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ON THE THEORY OF NORMAL VARIATIONS

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1. Introduction

Let M^n be an n-dimensional submanifold of a Riemannian manifold M^m. An infinitesimal deformation of M^n in M^m along a normal vector field ξ is called a normal variation. In this paper we shall study some fundamental properties of nomal variations.

In § 3 we shall prove that the submanifold M^n is totally geodesic (respectively, totally umbilical or minimal) if and only if every normal variation of M^n is isometric (respectively, conformal or volume-preserving). In § 4 we shall prove that the normal variation given by ξ is affine if and only if the second fundamental tensor with respect to ξ is parallel. In § 5 we shall show that the normal variation given by ξ carries a totally geodesic (respectively, totally umbilical or minimal) submanifold into a totally geodesic (respectively, totally umbilical or minimal) submanifold when and only when ξ satisfies certain second order differential equations. In the last section, we shall study H-variations and H-stable submanifolds, and obtain a characterization of H-stable submanifolds with some applications; for example, we prove that an H-stable submanifold of a positively curved manifold has parallel mean curvature vector if and only if the submanifold is minimal.

2. Preliminaries, [1]

Let M^m be an m-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$, and denote by g_{ji}, Γ^h_{ji}, ∇_j, $K_{kji}{}^h$, K_{ji} and K the metric tensor, the Christoffel symbols formed with g_{ji}, the operator of covariant differentiation with respect to Γ^h_{ji}, the curvature tensor, the Ricci tensor and the scalar curvature of M^m respectively, where and in the sequel, the indices $h, i, j, k, \cdots$ run over the range $\{\bar{1}, \bar{2}, \cdots, \bar{m}\}$.

Let M^n be an n-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{V; y^a\}$, and denote by g_{cb}, Γ^a_{cb}, ∇_c, $K_{dcb}{}^a$, K_{cb} and K' the corresponding quantities of M^n, where and in the sequel the indices $a, b, c, d, \cdots$ run over the range $\{1, 2, \cdots, n\}$.

Suppose that M^n is isometrically immersed in M^m by the immersion $i: M^n \to M^m$, and identify $i(M^n)$ with M^n. Represent the immersion by

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$$x^h = x^h(y^a) , \tag{1}$$

and put

$$B_b{}^h = \partial_b x^h , \tag{2}$$

where $\partial_b = \partial/\partial y^b$. Then we have

$$g_{cb} = B_{cb}^{ji} g_{ji} , \tag{3}$$

where $B_{cb}^{ji} = B_c{}^j B_b{}^i$. We denote $m - n$ mutually orthogonal unit normals to M^n by $C_x{}^h$, where and in the sequel the indices x, y, z run over the range $\{n + 1, \cdots, m\}$. Then the metric tensor of the normal bundle of M^n is given by

$$g_{zy} = C_z{}^j C_y{}^i g_{ji} . \tag{4}$$

The equations of Gauss and those of Weingarten are respectively

$$\nabla_c B_b{}^h = h_{cb}{}^x C_x{}^h , \tag{5}$$

$$\nabla_c C_y{}^h = -h_c{}^a{}_y B_a{}^h , \tag{6}$$

where $\nabla_c B_b{}^h$ and $\nabla_c C_y{}^h$ denote the van der Waerden-Bortolotti covariant derivatives of $B_b{}^h$ and $C_y{}^h$ respectively along the submanifold M^n, that is,

$$\nabla_c B_b{}^h = \partial_c B_b{}^h + \Gamma_{ji}^h B_{cb}^{ji} - \Gamma_{cb}^a B_a{}^h , \tag{7}$$

$$\nabla_c C_y{}^h = \partial_c C_y{}^h + \Gamma_{ji}^h B_c{}^j C_y{}^i - \Gamma_{cy}^x C_x{}^h , \tag{8}$$

Γ_{cy}^x being the components of the connection induced in the normal bundle. We note that Γ_{cy}^x are skew-symmetric in x and y.

