same sign as $(\lambda_1 - \lambda_2)$, and $\beta^* \neq 0$. Our goal is to show that such an action $\bar{\rho}^* \in V(\bar{\rho})$ is topologically equivalent to $\bar{\rho}$. Let us consider the vector fields

$$X_1^*(x) = A_1^* x \qquad \text{where } A_1^* = \text{diag}\left(\begin{pmatrix} \alpha^* - \lambda_2^* & -\beta^* \\ \beta^* & \alpha^* - \lambda_2^* \end{pmatrix}, (\lambda_1^* - \lambda_2^*)\right),$$

$$Y_1^*(x) = B_1^* x \qquad \text{where } B_1^* = \text{diag}\left(\begin{pmatrix} \bar{\alpha}^* - \bar{\lambda}_2^* & -\bar{\beta}^* \\ \bar{\beta}^* & \bar{\alpha}^* - \bar{\lambda}_2^* \end{pmatrix}, (\bar{\lambda}_1^* - \bar{\lambda}_2^*)\right).$$

Here $\pi(X_1^*)$ and $\pi(Y_1^*)$ are generators of $\bar{\rho}^*$, (X_1^*) is a Morse-Smale vector field, and $e^{sA_1^*+tB_1^*}$ is a hyperbolic linear R^2-action. For simplicity let us adopt the following notation:

$$A_1 = \operatorname{diag}\left(\begin{pmatrix} \tilde{\alpha} & -\tilde{\beta} \\ \tilde{\beta} & \tilde{\alpha} \end{pmatrix}, \tilde{\lambda}\right), \qquad A_1^* = \operatorname{diag}\left(\begin{pmatrix} \tilde{\alpha}^* & -\tilde{\beta}^* \\ \tilde{\beta}^* & \tilde{\alpha}^* \end{pmatrix}, \tilde{\lambda}^*\right),$$

where $\tilde{\alpha} = \alpha - \lambda_2$, $\tilde{\lambda} = \lambda_1 - \lambda_2$, $\tilde{\alpha}^* = \alpha^* - \lambda_2^*$ and $\tilde{\lambda}^* = \lambda_1^* - \lambda_2^*$. We also denote $\pi(X_1)$ by $\hat{X}_1$, and $\pi(X_1^*)$ by $\hat{X}_1^*$. Let γ be the orbit of the action $\bar{\rho}$ which is homeomorphic to S^1. This coincides with the closed orbit of the vector field $\hat{X}_1$. In the local coordinates (U_1, φ_1), defined in §2, consider the coordinate plane Σ transversal to γ. The Poincaré map f_1 associated to this transversal section Σ is given by $f_1(x_1, x_2) = (e^{2\pi(\tilde{\lambda}-\tilde{\alpha})/\tilde{\beta}}x_1, e^{-2\pi\tilde{\alpha}/\tilde{\beta}}x_2)$. Observe that f_1 is the flow $\hat{X}_{1t}$ restricted to Σ for $t = 2\pi/\tilde{\beta}$, and f_1 is a contracting or an expanding diffeomorphism. We will proceed by considering f_1 expanding (hence $\tilde{\alpha} < 0$ and $(\tilde{\lambda} - \tilde{\alpha}) > 0$) and also $\tilde{\lambda} > 0$. The proofs on the other cases are similar.
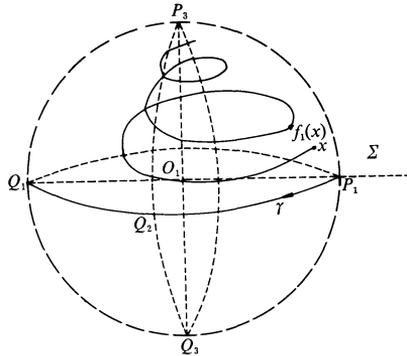
By taking the real number r such that $\tilde{\beta} + r\bar{\beta} = 0$, the operator $C_1 = A_1 + rB_1$ has the form $C_1 = \operatorname{diag}(\tilde{\tilde{\alpha}}, \tilde{\tilde{\alpha}}, \tilde{\tilde{\gamma}})$. Then the vector field $Z_1(x) = \pi(C_1 x)$ has Σ as an invariant set, and $P_1 = \gamma \cap \Sigma$ is a hyperbolic singularity for $Z_1|_\Sigma$. Note that the trajectories of Z_1 are contained in the orbits of $\bar{\rho}$, the flow $Z_{1t}|_\Sigma$ commutes with f_1, and the orbits of $\bar{\rho}$ are invariant by f_1. By the same procedure, consider the corresponding f_2 and Z_2 for the action $\bar{\rho}^*$. In order to construct the desired homeomorphism we have to analyse two cases according to the index of the singularity P_1 of $Z_1|_\Sigma$.

