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CURVATURE AND SPECTRUM OF COMPACT RIEMANNIAN MANIFOLDS

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0. Introduction

Let M be a compact and orientable Riemannian manifold of class C^∞ with positive definite metric g and $\dim M = n$. The eigenvalues λ, corresponding to the eigenvalue problem $\Delta\omega + \lambda\omega = 0$ for alternating, not necessarily homogeneous, differential forms ω which are regular everywhere on M, form a monotonically increasing (in a strict sense) sequence $\operatorname{Spec}(M) = \{\lambda_i\}_{i\geq 0}$. Let $\mathfrak{V}_i$ be the finite-dimensional eigenspace belonging to λ_i. The projection operator $P^{(p)}$ which gives the homogeneous part of degree p: $\omega^{(p)} = P^{(p)}\omega$ of any differential form ω maps $\mathfrak{V}_i$ onto $\mathfrak{V}_i^{(p)} = P^{(p)}\mathfrak{V}_i$. In $\mathfrak{V}_i^{(p)}$ we choose an orthonormal basis $\varphi_{il}^{(p)}$, $1 \leq l \leq \dim \mathfrak{V}_i^{(p)}$, and introduce

$$\Phi_i^{(p)}(x) = \sum_l \langle \varphi_{il}^{(p)}, \varphi_{il}^{(p)} \rangle (x) , \qquad \forall x \in M , \tag{0.1}$$

or $\Phi_i^{(p)}(x) = 0$ in case $\mathfrak{V}_i^{(p)} = \{0\}$, where $\langle \cdot, \cdot \rangle$ is defined by (1.6), (1.5). Now the following asymptotic expansion holds (for this cf. [19], [18], [3], [5], [10], [9]):

$$\sum_{i=0}^{\infty} e^{-\lambda_i t}\Phi_i^{(p)}(x) \underset{t\to+0}{\sim} (4\pi t)^{-n/2} \sum_{k=0}^{\infty} (\operatorname{Sp} V_k^{(p)})(x)(2t)^k , \tag{0.2}$$

where the $V_k^{(p)}(x, \xi)$ form a system of double differential forms[1] of degree p, defined in the neighborhood of the diagonal of $M \times M$ by recursion formulas (see (2.1), (2.2)). One gets the expansion (0.2) by applying the parametric method to Green's form for the heat equation of the manifold M. Integrating (0.2) over M yields

$$\sum_{i=0}^{\infty} e^{-\lambda_i t} \dim \mathfrak{V}_i^{(p)} \underset{t\to+0}{\sim} (4\pi t)^{-n/2} \sum_{k=0}^{\infty} (2t)^k \int_M (\operatorname{Sp} V_k^{(p)})(x) dv(x) . \tag{0.3}$$

According to the well-known theorems of Hodge [16] and De Rham [6] $\dim \mathfrak{V}_0^{(p)}$

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[1] In the normalization adopted here the $V_k^{(p)}(x, \xi)$ coincide with the coefficients of the Riesz kernel forms in [13], [15], [24], [25]. For the Riesz kernel forms and their coefficients further cf. [8], [4]. Note that only double differential forms of "bi-degree" (p, p) appear.

$= R_p$ coincides with the p-th Betti number, while for $i \geq 1$ a direct decomposition $\mathfrak{B}_i^{(p)} = \mathfrak{E}_i^{(p)} + \mathfrak{K}_i^{(p)}$ holds, where $\mathfrak{E}_i^{(p)}$ consists of exact forms and $\mathfrak{K}_i^{(p)}$ consists of coexact forms; $\mathfrak{E}_i^{(0)} = \{0\}$, $\mathfrak{K}_i^{(n)} = \{0\}$. The "théorème de télescopage" (this name was given to the theorem in [3]) by McKean Jr. and Singer [18] states $\dim \mathfrak{K}_i^{(p)} = \dim \mathfrak{E}_i^{(p+1)}$ for $p = 0, 1, \cdots, n-1$. Introducing these facts into (0.3) and taking the alternating sum with respect to p one gets for even n (cf. [18], [3]):

$$(0.4) \quad \chi(M) = \sum_{p=0}^{n} (-1)^p R_p = (2\pi)^{-n/2} \sum_{p=0}^{n} (-1)^p \int_M (\mathrm{Sp}\, V_{n/2}^{(p)})(x) dv(x) \,,$$

$$(0.5) \qquad \sum_{p=0}^{n} (-1)^p \int_M (\mathrm{Sp}\, V_k^{(p)})(x) dv(x) = 0 \,, \qquad \forall k \neq \frac{n}{2} \,,$$

where $\chi(M)$ denotes the Euler-Poincaré characteristic of M. For odd n one has $\chi(M) = 0$, and (0.5) is valid for all $k \geq 0$. Now it is highly desirable to find relations between the $V_k^{(p)}(x, \xi)$—or more exactly their traces—on the one hand and the curvature of M on the other hand. In this direction many partial results are known (cf. H. P. McKean Jr. and I. M. Singer [18], M. Berger [2], E. Combet [5], T. Sakai [22], V. K. Patodi [20], H. Donelly [7], where they considered small values of k and obtained some conclusions on isospectral manifolds). V. K. Patodi [21] has shown that for even n and $k < \frac{1}{2}n$ the integrand in (0.5) vanishes while the integrand in (0.4) just equals Chern's invariant occuring in the generalization of the Gauss-Bonnet integral theorem.[2] Another proof of these facts was given by P. B. Gilkey [11]; he studied the general nature of the invariants considered here and gave an interesting characterization of the Pontrjagin and the Chern forms. Generalizations of Gilkey's results which are closely connected with the index-theory of elliptic differential operators can be found in the papers: P. B. Gilkey [12] and M. Atiyah, R. Bott, V. K. Patodi [1].

In the present paper new results concerning the above mentioned problems are presented. Our method is quite different from those used by the authors listed above.

By a coincidence form we mean a double differential form the coefficients of which are defined only on the diagonal of $M \times M$. Every double form $U^{(p)}(x, \xi)$ defined in a neighborhood of the diagonal gives a coincidence form $U^{(p)}(x, x)$. Using the metric, the Ricci tensor and the curvature tensor (our convention for the sign of the curvature tensor coincides with that of J. A. Schouten [23]) we define some coincidence forms in the following way:[3]

[2] This is just the content of a conjecture of H. P. McKean, Jr. and I. M. Singer [18]; see also the paragraph after the corollary to Theorem II. A detailed proof of the Gauss-Bonnet integral formula can be found, e.g., in the book of R. Sulanke and P. Wintgen [26], where there is also an extensive list of literature.

[3] The multiplication of double differential forms is the exterior multiplication with respect to both groups of the variables, which is commutative and denoted by $\wedge$.

$$G^{(0)}(x) = 1\ , \quad G^{(1)}(x) = g_{i\alpha}dx^i d\xi^\alpha\ , \quad G^{(p)}(x) = \frac{1}{p!}[G^{(1)}(x)]^p\ ; \tag{0.6}$$
$$\forall x \in M\ ,$$

$$\Psi^{(1)}(x) = \tfrac{1}{2}R_{i\alpha}(x)dx^i d\xi^\alpha\ , \qquad \forall x \in M\ ,$$

$$\begin{aligned} \Psi^{(2)}(x) &= \tfrac{1}{2}R_{ij\alpha\beta}(x)dx^i \wedge dx^j d\xi^\alpha \wedge d\xi^\beta\ , \\ \Psi^{(3)}(x) &= \tfrac{1}{2}R^{h}_{\cdot\alpha ij}R_{hk\beta\gamma}(x)dx^i \wedge dx^j \wedge dx^k d\xi^\alpha \wedge d\xi^\beta \wedge d\xi^\gamma\ . \end{aligned} \tag{0.7}$$

By means of these forms we further construct, for $k \geq 1$,

$$\begin{aligned} Z^{(2k)} &= \frac{(-1)^k}{2^k k!}[\Psi^{(2)}]^k\ , \qquad Z^{(1)} = -\Psi^{(1)}\ , \\ Z^{(2k-1)} &= \frac{(-1)^k}{2^{k-1}(k-1)!}\left[\Psi^{(1)} \wedge \Psi^{(2)} - \frac{k-1}{2}\Psi^{(3)}\right] \wedge [\Psi^{(2)}]^{k-2}\ . \end{aligned} \tag{0.8}$$

Thus the coincidence forms Ψ, Z are defined explicitly in terms of the curvature tensor of (M, g).