The mean curvature vector H^h is given by $H^h = (1/n) g^{cb} \nabla_c B_b{}^h$. If C^h is a unit normal vector parallel to H^h, then $H^h = \alpha C^h$ for some function α. α is called the mean curvature of M^n. If α vanishes identically, M^n is said to be minimal. If α is nowhere zero, and the second fundamental tensor in the direction of H^h is proportional to the metric tensor, then M^n is said to be *pseudo-umbilical.*

A normal vector field $C^h = \xi^x C_x{}^h$ is said to be *parallel* if $\nabla_c \xi^x = 0$ identically, and to be *concurrent* if there exists a function γ such that $\nabla_c C^h = \gamma B_c{}^h$, [6].

3. Isometric, conformal and volume-preserving normal variations

We consider a *normal variation* of M^n in M^m given by

$$\bar{x}^h = x^h(y^a) + \xi^h(y^a)\varepsilon , \tag{9}$$

where

(10) $$\xi^h = \xi^x C_x{}^h ,$$

and ε is an infinitesimal. From (9) we have

(11) $$\bar{B}_b{}^h = B_b{}^h + (\partial_b \xi^h)\varepsilon ,$$

where $\bar{B}_b{}^h = \partial_b \bar{x}^h$.

If we displace the vectors $B_b{}^h$ parallelly from the point (x^h) to $(\bar{x}^h)$, we obtain

(12) $$\tilde{B}_b{}^h = B_b{}^h - \Gamma^h_{ji} \xi^j B_b{}^i \varepsilon .$$

Thus putting

(13) $$\delta B_b{}^h = \bar{B}_b{}^h - \tilde{B}_b{}^h ,$$

we find

(14) $$\delta B_b{}^h = \nabla_b \xi^h \varepsilon ,$$

where

(15) $$\nabla_b \xi^h = \partial_b \xi^h + \Gamma^h_{ji} B_b{}^j \xi^i .$$

From (6), (10) and (15), it follows that

(16) $$\nabla_b \xi^h = -h_b{}^a{}_x \xi^x B_a{}^h + (\nabla_b \xi^x) C_x{}^h ,$$

where

(17) $$\nabla_b \xi^x = \partial_b \xi^x + \Gamma^x_{by} \xi^y .$$

Now a computation of the metric tensor $\bar{g}_{cb} = \bar{B}_c{}^j \bar{B}_b{}^i g_{ji}(\bar{x})$ of the deformed submanifold gives

$$\bar{g}_{cb} = g_{cb} - 2h_{cbx} \xi^x \varepsilon .$$

Thus putting $\delta g_{cb} = \bar{g}_{cb} - g_{cb}$, we have

(18) $$\delta g_{cb} = -2h_{cbx} \xi^x \varepsilon ,$$

from which we can easily obtain

(19) $$\delta g^{ba} = 2h^{ba}{}_x \xi^x \varepsilon ,$$

where $h^{ba}{}_x = g^{be} g^{ad} h_{edx}$. A normal variation (9) is said to be *isometric* (respectively, *conformal*) if $\delta g_{cb} = 0$ (respectively, $\delta g_{cb} = \alpha g_{cb}$ for some function α). From (18) we thus reach

Proposition 1. *A normal variation* (9) *is isometric if and only if* $h_{cbx}\xi^x = 0$, *that is, if and only if the submanifold is geodesic with respect to the direction of the normal variation.*

Proposition 2. *A normal variation* (9) *is conformal if and only if* $h_{cbx}\xi^x = \alpha g_{cb}$, α *being a certain function, that is, if and only if the submanifold is umbilical with respect to the direction of the normal variation.*

If we denote the determinant $|g_{cb}|$ by g, then the volume element of the submanifold M^n is given by

$$dV = \sqrt{g}\ dy^1 \wedge dy^2 \wedge \cdots dy^n . \tag{20}$$

Since we see from (18) that

$$\delta\sqrt{g} = -\sqrt{g}\, h_t{}^t{}_x\xi^x\varepsilon ,$$

we have

$$\delta dV = -h_t{}^t{}_x\xi^x dV\varepsilon . \tag{21}$$

Hence

Proposition 3. *A normal variation* (9) *is volume-preserving if and only if* $h_t{}^t{}_x\xi^x = 0$, *that is, if and only if the submanifold is minimal with respevt to the direction of the normal variation.*

From Propositions 1, 2 and 3 we obtain the following theorems.

