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ON THE PRINCIPLE OF UNIFORMIZATION

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The uniformization theorem for Riemann surfaces is a milestone in the classical function theory which has led to several developments in different branches of mathematics. In topology one usually associates the concept of a covering space as directly arising from uniformization. While this is undoubtedly true the classical uniformization theory has a different group theoretical aspect which goes substantially beyond the usual covering space theory. What we have in mind is the special role played by the group of Möbius transformations in the classical uniformization. In several respects it comes close to the more elementary idea of "development" used in differential geometry, and in this form it seems worthwhile to generalize it in other situations. In the classical case there are two broad classes of groups appearing in uniformization. The first class is the class of Fuchsian groups which act discontinuously on the unit disc. The second class is that of Kleinian groups which act discontinuously on some region on the 2-sphere. Even in this classical case it is only in the last decade that Kleinian groups have been vigorously studied thanks to Ahlfors, Bers, Maskit, Kra and others. Higher dimensional generalization of Fuchsian groups have been studied in considerable detail as they arise naturally in the arithmetic problems and the modulii problems. On the other hand the Kleinian case has received much less attention. The method of Kleinian groups leads to construction of new manifolds with far more complicated topology than those arising in the Fuchsian case. Relating the properties of these manifolds to those of the corresponding model spaces and groups is a problem of intrinsic interest.

Here is a brief outline of the contents of the paper.

In § 1 we develop the theory of uniformization. It is implicit in certain classical problems. For instance as mentioned above it appears in its most elementary form in the discussion of developable (i.e., curvature zero) surfaces, and more generally in the theory of space forms (i.e., Riemannian manifolds of constant curvature). In a deeper way it appears in the conformal development of Riemann surfaces. In differential geometry it is implicit in the concept of holonomy especially in the case of integrable G-structures. The author's motivation mainly came from Kuiper [12], [13] and Gunning [7]. The problem of uniformization is the same as that of studying special coordinate coverings proposed by

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Gunning [7]. It may be considered as a special (but interesting!) case of more general theories, e.g., Grothendieck's theory of fiber spaces [6]. It is worth emphasizing that we make only a mild quasianalyticity hypothesis—the condition (U) of § 1—about the group and no further differentiability conditions are imposed. A fuller account of the theory, although desirable, would have taken longer than intended here. We have tried to present the essential points and hope that it is sufficient to generate interests of geometers and topologists.

In § 2 we study uniformization in the context of G-structures. A basic problem in the theory of G-structures is to describe manifolds admitting a G-structure for "interesting" G's. In the case of G-structures of finite type (cf. § 2 for the definition) uniformization theory provides such a description. This result should be regarded as a natural generalization of the Liouville's theorm in conformal geometry in dimensions ≥ 3. This approach via the uniformization theory appears to be by far the more revealing than the purely connection-theoretic approach which the differential geometers are accustomed to use.

We illustrate this result in § 3 through § 5 by discussing a notion of "quaternionic structure" on a manifold. As an extension of the notion of a complex manifold this notion has attracted a good deal of attention of differential geometers; cf. Ehresmann [3], Obata [17], Ishihara [9], Fueter [5]. Although various variants can be formulated, one conceivable notion of a quaternionic structure on a $4n$-dimensional manifold is that of an integrable $GL_n(\boldsymbol{H})$-structure, where $\boldsymbol{H}$ stands for the field of quaternions. It is known from Obata [17] that this notion is too rigid to allow even $\boldsymbol{P}^1(\boldsymbol{H})$, i.e., the one-dimensional quaternionic projective space to be "quaternionic". In this paper a quaternionic manifold is defined to be one with integrable $GL_n(\boldsymbol{H})\cdot GL_1(\boldsymbol{H})$-structure; cf. § 3. This modification allows $\boldsymbol{P}^n(\boldsymbol{H})$ to be quaternionic. The main result whose proof extends over § 3 thorugh § 5 is

Theorem 3.4. *A differentiable manifold M^{4n} is quaternionic if and only if it is uniformizable over $\boldsymbol{P}^n(\boldsymbol{H})$ with respect the group of quaternionic projective transformations.*

It appears remarkable that just as *a 1-dimensonal complex manifold is the same as an orientable 2-dimensional real manifold with conformally Euclidean geometry, so also a 1-dimensional quaternionic manifold is the same as an orientable 4-dimensional real manifold with conformally Euclidean geometry*; cf. § 6.

In § 7 we prove a certain "geometric surgery" theorem, a consequence of which is

Theorem 7.2. *A connected sum of conformally Euclidean manifolds admits a conformally Euclidean structure.*

This theorem should be compared to "combination theorems" of Klein in the theory of Kleinian groups which have been recently generalized by Maskit; cf. Ford [4, § 25], Maskit [15]. In these theorems one starts with groups Γ_i's of fractional linear transformations of the Riemann sphere such that each Γ_i acts properly discontinuously on a nonempty open subset of the Riemann sphere, and one shows that under certain conditions certain amalgamated products of Γ_i's also act properly discontinuously on some nonempty open subsets of the Riemann sphere. In Theorem 7.2 we deal directly with manifolds rather than groups. We note here, paranthetically, that the conformally Euclidean manifolds obtained by means of Theorem 7.2 are not in general obtainable by means of "combination theorems" even in the classical case of 2-dimensional conformal geometry.

From a differential geometric standpoint the possibility of combination theorems as well as of Theorem 7.2 may be traced to the possibility of "*inversions*" in conformal geometry. In 2-(resp. 4-) real dimensions identifying a conformally Euclidean structure (locally) with a complex (resp. quaternionic) structure such an inversion may be expressed as a holomorphic (resp. quaternionic) transformation. So *one can put a complex* (*resp. quaternionic*) *structure on a connected sum of* 1-*dimensional complex* (*resp. quaternionic*) *manifolds*!

In 2-dimensional conformal geometry the only possibility of inversion is that in a circle. In dimension > 2 one may ask the possibility of inversion not only in 1-codimensional spheres but also in other 1-codimensional submanifolds. In Theorem 8.2 we show that many such examples can be constructed. Curiously, the procedure is *real-algebraic*. We show moreover

Theorem 8.4. *For any* $g = 0, 1, 2 \cdots$ *there exists a conformal inversion of the standard* 3-*sphere which turns a closed oriented surface of genus g inside out.*

This paper is an outgrowth of the author's work on conformal geometry in higher dimensions which was inspired by Kuiper [12], [13] and the conformal nonimmersion theorems started by Chern and Simons [2]. The present work and the work on conformal geometry (cf. [14]) has been reported in the Bers seminar at Columbia; the author gratefully acknowledges the comments by the members of the seminar.

1. Uniformization

Let S be a topological space, and G a group of homeomorphisms of S. We shall assume that G satisfies the following condition:

(U) For any g_1, g_2 in G if their action coincides on a nonempty open subset of S, then $g_1 = g_2$.

The condition may be rephrased by saying that an element of G is determined uniquely by its action on an open set. The pair (S, G) will be fixed once and for all and will be referred to as the model space.

Definition 1.1. A topological space M is said to be uniformized by (S, G) if the following holds:

There exist a covering $\{U_\alpha\}_{\alpha\in A}$ of M and homeomorphisms $\varphi_\alpha\colon U_\alpha \to S$ such that for every pair α, β in A such that $U_\alpha \cap U_\beta \neq \emptyset$ the mapping

$$\varphi_\alpha \circ \varphi_\beta^{-1}\colon \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

is a restriction of an element $g_{\alpha\beta}$ of G.

We shall say in this case that $\{U_\alpha, \varphi_\alpha\}$ define a uniformization of M with respect to (S, G). Let $\{V_{\alpha'}\}_{\alpha' \in A'}$ be a refinement of $\{U_\alpha\}_{\alpha \in A}$. Let $j: A' \to A$ be a refinement map. Suppose that there exist homeomorphisms $\psi_{\alpha'}: V_{\alpha'} \to S$ such that $\{V_{\alpha'}, \psi_{\alpha'}\}$ also define a uniformization with respect to (S, G). We shall say that $\{V_{\alpha'}, \psi_{\alpha'}\}$ is an *admissible* refinement of $\{U_\alpha, \varphi_\alpha\}$ if there exist elements $\{g_{\alpha'}\}_{\alpha' \in A'}$ in G such that $\psi_{\alpha'} = g_{\alpha'} \circ \varphi_{j(\alpha')}$ restricted to $V_{\alpha'}$. If this is the case, then

$$(1.2) \qquad g_{\alpha'\beta'} = \psi_{\alpha'} \circ \psi_{\beta'}^{-1} = g_{\alpha'} \circ \varphi_{j(\alpha')} \circ \varphi_{j(\beta')}^{-1} \circ g_{\beta'}^{-1} .$$

Let $\{U_\alpha, \varphi_\alpha\}_{\alpha \in A}$, $\{V_{\alpha'}, \psi_{\alpha'}\}_{\alpha' \in A'}$ be two uniformizations of M with respect to (S, G). These uniformizations are said to be *equivalent* if there exists a uniformization $\{W_{\alpha''}, \theta_{\alpha''}\}_{\alpha'' \in A''}$ which is an admissible refinement common to both $\{U_\alpha, \varphi_\alpha\}$ and $\{V_{\alpha'}, \psi_{\alpha'}\}$.

Definition 1.3. An equivalence class of uniformizations will be called an (S, G)-structure on M.

Suppose that S is a differentiable manifold and G is a group of diffeomorphisms of S. An (S, G)-structure on a differentiable manifold M is said to be *differentiable* if each of the maps φ_α appearing in some uniformization belonging to the (S, G)-structure is a diffeomorphism. Similarly, the notions of real analytic (S, G)-structure etc. may be defined. For convenience we shall not keep the logical distinction between a uniformization and the underlying (S, G)-structure if there is no possibility of confusion.

The condition (U) of (1.1) on the group G implies that

$$(1.4) \qquad g_{\alpha\alpha} = 1 , \qquad g_{\alpha\beta} \circ g_{\beta\gamma} = g_{\alpha\gamma} .$$

This defines a 1-cocycle on M with values in the sheaf of germs of continuous G-valued functions where G is given the discrete topology. In cohomological terms (1.2) and (1.4) together imply that an (S, G)-structure on M canonically defines a cohomology class in $H^1(M, G)$.

The bundle ξ and the section σ. Let M have an (S, G)-structure. Let $\{U_\alpha, \varphi_\alpha\}_{\alpha \in A}$ define a uniformization of M which defines the (S, G)-structure. We construct a bundle ξ with base M, fiber S and group G with discrete topology as follows. The total space E_ξ is obtained by taking a disjoint union $\{U_\alpha \times S\}_{\alpha \in A}$ and factoring by the equivalence relation: for $x \in U_\alpha \cap U_\beta$

$$(1.5) \qquad (x, s_\alpha) \sim (x, s_\beta) \qquad \text{iff } s_\alpha = g_{\alpha\beta} s_\beta .$$

It is easy to see that since by construction G acts effectively on S the bundle ξ depends only on the (S, G)-structure and not on the choice of the uniformization $\{U_\alpha, \varphi_\alpha\}$. Consider the section

$$(1.6) \qquad \sigma: M \to E_\xi$$

defined as follows: if $x \in U_\alpha$, set $\sigma(x) = (x, \varphi_\alpha(x))$. Since $\varphi_\alpha = g_{\alpha\beta}\varphi_\beta$ this is well defined. σ depends on a choice of the uniformization.