Now we can state our theorems.

Theorem I. *For $k \geq 0$ and $0 \leq l \leq n-1$ one has*

$$\begin{aligned} &\sum_{p=0}^{n-l}(-1)^p V_k^{(p)}(x, x) \wedge G^{(n-l-p)} \\ &= \begin{cases} 0\ , & \text{for } k < \left[\dfrac{n-l+1}{2}\right]\ , \\ (-1)^{n-l}Z^{(n-l)}\ , & \text{for } k = \left[\dfrac{n-l+1}{2}\right]\ . \end{cases} \end{aligned} \tag{0.9}$$

By taking the trace Theorem I becomes

Theorem II. *For $k \geq 0$ and $0 \leq l \leq n-1$ one has*

$$\begin{aligned} &\sum_{p=0}^{n-l}(-1)^p \binom{n-p}{l} (\operatorname{Sp} V_k^{(p)})(x) \\ &= \begin{cases} 0\ , & \text{for } k < \left[\dfrac{n-l+1}{2}\right]\ , \\ (-1)^{n-l}(\operatorname{Sp} Z^{(n-l)})(x)\ , & \text{for } k = \left[\dfrac{n-l+1}{2}\right]\ . \end{cases} \end{aligned} \tag{0.10}$$

The traces of the $Z^{(p)}$ which appear in Theorem II are given more explicitly by

Corollary to Theorem II. *For $k \geq 1$ one has*

$$
\begin{aligned}
\operatorname{Sp} Z^{(2k)} &= -\frac{1}{k}\operatorname{Sp} Z^{(2k-1)} \\
&= \frac{(-1)^k(2k)!}{2^{2k}k!} R^{[i_1 i_2}{}_{\cdot\,\cdot\,[i_1 i_2} R^{i_3 i_4}{}_{\cdot\,\cdot\,i_3 i_4} \cdots R^{i_{2k-1} i_{2k}]}{}_{\cdot\,\cdot\,i_{2k-1} i_{2k}]} \,.
\end{aligned} \tag{0.11}
$$

For $l = 0$ and even n Theorem II just gives the above mentioned result of V. K. Patodi [21] and P. B. Gilkey [11], i.e., the conjecture of H. P. McKean Jr. and I. M. Singer [18]. (After seeing our manuscript Professor P. B. Gilkey kindly communicated to us that he was able to prove by means of his method our formulas (0.10) and also the upper bound for the number of invariants following from Theorem V.) For $l = 0$ and odd n Theorem II and the following two theorems give no new information if one takes the duality properties into consideration.

Using Theorem II we obtain the following asymptotic formula.

Theorem III. *For $0 \leq l \leq n-1$ one has*

$$
\begin{aligned}
&\sum_{i=0}^{\infty}\sum_{p=0}^{n-l}(-1)^p\binom{n-p}{l}e^{-\lambda_i t}\Phi_i^{(p)}(x) \\
&= t^{[(n-l+1)/2]-n/2}\{(-1)^{n-l}2^{[(n-l+1)/2]-n}\pi^{-n/2}(\operatorname{Sp} Z^{(n-l)})(x) + O(t)\} \,.
\end{aligned} \tag{0.12}
$$

Integrating over M and applying the "télescopage" theorem at last one gets

Theorem IV. *For $0 \leq l \leq n-1$ one has*

$$
\begin{aligned}
&\sum_{p=0}^{n-l}(-1)^p\binom{n-p}{l}R_p + \sum_{i=1}^{\infty}\sum_{p=0}^{n-l}(-1)^p\binom{n-p-1}{l-1}e^{-\lambda_i t}\dim \mathfrak{R}_i^{(p)} \\
&= t^{[(n-l+1)/2]-n/2}\Big\{(-1)^{n-l}2^{[(n-l+1)/2]-n}\pi^{-n/2} \\
&\qquad \times \int_M (\operatorname{Sp} Z^{(n-l)})(x)dv(x) + O(t)\Big\} \,.
\end{aligned} \tag{0.13}
$$

The main tool for the proof of these theorems, to be given in §§ 1–2, is a rearrangement of the recursion system defining the double differential forms $V_k^{(p)}$, called "expansion to transport forms". This method has already been applied in other investigations connected with Huygens' principle (cf. R. Schimming [24], [25], P. Günther [13]). An analogous expansion applies also to other geometrical objects forming a graded algebra and allowing the definition of a Laplacian Δ, e.g., tensors or spinors.

The expansion to transport forms possesses a dual formulation to be treated in § 3, and enables us to draw some more conclusions.

As a matter of fact the coincidence forms $V_k^{(p)}(x, x)$ are universal in the sense that their components are polynomials of the components g^{ij}, g_{ij}, R_{ijhk}, $\nabla_{i_1} \cdots \nabla_{i_m} R_{ijhk}$ of the metric tensor and its inverse, the curvature tensor, and the covariant derivatives of the latter, the polynomial coefficients being independent of $n = \dim M$. The traces $\operatorname{Sp} V_k^{(p)}$ then are scalar invariants and polynomials in g^{ij}, $\nabla_{i_1} \cdots \nabla_{i_m} R_{ijhk}$ with coefficients depending on n. With regard to the duality this first gives $1 + [n/2]$ invariants for fixed k. The following theorem proven in § 3 reduces this number to $1 + \min\{k, [n/2]\}$ and at the same time explains the way in which the $\operatorname{Sp} V_k^{(p)}$ depend on n and p.

Theorem V. *There are universal scalar invariants σ_k^r for $0 \leq r \leq k$ with integers r and k, i.e., polynomials in the contravariant fundamental tensor and the covariant derivatives of the curvature tensor with coefficients independent of the dimension n, such that*

$$\operatorname{Sp} V_k^{(p)} = \sum_{r=0}^{\min\{k,p,n-p\}} \binom{n-2r}{p-r} \sigma_k^r , \qquad \forall k \geq 0, \quad \forall p \text{ with } 0 \leq p \leq n . \tag{0.14}$$

Corollary to Theorem V. *For $k \geq 1$ one has*

$$\sigma_k^k = (-1)^k \operatorname{Sp} Z^{(2k)} . \tag{0.15}$$

For small values of k the invariants σ_k^r can be given explicitly as follows:

$$\begin{aligned}
&\sigma_0^0 = 1 , \\
&\sigma_1^0 = \operatorname{Sp} W_1^{(0)} = \frac{R}{12} , \qquad \sigma_1^1 = \operatorname{Sp} W_1^{(1)} = -\frac{R}{2} , \\
&\sigma_2^0 = \operatorname{Sp} W_2^{(0)} \\
&\quad = \frac{1}{4}\left\{\frac{1}{30}\nabla^l \nabla_l R + \frac{1}{180} R^{ijkm} R_{ijkm} - \frac{1}{180} R^{ij} R_{ij} + \frac{1}{72} R^2\right\} , \\
&\sigma_2^1 = \operatorname{Sp} W_2^{(1)} \\
&\quad = \frac{1}{4}\left\{-\frac{1}{6}\nabla^l \nabla_l R - \frac{1}{12} R^{ijkm} R_{ijkm} + \frac{1}{2} R^{ij} R_{ij} - \frac{1}{6} R^2\right\} , \\
&\sigma_2^2 = \operatorname{Sp} Z^{(4)} = \frac{1}{8}\{R^{ijlm} R_{ijlm} - 4R^{ij} R_{ij} + R^2\} .
\end{aligned} \tag{0.16}$$

For the derivation of (0.16) and of course (0.15), developments in a normal coordinate system are used (cf. [13], [14], [15]). The formulas (0.16) agree with a result of V. K. Patodi [20].