Theorem 1. *A submanifold is totally geodesic if and only if every normal variation of the submanifold is isometric.*

Theorem 2. *A submanifold is totally umbilical if and only if every normal variation of the submanifold is conformal.*

Theorem 3. *A submanifold is minimal if and only if every normal variation of the submanifold is volume-preserving.*

4. Affine normal variations

We introduce the notation

$$B^a{}_i = g^{ab}B_b{}^j g_{ji} , \qquad C^x{}_i = g^{xy}C_y{}^j g_{ji} . \tag{22}$$

Then the relation between Γ^a_{cb} and Γ^h_{ji} can be written as

$$\Gamma^a_{cb} = (\partial_c B_b{}^h + \Gamma^h_{ji}B^{ji}_{cb})B^a{}_h , \tag{23}$$

and that between Γ^x_{cy} and Γ^h_{ji} as

$$\Gamma^x_{cy} = (\partial_c C_y{}^h + \Gamma^h_{ji}B_c{}^j C_y{}^i)C^x{}_h . \tag{24}$$

We denote by $\bar{C}_y{}^h$, $\bar{B}^a{}_i$ and $\bar{C}^x{}_i$ the values at the point $(\bar{x}^h)$ of $C_y{}^h$, $B^a{}_i$ and $C^x{}_i$, and by $\tilde{C}_y{}^h$, $\tilde{B}^a{}_i$ and $\tilde{C}^x{}_i$ the components of the vectors obtained from $C_y{}^h$, $B^a{}_i$ and $C^x{}_i$ by replacing them parallelly from the point (x^h) to $(\bar{x}^h)$, respectively. We then have

$$\begin{aligned} \tilde{C}_y{}^h &= C_y{}^h - \Gamma^h_{ji}\xi^j C_y{}^i\varepsilon , \\ \tilde{B}^a{}_i &= B^a{}_i + \Gamma^h_{ji}\xi^j B^a{}_h\varepsilon , \\ \tilde{C}^x{}_i &= C^x{}_i + \Gamma^h_{ji}\xi^j C^x{}_h\varepsilon . \end{aligned} \tag{25}$$

Put

$$\delta C_y{}^h = \bar{C}_y{}^h - \tilde{C}_y{}^h , \quad \delta B^a{}_i = \bar{B}^a{}_i - \tilde{B}^a{}_i , \quad \delta C^x{}_i = \bar{C}^x{}_i - \tilde{C}^x{}_i . \tag{26}$$

By assuming that $\delta C_y{}^h$ is given by

$$\delta C_y{}^h = \eta_y{}^h\varepsilon = (\eta_y{}^a B_a{}^h + \eta_y{}^x C_x{}^h)\varepsilon , \tag{27}$$

applying the operator δ to $B_b{}^j C_y{}^i g_{ji} = 0$, and using $\delta g_{ji} = 0$, we obtain

$$(\nabla_b\xi^j)C_y{}^i g_{ji} + B_b{}^j(\eta_y{}^a B_a{}^i + \eta_y{}^x C_x{}^i)g_{ji} = 0 .$$

From the above equation it follows that $\nabla_b\xi_y + \eta_{yb} = 0$, where $\xi_y = \xi^z g_{zy}$ and $\eta_{yb} = \eta_y{}^c g_{cb}$, and therefore that

$$\eta_y{}^a = -\nabla^a\xi_y , \tag{28}$$

where $\nabla^a = g^{ae}\nabla_e$.

Applying δ to $B_b{}^h B^a{}_h = \delta^a{}_b$ and $C_y{}^h B^a{}_h = 0$ gives respectively

$$(\nabla_b\xi^h)B^a{}_h\varepsilon + B_b{}^h(\delta B^a{}_h) = 0 , \qquad \eta_y{}^a\varepsilon + C_y{}^h(\delta B^a{}_h) = 0 ,$$

from which we have, taking account of (16) and (28),

$$\delta B^a{}_i = [h_c{}^a{}_x\xi^x B^c{}_i + (\nabla^a\xi_x)C^x{}_i]\varepsilon . \tag{29}$$