If N is uniformizable on (S, G) and $f: M \to N$ a local homeomorphism, then pulling back a uniformization of N via f defines a uniformization of M.

We can now make spaces uniformizable on (S, G) into a category. If M and N are two such spaces, a *morphism*, or an (S, G)-*map*, or a G-*map* (if the reference to S is clear) is a local homeomorphism $f: M \to N$ such that the pullback of the uniformization on N by f is a uniformization equivalent to the given uniformization of M. In particular, a homeomorphism $f: M \to M$ which is a G-map is called a G-automorphism.

Holonomy. It is standard, cf. e.g. Steenrod [18], that $H^1(M, G)$ is in 1-1 correspondence with homomorphisms of the fundamental group of M into G:

$$\rho: \pi_1(M) \to G \tag{1.7}$$

up to inner automorphisms of G. If a cohomology class in $H^1(M, G)$ arises from a uniformization, the corresponding ρ will be called the *holonomy representation* of the uniformization.

From the above discussion the following propositions are obvious.

Proposition 1.8. *If M is uniformizable on (S, G), so is any covering space M_1 of M. Conversely, if M_1 is a reqular covering space which is uniformizable on (S, G), and the group of covering transformations of M_1 over M consists of G-automorphisms, then M is uniformizable on (S, G).*

Proposition 1.9. *If M, N are uniformizable on (S, G), $f: M \to N$ is a G-map, and $\rho_N: \pi_1(N) \to G$ is a holonomy representation of N, then the induced map $\rho_N \circ f_*$ is a holonomy representation of M.*

We apply these considerations to the following situation. Let M be uniformizable on (S, G), and $\rho: \pi_1(M) \to G$ be a corresponding holonomy representation. Let k be the kernel of ρ, and M_k the corresponding regular covering space of M. Let $p: M_k \to M$ be the projection map. Then $\rho_k = \rho \circ p_*$, which is a holonomy representation of M_k, is obviously trivial. Let ξ be the bundle on M constructed as above. Then $p^*\xi$ is obviously the corresponding bundle on M_k. Since the holonomy representation ρ_k is trivial, the bundle $p^*\xi$ is isomorphic to the product bundle so that the total space

$$E_{p^*\xi} \approx M_k \times S\,. \tag{1.10}$$

Let $\sigma_k: M_k \to E_{p^*\xi}$ be the section constructed as above. Then $\delta_k = pr_2 \circ \sigma_k$, where pr_2 is the projection into the second factor by the isomorphism (1.10) is a local homeomorphism of M_k into S. If $\tilde{p}: \tilde{M} \to M_k$ is any covering space, then $\delta_k \circ \tilde{p}$ is also a local homeomorphism. We shall use the same letter δ for any map of the type $\delta_k \circ \tilde{p}$ unless there is a possibility of confusion.

Definition 1.11. δ is called a development map of the uniformization.

Apart from any other geometrical structure which may be available, the representation ρ and the development map δ contain the most important information about the uniformization.

We shall now prove two basic theorems governing uniformization.

Theorem 1.12. *Let M be uniformizable on (S, G). Suppose that M and S are connected, arc connected and locally simply connected. If M is compact and simply connected, then M covers S. In particular, any two (S, G)-structures on M are isomorphic. Moreover, if S is also simply connected, then $M \approx S$.*

Proof. The development map $\delta\colon M \to S$ is a local homeomorphism. Let q be a point in S. The set $\delta^{-1}(q)$ must be finite, for otherwise since M is compact, $\delta^{-1}(q)$ would have a limit point p and δ would fail to be a local homeomorphism in a neighborhood at p. For each p_i in $\delta^{-1}(q)$ let U_i be a connected neighborhood on which δ is a local homeomorphism and $U_i \cap U_j = \emptyset$ if $p_i \neq p_j$. Then $V = \cap\, \delta(U_i)$ is a neighborhood of q. Let V_0 be the component of V containing q. Then each component of $\delta^{-1}(V_0)$ is mapped homeomorphically onto V_0, i.e., δ is a covering map. The last statement is clear. q.e.d.

For the next theorem we shall assume that S is a differentiable manifold, and we shall consider differentiable uniformizations only.

Theorem 1.13. *Let S be a Riemannian manifold of class C^1, and G a group of isometries of S. Let M be a differentiable manifold differentiably uniformizable on (S, G), and $p\colon \tilde{M} \to M$ its simply connected covering space. Then $M, \tilde{M}$ admit Riemann metrics such that p and δ are local isometries. If moreover M is complete, then $\tilde{M}$ covers S.*

Proof. Let $\{U_\alpha, \varphi_\alpha\}$ be a uniformization of M. We introduce Riemann metrics on U_α by pulling back the metric on $\varphi_\alpha(U_\alpha)$. Two metrics on $U_\alpha \cap U_\beta$ agree since G is a group of isometries. Thus M admits a Riemann metric and by pullback by p, $\tilde{M}$ admits a Riemann metric. By construction the development map δ is a local isometry. If M is complete, so is $\tilde{M}$. To show that δ is a covering map for a fix $q \in S$. $\delta^{-1}(q)$ is discrete since δ is a local homeomorphism. Let m_i be a point in $\delta^{-1}(q)$, and choose an ε-ball U_i around m_i, which is mapped isometrically by δ on the ε-ball V around q where ε is a sufficiently small positive number. $\delta^{-1}(V)$ clearly is a union of ε-balls U_j around each m_j in $\delta^{-1}(q)$. Since $\tilde{M}$ is complete, δ maps each U_j isometrically on V. Also if $m_j \neq m_k$, then $U_j \cap U_k$ must be empty for otherwise it will contain a point x such that the distance $d(x, m_j)$ is different from $d(x, m_k)$. This is a contradiction for each $d(x, m_j)$ and $d(x, m_k)$ must equal $d(\delta(x), q)$. So δ is a covering map.

Remark. In the above theorem in particular if S is simply connected, then $\tilde{M} \approx S$. This theorem essentially covers the "Fuchsian case." In this case uniformization does not give much beyond the usual covering space theory. When G cannot be made into a group of isometries—which is the case if the isotropy subgroup at some point is noncompact—then uniformization goes further. An example of this situation which arises naturally is that of conformally Euclidean manifolds of dimension n, which are precisely the manifolds uniformizable on the standard n-sphere S^n with respect to the full group of conformal transformation. When $n = 2$ and we restrict to orientation preserving conformal trnsformations, this is the classical uniformization for Riemann surfaces. Similarly it is not too difficult to see that every paracompact 2-dimensional manifold may also be uniformized on $\boldsymbol{RP}^2$—the real projective plane with respect to the group of projective transformations. Similar problems may be posed in several complex variables. Another interesting set-up is: $S = G/P$ where G is a real semisimple noncompact Lie group and P is a parabolic subgroup of G. This is a natural set-up in which a generalization of the classical theory of Kleinian groups may be sought.

2. Uniformization and G-structures

Let G be a subgroup of $GL_n(\boldsymbol{R})$, and ξ a bundle of rank n on a manifold M. We say that ξ carries a G-structure if the structure group of ξ can be reduced from $GL_n(\boldsymbol{R})$ to G. Equivalently, a G-structure amounts to specifying a principal G-bundle which is a subbundle of the bundle of frames associated to ξ. If such a bundle is specified, a frame belonging to this subbundle will be called a *G-frame*.

Let ${}^tG$ be the group obtained by taking transposes of elements of G. If ξ admits a G-structure, then the dual bundle ξ^* admits a ${}^tG$-structure. In particular for $G = {}^tG$, ξ carries a G-structure iff ξ^* does. This observation will be used later.

Let M^n be a differentiable manifold, and $\xi = \tau(M)$ the tangent bundle of M. *A* G-structure on $\tau(M)$ is called a G-structure on M. Moreover, a G-structure on M is said to be *integrable* if for every $p \in M$ there exists a coordinate neighborhood of p with coordinates $\{x^1, \cdots, x^n\}$ such that the coordinate frames $\{\partial/\partial x^1, \cdots, \partial/\partial x^n\}$ are G-frames.

While the language of G-structures is very convenient to formulate geometrical notions, it is usually a difficult problem for a given G to describe manifolds carrying a G-structure or an integrable G-structure. The following proposition gives a source of examples of manifolds carrying a G-structure or an integrable G-structure.

Proposition 2.1. *Let K be a Lie group acting effectively, transitively, and differentiably on a differentiable manifold S. Fix a base point $s \in S$. Let H be the isotropy subgroup at s, and $\iota: H \to \tau_s(S)$ be the isotropy representation. Let $G = \iota(H)$. Then S carries a G-structure, and any manifold uniformized on (S, K) carries a G-structure. If the G-structure on S is integrable, so is the G-structure on any manifold uniformized on (S, K).*

Proof. Fix a frame f at s and operating by G a set F_s of frames which we shall take to be the G-frames at s. For $k \in K$ we take $k_*(F_s)$ to be the G-frames at ks. This is well defined, for if $k_1 s = k_2 s$, then $k_1^{-1}k_2$ is in H and $(k_1^{-1}k_2)_* F_s = F_s$ so $k_{1*}(F_s) = k_{2*}(F_s)$. The frames $\{k_* F_s \mid k \in K\}$ clearly define a principal G-subbundle of the bundle of frames. So S carries a G-structure. Also K is a group of automorphisms of this G-structure. If M is a manifold uniformized on (S, K), choose a uniformization $\{U_\alpha, \varphi_\alpha\}$. Define a G-structure on U_α as a pullback of the G-structure on $\varphi_\alpha(U_\alpha)$ by $\varphi_{\alpha*}^{-1}$. Two G-structures on $U_\alpha \cap U_\beta$ if nonempty, match since $\varphi_\alpha \circ \varphi_\beta^{-1} \in K$. The final observation is also clear.

Remark. To say that a manifold uniformized on (S, K) carries a G-structure, we only need that K is a group of G-automorphisms. The full strength of $S = K/H$ is not needed.

In the following we are interested in the problem: given a subgroup G of $GL_n(\boldsymbol{R})$ describe the differentiable manifolds carrying integrable G-structures. In general this is a difficult problem. We shall deal with the case where the G-structure is of *finite type* in the sense described below. For a fuller account of this notion cf. Sternberg [9] which also deals with the nonintegrable case.

Fix a subgroup G of $GL_n(\boldsymbol{R})$. Consider $\boldsymbol{R}^n$ with standard Cartesian coordinates. We can equip $\boldsymbol{R}^n$ with a canonical integrable G-structure. We are actually interested only in the "germ" of this G-structure at the origin. A local diffeomorphism $x \to f(x)$ is an automorphism of the G-structure if it carries G-frames into G-frames. This is so iff the Jacobian matrix $(\partial f^j/\partial x^i)_p$ for any p in a sufficiently small neighborhood of the origin belongs to G. In particular if $\{f_t\}$ is a local 1-parameter group of G-automorphisms and $X^j = \partial f_t^j/\partial t|_{t=0}$ then $(\partial X^j/\partial x^i)_p$ for any p in a sufficiently small neighborhood of the origin belongs to the Lie algebra $\mathfrak{g}$ of G where $\mathfrak{g}$ is considered as a matrix subalgebra of $\mathfrak{gl}_n(\boldsymbol{R})$ in a canonical way. A vector field of class C^1

$$\xi = \sum_{j=1}^{n} X^j \frac{\partial}{\partial x^j}$$

defined on a neighborhood of the origin with the property $(\partial X^j/\partial x^i) \in \mathfrak{g}$ is called an infinitesimal automorphism of the G-structure.