In another paper the present results will be applied to Kählerian manifolds.

1. Conventions and preliminaries

For two alternating differential forms $\varphi^{(p)}$, $\psi^{(p)}$ of degree p with the local representation

$$\varphi^{(p)} = \varphi_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} , \tag{1.1}$$

$$\psi^{(p)} = \psi_{i_1\cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} , \tag{1.2}$$

$\varphi^{(p)}(x) \otimes \psi^{(p)}(\xi)$ will denote the double differential form with the local representation

$$\begin{aligned} \varphi^{(p)}(x) &\otimes \psi^{(p)}(\xi) \\ &= \varphi_{i_1\cdots i_p}(x)\psi_{\alpha_1\cdots\alpha_p}(\xi) dx^{i_1} \wedge \cdots \wedge dx^{i_p} d\xi^{\alpha_1} \wedge \cdots \wedge d\xi^{\alpha_p} . \end{aligned} \tag{1.3}$$

The trace of a double differential form $U^{(p)}$ given by

$$U^{(p)}(x, \xi) = U_{i_1\cdots i_p \alpha_1\cdots\alpha_p}(x, \xi) dx^{i_1} \wedge \cdots \wedge dx^{i_p} d\xi^{\alpha_1} \wedge \cdots \wedge d\xi^{\alpha_p} \tag{1.4}$$

is the scalar quantity

$$(\mathrm{Sp}\, U^{(p)})(x) = p!\, g^{i_1\alpha_1}(x) \cdots g^{i_p\alpha_p}(x) U_{i_1\cdots i_p\alpha_1\cdots\alpha_p}(x, x) . \tag{1.5}$$

Further we define

$$\langle \varphi^{(p)}, \psi^{(p)} \rangle(x) = \mathrm{Sp}\, (\varphi^{(p)} \otimes \psi^{(p)})(x) , \tag{1.6}$$

$$(\varphi^{(p)}, \psi^{(p)}) = \int_M \langle \varphi^{(p)}, \psi^{(p)} \rangle(x) dv(x) , \tag{1.7}$$

where the volume element is given by $dv(x) = {}^*1$. Let the dual of the form (1.1) be defined in the usual way by

$$({}^*\varphi^{(p)})(x) = \frac{1}{(n-p)!} e_{i_1\cdots i_p j_1\cdots j_{n-p}}(x) \varphi^{i_1\cdots i_p}(x) dx^{j_1} \wedge \cdots \wedge dx^{j_{n-p}} . \tag{1.8}$$

Then instead of (1.7) one can write

$$(\varphi^{(p)}, \psi^{(p)}) = \int_M \varphi^{(p)} \wedge {}^*\psi^{(p)} . \tag{1.7'}$$

The terms "orthogonal" or "orthonormal" with respect to differential forms refer to the scalar product defined by (1.7) or (1.7′). The Laplacian $\Delta = d\delta + \delta d$ for alternating differential forms obeys a product rule

$$\Delta(\alpha^{(p)} \wedge \beta^{(q)}) = (\Delta\alpha^{(p)}) \wedge \beta^{(q)} + \alpha^{(p)} \wedge \Delta\beta^{(q)} - 2L(\alpha^{(p)}, \beta^{(q)}) , \tag{1.9}$$

where $\alpha^{(p)}$, $\beta^{(q)}$ are two forms with $p + q \leq n$, and

$$L(\alpha^{(p)}, \beta^{(q)}) = \nabla^i \alpha^{(p)} \wedge \nabla_i \beta^{(q)} + L_0(\alpha^{(p)}, \beta^{(q)}) . \tag{1.10}$$

As one knows, the expression $L_0(\alpha^{(p)}, \beta^{(q)})$ is bilinear in the components of the forms $\alpha^{(p)}$, $\beta^{(q)}$ with the curvature tensor as the coefficients of its terms. We introduce

$$\Omega^{jl} = \tfrac{1}{2} R^{jl}_{\cdot\cdot i_1 i_2} dx^{i_1} \wedge dx^{i_2} \tag{1.11}$$

and the operation[4]

$$e_j(\varphi) = p\varphi_{j i_2 \cdots i_p} dx^{i_2} \wedge \cdots \wedge dx^{i_p} . \tag{1.12}$$

Then the well-known representation of Δ (cf., for example, [6]) leads to

$$L_0(\alpha^{(p)}, \beta^{(q)}) = (-1)^p \Omega^{jl} \wedge e_j(\alpha^{(p)}) \wedge e_l(\beta^{(q)}) , \tag{1.13}$$

and this formula remains valid if $\alpha^{(p)}$, $\beta^{(q)}$ are double forms.

Lemma 1.1. *For three forms $\alpha^{(p)}$, $\beta^{(q)}$, $\gamma^{(r)}$ with $p + q + r \leq n$ one has*

$$\begin{aligned} &L_0(\alpha^{(p)}, \beta^{(q)} \wedge \gamma^{(r)}) \\ &\quad = L_0(\alpha^{(p)}, \beta^{(q)}) \wedge \gamma^{(r)} + (-1)^{qr} L_0(\alpha^{(p)}, \gamma^{(r)}) \wedge \beta^{(q)} . \end{aligned} \tag{1.14}$$

An analogous formula holds for double forms $\alpha^{(p)}$, $\beta^{(q)}$, $\gamma^{(r)}$ with $(-1)^{qr}$ replaced by $+1$.

Proof. Use (1.13) and the formula

$$e_l(\beta^{(q)} \wedge \gamma^{(r)}) = e_l(\beta^{(q)}) \wedge \gamma^{(r)} + (-1)^q \beta^{(q)} \wedge e_l(\gamma^{(r)}) , \tag{1.15}$$

which is easy to verify. In the case of double forms one has to take into consideration the commutativity of their multiplication.

In the following the transport form $T^{(1)}$ plays an important role. Let $x \in M$ and $\xi \in M$ be two points of M sufficiently near to each other, and denote by $\Gamma(x, \xi)$ the square of their geodesic distance. Then the double differential form $T^{(1)}$ satisfies the differential equation

$$L_x(\Gamma, T^{(1)}) = 0 \tag{1.16}$$

and the initial condition

$$T^{(1)}(\xi, \xi) = G^{(1)}(\xi) . \tag{1.17}$$

In (1.16) the differentiations refer to the variable x, while ξ remains fixed. (1.16) can be interpreted as ordinary differential equations for the coefficients of $T^{(1)}$ along the geodesic lines issuing from ξ. Thus $T^{(1)}(x, \xi)$ is defined in a neighborhood of the diagonal of $M \times M$ and is of class C^∞. We further define $T^{(p)}$ by

$$T^{(p)} = \frac{1}{p!}[T^{(1)}]^p , \qquad T^{(0)} = 1 , \tag{1.18}$$

[4] The definition (1.12) is, as far as we know, due to E. Kähler [17]. Also the relation (1.15) was already given in [17], but it should be noted that our inner multiplication defined in §3 differs from the inner multiplication of Kähler.

and easily get the composition rule

$$T^{(p)} \wedge T^{(q)} = \binom{p+q}{p} T^{(p+q)} . \tag{1.19}$$

Lemm 1.2. *For $p \geq 2$ one has*

$$\varDelta_x T^{(p)} = \varDelta_x T^{(1)} \wedge T^{(p-1)} - L_x(T^{(1)}, T^{(1)}) \wedge T^{(p-2)} . \tag{1.20}$$

Proof. For $p = 2$ the relation (1.20) readily follows from (1.18) and (1.9). Now assume it to be true for a certain p. Then substitute it in

$$\varDelta_x T^{(p+1)} = \frac{1}{p+1}\{\varDelta_x T^{(1)} \wedge T^{(p)} + T^{(1)} \wedge \varDelta_x T^{(p)} - 2L_x(T^{(1)}, T^{(p)})\} \tag{1.21}$$

together with

$$L_x(T^{(1)}, T^{(p)}) = L_x(T^{(1)}, T^{(1)}) \wedge T^{(p-1)} , \tag{1.22}$$

which in turn follows from Lemma 1.1. Thus we arrive at our relation with p replaced by $p + 1$.