Applying δ to $B_b{}^h C^x{}_h = 0$ and $C_y{}^h C^x{}_h = \delta^x{}_y$ gives respectively

$$(\nabla_b\xi^h)C^x{}_h\varepsilon + B_b{}^h(\delta C^x{}_h) = 0 , \qquad \eta_y{}^z C_z{}^h C^x{}_h + C_y{}^h(\delta C^x{}_h) = 0 ,$$

from which we have, taking account of (16),

$$\delta C^x{}_i = -[(\nabla_c\xi^x)B^c{}_i + \eta_y{}^x C^y{}_i]\varepsilon . \tag{30}$$

Thus by (12), (13), (14), (25), (26), (27), (29) and (30) we obtain

$$\begin{aligned} \bar{B}_b{}^h &= B_b{}^h - \Gamma^h_{ji}\xi^j B_b{}^i\varepsilon + (\nabla_b\xi^h)\varepsilon , \\ \bar{C}_y{}^h &= C_y{}^h - \Gamma^h_{ji}\xi^j C_y{}^i\varepsilon + \eta_y{}^h\varepsilon , \\ \bar{B}^a{}_i &= B^a{}_i + \Gamma^h_{ji}\xi^j B^a{}_h\varepsilon + [h_c{}^a{}_x\xi^x B^c{}_i + (\nabla^a\xi_x)C^x{}_i]\varepsilon . \\ \bar{C}^x{}_i &= C^x{}_i + \Gamma^h_{ji}\xi^j C^x{}_h\varepsilon - [(\nabla_c\xi^x)B^c{}_i + \eta_y{}^x C^y{}_i]\varepsilon . \end{aligned}$$

Put

$$\bar{\Gamma}^a_{cb} = (\partial_c \bar{B}_b{}^h + \Gamma^h_{ji}(\bar{x})\bar{B}_c{}^j \bar{B}_b{}^i)\bar{B}^a{}_h \,, \tag{31}$$

$$\delta\Gamma^a_{cb} = \bar{\Gamma}^a_{cb} - \Gamma^a_{cb} \,. \tag{32}$$

Then a straightforward computation yields

$$\delta\Gamma^a_{cb} = [(\nabla_c\nabla_b\xi^h + K_{kji}{}^h\xi^k B^{ji}_{cb})B^a{}_h + h_{ji}{}^x\nabla^h\xi_x]\varepsilon \,, \tag{33}$$

from which together with $\xi^h = \xi^x C_x{}^h$ and equation of Codazzi it follows that

$$\delta\Gamma^a_{cb} = -[\nabla_c(h_{bex}\xi^x) + \nabla_b(h_{cex}\xi^x) - \nabla_e(h_{cbx}\xi^x)]g^{ea}\varepsilon \,. \tag{34}$$

Since we can easily see from (34) that $\delta\Gamma^a_{cb} = 0$ and $\nabla_c(h_{bex}\xi^x) = 0$ are equivalent, we have

Theorem 4. *The normal variation* (9) *is affine if and only if* $h_{cbx}\xi^x$ *is parallel.*

5. Normal variations which carry umbilical submanifolds to umbilical submanifolds

By putting

$$\bar{\Gamma}^x_{cy} = (\partial_c \bar{C}_y{}^h + \Gamma^h_{ji}(\bar{x})\bar{B}_c{}^j \bar{C}_y{}^i)\bar{C}^x{}_h \,, \tag{35}$$

$$\delta\Gamma^x_{cy} = \bar{\Gamma}^x_{cy} - \Gamma^x_{cy} \,, \tag{36}$$

we obtain

$$\delta\Gamma^x_{cy} = [(\nabla_c\eta_y{}^h + K_{kji}{}^h\xi^k B_c{}^j C_y{}^i)C^x{}_h + h_c{}^a{}_y\nabla_a\xi^x]\varepsilon \,. \tag{37}$$

Suppose that v^h is a vector field of M^m defined intrinsically along the submanifold M^n. When we displace the submanifold by $\bar{x}^h = x^h + \xi^h\varepsilon$ in the direction ξ^h normal to it, we obtain a vector field $\bar{v}^h$ which is defined also intrinsically along the deformed submanifold. If we displace v^h parallelly from the point (x^h) to $(\bar{x}^h)$, we obtain $\tilde{v}^h = v^h - \Gamma^h_{ji}\xi^j v^i\varepsilon$ and hence forming $\delta v^h = \bar{v}^h - \tilde{v}^h$, so that

$$\delta v^h = \bar{v}^h - v^h + \Gamma^h_{ji}\xi^j v^i\varepsilon \,. \tag{38}$$