Definition 2.2. G is said to be of finite type if the Lie algebra of infinitesimal automorphisms of the above G-structure is finite dimensional.

With this definition we formulate

Theorem 2.3 (*A generalized Liouville's theorem*). *Let G be a subgroup of $GL_n(\boldsymbol{R})$ of finite type. Then there exists an n-dimensional differentiable manifold S together with a transitive effective Lie group K acting on S so that S carries an integrable G-structure, and a manifold M^n admits an integrable G-structure iff it is uniformizable on (S, K).*

Proof. Consider $\boldsymbol{R}^n$ with standard Cartesian coordinates, and consider the germ of the standard integrable G-structure at the origin. Let $\mathfrak{g}$ be the Lie algebra of G, and κ the Lie algebra of infinitesimal G-automorphisms which is finite dimensional since G is of finite type. Let $\mathfrak{h}$ be the isotropy subalgebra of κ at the origin, i.e.,

$$\mathfrak{h} = \{X \in \kappa \mid X|_0 = 0\} .$$

Let $\iota\colon \mathfrak{h} \to \mathfrak{gl}_n(\boldsymbol{R})$ be the linear isotropy representation. κ clearly contains $\mathfrak{g}$ and the algebra $\mathscr{T}$ of translations $\approx \boldsymbol{R}^n$. Also $\mathfrak{g} \subseteq \mathfrak{h}$, and if $\mathfrak{g}$ is identified with the subalgebra of $\mathfrak{gl}_n(\boldsymbol{R})$ then $\iota(\mathfrak{h}) = \mathfrak{g}$. Let K' be the simply connected Lie group with the Lie algebra κ, and let H' be the connected integral subgroup of K' with the Lie algebra $\mathfrak{h}$. Then H' is closed. For if it were not closed then, its closure $\bar{H}'$ will be of dimension greater than that of H', and the same is true for their Lie algebras $\bar{\mathfrak{h}}$ and $\mathfrak{h}$. Sufficiently small neighborhoods of the identity in H' and $\bar{H}'$ act on a neighborhood of the origin in $\boldsymbol{R}^n$, and by construction the origin is the fixed point of this action. So $\bar{\mathfrak{h}}$ is contained in the isotropy subalgebra $\mathfrak{h}$. Hence dim $\bar{\mathfrak{h}}$ equals dim $\mathfrak{h}$ which is a contradiction. Thus H' is closed and $S = K'/H'$ is a manifold. If N is the (necessarily discrete) subgroup of K' acting trivially on S, then N is normal and closed in both K' and H'. Set $K'' = K'/N$, $H'' = H'/N$ so that we have also $S = K''/H''$. Let s be the base point corresponding to the coset (H''). A sufficiently small neighborhood of the origin in $\boldsymbol{R}^n$ may be mapped homeomorphically and K''-equivariantly onto a neighborhood of the base point s in S. Using this isomorphism consider the isotropy representation $\iota\colon H'' \to GL_n(\boldsymbol{R}) \approx \operatorname{Aut} \tau_s(S)$. Then $\iota(H'') =$ say G'' is clearly a subgroup of G which is in fact the connected component of the identity of G. By Proposition 2.1, S carries an integrable G''-structure, in particular an integrable G-structure. Let K be the full group of G-automorphisms of S. We claim that every locally defined G-isomorphism is a restriction of an element of K. Indeed let V, W be open nonempty subsets, and $g\colon V \to W$ a G-isomorphism. Composing with suitable elements of K'' we may assume that V, W are neighborhoods of the base point s. Then for k lying in a sufficiently small neighborhood of the identity in K', gkg^{-1} is defined and is an element of K'. Thus g induces a local automorphism of a neighborhood of the identity in K' which extends to an automorphism of K' since K' is connected and simply connected. We denote this automorphism by $k \to k^g$. Clearly $N^g = N$ so that we get an automorphism again denoted by $k \to k^g$ of K''. If p is any point of S, then choose $k \in K''$ such that $ks = p$ and set $gp = k^g(gs)$. As H'' is connected it follows that $gH''g^{-1}$ is the isotropy subgroup of K'' at gs, and this fact easily shows that the action of g on S described above is well defined. Thus g and in the same way g^{-1} extend to maps of S into S. Using the well-known fact that a local automorphism of an integrable G-structure of finite type is analytic, we see that $gg^{-1} = 1$ everywhere since it is so on a nonempty open subset. So g extends to a G-automorphism of S.

Let now M^n be a manifold with an integrable G-structure. Then every point $p \in M$ has a neighborhood U_p and a G-isomorphism $\varphi_p\colon U_p \to S$. By what has been said above if $U_p \cap U_q \neq \emptyset$ then $\varphi_p \circ \varphi_q^{-1}$ is a restriction of an element of K, i.e., M is uniformizable over (S, K). This completes the proof.

A generalization. In the above theorem we have restricted to "first order" G-structures, i.e., where G is a subgroup of $GL_n(\boldsymbol{R})$. We shall briefly outline a generalization to the higher order structures. As the main interest here is in the integrable case we restrict the discussion to this case only. For more details concerning higher order structures see e.g., Kobayashi [10] and Sternberg [19].

Consider all C^∞ diffeomorphisms $x \to f(x)$ defined in some neighborhood of the origin of $\boldsymbol{R}^n$ such that $f(0) = 0$. Two such diffeomorphisms are considered equivalent if they coincide in some neighborhood of the origin. An equivalence class of diffeomorphisms is called a germ of a diffeomorphism. Under composition this obviously defines a group structure on the set of germs. Let us denote this group by $G(n)$. If we fix a coordinate system at the origin, the subgroup of $G(n)$ defined by linear automorphisms can obviously be identified with $GL_n(\boldsymbol{R})$. Let U be an open subset of $\boldsymbol{R}^n$ (with standard coordinates). Let f: $U \to \boldsymbol{R}^n$ be a diffeomorphism. Define Df: $U \to G(n)$ as follows: if $f(p) = q$, and τ_p (resp. τ_q) is the translation carrying the origin to p (resp. q), then $Df(p)$ is the equivalence class of $\tau_q^{-1} \circ f \circ \tau_p$. Let G be a subgroup of $G(n)$, and M^n an n-dimensional differentiable manifold. We shall now define the notion of an integrable G-structure on M as follows. M^n is said to admit an integrable G-structure if there exists a covering $\{U_\alpha\}$ by coordinate neighborhoods such that if $\{g_{\alpha\beta}\}$ are the coordinate transition functions then for each p in $U_\alpha \cap U_\beta$, $Dg_{\alpha\beta}(p)$ is an element of G. Two systems of coordinate neighborhoods defining an integrable G-structure are *compatible* if their union also defines an integrable G-structure. The union of all compatible coordinate systems is called the *maximal atlas* of the G-structure. A diffeomorphism of M is a *G-automorphism* if it carries the maximal atlas into itself. Similarly an infinitesimal G-automorphism may be defined. Consider the "germ" of the canonical G-structure defined at the origin in $\boldsymbol{R}^n$. G is said to be *finite type* if the infinitesimal G-automorphisms of this G-structure at the origin in $\boldsymbol{R}^n$ form a Lie algebra which is finite dimensional. With minor modifications Theorem 2.3 is generalized to

Theorem 2.3′. *Let $G \subseteq G(n)$ be of finite type. Then there exists a manifold S with an integrable G-structure such that an n-dimensional differentiable manifold M^n admits an integrable G-structure iff it is uniformizable on (S, K) where K is its full group of G-automorphisms.*

As an example where Theorem (2.3)′ applies but Theorem (2.3) does not, we may consider $G \subseteq G(n)$ defined by the germs of real respective complex projective transformations of the real respective complex projective space. If the dimension of the projective space is ≥ 2, in fact 2-jets of the germs suffice. If dimension $= 1$, one needs 3-jets which corresponds to the third order differential equation ("Schwarzian derivative") satisfied by the projective transformations.

The group K. The structure of the group K appearing in Theorem 2.3 or 2.3′ is in general complicated. In particular it may have infinitely many components. In this connection in the important case when S is *compact*, the following results based on some standard results from Lie theory are quite useful.

Theorem 2.4. *Let $S = K/H$ be as in Theorem 2.3 or 2.3′. Let K_0 (resp. H_0) be the connected component of the identity in K (resp. H). Let P (resp. Q) be a maximal compact subgroup of K_0 (resp. H_0) such that $P \supseteq Q$. Assume that S is compact. Then the following hold*:

(a) $S \approx P/Q$.

(b) *If K_0 is semisimple, then K has finitely many components.*

(c) *If the Euler characteristic of S is nonzero, then H_0 and Q have only finitely many fixed points.*

Proof. (a) By the Lie theory P (resp. Q) is homotopic to K_0 (resp. H_0). Hence the homotopy sequence

$$\begin{tikzcd}
\arrow[r] & \pi_i(Q) \arrow[r] \arrow[d, "\approx"] & \pi_i(P) \arrow[r] \arrow[d, "\approx"] & \pi_i(P/Q) \arrow[r] \arrow[d] & \pi_{i-1}(Q) \arrow[r] \arrow[d, "\approx"] & {} \\
\arrow[r] & \pi_i(H_0) \arrow[r] & \pi_i(K_0) \arrow[r] & \pi_i(K_0/H_0) \arrow[r] & \pi_{i-1}(H_0) \arrow[r] & {}
\end{tikzcd}$$

the five lemma and the Whitehead theorems show that P/Q is homotopic to $K_0/H_0 = S$. Since P/Q and S are compact manifolds, the canonical inclusion $P/Q \to S$ must be a homeomorphism.

(b) Consider the map $\varphi: K \to \operatorname{Aut} K_0$ defined by $\varphi(k)x = kxk^{-1}$. Since K_0 is semisimple, the qroup of inner automorphisms $\operatorname{Inn} K_0$ is of finite index in $\operatorname{Aut} K_0$. So for our purpose without loss of generality we may assume that $\varphi(K)$ is contained in $\operatorname{Inn} K_0$. Let Z be the centralizer of K_0 in K. Given k in K there exists k_0 in K_0 such that $\varphi(k) = \varphi(k_0)$, i.e., kk_0^{-1} belongs to Z. Hence $K = ZK_0$. Since

$$K/K_0 \approx ZK_0/K_0 \approx Z/Z \cap K_0$$

is discrete and $Z \cap K_0$ is also discrete, since $Z \cap K_0$ is a central subgroup and K_0 is semisimple, it follows that Z is discrete. We assert that Z acts without fixed points on S. For suppose p is a fixed point of an element z of Z. Then for every k_0 in K_0, $zk_0p = k_0zp = k_0p$. Since K_0 is transitive on S we see that $z = 1$. Since Z is discrete and acts without fixed points on the compact space S, it follows that Z is finite, and hence K has finitely many components.