Lemma 1.3. *The transport forms obey the duality relation*

$$(*T^{(p)})(x, \xi) = T^{(n-p)}(x, \xi) . \tag{1.23}$$

(The $*$-operation for double forms refers to both groups of indices and variables as long as no other convention is made.)

Proof. Since both sides of (1.23) satisfy the same differential equation (1.16) of geodesic parallel translation

$$L_x(\Gamma, *T^{(p)}) = 0 , \qquad L_x(\Gamma, T^{(n-p)}) = 0 , \tag{1.24}$$

all one has to do is to perform the routine verification of (1.23) for the initial values, i.e., for the coincidence forms $G^{(p)}(\xi)$.

Lemma 1.4. *For a double differential form $U^{(p)}$ of degree p, $0 \leq p \leq n$, and $0 \leq q \leq n$, one has*

$$\mathrm{Sp}\,(U^{(p)} \wedge T^{(q)}) = \binom{n-p}{q} \mathrm{Sp}\, U^{(p)} . \tag{1.25}$$

Proof. The relation (1.25) obviously holds for $q = 0$ and arbitrary p. Assuming it to be true for a certain q and arbitrary p one has

$$\begin{aligned} \mathrm{Sp}\,(U^{(p)} \wedge T^{(q+1)}) &= \frac{1}{q+1} \mathrm{Sp}\,((U^{(p)} \wedge T^{(1)}) \wedge T^{(q)}) \\ &= \binom{n-p-1}{q} \mathrm{Sp}\,(U^{(p)} \wedge T^{(1)}) . \end{aligned} \tag{1.26}$$

An elementary consideration shows

$$\mathrm{Sp}\,(U^{(p)} \wedge T^{(1)}) = (n - p)\,\mathrm{Sp}\,U^{(p)} \,. \tag{1.27}$$

Substituting (1.27) in (1.26) yields our relation with q replaced by $q + 1$.
In particular, taking $p = 0$ and $U^{(0)} = 1$ in (1.25) one gets

$$\mathrm{Sp}\,T^{(q)} = \binom{n}{q} \,. \tag{1.28}$$

2. Expansion to transport forms

The recursion system for the determination of the double forms $V_k^{(p)}$, occurring in the asymptotic expansion (0.2), reads as follows (cf., for example, [13]):

$$L_x(\Gamma, V_0^{(p)}) + MV_0^{(p)} = 0 \,, \tag{2.1}$$

$$L_x(\Gamma, V_k^{(p)}) + [M + 2k]V_k^{(p)} = -\Delta_x V_{k-1}^{(p)} \,, \qquad k \geq 1 \,, \tag{2.2}$$

where $M(x, \xi) = \frac{1}{2}\Delta_x \Gamma(x, \xi) - n$. The differential equations (2.1) are to be adjoined by the initial conditions

$$V_0^{(p)}(\xi, \xi) = G^{(p)}(\xi) \,. \tag{2.3}$$

The $V_k^{(p)}$ with $k \geq 1$ are uniquely determined by (2.2) and the regularity condition of the corresponding coincidence forms. Note that in general the $V_k^{(p)}$ are defined not over the whole, but only in a suitable neighborhood of the diagonal, of $M \times M$.

Theorem 2.1. There are regular double differential forms $W_k^{(q)}$ of degree q, $0 \leq q \leq n$, $k \geq 0$, defined in a neighborhood of the diagonal of $M \times M$ such that

$$V_k^{(p)} = \sum_{q=0}^{\min\{2k,p\}} W_k^{(q)} \wedge T^{(p-q)} \,. \tag{2.4}$$

Proof. For the sake of a formal simplification we set

$$W_k^{(q)} = 0,\ T^{(q)} = 0 \,, \qquad \forall q < 0,\ k \geq 0 \,. \tag{2.5}$$

Now, to solve (2.1), (2.2), by introducing

$$V_k^{(p)} = \sum_{q=0}^{p} W_k^{(q)} \wedge T^{(p-q)} \tag{2.6}$$

into (2.2) and making use of $L_x(\Gamma, T^{(p-q)}) = 0$, we obtain

$$L_x(\Gamma, V_k^{(p)}) + [M + 2k]V_k^{(p)} + \Delta_x V_{k-1}^{(p)}$$

$$= \sum_{q=0}^{p} \{L_x(\Gamma, W_k^{(q)}) + [M + 2k]W_k^{(q)}\} \wedge T^{(p-q)} \tag{2.7}$$
$$+ \sum_{q=0}^{p} \{\Delta_x W_{k-1}^{(q)} \wedge T^{(p-q)}$$
$$+ W_{k-1}^{(q)} \wedge \Delta_x T^{(p-q)} - 2L_x(W_{k-1}^{(q)}, T^{(p-q)})\} .$$

Into the last sums substitution of (1.20), as well as the conclusion following from Lemma 1.1, gives

$$L_x(W_{k-1}^{(q)}, T^{(p-q)}) = L_x(W_{k-1}^{(q)}, T^{(1)}) \wedge T^{(p-q-1)} . \tag{2.8}$$

By using a trivial change of the second summation index and considering (2.5) at last, we can convert (2.2) to

$$\sum_{q=0}^{p} \{L_x(\Gamma, W_k^{(q)}) + [M + 2k]W_k^{(q)} + \Delta_x W_{k-1}^{(q)} - 2L_x(W_{k-1}^{(q-1)}, T^{(1)}) \tag{2.9}$$
$$+ W_{k-1}^{(q-1)} \wedge \Delta_x T^{(1)} - W_{k-1}^{(q-2)} \wedge L_x(T^{(1)}, T^{(1)})\} \wedge T^{(p-q)} = 0 .$$

One gets an analogous equation for $k = 0$. In order to establish (2.1), (2.2) we impose on the forms $W_k^{(q)}$ the following system:

$$L_x(\Gamma, W_0^{(q)}) + MW_0^{(q)} = 0 , \tag{2.10}$$

$$L_x(\Gamma, W_k^{(q)}) + [M + 2k]W_k^{(q)}$$
$$= -\Delta_x W_{k-1}^{(q)} + 2L_x(W_{k-1}^{(q-1)}, T^{(1)}) - W_{k-1}^{(q-1)} \wedge \Delta_x T^{(1)} \tag{2.11}$$
$$+ W_{k-1}^{(q-2)} \wedge L_x(T^{(1)}, T^{(1)}) .$$

The initial conditions (2.3) are fulfilled if we require

$$W_0^{(0)}(\xi, \xi) = 1 , \tag{2.12}$$

$$W_0^{(q)}(\xi, \xi) = 0 , \qquad \forall q \geq 1 . \tag{2.13}$$

(2.10), (2.11) represent a recursion system with respect to the increasing k for the $W_k^{(q)}$. In particular, the equations with $q > 2k$ give a separate recursion system for the $W_k^{(q)}$ with $q > 2k$. This separate system is solved by taking into account the initial conditions (2.13) and requiring

$$W_k^{(q)} = 0 , \qquad \forall k \geq 0, \forall q \text{ with } q > 2k . \tag{2.14}$$

Accordingly, (2.6) changes to (2.4). (2.10), (2.11) for $q \leq 2k$ together with the initial condition (2.12) for $k = 0$ and with the regularity condition of the coincidence forms $W_k^{(q)}(\xi, \xi)$ for $k \geq 1$ now uniquely determine the $W_k^{(q)}$ with $q \leq 2k$. Thus Theorem 2.1 is proved.