Similarly, we have

$$\delta\nabla_c v^h = \bar{\nabla}_c\bar{v}^h - \nabla_c v^h + \Gamma^h_{ji}\xi^j\nabla_c v^i\varepsilon \,,$$

that is,

$$\begin{aligned} \delta\nabla_c v^h = {} & \nabla_c\bar{v}^h - \nabla_c v^h + (\partial_k\Gamma^h_{ji} + \Gamma^h_{kt}\Gamma^t_{ji})\xi^k B_c{}^j v^i\varepsilon \\ & + (\Gamma^h_{ji}\partial_c\xi^j v^i + \Gamma^h_{ji}\xi_j\partial_c v^i)\varepsilon \,. \end{aligned} \tag{39}$$

On the other hand, from (38) it follows that

$$\begin{aligned}\nabla_c\delta v^h = \nabla_c\bar{v}^h - \nabla_c v^h + (\partial_j\Gamma^h_{ki} + \Gamma^h_{jt}\Gamma^t_{ki})\xi^k B_c{}^j v^i\varepsilon \\ + (\Gamma^h_{ji}\partial_c\xi^j v^i + \Gamma^h_{ji}\xi^j\partial_c v^i)\varepsilon\ .\end{aligned} \tag{40}$$

Thus by (39) and (40) we find

$$\delta\nabla_c v^h - \nabla_c\delta v^h = K_{kji}{}^h\xi^k B_c{}^j v^i\varepsilon\ . \tag{41}$$

Similarly, for a covector w_i we have

$$\delta\nabla_c w_i - \nabla_c\delta w_i = -K_{kji}{}^h\xi^k B_c{}^j w_h\varepsilon\ . \tag{42}$$

For a tensor field carrying three kinds of indices, say, $T_{by}{}^h$, we have

$$\delta\nabla_c T_{by}{}^h - \nabla_c\delta T_{by}{}^h = K_{kji}{}^h\xi^k B_c{}^j T_{by}{}^i - (\delta\Gamma^a_{cb})T_{ay}{}^h - (\delta\Gamma^x_{cy})T_{bx}{}^h\ . \tag{43}$$

Applying (43) to $B_b{}^h$ gives

$$\delta\nabla_c B_b{}^h - \nabla_c\delta B_b{}^h = K_{kji}{}^h\xi^k B_c{}^j B_b{}^i\varepsilon - B_a{}^h\delta\Gamma^a_{cb}\ ,$$
$$\delta(h_{cb}{}^x C_x{}^h) = (\nabla_c\nabla_b\xi^h + K_{kji}{}^h\xi^k B_c{}^j B_b{}^i)\varepsilon - B_a{}^h\delta\Gamma^a_{cb}\ ,$$

from which follows

$$\delta h_{cb}{}^x = [h_{cb}{}^z\eta_z{}^x + (\nabla_c\nabla_b\xi^h + K_{kji}{}^h\xi^k B_c{}^j B_b{}^i)C^x{}_h]\varepsilon\ . \tag{44}$$

Substituting $\xi^h = \xi^x C_x{}^h$ in (44) we find

$$\delta h_{cb}{}^x = [h_{cb}{}^z\eta_z{}^x - h_{ce}{}^x h_b{}^e{}_y\xi^y + \nabla_c\nabla_b\xi^x + K_{kji}{}^h C_y{}^k B_c{}^j B_b{}^i C^x{}_h\xi^y]\varepsilon\ . \tag{45}$$

Thus we obtain the following theorems.

Theorem 5. *The normal variation given by $\xi^x C_x{}^h$ carries a totally geodesic submanifold into a totally geodesic submanifold if and only if*

$$\nabla_c\nabla_b\xi^x + K_{kji}{}^h C_y{}^k B_c{}^j B_b{}^i C^x{}_h\xi^y = 0\ . \tag{46}$$

Theorem 6. *The normal variation given by $\xi^x C_x{}^h$ carries a totally umbilical submanifold into a totally umbilical submanifold if and only if*

$$\nabla_c\nabla_b\xi^x + K_{kji}{}^h C_y{}^k B_c{}^j B_b{}^i C^x{}_h\xi^y = g_{cb}\alpha^x\ , \tag{47}$$

α^x being certain functions.