(1) P_1 is source or a sink for $Z_1|_\Sigma$. In a neighborhood of P_1 in the transversal section Σ, fix one orbit of Z_1 in each quadrant, and its image by f_1 and a circle S transversal to Z_1. Do the same for Z_2.
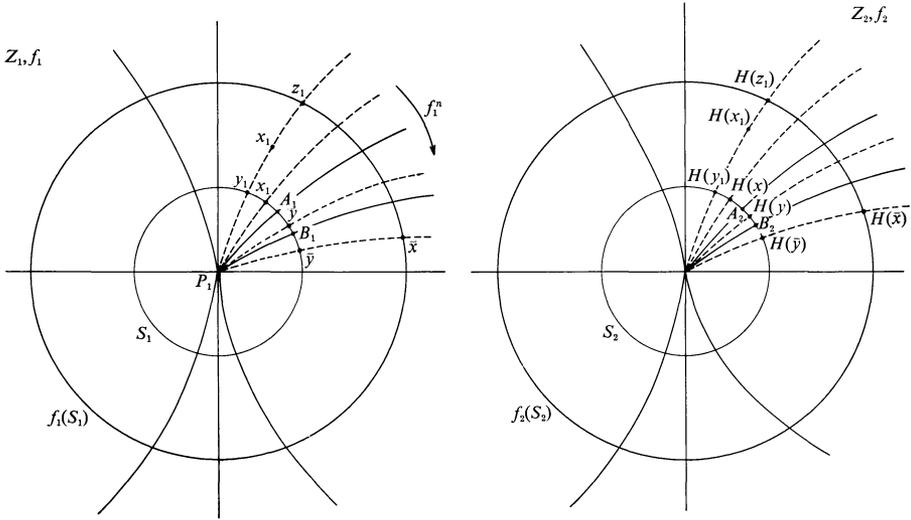
We will restrict to a quadrant, for the others proceed similarly. Define an orientation preserving homeomorphism $H: \widehat{A_1B_1} \to \widehat{A_2B_2}$ where A_1 and B_1 are the points of intersection of the above orbits of Z_1 with S_1. To obtain an extension for S_1, set $H(x) = f_2^{-n}(\mathcal{O}_{H(y)}(Z_2)) \cap S_2$. Here x does not belong to the coordinate axis (in which case the definition is the natural one) and n is such that $y = f_1^n(\mathcal{O}_x(Z_1)) \cap A_1P_1B_1$. In order to define H in $f_1(S_1)$, let $\bar{x} \in f_1(S_1)$ and take $\bar{y} = \mathcal{O}_{\bar{x}}(Z_1) \cap S_1$. Then put $H(\bar{x}) = \mathcal{O}_{H(\bar{y})}(Z_1) \cap f_2(S_2)$. Now we may extend H to the interior of the annulus with boundaries S_1 and $f_1(S_1)$, proportionally to the arc length of trajectories of Z_1 and Z_2. If x_1 belongs to the interior of the annulus, consider the points of intersection $y_1 = \mathcal{O}_x(Z_1) \cap S_1$ and $z_1 = \mathcal{O}_x(Z_1)$

$\cap f_1(S_1)$. Define $H(x_1)$ by $H(x_1) \in \mathcal{O}_{H(y_1)}(Z_2)$ and $\dfrac{\widehat{x_1y_1}}{\widehat{z_1y_1}} = \dfrac{\widehat{H(x_1)H(y_1)}}{\widehat{H(z_1)H(y_1)}}$. For x in

the interior of the disc bounded by S_1 there exists n such that $f_1^n(x)$ belongs to the annulus. Let $H(x) = f_2^{-n}Hf_1^n(x)$. Thus we have a homeomorphism defined in the disc bounded by $f_1(S_1)$ taking orbits of $\bar{\rho}$ onto orbits of $\bar{\rho}^*$ and such that $Hf_1 = f_2H$.