Theorem 2.2. *For integer l and $0 \leq l \leq n$ one has*

$$(2.15)\quad \sum_{p=0}^{n-l}(-1)^p V_k^{(p)} \wedge T^{(n-l-p)} = \begin{cases} 0\,, & \textit{for } 2k < n-l\,, \\ (-1)^{n-l} W_k^{(n-l)}\,, & \textit{for } 2k \geq n-l\,. \end{cases}$$

Proof. Applying Theorem 2.1 (in its modification (2.6)) and using (1.19) and (2.14) we get

$$\begin{aligned}\sum_{p=0}^{n-l}(-1)^p V_k^{(p)} \wedge T^{(n-l-p)} &= \sum_{p=0}^{n-l}\sum_{q=0}^{p}(-1)^p W_k^{(q)} \wedge T^{(p-q)} \wedge T^{(n-l-p)} \\ &= \sum_{p=0}^{n-l}\sum_{q=0}^{p}(-1)^p \binom{n-l-q}{p-q} W_k^{(q)} \wedge T^{(n-l-q)}\,.\end{aligned}$$

An interchange of the successive summations in the last double sum gives

$$\sum_{q=0}^{n-l}\sum_{p=q}^{n-l}(-1)^p \binom{n-l-q}{p-q} W_k^{(q)} \wedge T^{(n-l-q)}\,,$$

which reduces to $(-1)^{n-l}W_k^{(n-l)}$ by a well-known summation formula for the binomial coefficients.Thus we establish our assertion on account of (2.14).

Remark. By means of (2.15) the $W_k^{(q)}$ are expressed in terms of the $V_k^{(q)}$, and in addition to this one obtains certain remarkable relations among the double forms $V_k^{(p)}$ themselves. The coincidence forms $Z^{(p)}$ defined in the introduction will turn out to be the coincidence values of some of the $W_k^{(q)}$.

Lemma 2.1. *The double differential form $T^{(1)}$ obeys*

$$(2.16)\qquad \nabla_i T^{(1)}(\xi,\xi) = 0\,,$$

$$(2.17)\qquad \nabla^i\nabla_i T^{(1)}(\xi,\xi) = 0\,,$$

$$(2.18)\qquad (\Delta_x T^{(1)})(\xi,\xi) = 2\Psi^{(1)}(\xi)\,,$$

where the covariant differentiations ∇_i refer to the first argument x, and the coincidence form $\Psi^{(1)}$ was defined in (0.7).

Proof. Covariant differentiations of the defining equation of $T^{(1)}$,

$$L_x(\Gamma, T^{(1)}) = (\nabla_j\Gamma)\nabla^j T^{(1)} = 0\,,$$

give

$$(\nabla_i\nabla_j\Gamma)\nabla^j T^{(1)} + (\nabla_j\Gamma)\nabla_i\nabla^j T^{(1)} = 0\,,$$

$$(\nabla^i\nabla_i\nabla_j\Gamma)\nabla^j T^{(1)} + 2(\nabla_i\nabla_j\Gamma)(\nabla^i\nabla^j T^{(1)}) + (\nabla_j\Gamma)\nabla^i\nabla_i\nabla^j T^{(1)} = 0\,.$$

Passage to the coincidence values $x \to \xi$ establishes (2.16), (2.17) if one takes into account

$$(\nabla_j\Gamma)(\xi,\xi) = 0\,,\quad (\nabla_i\nabla_j\Gamma)(\xi,\xi) = 2g_{ij}(\xi)\,,\quad (\nabla_i\nabla_j\nabla_k\Gamma)(\xi,\xi) = 0\,.$$

Now if we set

$$T^{(1)}(x,\xi) = t_{i\alpha}(x,\xi)dx^i d\xi^\alpha\,,$$

then, as is well known, the Laplacian of $T^{(1)}$ can be expressed as

$$\Delta_x T^{(1)}(x, \xi) = \{-\nabla^j \nabla_j t_{i\alpha}(x, \xi) + R_i^l(x) t_{l\alpha}(x, \xi)\} dx^i d\xi^\alpha ,$$

and the corresponding coincidence form reduces to

$$(\Delta_x T^{(1)})(\xi, \xi) = R_i^l(\xi) t_{l\alpha}(\xi, \xi) dx^i d\xi^\alpha = R_{i\alpha}(\xi) dx^i d\xi^\alpha = 2\Psi^{(1)}(\xi) ,$$

which completes the proof.

Theorem 2.3. *For integer l, $0 \leq l \leq n-1$, the coincidence forms $Z^{(p)}$ defined by* (0.8) *obey*

$$W^{(n-l)}_{[(n-l+1)/2]}(\xi, \xi) = Z^{(n-l)}(\xi) . \tag{2.19}$$

Proof. (a) We start with the case where $n - l$ is even. Put $n - l = 2k$. Then $[(n-l+1)/2] = k$. In view of $W_k^{(q)} = 0$ for $q > 2k$ the recursion system (2.10), (2.11) gives

$$L_x(\Gamma, W_k^{(2k)}) + [M + 2k] W_k^{(2k)} = W_{k-1}^{(2k-2)} \wedge L_x(T^{(1)}, T^{(1)}) , \qquad \forall k \geq 1 . \tag{2.20}$$

Now passing to the coincidence values on both sides and considering

$$M(\xi, \xi) = 0 , \qquad L_x(\Gamma, W_k^{(2k)})(\xi, \xi) = 0$$

we get

$$2k W_k^{(2k)}(\xi, \xi) = (W_{k-1}^{(2k-2)} \wedge L_x(T^{(1)}, T^{(1)}))(\xi, \xi) , \qquad \forall k \geq 1 , \tag{2.21}$$

which is solved by in view of (2.12),

$$W_k^{(2k)}(\xi, \xi) = \frac{1}{2^k k!} [L_x(T^{(1)}, T^{(1)})(\xi, \xi)]^k , \qquad \forall k \geq 1 . \tag{2.22}$$

Owing to (2.16), here one can replace the operator L by the operator L_0. Further using formula (1.13) we readily get

$$\begin{aligned} L_0(T^{(1)}, T^{(1)})(\xi, \xi) &= -\Omega^{jl}(\xi) g_{j\alpha}(\xi) g_{l\beta}(\xi) d\xi^\alpha \wedge d\xi^\beta \\ &= -\Omega_{\alpha\beta}(\xi) d\xi^\alpha \wedge d\xi^\beta = -\Psi^{(2)}(\xi) . \end{aligned}$$

Hence

$$W_k^{(2k)}(\xi, \xi) = \frac{(-1)^k}{2^k k!} [\Psi^{(2)}(\xi)]^k = Z^{(2k)}(\xi) , \qquad \forall k \geq 1 , \tag{2.23}$$

which is our assertion in the considered case.