(c) We show the statement for H_0. The proof for Q is similar. The set $F(H_0)$ of fixed points of H_0 is clearly closed and hence compact. Let N be the normalizer of H_0 in K_0. Then N is transitive on $F(H_0)$ so that $N/H_0 \approx F(H_0)$ is a compact Lie group. If $N/H \approx F(H_0)$ is not finite, its dimension is greater than one and its Euler characteristic is zero since a Lie group is parallelizable. But then $S = K_0/H_0$ is the total space of the fiber bundle with base K_0/N and fiber N/H_0. Hence, if N/H_0 is not finite the Euler characteristic of S would be zero.

Remarks. (i) The part (a) is a consequence of Mostow [16]. See [16, § 3] for a different proof.

(ii) Under the hypothesis of part (c) it can be proved that P must be semisimple. See Kobayashi and Nomizu [11, Vol. II, p. 336].

3. Quaternionic structure

In § 3 through § 5 we develop the notion of a quaternionic manifold which is just "tailormade" for the procedure described in § 3. We need the following definition.

Definition 3.1. A differentiable manifold M^{4n} is quaternionic if it has an integrable $GL_n(\boldsymbol{H}) \cdot GL_1(\boldsymbol{H})$-structure.

We shall discuss this definition below. There is an interesting relationship between complex manifolds, quaternionic manifolds and conformally Euclidean manifolds. There are different levels of rigidity in these structures which will be clear later.

In the above definition $\boldsymbol{H}$ stands for the noncommutative field of quaternion with center $\boldsymbol{R}$. Let $\boldsymbol{R}^{4n}$ be identified with $\boldsymbol{H}^n$ as a *right* $\boldsymbol{H}$-vector space of $n \times 1$ coulmn vectors with quaternion entries. Let $GL_n(\boldsymbol{H})$ denote the group of $\boldsymbol{H}$-linear invertible transformations of $\boldsymbol{H}^n$ acting on the left which, after a choice of a base, may be identified with $(n \times n)$ invertible matrices with quaternion entries. $GL_1(\boldsymbol{H}) = \boldsymbol{H}^*$ acts on $\boldsymbol{H}^n$ as acalar multiplication on the right. Two actions commute and are $\boldsymbol{R}$-linear. Let $GL_n(\boldsymbol{H}) \cdot GL_1(\boldsymbol{H})$ denote the image of $GL_n(\boldsymbol{H}) \times GL_1(\boldsymbol{H})$ in $GL_{4n}(\boldsymbol{R})$ via this action. It is easy to see that

$$\{(r, r^{-1}) \mid r \in \boldsymbol{R} - (0)\} \subseteq GL_n(\boldsymbol{H}) \times GL_1(\boldsymbol{H}) ,$$

the kernel of this action.

Some explanation of the definition is in order. In comparison with the complex structure which is the same as an integrable $GL_n(\boldsymbol{C})$-structure it may seem more natural to define a quaternionic structure as an integrable $GL_n(\boldsymbol{H})$-structure. However this notion leads to a very restricted class of manifolds, e.g., let $(q_1, \cdots, q_n)$ be a system of local admissible quaternionic coordinates. Let σ: $H \to H$ be a field automorphism; then $(q_1^\sigma, \cdots, q_n^\sigma)$ is *not* in general admissible with respect to $GL_n(\boldsymbol{H})$-structure. Now all field automorphisms of $\boldsymbol{H}$ are of the form $q \to \lambda q \lambda^{-1}$, $\lambda \in H^*$. So, if we want to allow $(\lambda q_1 \lambda^{-1}, \cdots, \lambda q_n \lambda^{-1})$ to be an admissible coordinate system, $GL_n(\boldsymbol{H}) \cdot GL_1(\boldsymbol{H})$ is forced on us. Similarly a $GL_n(\boldsymbol{H})$-structure, integrable or not, would imply the existence of three automorphisms I, J, K of the tangent bundle satisfying

$$I^2 = J^2 = K^2 = -1 , \quad IJ = K , \quad JK = I , \quad KI = J .$$

Any of I, J, K is an almost complex structure. Thus e.g., $\boldsymbol{P}^1(\boldsymbol{H})$ which is diffeomorphic to S^4 would not carry a quaternionic structure, for as is well known, S^4 does not carry any almost complex structure. The condition that M^{4n} carries I, J, K satisfying the above relations and some variants of this condition have been investigated previously by Ehresman [3], obata [17] and others. See [17] for further references before 1956 and the paper by S. Ishihara [9] for recent work. Also for a 'quaternion calculus' see Fueter [5]. Considering the rigidity in these higher dimensional phenomena—basically arising from the overdetermined nature of the underlying differential equations—the function-theoretic interest must be considered as very special to Riemann surfaces. The Fuchsian or Kleinian group theoretic interest however remains the same.

I wish to thank T. Smith for telling me about quaternionic structures and related literature.

In view of the above discussion it is important to observe the following:

Lemma 3.2. *The quaternionic projective space $P^n(H)$ of real dimension $4n$ carries an integrable $GL_n(H)\cdot GL_1(H)$ structure.*

Proof. It suffices to show that the cotangent bundle carries an integrable structure since $GL_n(H)\cdot GL_1(H)$ is invariant under the operation of taking a transpose. Let

$$U_\lambda = \{q = (q_0, \cdots, q_n) \mid q_\lambda \neq 0\}$$

be the usual covering. On U_λ we may choose the quaternionic coordinates $\zeta_i^\lambda = q_i q_\lambda^{-1}$, $i \neq \lambda$. Setting $\zeta_i^i = 1$ we see that these coordinates are related on the overlap $U_\lambda \cap U_\mu$ by the relations

$$\zeta_i^\lambda = \zeta_i^\mu \zeta_\mu^\lambda = \zeta_i^\mu (\zeta_\lambda^\mu)^{-1} .$$

Hence

$$d\zeta_i^\lambda = d\zeta_i^\mu \zeta_\mu^\lambda + \zeta_i^\mu d\zeta_\mu^\lambda = \{d\zeta_i^\mu - \zeta_i^\mu (\zeta_\lambda^\mu)^{-1} d\zeta_\lambda^\mu\}(\zeta_\lambda^\mu)^{-1} ,$$

which clearly implies the lemma. q.e.d.

Also note the following useful fact.

Lemma 3.3. *A quaternionic manifold is orientable.*

Proof. $GL_n(H)\cdot GL_1(H)$ is connected. q.e.d.

The main theorem about quaternionic manifolds is the following.

Theorem 3.4. *A differentiable manifold M^{4n} is quaternionic if and only if it is unformizable with respect to $(P^n(H), PGL_{n+1}(H))$.*

The proof extends over the next two sections. In § 4 and § 5 we shall determine, respectively, the infinitesimal automorphisms of a quaternionic structure and the full group of quaternionic transformations of $P^n(H)$. Some Lie-theoretic generalizations of this proof seem plausible.

4. Theorem 3.4, first part

We first need a matrix representation of the Lie algebra of $GL_n(H)\cdot GL_1(H)$. Write

$$q^\lambda = x^\lambda + y^\lambda{}_i + z^\lambda{}_j + t^\lambda{}_k , \qquad \lambda = 1, 2, \cdots, n ,$$

where $1, i, j, k$ is the usual basis of H over R. We shall use $\{x^1, \cdots, x^n, y^1, \cdots, y^n, z^1, \cdots, z^n, t^1, \cdots, t^n\}$ as coordinates for R^{4n} in this specific order. An $n \times n$ matrix A with quaternionic entries may be written as

$$A = A_x + A_y i + A_z j + A_t k ,$$

where A_x, A_y, A_z, A_t are $n \times n$ matrices with real entries. A quaternion $n \times 1$ column vector may be written as

$$Q = X + Yi + Zj + Tk ,$$

where X, Y, Z, T are $n \times 1$ real column vectors. Then

$$\begin{aligned} AQ = &\{A_xX - A_yY - A_zZ - A_tT\} + \{A_xY + A_yX + A_zT - A_tZ\}i \\ &+ \{A_xZ + A_zX - A_yT + A_tY\}j + \{A_xT + A_tX + A_yZ - A_zY\}k , \end{aligned}$$

so that A may be identified with the $4n \times 4n$ real matrix

$$\begin{bmatrix} A_x & -A_y & -A_z & -A_t \\ A_y & A_x & -A_t & A_z \\ A_z & A_t & A_x & -A_y \\ A_t & -A_z & A_y & A_x \end{bmatrix} .$$

Similarly if Λ is an $n \times n$ scalar quaternion matrix acting on the *right*

$$\Lambda = \lambda_x + \lambda_y i + \lambda_z j + \lambda_t k .$$

Then a similar calculation shows that Λ may be identified with the matrix

$$\begin{bmatrix} \lambda_x & -\lambda_y & -\lambda_z & -\lambda_t \\ \lambda_y & \lambda_x & \lambda_t & -\lambda_z \\ \lambda_z & -\lambda_t & \lambda_x & \lambda_y \\ \lambda_t & \lambda_z & -\lambda_y & \lambda_x \end{bmatrix}$$

in $M_{4n}(\boldsymbol{R})$. The Lie algebra of $GL_n(\boldsymbol{H})\cdot GL_1(\boldsymbol{H})$ is therefore given by

$$\begin{bmatrix} A_x + \lambda_x & -A_y - \lambda_y & -A_z - \lambda_z & -A_t - \lambda_t \\ A_y + \lambda_x & A_x + \lambda_x & -A_t + \lambda_t & A_z - \lambda_z \\ A_z + \lambda_z & A_t - \lambda_t & A_x + \lambda_x & -A_y + \lambda_y \\ A_t + \lambda_t & -A_z + \lambda_z & A_y - \lambda_y & A_x + \lambda_x \end{bmatrix} . \tag{4.1}$$

Let $\xi = \sum_\lambda \xi^\lambda$ where $1 \leq \lambda \leq n$, and

$$\xi^\lambda = X^\lambda \frac{\partial}{\partial x^\lambda} + Y^\lambda \frac{\partial}{\partial y^\lambda} + Z^\lambda \frac{\partial}{\partial z^\lambda} + T^\lambda \frac{\partial}{\partial t^\lambda} \tag{4.2}$$

is a vector field on a neighborhood of the origin in $\boldsymbol{R}^{4n}$ on which a quaternionic structure is defined. Then (4.1) implies the following.

Proposition 4.3. *Let $\xi = \sum_\lambda \xi^\lambda$, $1 \leq \lambda \leq n$, be as in* (4.2). *Then ξ is an infinitesimal automorphism of the quaternionic structure if and only if*

(1) $X^\lambda_{x_\mu} = Y^\lambda_{y_\mu} = Z^\lambda_{z_\mu} = T^\lambda_{t_\mu}$,

(2) $X^\lambda_{y_\mu} = -Y^\lambda_{x_\mu}$, $X^\lambda_{z_\mu} = -Z^\lambda_{x_\mu}$, $X^\lambda_{t_\mu} = -T^\lambda_{x_\mu}$,
$Y^\lambda_{z_\mu} = -Z^\lambda_{y_\mu}$, $Y^\lambda_{t_\mu} = -T^\lambda_{y_\mu}$, $Z^\lambda_{t_\mu} = -T^\lambda_{z_\mu}$,

(3) $X^\lambda_{y_\mu} = Z^\lambda_{t_\mu}$, $X^\lambda_{z_\mu} = T^\lambda_{y_\mu}$, $X^\lambda_{t_\mu} = Y^\lambda_{z_\mu}$, for $\lambda \neq \mu$.