(2) P_1 is a saddle point for $Z_1|_{\Sigma}$. Here fix an orbit of Z_1 in each quadrant, its image by f_1, circles S_1 and $f_1(S_1)$ which are tangent to these orbits, and the analogous f_2 and S_2 for Z_2.
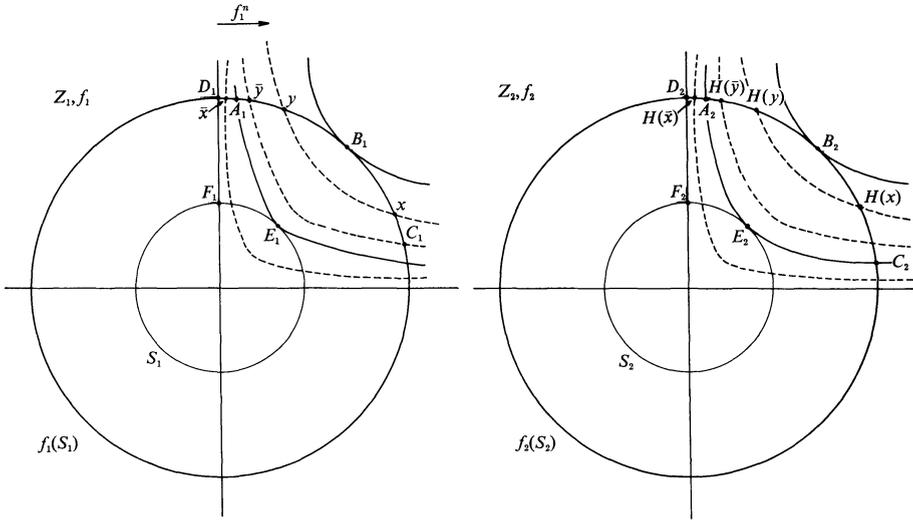
As before let us restrict ourselves to one quadrant, the procedure for the others being similar. Define first an orientation preserving homeomorphism $H \colon \widehat{A_1 B_1} \to \widehat{A_2 B_2}$. In order to get an extension to $f_1(S_1)$, for $x \in B_1 C_1$ set $H(x) = \mathcal{O}_{H(y)}(Z_2) \cap \widehat{B_2 C_2}$ where $y = \mathcal{O}_x(Z_1) \cap \widehat{A_1 B_1}$. For $\bar{x} \in \widehat{D_1 A_1}$ define $H(\bar{x}) = f_2^{-n}(\mathcal{O}_{H(\bar{y})}(Z_2)) \cap \widehat{D_2 A_2}$ where n is such that $\bar{y} = f_1^n(\mathcal{O}_{\bar{x}}(Z_1)) \cap \widehat{A_1 B_1}$. Now we want to define H on S_1. Let $x_1 \in \widehat{E_1 F_1}$ and consider $y_1 = \mathcal{O}_{x_1}(Z_1) \cap \widehat{D_1 A_1}$. Then set $H(x_1) = \mathcal{O}_{H(y_1)}(Z_2) \cap E_2 F_2$. The extension of H to the interior of the disc bounded by $f_1(S_1)$ is done as in the previous case.

Up to this point, regardless of the index of the singularity P_1 of $Z_1|_{\Sigma}$, we have an orbit preserving homeomorphism defined in a neighborhood N of P_1 in the transversal section Σ. Now we want to extend H to a neighborhood V of γ. If $x \in V$, let t_x be minimum such that $\hat{X}_{1-t_x}(x) = y$ belongs to N and set $H(x) = \hat{X}_{1t}^*(H(y))$ where $t = \tilde{\beta} t_x / \tilde{\beta}^*$. Restrict H to a neighborhood of γ whose boundary is a torus T transversal to $\hat{X}_1$.
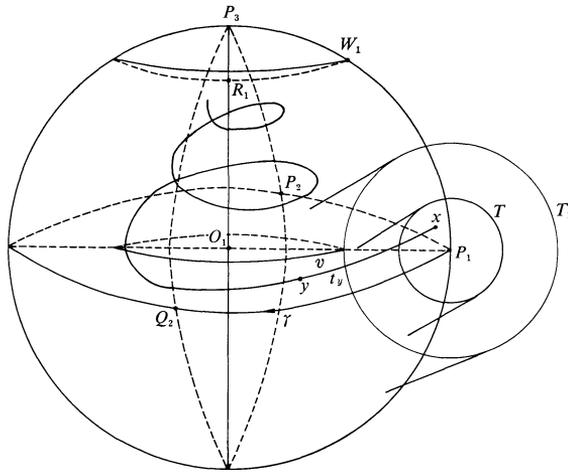
Consider now a circle C_1 in the plane $O_1P_1Q_2$ such that all positive trajectories of $\hat{X}_1$ beginning at points of C_1 have arc length k, and a point R_1 on the axis O_1P_3 such that the arc length of the trajectory of $\hat{X}_1$ from O_1 to R_1 is l. Assume V_1 is a neighborhood of γ whose boundary is a torus T_1 transversal to $\hat{X}_1$ and which intercepts the plane $O_1P_1Q_2$ on C_1, and W_1 is a sphere centered at P_3 also transversal to $\hat{X}_1$ and intercepting O_1P_3 in R_1. Similarly take C_2, R_2, T_2 and W_2 corresponding to $\hat{X}_1^*$. Next we define H on T_1. If $x \in T_1$, consider $y = \mathcal{O}_x(\hat{X}_1) \cap T$ and set $H(x) = \mathcal{O}_{H(y)}(\hat{X}_1^*) \cap T_2$. The extension of H to $(T_1 - T)$ can be done proportionally to the arc length of trajectories of $\hat{X}_1$ and $\hat{X}_1^*$. The definition of H in W_1 can be obtained similarly as it was set for T_1, the extension to the interior of W_1 can be obtained by the time parameter on the trajectories of $\hat{X}_1$ and $\hat{X}_1^*$ and between T_1 and W_1 in an analogous manner as in $(T_1 - T)$. Therefore H takes orbits of $\bar{\rho}$ onto orbits of $\bar{\rho}^*$.