(b) Now let us turn to the case $n - l = 1$, where $[(n-l+1)/2] = 1$. By (2.11), (2.5) and (2.14) we have, for the determination of $W_1^{(1)}$,

$$L_x(\Gamma, W_1^{(1)}) + [M+2]W_1^{(1)} = 2L_x(W_0^{(0)}, T^{(1)}) - W_0^{(0)} \wedge \Delta_x T^{(1)} . \tag{2.24}$$

Passing to the coincidence values one can replace L by L_0 owing to (2.16), but the resulting expression vanishes because the first argument in L_0 is a form of degree 0. On account of (2.12) we get

$$2W_1^{(1)}(\xi, \xi) = -\Delta_x T^{(1)}(\xi, \xi) , \tag{2.25}$$

and hence, by Lemma 2.1,

$$W_1^{(1)}(\xi, \xi) = -\Psi^{(1)}(\xi) = -Z^{(1)}(\xi) . \tag{2.26}$$

(c) At last we consider the case where $n-l$ is odd. Put $n-l = 2k-1$ with $k \geq 2$. Then $[(n-l+1)/2] = k$. By considering $W_k^{(q)} = 0$ for $q > 2k$, from (2.11) it follows

$$\begin{aligned} &L_x(\Gamma, W_k^{(2k-1)}) + [M+2k]W_k^{(2k-1)} \\ &\quad = 2L_x(W_{k-1}^{(2k-2)}, T^{(1)}) - W_{k-1}^{(2k-2)} \wedge \Delta_x T^{(1)} \\ &\qquad + W_{k-1}^{(2k-3)} \wedge L_x(T^{(1)}, T^{(1)}) . \end{aligned} \tag{2.27}$$

Passing to the coincidence values on both sides and taking account of the results of (a) and (b) we get

$$\begin{aligned} 2kW_k^{(2k-1)}(\xi, \xi) = &\ \frac{(-1)^{k-1}}{2^{k-2}(k-1)!} L_0((\Psi^{(2)})^{k-1}, T^{(1)})(\xi, \xi) \\ &- \frac{(-1)^{k-1}}{2^{k-2}(k-1)!} (\Psi^{(2)})^{k-1}(\xi) \wedge \Psi^{(1)}(\xi) \\ &- W_{k-1}^{(2k-3)}(\xi, \xi) \wedge \Psi^{(2)}(\xi) . \end{aligned} \tag{2.28}$$

Now owing to Lemma 1.1 and (1.13)

$$\begin{aligned} &L_0((\Psi^{(2)})^{k-1}, T^{(1)})(\xi, \xi) \\ &\quad = (k-1)L_0(\Psi^{(2)}, T^{(1)})(\xi, \xi) \wedge (\Psi^{(2)})^{k-2}(\xi) \\ &\quad = (k-1)\Omega^{jl}(\xi) \wedge e_j(\Psi^{(2)})(\xi) \wedge e_l(G^{(1)})(\xi) \wedge (\Psi^{(2)})^{k-2}(\xi) \\ &\quad = (k-1)\Psi^{(3)}(\xi) \wedge (\Psi^{(2)})^{k-2}(\xi) . \end{aligned} \tag{2.29}$$

Substituting this in (2.28) one has

$$\begin{aligned} 2kW_k^{(2k-1)}(\xi, \xi) = &-W_{k-1}^{(2k-3)}(\xi, \xi) \wedge \Psi^{(2)}(\xi) \\ &+ \frac{(-1)^k}{2^{k-2}(k-1)!} [\Psi^{(1)}(\xi) \wedge \Psi^{(2)}(\xi) \\ &\qquad - (k-1)\Psi^{(3)}(\xi)] \wedge [\Psi^{(2)}(\xi)]^{k-2} . \end{aligned} \tag{2.30}$$

Solving (2.30) by an elementary induction with respect to k, the beginning case $k = 1$ being given by (2.26), we get

$$\begin{aligned} W_k^{(2k-1)}(\xi, \xi) &= Z^{(2k-1)}(\xi) \\ &= \frac{(-1)^k}{2^{k-1}(k-1)!}\Big[\Psi^{(1)}(\xi) \wedge \Psi^{(2)}(\xi) \\ &\qquad - \frac{k-1}{2}\Psi^{(3)}(\xi)\Big] \wedge [\Psi^{(2)}(\xi)]^{k-2} , \end{aligned} \tag{2.31}$$

which completes the proof of Theorem 2.3.

Now we are in a position to prove Theorems I–IV mentioned in the introduction.

Proof of Theorem I. Pass to the coincidence values on both sides of formula (2.15) of Theorem 2.2 and apply formula (2.19) of Theorem 2.3.

Proof of Theorem II. Take the trace on both sides of formula (0.9) of Theorem I and consider Lemma 1.4 and the fact that the $G^{(p)}$ are the coincidence values of the $T^{(p)}$.

Proof of the corollary to Theorem II. Let dx^i, $d\xi^\alpha$ be the differentials appearing in a local representation of a double form. Let e_j be the operation defined by (1.12) with respect to the differentials dx^i, and $\hat{e}_\alpha$ an analogous operation with respect to the differentials $d\xi^\alpha$; both operations commute with each other. Then for a double differential form $U^{(p)}$ one can write

$$\operatorname{Sp} U^{(p)} = \frac{1}{p} \operatorname{Sp} (g^{i\alpha} e_i \hat{e}_\alpha (U^{(p)})) . \tag{2.32}$$

Now an easy computation using (1.15) shows that

$$\begin{aligned} g^{i\alpha} e_i \hat{e}_\alpha([\Psi^{(2)}]^k) &= k g^{i\alpha} \hat{e}_\alpha (e_i(\Psi^{(2)}) \wedge [\Psi^{(2)}]^{k-1}) \\ &= k g^{i\alpha} \hat{e}_\alpha e_i(\Psi^{(2)}) \wedge [\Psi^{(2)}]^{k-1} \\ &\quad + k(k-1) g^{i\alpha} e_i(\Psi^{(2)}) \wedge \hat{e}_\alpha(\Psi^{(2)}) \wedge [\Psi^{(2)}]^{k-2} . \end{aligned} \tag{2.33}$$

Further, from the definition (0.7) it follows

$$g^{i\alpha} \hat{e}_\alpha e_i(\Psi^{(2)}) = -4\Psi^{(1)} , \qquad g^{i\alpha} e_i(\Psi^{(2)}) \wedge \hat{e}_\alpha(\Psi^{(2)}) = 2\Psi^{(3)} .$$

Thus by (2.32), (2.33) one has the assertion of the corollary:

$$\begin{aligned} \operatorname{Sp} Z^{(2k)} &= \frac{(-1)^k}{2^k k!} \operatorname{Sp} [\Psi^{(2)}]^k \\ &= \frac{(-1)^{k-1}}{2^{k-1} k!} \operatorname{Sp} \left(\left[\Psi^{(1)} \wedge \Psi^{(2)} - \frac{k-1}{2} \Psi^{(3)} \right] \wedge [\Psi^{(2)}]^{k-2} \right) \\ &= \frac{-1}{k} \operatorname{Sp} Z^{(2k-1)} . \end{aligned}$$

Proof of Theorem III. Multiply the expansion (0.2) by $(-1)^p \binom{n-p}{l}$, $0 \leq p \leq l$, and add the resulting expression. Substituting the results of Theorem II in the right of the obtained equation we arrive at (0.12).

Proof of Therem IV. First, one has to integrate formula (0.12) over M. For $i \geq 1$ one takes into consideration

$$\begin{aligned}
&\sum_{p=0}^{n-l} (-1)^p \binom{n-p}{l} \dim \mathfrak{B}_i^{(p)} \\
&\quad = \sum_{p=0}^{n-l} (-1)^p \binom{n-p}{l} \{\dim \mathfrak{E}_i^{(p)} + \dim \mathfrak{K}_i^{(p)}\} \\
&\quad = \sum_{p=1}^{n-l} (-1)^p \binom{n-p}{l} \dim \mathfrak{K}_i^{(p-1)} + \sum_{p=0}^{n-l} (-1)^p \binom{n-p}{l} \dim \mathfrak{K}_i^{(p)} \\
&\quad = \sum_{p=0}^{n-l} (-1)^p \left\{\binom{n-p}{l} - \binom{n-p-1}{l}\right\} \mathfrak{K}_i^{(p)} \\
&\quad = \sum_{p=0}^{n-l} (-1)^p \binom{n-p-1}{l-1} \dim \mathfrak{K}_i^{(p)} \, ,
\end{aligned}$$

which establishes Theorem IV.