Proof. The part (1) follows from the equality of the diagonal terms in (4.1). The proof of (2) and (3) is similar. q.e.d.

The first step in the proof of Theorem 3.4 is the determination of the algebra of infinitesimal automorphisms of a quaternionic structure. We state the main conclusion.

Theorem 4.4. *A quaternionic structure is of finite type, and the dimension* (*over* $\boldsymbol{R}$) *of the algebra of infinitesimal automorphisms of the germ of a quaternionic structure on* $\boldsymbol{R}^{4n}$ *is* $4(n+1)^2 - 1$.

Proof. We first prove

Lemma 4.5.

(a) $X^\lambda_{y_\mu z_\mu} = 0 = \cdots$.

(b) *If* $\lambda \neq \mu$, $\lambda \neq \rho$, *then* $X^\lambda_{x_\mu x_\rho} = X^\lambda_{y_\mu y_\rho} = X^\lambda_{y_\mu z_\rho} = 0 = \cdots$.

(c) *All third order derivatives of* $X^\lambda, \cdots$ *are zero.*

Here in (a), (b), (c) the dots denote that similar expressions involving other coefficients of ξ and other coordinates obtainable from symmetry considerations are zero.

Proof of (*a*). Repeated use of (2) of Proposition 4.3 gives

$$X^\lambda_{y_\mu z_\mu} = -Y^\lambda_{x_\mu z_\mu} = Z^\lambda_{x_\mu y_\mu} = -X^\lambda_{z_\mu y_\mu} \qquad \text{so } X^\lambda_{y_\mu z_\mu} = 0\,.$$

Proof of (*b*). Using (1) and (2) of Proposition 4.3, and from symmetry considerations we get

$$X^\lambda_{x_\mu x_\rho} = Y^\lambda_{y_\mu x_\rho} = -X^\lambda_{y_\mu y_\rho} = -X^\lambda_{z_\mu z_\rho} = -X^\lambda_{t_\mu t_\rho}\,.$$

Also by (2) and (3) of Proposition 4.3, assuming $\lambda \neq \mu$, $\lambda \neq \rho$ we have

$$X^\lambda_{y_\mu y_\rho} = Z^\lambda_{y_\mu t_\rho} = -Y^\lambda_{z_\mu t_\rho} = -X^\lambda_{t_\mu t_\rho}\,,$$

which clearly imply that

$$X^\lambda_{x_\mu x_\rho} = -X^\lambda_{y_\mu y_\rho} = 0\,.$$

In the process we have also shown that $Z^\lambda_{y_\mu t_\rho} = 0$. So (b) follows from symmetry considerations.

Proof of (*c*). In view of (a) and (b) and the symmetry we need only to show that

$$X^\lambda_{x_\lambda x_\lambda x_\rho} = 0 = X^\lambda_{x_\lambda x_\lambda y_\rho}\,.$$

Now

$$X^\lambda_{x_\mu x_\nu x_\rho} = Y^\lambda_{y_\mu x_\nu x_\rho} = -X^\lambda_{y_\mu y_\nu x_\rho} = -Y^\lambda_{y_\mu y_\nu y_\rho} = -Z^\lambda_{z_\mu z_\nu z_\rho}\,.$$

Similarly

$$Y^\lambda_{y_\mu y_\nu y_\rho} = -Z^\lambda_{z_\mu z_\nu z_\rho}\,.$$

Combining the two equations we get in particular $X^\lambda_{x_\lambda x_\lambda x_\rho} = 0$. To prove $X^\lambda_{x_\lambda x_\lambda y_\rho} = 0$ we consider two cases. If $\lambda = \rho$ then by (a)

$$X^\lambda_{x_\lambda x_\lambda y_\lambda} = Z_{z_\lambda x_\lambda y_\lambda} = 0 \, .$$

If $\lambda \neq \rho$, then

$$X^\lambda_{x_\lambda x_\lambda y_\rho} = -X^\lambda_{y_\lambda y_\lambda y_\rho} = -Z^\lambda_{y_\lambda y_\lambda t_\rho} = Z^\lambda_{z_\lambda z_\lambda t_\rho} \, .$$

But also

$$X^\lambda_{x_\lambda x_\lambda y_\rho} = Z^\lambda_{x_\lambda x_\lambda t_\rho} = -Z^\lambda_{z_\lambda z_\lambda t_\rho} \, .$$

Combining these two sets of equations we get $X^\lambda_{x_\lambda x_\lambda y_\rho} = 0$. This finishes the proof of the lemma.

The part (c) of the lemma clearly implies that the coefficients $\{X^\lambda, \cdots\}$ of ξ are polynomials of degree ≤ 2 in the coordinates. We shall now compute a maximal set of linearly independent infinitesimal transformations. First note that the parts (1) and (2) of Proposition 4.3 easily imply that the only ξ for which the coefficients are of degree ≤ 1 are given by

$$\begin{aligned}
X^\lambda &= a^\lambda_x + \textstyle\sum_\mu a^\lambda_\mu x_\mu + \sum_\mu b^\lambda_\mu y_\mu + \sum_\mu c^\lambda_\mu z_\mu + \sum d^\lambda_\mu t_\mu \, , \\
Y^\lambda &= a^\lambda_y - \textstyle\sum_\mu b^\lambda_\mu x_\mu + \sum_\mu a^\lambda_\mu y_\mu + \sum_\mu (d^\lambda_\mu + \eta\delta^\lambda_\mu) z_\mu - \sum_\mu (c^\lambda_\mu + \gamma\delta^\lambda_\mu) t_\mu \, , \\
Z^\lambda &= a^\lambda_z - \textstyle\sum_\mu c^\lambda_\mu x_\mu - \sum_\mu (d^\lambda_\mu + \eta\delta^\lambda_\mu) y_\mu + \sum_\mu a^\lambda_\mu x_\mu + \sum_\mu (b^\lambda_\mu + \beta\delta^\lambda_\mu) t_\mu \, , \\
T^\lambda &= a^\lambda_t - \textstyle\sum_\mu d^\lambda_\mu x_\mu + \sum_\mu (c^\lambda_\mu + \gamma\delta^\lambda_\mu) y_\mu - \sum_\mu (b^\lambda_\mu + \beta\delta^\lambda_\mu) z_\mu + \sum_\mu a^\lambda_\mu t_\mu \, ,
\end{aligned} \tag{4.6}$$

where δ^λ_μ is the Kronecker delta, and

$$a^\lambda_x \cdots, a^\lambda_t, a^\lambda_\mu \cdots d^\lambda_\mu, \beta, \gamma, \eta$$

are arbitrary real constants. Thus this is a family of dimension $4n^2 + 4n + 3$.

The determination of ξ whose coefficients are of degree 2 is more complicated but straightforward. We shall noly briefly indicate it. Suppose X^1 contains a second degree term. Then by the previous lemma, the term must be of the form x_1^2 or $x_1 x_\mu$ $(1 \neq \mu)$. For definiteness suppose that

$$X^1 = x_1^2 + \cdots \, .$$

We show that the term $x_1^2 \partial/\partial x_1$ uniquely determines an infinitesimal transformation. We have observed in the process of proving the lemma that

$$X^\lambda_{x_\lambda x_\lambda} = -X^\lambda_{y_\lambda y_\lambda} = -X^\lambda_{z_\lambda z_\lambda} = -X^\lambda_{t_\lambda t_\lambda} \, .$$

Using this and the parts (1), (2) of Proposition 4.3 we have

$$\begin{aligned}
&X^1 = x_1^2 - y_1^2 - z_1^2 - t_1^2 + \cdots \, , \\
&Y^1 = 2x_1 y_1 + \cdots \, , \quad Z^1 = 2x_1 z_1 + \cdots \, , \quad T^1 = 2x_1 t_1 + \cdots \, .
\end{aligned} \tag{4.7}$$

Since

$$X^1_{y_1} - Z^1_{t_1} = -2y_1 + \cdots = X^\lambda_{y_\lambda} - Z^\lambda_{t_\lambda}\,, \qquad \lambda = 2, 3, \cdots\,,$$

we see that X^λ contains a term involving $y_1 y_\lambda$, or Z^λ contains a term involving $y_1 t_\lambda$. Suppose for definiteness that

$$X^\lambda = c\{-y_1 y_\lambda + \cdots\}\,,$$

where c is a constant to be determined. Again noting that

$$X^\lambda_{x_\lambda x_\mu} = -X^\lambda_{y_\lambda y_\mu} = \cdots$$

we see by applying Proposition 4.3 that

$$\begin{aligned} X^\lambda &= c\{x_1x_\lambda - y_1y_\lambda - z_1z_\lambda - t_1t_\lambda\} + \cdots\,, \\ Y^\lambda &= c\{y_1x_\lambda + x_1y_\lambda + t_1z_\lambda - z_1t_\lambda\} + \cdots\,, \\ Z^\lambda &= c\{z_1x_\lambda - t_1y_\lambda + x_1z_\lambda + y_1t_\lambda\} + \cdots\,, \\ T^\lambda &= c\{t_1x_\lambda + z_1y_\lambda - y_1z_\lambda + x_1t_\lambda\} + \cdots\,. \end{aligned} \tag{4.8}$$

Now

$$X^1_{y_1} - Z^1_{t_1} = -2y_1 + \cdots = X^\lambda_{y_\lambda} - Z^\lambda_{t_\lambda} = -2cy_1 + \cdots$$

implies that $c = 1$. If in (4.7) and (4.8) we put the terms denoted by dots equal to zero, we see that we have obtained an infinitesimal transformation preserving the quaternionic structure. In this construction if instead of choosing $X^\lambda = c\{-y_1y_\lambda + \cdots\}$ we had started with $Z^\lambda = c\{y_1t_\lambda + \cdots\}$, it is easy to see that we would have obtained the same infinitesimal transformation. Finally we could have started the construction with

$$X^1 = \{x_1x_{\lambda_0} + \cdots\}$$

instead of $X^1 = x_1^2 + \cdots$. It is equally easy to see that in this case we would have obtained a vector field similar to the one obtained where the roles of 1 and λ_0 are interchanged. Thus in all there are $4n$ infinitesimal transformations whose coefficients are of degree 2. Hence the total number of linearly independent infinitesimal quaternionic transformations is

$$4n^2 + 4n + 3 + 4n = 4(n+1)^2 - 1\,.$$

This finishes the proof of the theorem.

5. Theorem 3.4, second part

We shall prove

Theorem 5.1. *$PGL_{n+1}(\boldsymbol{H})$ is the full group of quaternionic transformations of $\boldsymbol{P}^n(\boldsymbol{H})$.*

Proof. The group $PGL_{n+1}(\boldsymbol{H})$ is connected. So to see that it is a group of quaternionic transformations we may restrict to a small neighborhood of the identity. Choose a neighborhood on $\boldsymbol{P}^n(\boldsymbol{H})$, say

$$U = \{q = (q_0, \cdots, q_n) \mid q_0 \neq 0\} .$$

On U we may take coordinates $\zeta_j = q_j q_0^{-1}$, $j = 1, 2, \cdots n$. Set $\zeta_0 = 1$. Represent an element of $PGL_{n+1}(\boldsymbol{H})$ by an $(n + 1) \times (n + 1)$ invertible matrix (a_{ij}) of quaternions $0 \leq i, j \leq n$. Then the corresponding projective transformation is given by

$$\zeta_i' = \Big(\sum_{j=0}^{n} a_{ij}\zeta_j\Big)\Big(\sum_j a_{0j}\zeta_j\Big)^{-1} .$$

Thus

$$d\zeta_i' = \Big\{\sum_j a_{ij}d\zeta_j - \Big(\sum_j a_{ij}\zeta_j\Big)\Big(\sum_j a_{0j}\zeta_j\Big)^{-1} \sum_j a_{0j}d\zeta_j\Big\}\Big(\sum_j a_{0j}\zeta_j\Big)^{-1} ,$$

and $PGL_{n+1}(\boldsymbol{H})$ consists of quaternionic transformations.