It remains to prove continuity for points in the interior of the disc bounded by C_1 and in the line segment O_1R_1. This proof depends on the following lemma.

Lemma. *Let J be a set with accumulation point x_0 in the interior of the disc bounded by C_1. For $x \in J$ let $L(x)$ be the arc length of the trajectory of $\hat{X}_1$ starting at x and with final point in W_1. Then $\lim_{x \to x_0} L(x) = k_0 + l$ where k_0 is the arc length of the positive trajectory of $\hat{X}_1$ starting at x_0.*
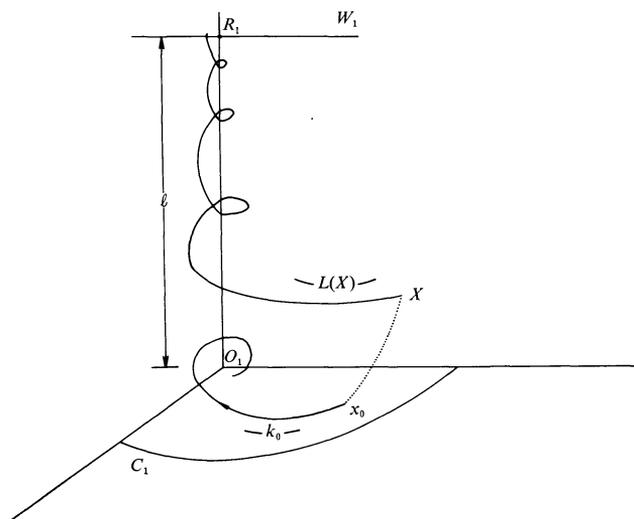
Proof of the lemma. Let $T(x)$ be such that $\hat{X}_{1T(x)}(x) \in W_1$. Then $T(x) = (1/\tilde{\lambda})\lg(l/x_3)$. Put

$$A = A(x, t) = \sqrt{(x_1^2 + x_2^2)(\tilde{\alpha}^2 + \tilde{\beta}^2)}e^{\tilde{\alpha}t},$$
$$B = B(x, t) = x_3\tilde{\lambda}e^{\tilde{\lambda}t}.$$

Then

$$L(x) = \int_0^{T(x)} \sqrt{A^2 + B^2}dt = \int_0^{T(x)} \sqrt{(A + B)^2 - 2AB}dt.$$

Let

$$\bar{L}(x) = \int_0^{T(x)} (A + B)dt - \int_0^{T(x)} \sqrt{A^2 + B^2}dt = \int_0^{T(x)} \frac{2AB}{A + B + \sqrt{A^2 + B^2}}dt.$$

One can verify that $\lim_{x \to x_0} \bar{L}(x) = 0$ and that $\lim_{x \to x_0} \int_0^{T(x)} (A + B)dt = k_0 + l$. Therefore $\lim_{x \to x_0} L(x) = k_0 + l$ which proves the lemma.