3. Dualisation and Conclusions

Definition. Let $X^{(p)}$, $Y^{(p+q)}$ be double differential forms of degrees p, $p+q$ $(q \geq 0)$ respectively. A double form $X^{(p)} \vee Y^{(p+q)}$ of degree q is defined by

$$X^{(p)} \vee Y^{(p+q)} = *(X^{(p)} \wedge *Y^{(p+q)}) \, . \tag{3.1}$$

The following theorem gives an expansion dual to the expansion to transport forms of Theorem 2.1.

Theorem 3.1. *For $0 \leq p \leq n$ and the double forms $W_k^{(q)}$ considered in § 2 one has*

$$V_k^{(n-p)} = \sum_{q=0}^{\min\{2k,p\}} W_k^{(q)} \vee T^{(n-p+q)} \, . \tag{3.2}$$

Proof. Taking the dual of both sides of (2.4) yields

$$*V_k^{(p)} = \sum_{q=0}^{\min\{2k,p\}} *(W_k^{(q)} \wedge T^{(p-q)}) \, . \tag{3.3}$$

By Lemma 1.3 we can write

$$T^{(p-q)} = *T^{(n-p+q)} \, . \tag{3.4}$$

Further we use the well-known relation

$$*V_k^{(p)} = V_k^{(n-p)}\,, \tag{3.5}$$

which originates from the duality properties of Green's form of the heat e-quation. Substituting (3.4) and (3.5) in (3.3) one can apply the above definition and thus derive the assertion of Theorem 3.1.

Lemma 3.1. *For a double differential form $U^{(p)}$ of degree p one has*

$$\mathrm{Sp}\,(*U^{(p)}) = \mathrm{Sp}\,U^{(p)}\,. \tag{3.6}$$

The proof is a matter of routine.

Lemma 3.2. *For a double differential form $U^{(p)}$ of degree p, $0 \le p \le n$, one has*

$$\mathrm{Sp}\,(U^{(q)} \vee T^{(n-p+q)}) = \binom{n-q}{p-q}\,\mathrm{Sp}\,U^{(q)}\,. \tag{3.7}$$

The proof follows from our definition and Lemmas 3.1, 1.3 and 1.4.

Theorem 3.2. *For the double forms $W_k^{(q)}$ considered in* § 2 *one has,* $\forall k \ge 0$, $\forall p$ with $0 \le p \le n$,

$$\sum_{q=0}^{\min\{2k,n-p\}} \binom{n-q}{p}\,\mathrm{Sp}\,W_k^{(q)} = \sum_{q=0}^{\min\{2k,p\}} \binom{n-q}{p-q}\,\mathrm{Sp}\,W_k^{(q)}\,. \tag{3.8}$$

Proof. We take the trace on both sides of (3.2) in Theorem 3.1, and obtain by applying Lemma 3.2

$$\mathrm{Sp}\,V_k^{(n-p)} = \sum_{q=0}^{\min\{2k,p\}} \binom{n-q}{p-q}\,\mathrm{Sp}\,W_k^{(q)}\,. \tag{3.9}$$

Further we replace p by $n-p$ and take the trace once more on both sides of (2.4) in Theorem 2.1. This yields

$$\mathrm{Sp}\,V_k^{(n-p)} = \sum_{q=0}^{\min\{2k,n-p\}} \binom{n-q}{p}\,\mathrm{Sp}\,W_k^{(q)}\,. \tag{3.10}$$

The comparison between (3.9) and (3.10) leads to the assertion.

Lemma 3.3. *For integers p and r with $0 \le r \le p \le n$ and $0 \le r \le \frac{1}{2}n$ one has*

$$\binom{n-2r}{p-r} = \sum_{q=r}^{p} (-1)^{q+r}\binom{n-q}{p-q}\binom{r}{q-r}\,. \tag{3.11}$$

The proof is carried out by induction with respect to r.

Now we come to the

Proof of Theorem V. To begin with we consider for fixed $k \ge 0$ the linear equation system for quantities σ_k^r, $0 \le r \le [\frac{1}{2}n]$:

(3.12) $\operatorname{Sp} W_k^{(p)} = \sum_{r=0}^{p} (-1)^{p+r} \binom{r}{p-r} \sigma_k^r , \qquad \forall p \text{ with } 0 \leq p \leq \left[\frac{n}{2}\right] .$

The matrix of the coefficients of the linear system has triangular shape with principal diagonal consisting of numbers equal to one. Hence the σ_k^r, $0 \leq r \leq [\frac{1}{2}n,]$ are uniquely determined as linear combinations of the $\operatorname{Sp} W_k^{(p)}$, $0 \leq p \leq [\frac{1}{2}n]$, with integer coefficients. We now show that the quantities σ_k^r thus defined obey

(3.13) $\operatorname{Sp} W_k^{(p)} = \sum_{r=0}^{\min\{p,[n/2]\}} (-1)^{p+r} \binom{r}{p-r} \sigma_k^r , \qquad \forall p \text{ with } 0 \leq p \leq n .$

To this end we form the sums

(3.14) $I_p = \sum_{q=0}^{p} \binom{n-q}{p-q} \left\{ \operatorname{Sp} W_k^{(q)} - \sum_{r=0}^{\min\{q,[n/2]\}} (-1)^{r+q} \binom{r}{q-r} \sigma_k^r \right\} .$

Obviously $I_p = 0$ for $0 \leq p \leq [\frac{1}{2}n]$ owing to (3.12). On the other hand for $p > \frac{1}{2}n$ we get, setting $p' = n - p$ and formally taking into account Theorem 3.2,

$$I_p = \sum_{q=0}^{p'} \binom{n-q}{n-p'} \operatorname{Sp} W_k^{(q)} - \sum_{r=0}^{[n/2]} \sum_{q=r}^{p} (-1)^{r+q} \binom{r}{q-r} \binom{n-q}{p-q} \sigma_k^r .$$

Here we can apply (3.12) and Lemma 3.3 so that

$$\begin{aligned} I_p &= \sum_{q=0}^{p'} \sum_{r=0}^{q} (-1)^{q+r} \binom{n-q}{p'-q} \binom{r}{q-r} \sigma_k^r - \sum_{r=0}^{[n/2]} \binom{n-2r}{p-r} \sigma_k^r \\ &= \sum_{r=0}^{p'} \binom{n-2r}{p'-r} \sigma_k^r - \sum_{r=0}^{p'} \binom{n-2r}{p-r} \sigma_k^r = 0 . \end{aligned}$$

The equations $I_p = 0$ for $0 \leq p \leq n$, however, form a linear homogenous system for the quantities between the braces in (3.14), the matrix of which has triangular shape and has the numbers $\binom{n-p}{0} \neq 0$ in the diagonal. This establishes (3.13). Further we show

(3.15) $\sigma_k^r = 0 \qquad \text{for } k < r \leq \left[\frac{n}{2}\right] .$

To this end let $k < [\frac{1}{2}n]$. In (3.13) we choose $p = 2[\frac{1}{2}n]$ to begin with. Because of $W_k^{(q)} = 0$ for $q > 2k$ we get

$$\sum_{r=0}^{[n/2]} (-1)^r \binom{r}{2[\frac{1}{2}n] - r} \sigma_k^r = (-1)^{[n/2]} \sigma_k^{[n/2]} = 0 .$$