We have seen in Theorem 4.4 that the dimension (over $\boldsymbol{R}$) of the algebra of infinitesimal automorphisms of the germ of a quaternionic structure on $\boldsymbol{R}^{4n}$ is $4(n + 1)^2 - 1$ which is also the dimension of $PGL_{n+1}(\boldsymbol{H})$ over $\boldsymbol{R}$. It is now clear that the manifold S of Theorem 2.3 corresponding to $G = GL_n(\boldsymbol{H}) \cdot GL_1(\boldsymbol{H})$ is $\boldsymbol{P}^n(\boldsymbol{H})$. The only point which is not at all afortiori clear is that $PGL_{n+1}(\boldsymbol{H})$ is the *full* group of quaternionic transformations of $\boldsymbol{P}^n(\boldsymbol{H})$. The proof of this fact is not quite elementary. The function-theoretic methods used in similar questions for complex projective spaces are not available in our case. The author is indebted to W. Schmid for a helpful conversation on this point.

Let K be the full group of quaternionic transformations. By what has been said above $PGL_{n+1}(\boldsymbol{H})$ is the connected component of the identity of K. Let us call it K_0. We have to show $K = K_0$. We first need a technical lemma.

Lemma 5.2. *The automorphism group of $PGL_n(\boldsymbol{H})$ modulo the inner automorphisms is $\boldsymbol{Z}_2$ for $n \geq 2$, and consists of inner automorphisms only for $n = 1$.*

Proof. Let

$$SL_n(\boldsymbol{H}) = GL_n(\boldsymbol{H})/\boldsymbol{R} ,$$

where $\boldsymbol{R} \approx \{rI_n \mid r > 0\}$. $SL_n(\boldsymbol{H})$ has center $= \pm 1$ and is simply connected. Moreover $PGL_n(\boldsymbol{H}) = SL_n(\boldsymbol{H})/(\pm 1)$. Let $sl_n(\boldsymbol{H})$ be the Lie algebra of $SL_n(\boldsymbol{H})$. It is obvious that every automorphism of $SL_n(\boldsymbol{H})$ induces an automorphism of $PGL_n(\boldsymbol{H})$ and the automorphisms of $SL_n(\boldsymbol{H})$, the Lie algebra automorphisms of $sl_n(\boldsymbol{H})$, and the automorphisms of $PGL_n(\boldsymbol{H})$ are in a canonical 1–1 correspondence.

$sl_n(\boldsymbol{H})$ is a real simple Lie algebra and may be identified with $(n \times n)$ matrices (a_{ij}) with quaternionic entries such that $\operatorname{Re} \sum_i a_{ii} = 0$. Its complexification is $sl_{2n}(\boldsymbol{C})$ since $\boldsymbol{H} \otimes_{\boldsymbol{R}} \boldsymbol{C} \approx M_2(\boldsymbol{C})$, the full matrix ring of 2×2 matrices with complex entries. Explicitly the map $sl_n(\boldsymbol{H}) \to sl_{2n}(\boldsymbol{C})$ may be described as follows: write $A \in sl_n(\boldsymbol{H})$ as

$$A = Z + Wj ,$$

where Z, W are $n \times n$ matrices with complex entries. Map

$$A \to A^c = \begin{bmatrix} Z & W \\ -\overline{W} & \overline{Z} \end{bmatrix} .$$

Since $\operatorname{Re} \sum_i z_{ii} = 0$ we see that the trace $A^c = 0$ so $A^c \in sl_{2n}(\boldsymbol{C})$. The induced map

$$sl_n(\boldsymbol{H}) \underset{\boldsymbol{R}}{\otimes} \boldsymbol{C} \to sl_{2n}(\boldsymbol{C})$$

is easily seen to be a Lie algebra isomorphism.

Let σ be an automorphsim of $sl_n(\boldsymbol{H})$. By complexification it induces an automorphism $\sigma_{\boldsymbol{C}}$ of $sl_{2n}(\boldsymbol{C})$. The group of outer automorphisms *over* $\boldsymbol{C}$ of $sl_{2n}(\boldsymbol{C})$ can be constructed from its Dynkin diagram; cf. Helgason [8, Chap. 9, § 2]. For $n = 2$ all automorphisms of $sl_2(\boldsymbol{C})$ are inner, whereas for $n > 2$ the group of outer automorphisms is generated by

$$A \xrightarrow{\alpha} -{}^tA ,$$

where ${}^tA$ is the transpose of A. The lemma now follows in two steps:

(i) If an inner automorphism of $sl_{2n}(\boldsymbol{C})$ leaves $sl_n(\boldsymbol{H})$ invariant, it is induced by an inner automorphism of $sl_n(\boldsymbol{H})$.

(ii) The outer automorphism α is induced by an outer automorphism of $sl_n(\boldsymbol{H})$.

Proof of (*i*). The group of inner automorphisms of $SL_{2n}(\boldsymbol{C})$ is $PSL_{2n}(\boldsymbol{C})$ which is connected. So it suffices to show (i) for an inner automorphism induced by an element g sufficiently close to the identity, and so taking $\log g$ it suffices to show that if $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in sl_{2n}(\boldsymbol{C})$ is such that

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\begin{bmatrix} Z & W \\ -\overline{W} & \overline{Z} \end{bmatrix} - \begin{bmatrix} Z & W \\ -\overline{W} & \overline{Z} \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} Z^1 & W^1 \\ -\overline{W}^1 & \overline{Z}^1 \end{bmatrix}$$

for all Z, W, then $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in sl_n(\boldsymbol{H})$. This is seen easily by a simple calculation from the above equation.

Proof of (*ii*). The automorphism α: $A \to -{}^tA$ restricted to $sl_n(\boldsymbol{H})$ becomes

$$\begin{bmatrix} Z & W \\ -\overline{W} & \overline{Z} \end{bmatrix} \rightarrow \begin{bmatrix} -{}^tZ & {}^t\overline{W} \\ -{}^tW & {}^t\overline{Z} \end{bmatrix},$$

i.e., $Z + Wj \rightarrow -{}^tZ + {}^t\overline{W}$. For $n > 1$ this is easily seen to be an outer automorphism of $sl_{2n}(\boldsymbol{H})$. Thus the proof of the lemma is complete.

Remark 5.3. Modulo an inner automorphism by $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ this is the same as $A \rightarrow -{}^t\bar{A}$ where the bar denotes the quaternionic conjugation.

We return to the proof of the theorem. Recall that we have to show $K = K_0$. Let $k \in K$. By conjugation it induces an automorphism $\sigma_k : h \rightarrow khk^{-1}$ of K_0. First suppose that this automorphism is inner. Thus $\sigma_k = \sigma_{k_0}$ where $k_0 \in K_0$, so that $\sigma_{k_0 - 1_k} = 1$. Let P_0 be the isotropy subgroup of K_0 at the base point m of $\boldsymbol{P}^n(\boldsymbol{H})$. It may be represented by a matrix of the form

$$\left\{\begin{array}{c|ccc} * & * & & * \\ \hline 0 & & & \\ \vdots & & * & \\ 0 & & & \end{array}\right\} \in SL_{n+1}(\boldsymbol{H}) .$$

Then for every $p \in P_0$

$$k_0^{-1}kpm = k_0^{-1}km = pk_0^{-1}km ,$$

i.e., $k_0^{-1}km$ is also a fixed point of P_0. But by the above matrix representation of P_0 it is clear that m is the only fixed point P_0. So $k_0^{-1}km = m$. Hence

$$k_0^{-1}khm = hk_0^{-1}km = hm \qquad \text{for all } h \text{ in } K_0 ,$$

i.e., $k_0^{-1}k = 1$ or $k = k_0$.

It remains to consider the case where σ_k is an outer automorphism. By the lemma above and the remark following it we may assume that

$$\sigma_k : A \rightarrow {}^t\bar{A}^{-1} .$$

Thus $\sigma_k(P_0)$ is represented by matrices

$$\left[\begin{array}{c|ccc} * & 0 & & 0 \\ \hline * & & & \\ & & \delta & \\ * & & & \end{array}\right],$$

where $\delta \in GL_n(\boldsymbol{H})$. Since for every $p \in P_0$, $kpk^{-1}(km) = km$ is a fixed point of $\sigma_k(P_0)$. If $n \geq 2$ it is not difficult to see that $\sigma_k(P_0)$ has no common fixed point, i.e., σ_k cannot be an outer automorphism. This finishes the proof for $n \geq 2$.

The case $n = 1$ needs a separate argument which is based on the following proposition of independent interest.

Proposition 5.4. *M^4 admits a quaternionic structure if and only if it is orientable and admits a conformally Euclidean structure.*

Proof. We have already seen that a quaternionic manifold is orientable; cf. Lemma 3.3. On the other hand $GL_1(\boldsymbol{H})\cdot GL_1(\boldsymbol{H}) = SO(4) \times \boldsymbol{R}_{>0}$. Thus an integrable $GL_1(\boldsymbol{H})\cdot GL_1(\boldsymbol{H})$ structure means the existence of coordinate charts (x, y, z, t) such that the Jacobian of the transition function are conformal matrices. This clearly means that an integrable $GL_1(\boldsymbol{H})\cdot GL_1(\boldsymbol{H})$ structure is the conformally Euclidean structure with orientation. q.e.d.

To complete the proof of the theorem we only note the well known fact that the group of orientation preserving conformal transformations of $\boldsymbol{P}^1(\boldsymbol{H}) \approx S^4$ is the connected component of the identity of the group $SO(5, 1)$.

This finishes the proof of the Theorem 5.1 and so by §§ 2 and 3 also the proof of Theorem 4.3.

Remark 5.5. It is remarkable that in the above proof we needed the fact that K preserves the quaternionic structure only in the case $n = 1$. For $n \geq 2$ we only need the *group structure* of G. The underlying reason for this is to be traced to the existence of special transformations, namely, $q \to (\bar{q})^{-1}$ of $\boldsymbol{P}^1(\boldsymbol{H})$. These would be "quaternionic transformations" if we ask for integrability only with respect to $O(n) \times \boldsymbol{R}_{>0}$. There are no such transformations for $n > 1$. It may also be remarked that since $\overline{q_1q_2} = \bar{q}_2\bar{q}_1$ the formula $(q_0, \cdots, q_n) \to (\bar{q}_0, \cdots, \bar{q}_n)$ does *not* determine a map of $\boldsymbol{P}^n(\boldsymbol{H})$.