We return to the proof of the theorem. Notice that if $T(x)$ is such that $\hat{X}_{1-T(x)}(x) = y \in T_1$, $L(x)$ is the arc length between x and y through a trajectory of $\hat{X}_1$, and x_0 belongs to the interior of the disc bounded by C_1, then $\lim_{x \to x_0} y = y_0 \in C_1$, and $\lim_{x \to x_0} L(x)$ is the arc length of the trajectory of $\hat{X}_1$ between x_0 and y_0. We apply the above arguments to convergent sequences to a point in the interior of the disc bounded by C_1 and to a point in O_1R_1. We have thus obtained an orbit preserving homeomorphism in part of S^3. But by considering the other fixed points of $\bar{\rho}$ we can make the same construction simultaneously for the whole S^3. Thus we obtain a homeomorphism of S^3 taking orbits of $\bar{\rho}$ onto orbits of $\bar{\rho}^*$, proving the theorem.

Let $\mathcal{X}^r(R^2, S^3)$ be the set of actions of R^2 on S^3 with the C^r $(r \geq 1)$ topology. The method used to prove the previous theorem yields the following stronger result.

Theorem 9. *A linearly induced R^2-action on S^3 having only hyperbolic compact orbits is structurally stable in $\mathcal{X}^r(R^2, S^3)$.*

Proof. Let $\bar{\rho}: R^2 \to \mathrm{Dif}\,(S^3)$ be a linearly induced action of R^2 on S^3. As in the proof of Theorem 8 we can consider a Morse-Smale generator $\hat{X}_1$ of $\bar{\rho}$, the Poincaré map f_1 corresponding to the transversal section Σ, and the vector field Z_1. By a theorem in [8], there exists a neighborhood $V(\bar{\rho}) \subset \mathcal{X}(R^2, S^3)$ such that if $\eta \in V(\bar{\rho})$ then η has a Morse-Smale generator $\hat{X}_1^*$ with the same properties as $\hat{X}_1$. Let f_2 be the Poincaré map corresponding to the section Σ, and suppose $\hat{X}_{1T}^*(P_1) = P_1$ for P_1 in the intersection of Σ with the closed orbit of $\hat{X}_1^*$. We can reparametrize $\hat{X}_{1t}^*$ in such a way that the new Poincaré map $\tilde{f}_2 = \tilde{X}_{1T}^*$, where $\tilde{X}_{1t}^*$ is the reparametrization. To obtain the corresponding vector field Z_2 let $Z_2(x) = a(x)\tilde{X}_1^*(x) + \tilde{Y}_1^*(x)$ where $\tilde{X}_1^*(x)$ and $\tilde{Y}_1^*(x)$ are generators of η. The function $a(x)$ can be chosen in such a way that Z_2 has Σ as an invariant set. With $\tilde{X}_{1t}^*$, $\tilde{f}_2$ and Z_2 so defined the proof is the same as in the previous theorem.

References

[1]   C. Camacho, *On $R^k \times Z^l$-actions*, Dynamical Systems, Salvador Symposium, Editor M. Peixoto, Academic Press, New York, 1973, 23–70.

[2]   C. Camacho, *Morse-Smale R^2-actions on two-manifolds*, Dynamical Systems, Editor M. Peixoto, Academic Press, New York, 1973, 71–74.

[3]   T. V. Davies & E. M. James, *Nonlinear differential equations*, Addison-Wesley, Reading, Mass., 1966, 42–46.

[4]   F. Gantmacher, *The theory of matrices*, Chelsea, New York, 1959.

[5]   E. A. Gonzalez Velasco, *Generic properties of polynomial vector fields at infinity*, Trans. Amer. Math. Soc. **143** (1969) 201–222.

[6]   S. Lefschetz, *Differential equations: Geometric theory*, Interscience, New York, 1963, 201–208.

[7]   G. Palis, *Linearly induced vector fields and R^2-actions on spheres*, Dynamical Systems—Warwick, Springer, Berlin, 1975, 13–17.

[8]   J. Palis, *On Morse-Smale dynamical systems*, Topology **8** (1969) 385-405.

[9]   J. Palis & S. Smale, *Structural stability theorems*, Global Analysis, Proc. Sympos. Pure Math., Vol. XIV, Amer. Math. Soc., 1970, 223–231.

[10]   M. Peixoto, *Classification of vector fields in two-dimensional manifolds*, Dynamical Systems, Salvador Symposium, Editor M. Peixoto, Academic Press, New York, 1973, 389–420.

[11]   E. Picard, *Traité d'analyse*, Vol. III, Gauthier-Villars, Paris, 1896, 228–232.

[12]   H. Poincaré, *Mémoire sur les courbes définies par une équation différentielle*, J. Mathématique **7** (1881) 375–422.

[13]   G. Sansone & R. Conti, *Equazioni differentiali non lineari*, Edizione Cremonese, Rome, 1956, 122–133.

[14]   S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967) 747–817.