Theorem 7. *The normal variation given by $\xi^x C_x{}^h$ carries a minimal submanifold into a minimal submanifold if and only if*

$$g^{cb}\nabla_c\nabla_b\xi^x + K_{kji}{}^h C_y{}^k B^{ji} C^x{}_h\xi^y - h^t{}_e{}^x h_t{}^e{}_y\xi^y = 0\ , \tag{48}$$

where $B^{ji} = g^{cb}B^{ji}_{cb}$. In particular, the normal variation given by $\xi^x C_x{}^h$ carries *a totally geodesic submanifold into a minimal submanifold if and only if* $g^{cb}\nabla_c\nabla_b\xi^x + K_{kji}{}^h C_y{}^k B^{ji} C^x{}_h \xi^y = 0$.

6. *H*-variations

The *mean curvature vector* of M^n in M^m is given by

$$H^h = \frac{1}{n} g^{cb}\nabla_c B_b{}^h .$$

For the normal variation (9), if the normal vector field $\xi^x C_x{}^h$ is parallel to the mean curvature vector along M^n, then the normal variation (9) is called an H-variation. In this section, we shall choose the first unit normal vector $C_{n+1}{}^h$ in the direction of the mean curvature vector. Thus

$$\frac{1}{n} g^{cb}\nabla_c B_b{}^h = \alpha C_{n+1}{}^h , \tag{49}$$

where α is the *mean curvature* of M^n. From (5) it follows that

$$g^{cb} h_{cb}{}^x = 0 , \qquad (x = n+2, \cdots, m) . \tag{50}$$

We consider an H-variation and hence

$$\xi^{n+1} = \phi , \qquad \xi^{n+2} = \cdots = \xi^m = 0 , \tag{51}$$

ϕ being the length of the variation vector.

Substituting (51) in (45) gives

$$\begin{aligned} \delta h_{cb}{}^{n+1} = [&h_{cb}{}^x \eta_x{}^{n+1} - \phi h_{ce}{}^{n+1} h_b{}^e{}_{n+1} + \phi \Gamma_c{}^{n+1}{}_y \Gamma_b{}^y{}_{n+1} \\ &+ \nabla_c \nabla_b \phi + K_{kjih} C_{n+1}{}^k B^{ji}_{cb} C_{n+1}{}^h]\varepsilon , \end{aligned} \tag{52}$$

from which, transvecting with g^{cb} and using (15) and (19), we find

$$n\delta\alpha = \Delta\phi - \phi l^2 + \phi h_{cb} h^{cb} + \phi K_{kjih} C^k B^{ji} C^h , \tag{53}$$

where α is the mean curvature, and

$$l^2 = g^{cb}(\Gamma_c{}^{n+1}{}_y \Gamma_b{}^{n+1}{}_y) , \quad h_{cb} = h_{cb}{}^{n+1} , \quad C^h = C_{n+1}{}^h , \quad B^{ji} = B^{ji}_{cb} g^{cb} .$$

For the normal variation of the integral $\int_M \alpha^c \alpha V$, c being any nonegative number, we have

$$\delta \int_M \alpha^c dV = \int_M c\alpha^{c-1}\delta\alpha dV + \int_M \alpha^c \delta dV ,$$

and therefore, in consequence of (21) and (53),

$$\begin{aligned}&\delta\int_M \alpha^c dV \\ (54)\qquad &= \int_M \left[\frac{c}{n}\alpha^{c-1}(\Delta\phi - \phi l^2 + \phi h_{cb}h^{cb} + \phi K_{kjih}C^k B^{ji} C^h) - n\alpha^{c+1}\phi\right] dV\,.\end{aligned}$$

We assume that the normal variation leaves the boundary ∂M of M strongly fixed in the sense that both ϕ and its gradient vanish on ∂M. Then

$$\int_M (\alpha^{c-1}\Delta\phi)dV = \int_M \phi(\Delta\alpha^{c-1})dV\,,$$

which together with (54) implies that

$$\begin{aligned}\delta\int_M \alpha^c dV = \int_M \frac{c}{n}\phi\Big[\Delta\alpha^{c-1} - \alpha^{c-1}l^2 - \frac{n^2}{c}\alpha^{c+1} \\ + \alpha^{c-1}h_{cb}h^{cb} + \alpha^{c-1}K_{kjih}C^k B^{ji} C^h\Big] dV\,.\end{aligned}$$