6. Complements—topology of quaternionic manifolds

In § 5 we have already noted that a 1-dimensional quaternionic manifold is the same as an oriented conformally Euclidean 4-dimensional manifold. It is amusing to note that a complex 1-dimensional manifold, i.e., a Riemann surface, is also the same as an oriented conformally Euclidean 2-dimensional manifold. This analogy persists in some gross form. A Riemann surface is uniformizable on $\boldsymbol{P}^1(\boldsymbol{C})$ with group of Möbius transformations

$$Z \to \frac{az + b}{cz + d} .$$

A quaternionic 1-dimensional manifold is uniformizable on $\boldsymbol{P}^1(\boldsymbol{H})$ with respect to the group of quaternionic transformations which again may be represented as

$$\zeta \to (a\zeta + b)(c\zeta + d)^{-1} ,$$

where quaternionic multiplication is intended.

As in the case of Riemann surfaces we have

Theorem 6.1. *A connected sum of* 1*-dimensional quaternionic manifolds admits a quaternionic structure.*

This will follow by a surgery theorem in the next section.

Theorem 6.2. *A quaternionic* 1*-dimensional compact manifold bounds an orientable manifold.*

Proof. We shall use the Thom Pontryagin's basic theorem that an oriented compact manifold bounds iff its Stiefel Whitney numbers and the Pontryagain numbers are zero. For an oriented real 4-dimensional manifold the only possibly nonzero Stiefel Whitney numbers are w_2^2 and w_4, and $w_2^2 = w_4$ by the Wu relations. $w_4 = 0$ iff the Euler characteristic is even. By Chern and Simons [2] one knows that the Pontryagin forms are conformally invariant. So for a conformally Euclidean manifold and in particular for a 1-dimensional quaternionic manifold the first Pontryagin number is zero. Hence the signature is zero, so the middle Betti number is even. It follows that the Euler characteristic is even also. q.e.d.

It is of interest to ask when is the second Stiefel Whitney class $w_2 = 0$? One knows from differential topology that vanishing of w_2 is a necessary and sufficient condition for the existence of a spin structure. The following theorem answers this question.

Theorem 6.3. *Let $SL_{n+1}(\boldsymbol{H}) \xrightarrow{p} PGL_{n+1}(\boldsymbol{H})$ be the canonical projection. Let M^n be a quaternionic manifold, and $\rho\colon \pi_1(M) \to PGL_{n+1}(\boldsymbol{H})$ its holonomy representation. Then $w_2 = 0$ iff ρ lifts to $\tilde{\rho}\colon \pi_1(M) \to SL_{n+1}(\boldsymbol{H})$ such that $\rho = p \cdot \tilde{\rho}$. If a lifting exists, all possible liftings are parametrized by $H^1(M\boldsymbol{Z}_2)$.*

Proof. Consider the bundle ξ of quaternionic frames on M. Its fiber is $G = GL_n(\boldsymbol{H}) \cdot GL_1(\boldsymbol{H})$. Let $\tilde{G}$ be its simply connected double cover. If $\tilde{K}$ is a maximal compact subgroup of $\tilde{G}$ projecting onto a maximal compact subgroup K of G, then we have a diagram

$$\begin{array}{ccc} \tilde{K} & \longrightarrow & \text{Spin}\,(4n) \\ \big\downarrow & & \big\downarrow \\ K & \longrightarrow & SO(4n) \end{array} \quad .$$

Thus $w_2 = 0$ is a necessary and sufficient condition for a lifting of the structure group from K to $\tilde{K}$, which implies a lifting from $SO(4n)$ to spin $(4n)$. For $P^n(\boldsymbol{H})$, w_2 vanishes and we can construct a bundle $\tilde{\xi}_0$ with fiber $\tilde{G}$ which covers the bundle ξ_0 with fiber G. Then $SL_{n+1}(\boldsymbol{H})$ is the group of bundle automorphisms of $\tilde{\xi}$.

Fix a uniformization $\{U_\alpha, \varphi_\alpha\}$ of M. Then it is not difficult to see that $\rho\colon \pi_1(M) \to PGL_{n+1}(\boldsymbol{H})$ lifts to $SL_{n+1}(\boldsymbol{H})$ iff the corresponding cocycle $\{g_{\alpha\beta}\}$ with values in $PGL_{n+1}(\boldsymbol{H})$ lifts to a cocycle $\{\tilde{g}_{\alpha\beta}\}$ with values in $SL_{n+1}(\boldsymbol{H})$. Finally the existence of the bundle $\tilde{\xi}$ covering ξ is clearly equivalent to the existence of the cocycle $(\tilde{g}_{\alpha\beta})$.

The final assertion in the theorem follows from the exact sequence (cf. [6])

$$\to H^1(M, \boldsymbol{Z}_2) \to H^1(M, \tilde{K}) \to H^1(M, K) \to H^2(M, \boldsymbol{Z}_2)\ .$$

Remarks. (1) In general $w_2 \neq 0$. If M is weakly Kleinian, i.e., if im ρ is discrete, then one may show that M has a finite covering for which $w_2 = 0$.

(2) The argument given above may be used in other uniformization problems, e.g., in the case of oriented conformally Euclidean manifolds, the connected group $SO_0(n+1, 1)$ has a double cover Spin $(n+1, 1)$ and the lifting of the holonomy representation $\rho\colon \pi_1 \to so_0(n+1, 1)$ to $\tilde{\rho}\colon \pi_1 \to \text{Spin}\,(n+1, 1)$ is again equivalent to $w_2 = 0$. This observation seems to be new even in the case of manifolds of constant curvature. For compact Riemannian flat manifolds M^n a finer statement may be made: $w_2 = 0$ if and only if the Riemannian holonomy group which is a finite subgroup of $SO(n)$ is an isomorphic image of a subgroup of Spin (n). Thus, e.g., if the Riemannian holonomy group contains an element

$$\sigma = \left[\begin{array}{cc|ccc} -1 & & & 0 & \\ & -1 & & & \\ \hline & & 1 & & 0 \\ & 0 & & \ddots & \\ & & 0 & & 1 \end{array}\right],$$

then $w_2 \neq 0$, for σ is of order 2 whereas the two elements of Spin (n) mapping onto σ are of order 4. This gives a simple proof of a therem of Auslander and Sczarba [1] to the effect that oriented solvmanifolds are not in general parallelizable. In fact in the explicit example they construct which is a flat Riemannian manifold, it is easily seen that the holonomy group contains an element of the type σ.

(3) It seems very likely that in general there are strong restictions on the real characteristic classes of all manofolds uniformizable on a given pair (S, K).

7. Complements—a surgery theorem

Given a model space (S, G) where G satisfies the uniformization condition (U), it is a problem of great interest to "construct" manifolds uniformizable on (S, G). The constructions will very much depend on the topological and analytic structure of S and the group structure of G. In the classical Fuchsian case the methods have been manily function theoretic, geometric and arithmetic. In the Kleinian case a powerful group theoretic method is that of "combination theorems" of Klein and Maskit; cf. [13], [14], [15]. The underlying geometric idea in this latter method is simple, and this section deals with its higher dimensional analogues. We first discovered it in the case of conformally Euclidean manifolds by a purely Riemann geometric method. We tend to feel that the present approach from the "Kleinian" point of view is more basic. It should also be remarked that this method may be used in the classical case of Riemann surfaces also, and the structures so obtained do not necessarily arise from the Klein Maskit constructions. This is because in the Klein Maskit constructions the image of the holonomy representation is always discontinuous, whereas this is not necessarily the case in what follows.

We wish to express our appreciation of R. Sharpe's comments on this section.

We need the following notion: Let A and B be n-dimensional manifolds such that their respective boundaries ∂A and ∂B are homeomorphic by a homeomorphism φ. Then the connected sum of A and B via φ denoted by $A \sharp_\varphi B$ is the manifold obtained from $A \cup B$ by identifying ∂A with ∂B via φ. If φ_t, $0 \leq t \leq 1$, is a 1-parameter family of homeomorphisms of ∂A with ∂B, it is clear that $A \sharp_\varphi B$ and $A \sharp_{\varphi 1} B$ are homeomorphic.

Let G be a group of homeomorphisms of the n-sphere S^n, and W^{n-1} a connected compact submanifold of S^n which has a neighborhood $V \approx W \times [-1, 1]$. If W^{n-1} is embedded in an open disc D^n, it disconnects D^n into two components. One of the components which is relatively compact in D^n will be called the *inside* of W.

Theorem 7.1. *Let G, W be as above. Suppose that* (a) *given a compact subset $K \subsetneqq S^n$ and a nonempty open subset U there exists $g \in G$ such that $K \subseteq gU$;*

(b) *there exists $g_0 \in G$ leaving $V \approx W \times [-1, 1]$ invariant so that*

$$g_0(x, t) = (f_t(x), -t)\ , \quad x \in W\ , \quad t \in [-1, 1]\ .$$

Let M^n, N^n be two manifolds uniformizable on (S^n, G). Then there exists an embedding of W in a small disc contained in M (resp. N) so that if M_1 (resp. N_1) is the manifold with boundary $\approx W$ obtained by removing the inside of W, then $M_1 \sharp_{f_0} N_1$ is also uniformizable on (S^n, G).

Proof. Fix a uniformization $(U_\alpha, \varphi_\alpha)$ on M. Choose a point p and disc D' around p, which is mapped homeomorphically by some $\varphi = \varphi_\alpha$ onto a disc D is S. Changing φ by an element of G whose existence is asusmed in condition (a), we may assume that D contains $V \approx W \times [-1, 1]$. By changing the parametrization of $[-1, 1]$ by $t \to -t$ if necessary we may also assume that the inside of $W \times \{1\}$ contains the inside of $W \times \{-1\}$. Let U be the inside of $W \times \{-\frac{1}{2}\}$ and set

$$M_1 = M - \varphi^{-1}(U)\ .$$

By a similar construction we obtain N_1. Then M_1, N_1 are manifolds with boundary $\approx W$ such that each boundary has a neighborhood G-isomorphic to $W \times [-\frac{1}{2}, 1] \subseteq V$. We now identify the part of the neighborhood $\approx W \times [-\frac{1}{2}, \frac{1}{2}]$ in M_1 to the similar part in N_1 by the map g_0 whose existence is guaranteed in (b). We clearly obtain a manifold $\approx M_1 \sharp_{f_0} N_1$ with a well defined uniformization.

Remarks. In the above proof we have not explicitly used the fact that the ambient manifold on which G acts is the n-sphere. But the condition (a) in the theorem already implies that the ambient manifold can be covered by two discs, hence it must be S^n. Topologically $M \sharp_{f_0} N$ as constructed in the theorem is easily seen to be homeomorphic to the usual connected sum. The specific uniformization obtained on $M \sharp_{f0} N$ *depends* on the map g_0. This has bearing on the "deformation" of this structure and roughly explains, e.g., how the deformations of a compact Riemann surface increase with the genus. In the proof the condition (a) is used only to insure that there exist some appropriate embeddings of W in any arbitrary M and N. If instead we knew that for particular M and N embeddings of W exist with neighborhoods G-isomorphic to $V \approx W \times [-1, 1]$, then the condition (a) can be dropped and the ambient manifold need not be S^n.

In the case of the group $\mathfrak{M}_n$ of conformal transformations of the sphere the condition (a) of the surgery theorem 7.1 is satisfied, e.g., consider the homotheties $x \to \lambda x$, $x \in \boldsymbol{R}^n$, $\lambda \in \boldsymbol{R}_{>0}$ pulled back to S^n via the stereographic projection which maps an arbitrary neighborhood of the origin onto an open subset containing any given compact subset not containing the point at infinity.