Assume $\sigma_k^r = 0$ to be shown already for $[\frac{1}{2}n] - l + 1 \leq r \leq [\frac{1}{2}n]$, where additionally $[\frac{1}{2}n] - l > k$, otherwise the proof is finished. Then in (3.13) we set $p = 2([\frac{1}{2}n] - l)$. Again with $p > 2k$ one has

$$\sum_{r=0}^{\min\{2[n/2]-2l,[n/2]\}} (-1)^r \binom{r}{2[\frac{1}{2}n] - 2l - r} \sigma_k^r = 0 \ . \tag{3.16}$$

By the induction hypothesis the summation index r can be limited according to $r \leq [\frac{1}{2}n] - l$. Then, however, $\sigma_k^{[n/2]-l} = 0$ follows immediately from (3.16). Thus (3.15) is proved. Return now to (3.9) written in the equivalent form

$$\operatorname{Sp} V_k^{(p)} = \sum_{q=0}^{p} \binom{n-q}{p-q} \operatorname{Sp} W_k^{(q)} \ . \tag{3.17}$$

Substituting (3.13) in (3.17) taking account of (3.15), interchanging the successive summations and applying Lemma 3.3, we thus get (0.14) of Theorem V.

It remains to show that the σ_k^r are universal invariants. To this end we think the coefficients of the double forms $W_k^{(q)}(x, \xi)$ as functions of x expanded to finite Taylor's series about ξ with a remainder of a sufficiently high order. Performing the expansion in a normal coordinate system with origin ξ the individual Taylor terms can be expressed according to a standard procedure as polynomials, with coefficients independent of the dimension, in the metric tensor and its inverse, the curvature tensor and the covariant derivatives of the latter.[5] This can be arranged in such a way that only the g^{ij} and $\nabla_{i_l} \cdots \nabla_{i_l} R_{ijhk}$ occur, not the g_{ij}. For the lowest double form $W_0^{(0)}$ it can be seen from the formula (in normal coordinates)

$$W_0^{(0)}(x, \xi) = \{g(x)/g(\xi)\}^{-1/4} \ , \qquad g = \operatorname{Det}(g_{ij}) \ . \tag{3.18}$$

For the higher double forms $W_k^{(q)}$ it can be seen by an induction with respect to k using the recursion system (2.11). The g_{ij} occur only in the coincidence value, that is, only in the Taylor term of zero order, of the transport form $T^{(1)}$. Since no terms g_{ij} occur in the coincidence forms $W_k^{(q)}(\xi, \xi)$, no factors $n = g^{ij}g_{ij}$ can appear in the traces $\operatorname{Sp} W_k^{(q)}$. Thus the $\operatorname{Sp} W_k^{(q)}$ turn out to be universal invariants, and the same holds for the σ_k^r, because the linear equation system (3.12) for the determination of the σ_k^r by the $\operatorname{Sp} W_k^{(q)}$ is independent of n. This completely establishes Theorem V.

Proof of the corollary to Theorem V. Owing to (3.15) we can write (3.13) in the form

$$\operatorname{Sp} W_k^{(p)} = \sum_{r=0}^{\min\{k,p,[n/2]\}} (-1)^{p+r} \binom{r}{p-r} \sigma_k^r \ .$$

[5] In [14] explicit formulas are given by means of which the derivatives of the metric tensor in normal coordinates are expressed by the curvature tensor and its covariant derivatives.

Now put $p = 2k$. Then the sum on the right-hand side contains at most one term, while to the left-hand side we apply (2.23) so that we get (0.15) and, in addition,

$$\sigma_k^{[n/2]+1} = \cdots = \sigma_k^k = 0 \qquad \text{for } k > \left[\frac{n}{2}\right].$$

References

[1] M. Atiyah, R. Bott & V. K. Patodi, *On the heat equation and the index theorem,* Invent. Math. **19** (1973) 279–330.

[2] M. Berger, *Le spectre des variétés Riemanniennes,* Rev. Roumaine Math. Phys. Appl. **13** (1968) 915–931.

[3] M. Berger, P. Gauduchon & E. Mazet, *Le spectre d'une variété Riemannienne,* Lecture Notes in Math. Vol. 194, Springer, Berlin, 1971.

[4] E. Combet, *Solutions élémentaires des d'Alembertiens généralisés,* Mémor. Sci. Math. Vol. 160, Gauthier-Villars, Paris, 1965.

[5] ——, *Parametrix et invariants sur les variétés compactes,* Ann. Sci. École Norm. Sup. **3** (1970) 247–271.

[6] G. de Rham, *Variétés differentiables,* Hermann, Paris, 1955.

[7] H. Donnelly, *Symmetric Einstein spaces and spectral geometry,* Indiana Univ. Math. J. **24** (1974) 603–606.

[8] G. F. D. Duff, *Harmonic* p-*tensors on normal hyperbolic Riemannian spaces,* Canad. J. Math. **5** (1953) 57–80.

[9] A. Friedman, *The wave equation for differential forms,* Pacific J. Math. **11** (1961) 1267–1279.

[10] M. P. Gaffney, *Asymptotic distributions associated with the Laplacian for forms,* Comm. Pure Appl. Math. **11** (1958) 535–545.

[11] P. B. Gilkey, *Curvature and the eigenvalues of the Laplacian for elliptic complexes,* Advances in Math. **10** (1973) 344–382.

[12] ——, *Local invariants of a pseudo-Riemannian manifold,* Math. Scand. **36** (1975) 109–130.

[13] P. Günther, *Einige Sätze über huygenssche Differentialgleichungen,* Wiss. Z. Karl-Marx-Univ. Leipzig, Math.-Natur. Reihe **14** (1965) 497–507.

[14] ——, *Spinorkalkül und Normalkoordinaten,* Z. Angew. Math. Mech. **55** (1975) 205–210.

[15] P. Günther & V. Wünsch, *Maxwellsche Gleichungen und Huygenssches Prinzip.* I, Math. Nachr. **63** (1974) 97–121.

[16] W. V. D. Hodge, *Theorie und Anwendungen harmonischer Integrale,* B. G. Teubner, Leipzig, 1958. (The German translation from the English edition.)

[17] E. Kähler, *Innerer und äusserer Differentialkalkül,* Abh. Deutsch. Akad. Wiss. Berlin Kl. Math. Phys. Tech **4** (1960).

[18] H. P. McKean, Jr. & I. M. Singer, *Curvature and the eigenvalues of the Laplacian,* J. Differential Geometry **1** (1967) 43–69.

[19] S. Minakshisundaram & A. Pleijel, *Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds,* Canad. J. Math. **1** (1949) 242–256.

[20] V. K. Patodi, *Curvature and the fundamental solution of the heat operator,* J. Indian Math. Soc. **34** (1970) 269–285.

[21] ——, *Curvature and the eigenforms of the Laplace-operator,* J. Differential Geometry **5** (1971) 233–249.

[22] T. Sakai, *On eigenvalues of Laplacian and curvature of Riemannian manifolds,* Tôhoku Math. J. (2) **23** (1971) 589–603.

[23] J. A. Schouten, *Ricci-Calculus,* 2nd ed., Springer, Berlin, 1954.

[24] R. Schimming, *Riemannsche Metriken mit ebener Symmetrie und das Huygenssche Prinzip,* Thesis, Leipzig, 1971.

[25] ——, *Zur Gültigkeit des Huygensschen Prinzips bei einer speziellen Metrik,* Z. Angew, Math. Mech. **51** (1971) 201–208.

[26] R. Sulanke & P. Wintgen, *Differentialgeometrie und Faserbündel,* Deutsch. Verlag Wiss., Berlin, 1972.

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