From this we see that $\delta\int_M \alpha^c dV = 0$ for all H-variations which leave the boundary strongly fixed if and only if

$$\Delta\alpha^{c-1} = \alpha^{c-1}\left(l^2 + \frac{n^2}{c}\alpha^2 - h_{cb}h^{cb} - K_{kjih}C^k B^{ji} C^h\right)\,.$$

We say that a submanifold is *H-stable* if $\delta\int_M \alpha^n dV = 0$ for all H-variations which leave the boundary strongly fixed. From the above equation, we have

Theorem 8. *Let M^n be an n-dimensional submanifold of an m-dimensional Riemannian manifold M^m. Then M^n is H-stable if and only if*

$$(55)\qquad \Delta\alpha^{n-1} = \alpha^{n-1}(l^2 + n\alpha^2 - h_{cb}h^{cb} - K_{kjih}C^k B^{ji} C^h)\,.$$

We now assume that M^n is H-stable and has parallel mean curvature vector. Then $\nabla^c(\alpha C_{n+1}{}^h) = 0$, and therefore α is constant. If $\alpha \neq 0$, then $l^2 = 0$. Substituting this in (55) gives

$$(56)\qquad \frac{1}{n}\sum_{b<a}(\lambda_b - \lambda_a)^2 + K_{kjih}C^k B^{ji} C^h = 0\,,$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues of $h_c{}^a$.

Thus assuming that $K_{kjih}C^k B^{ji} C^h \geq 0$, we have $\lambda_1 = \lambda_2 = \cdots = \lambda_n$, that is, M^n is pseudo-umbilical, and $K_{kjih}C^k B^{ji} C^h = 0$, from which we find

$$(59)\qquad \nabla_c C^h = -\frac{1}{n}\alpha B_c{}^h\,,$$

that is, the mean curvature vector is *concurrent* along M^n. Conversely, if the mean curvature vector is concurrent, then it is parallel, M^n is pseudo-umbilical, and α is constant. Thus M^n is H-stable if and only if $K_{kjih}C^kB^{ji}C^h = 0$. Consequently, we have the following propositions.

Proposition 4. *Let M^n be an H-stable submanifold of M^n with $K_{kjih}C^kB^{ji}C^h \geq 0$. Then M^n has parallel mean curvature vector if and only if either M^n is minimal or $K_{kjih}C^kB^{ji}C^h = 0$ and the mean curvature vector is concurrent.*

Proposition 5. *Let M^n be a submanifold of M^m with concurrent mean curvature vector. Then M^n is H-stable if and only if $K_{kjih}C^kB^{ji}C^h = 0$.*

Assume that $K_{kjih}C^kB^{ji}C^h \leq 0$ and M^n is pseudo-umbilical. If M is compact and H-stable, then $\Delta\alpha^{n-1}$ does not change its sign. Hence, from Hopf's lemma, $\Delta\alpha^{n-1} = 0$, $l^2 = 0$, and $K_{kjih}C^kB^{ji}C^k = 0$, so that the mean curvature vector is parallel and therefore concurrent. Consequently, we have

Proposition 6. *Let M^n be a compact H-stable submanifold of M^m with $K_{kjih}C^kB^{ji}C^h \leq 0$. If M^n is pseudo-umbilical, then the mean curvature vector is concurrent and $K_{kjih}C^kB^{ji}C^h = 0$.*

In particular, Propositions 4 and 6 give immediately the following.

Theorem 9. *Let M^n be an H-stable submanifold of a positively curved manifold M^m. Then M^n has parallel mean curvature vector if and only if M^n is minimal.*

Theorem 10. *Let M^n be a compact pseudo-umbilical submanifold of a negatively curved manifold M^m. Then M^n is not H-stable.*

Theorem 11 (*Chen and Houh* [3]). *Let M^n be an H-stable submanifold of a euclidean space E^m. Then M^n has parallel mean curvature vector if and only if either M^n is minimal in E^m or M^n is a minimal submanifold of a hypersphere of E^m.*

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