As a first application we formulate

Theorem 7.2. *A connected sum of conformally Euclidean manifolds admits a conformally Euclidean structure. If the manifolds are oriented, then a connected sum can also be constructed with orientation which restricts to given orientation on the factors.*

Proof. By definition a connected sum of two manifolds M_1^n and M_2^n is obtained by choosing closed discs D_i^n in M_2^n, $i = 1, 2$, removing the inside $\mathring{D}_i^n$ of D_i^n and glueing $M_i^n - \mathring{D}_i^n$ and $M_2^n - \mathring{D}_2^n$ along the boundary $(n-1)$-spheres by some homeomorphism. Take W^{n-1} to be the equatorial $(n-1)$-sphere, and g_0 the reflection in the equatorial n-plane. By the surgery theorem the first assertion is proved. It is equally easy to choose g_0 which is an orientation-preserving map of S^n. If M_1^n and M_2^n are oriented conformally Euclidean manifolds, we may choose their uniformizing local homeomorphisms (cf. φ_α in Definition 1.2) to be orientation-preserving. Then a connected sum of M_1^n and M_2^n via an orientation preserving g_0 is clearly oriented. This proves the second assertion.

Corollary 7.3. *Every paracompact topological surface* (*i.e.*, dim $= 2$) *is uniformizable on* $(S^2, \mathfrak{M}_2)$ *where* $\mathfrak{M}_2$ *is the full group of conformal transformations. In particular, every orientable paracompact surface admits a complex structure.*

As is well known in the case of orientable paracompact surfaces, this follows from the classical uniformization theorem which proves it in a stronger form, namely, every *Riemann surface* is so uniformizable. On the other hand existence of a complex structure on a topological oriented 2-manifold is usually proved by using isothermal coordinates. Corollary 7.3 is a simple geometric proof of a finer assertion.

Proof. Note that $\mathfrak{M}_2$ consists of both orientation-preserving and orientation-reversing conformal transformations. Using the antipodal map (resp. a lattice of translations) we can uniformize the real projective plane (resp. the torus). Since all compact surfaces are connected sums of real projective planes and torii, they are also so uniformizable. A paracompact noncompact orientable surface immerses in S^2. Any such immersion provides a uniformization. Finally a nonorientable paracompact surface may be represented as a connected sum of an orientable surface and a certatin number of copies of a real projective plane. So any such surface is uniformizable also.

8. Inversions

Inversions in circles, e.g., the map $z \to 1/z$ is a distinctive feature of conformal geometry. In higher dimensions one has of course inversions in 1-codimensional spheres as in Theorem 7.1, but one can ask for other possibilities. Let us formulate the following notion.

Definition 8.1. Let W^{n-1} be a compact connected submanifold of S^n which disconnects S^n into two components S_+, S_- say. An inversion in W is an involution of S^n which maps S_+ (resp. S_-) into S_- (resp. S_+). W is called the trace of the inversion.

Any such inversion gives a choice of W in Theorem 7.1 and indicates a higher-dimensional "combination theorem." We shall now construct some inversions. The use of real algebraic submanifolds in these constructions seems to be noteworthy.

Theorem 8.2. *Let $f(x_1, \cdots, x_{n+1})$ be a homogeneous polynomial with real coefficients. Let*

$$W = \{x \in \boldsymbol{R}^{n+1} \,|\, f(x) = 0,\ \|x\| = 1\}\ .$$

Assume that W is a connected submanifold of the unit sphere so that grad $f \neq 0$ *on W. Suppose moreover that f considered as a polynomial in some of the variables say $x_1, \cdots, x_k$ is homogeneous and of odd degree. Then there exists an inversion in $\mathfrak{M}_n$* f*with trace W.*

Proof. By definition

$$\operatorname{grad} f = \sum_{i=1}^{n+1} \frac{\partial f}{\partial x_i}(x)\frac{\partial}{\partial x_i}\ .$$

Since f is homogeneous, grad f is tangential to S^n at x in W so that it gives a trivialization of the normal bundle of W in S^n. Consider σ given by

$$x_i \to \begin{cases} -x_i\ , & i \leq k\ , \\ x_i\ , & i > k\ . \end{cases}$$

Then

$$\begin{aligned} \operatorname{grad} f|_{\sigma(x)} &= \sum_{i=1}^{n+1} \frac{\partial f}{\partial x_i}(\sigma(x))\frac{\partial}{\partial x_i} \\ &= \sum_{i\leq k} \frac{\partial f}{\partial x_i}(x)\frac{\partial}{\partial x_i} - \sum_{i>k} \frac{\partial f}{\partial x_i}(x)\frac{\partial}{\partial x_i}\ , \end{aligned}$$

whereas

$$d\sigma\,(\operatorname{grad} f|_x) = -\sum_{i\le k} \frac{\partial f}{\partial x_i}(x)\frac{\partial}{\partial x_i} + \sum_{i>k} \frac{\partial f}{\partial x_i}(x)\frac{\partial}{\partial x_i}\ .$$

Thus $d\sigma$ is orientation-reversing on the normal bundle of W which clearly implies that σ is an inversion with trace W, and obviously σ is in $\mathfrak{M}_n$.

Example 8.3. If f is a linear polynomial, then the construction gives the antipodal map in the equator. If $n = 3$, and

$$f(x_1, x_2, x_3, x_4) = x_1x_3 + x_2x_4$$

considered as a homogeneous polynomial in x_1, x_2, then the construction gives an inversion in the Clifford torus.

In this context we shall prove

Theorem 8.4. *For any $g = 0, 1, 2, \cdots$ there exists an inversion σ of S^3 such that $\sigma \in \mathfrak{M}_3$ and the trace of σ is a surface of genus g.*

Proof. Consider the equation $f(x, y) = 0$, e. g.,

$$f(x, y) = x \prod_{i=1}^{m} \{(x - a_i)^2 + (y - b_i)^2 - c_i^2\}$$

with appropriate constants whose zero locus in $\boldsymbol{R}^2$ consists of a line and m circles exterior to each other and lying on one side of the line. See Fig. 1. Moreover

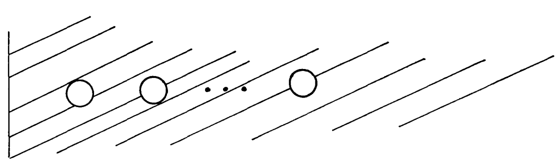

Fig. 1

we may assume that when $f = 0$, $\operatorname{grad} f \neq 0$ and that f is positive on the shaded part. Consider the affine surface

$$f(x, y) - z^{2m} = 0$$

and its projectivization $F(x, y, z, t) = 0$ obtained by homogenizing this equation. Intersection of the affine surface with a large enough ball is clearly homeomorphic to a surface of genus m with a disc removed so that its Euler characteristic is $1 - 2m$. It is easily checked that the projective surface is nonsingular, its intersection with the plane at infinity $t = 0$ is a projective line, and so the surface is a nonorientable surface with Euler characteristice $1 - 2m$. The surface

$$\sum = \{x \in S^3 \,|\, F(x) = 0\}$$

is clearly a double covering of this nonorientable projective surface. $\sum$ is clearly a surface of genus 2m defined by an odd degree polynomial. Thus by Theorem

8.2 we can do conformal surgery along $\sum$. This proves the theorem for surfaces with *even* genus.

We shall construct an appropriate surface for each odd genus. Consider the the projective surface P defined by the homogeneous equation

$$G(x, y, z, t) = \prod_{i=1}^{m} (x - a_i y)(y + a_i x) - t^{2m} + z^{2m} = 0 ,$$

where a_i are mutually distinct. It is easily checked that when $G = 0$, grad $G \neq 0$, so P is nonsingular. Consider its affine part where $t \neq 0$. Intersected with a sufficiently large ball it is seen that we get a sphere minus 2m discs; cf. Fig. 2

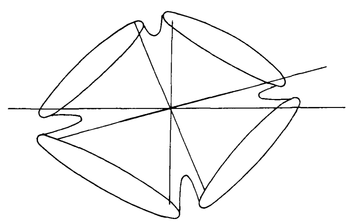

Fig. 2

for $m = 2$. Hence its Euler characteristic is 2 — 2m. The part of P on the plane at infinity $t = 0$ consists of m circles. G is of even degree, so G regarded as a function on the unit 3-sphere descends to a well defined function on the real projective 3-space. Since grad $G \neq 0$ on P, it follows that P is orientable and is indeed a surface of genus m. Let

$$\tilde{P} = \{x \in S^3 \mid G(x) = 0\} .$$

P contains the circle C

$$x - a_1 y = 0 , \qquad t = z ,$$

which does not disconnect P. $\tilde{P}$ is obtained as follows. Let P_1 be the surface with boundary obtained by cutting P along C. Take two copies of P_1 and glue the four boundary components pairwise to obtain $\tilde{P}$. This description shows that $\tilde{P}$ is a surface of genus 2m — 1. Let g_0 be the map $(x, y, z, t) \to (y, -x, t, z)$. It is easy to see that g_0 leaves $\tilde{P}$ invariant. Moreover, at a point $(0, 0, z, t)$ the induced map on the normal boundle is clearly -1, hence by analyticity it is identically -1. So we can do surgery along $\tilde{P}$. This finishes the proof of the theorem.

9. Concluding remarks

It is worth remarking that there is *no* analogue of Theorem 7.1 in the category of complex manifolds. More precisely, we have

Proposition 9.1. *Let M^{2n-1} be a compact connected submanifold of $\boldsymbol{C}^n$ with a* *tubular neighborhood N, $n \geq 2$. Then there does not exist a holomorphic map leaving N invariant, which maps the inner boundary of N into the outer boundary.*

Proof. Any holomorphic map of N into itself by Hartog's phenomenon extends to the inside of M, so by the maximum principle it cannot map the inner boundary into the outer boundary. q.e.d.

Thus, for instance, in general a connected sum of two complex manifolds does not carry a complex structure, e.g., it is not difficult to see that a connected sum of two Hopf surfaces each homeomorphic to $S^3 \times S^1$ does not carry even an almost complex structure.

Remark 9.2. In the examples of uniformization discussed in this paper the group under consideration was always a Lie group. While this is certainly an important case it is important to note that the uniformization condition (U) requires only a weak type of analyticity. We shall discuss one example here which was suggested by P. Shalen.

Proposition 9.3. *Let G be the group of homeomorphisms of S^n which are real-analytic on an open dense subset. Let M^n be an analytic manifold covered by $\boldsymbol{R}^n$. Then M^n is uniformizable on (S^n, G).*

Proof. Note first that clearly G satisfies the condition (U). The fundamental group acts on $\boldsymbol{R}^n$ as an analytic homeomorphism. This action clearly extends to S^n and defines elements of G. Hence the result follows. q.e.d.

The proposition is of special interest in the three-dimensional topology where a large class of compact manifolds are known to be covered by $\boldsymbol{R}^3$. It may be asked whether every compact 3-manifold is uniformizable on (S^3, G)? The depth of this question can be gauged when one notices that the affirmative answer would imply the Poincaré conjecture; cf. Theorem 2.1. To answer it negatively also seems difficult.

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