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e-FOLIATIONS OF CODIMENSION TWO

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Introduction

A codimension-q foliation $\mathscr{F}$ of the manifold M is an e-foliation if a framing of its normal bundle Q can be chosen to be invariant under the linear holonomy of each leaf. These structures occur as one extreme case in a general theory of transverse H-structures for foliations analogous to the theory of G-structures for manifolds.

In codimension one, e-foliations are defined by a nonsingular closed 1-form. There is a strong structure theorem for such foliations of compact manifolds due essentially to G. Reeb [7] (also see [1, (5.5)]). L. Conlon [1] has investigated the properties of e-foliations in higher codimension and has proven a partial analogue of Reeb's theorem in codimension two.

We view an e-foliation as a foliation with transverse structure modeled by a parallelizable manifold. In this spirit, we define a Lie foliation as a foliation with transverse structure modeled by a Lie group.

Evidently, every Lie foliation is an e-foliation. It is easy to see that the two notions coincide in codimension one, but differ in codimension greater than two.

The main result of this paper is that every codimension-two e-foliation of a closed manifold is a Lie foliation. This additional structure enables us to answer some questions left open by Conlon and essentially complete the structure theory for e-foliations of codimension two.

In § 1 we define e-foliations and Lie foliations as special cases of a general notion of transverse structures for foliations. § 2 is devoted to the proof of our main result and the remainder of the paper consists of remarks on the structure of e-foliations in codimension two. In particular, an example of a Lie foliation modeled on the affine group of transformations of $\boldsymbol{R}^1$ is constructed, and a theorem of D. Tischler [8] is used to draw several easy but pleasant corollaries of our main theorem.

Unless otherwise specified, all manifolds and mappings considered are assumed to be differentiable of class C^∞.

I would like to express my gratitude to my advisor, Lawrence Conlon, for his help in this work and indeed for his own work from which this is derived.

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The unified treatment of transverse structures for foliations sketched in § 1 of this paper was suggested by Professor Conlon.

1. Transverse structures for foliations

In order to develop a notion of a natural geometric structure for foliations, it is convenient to reformulate the description of a foliation in terms of a Haefliger cocycle by building into the cocycle the notion of a modeling manifold for the transverse structure of the foliation.

Definition. Let N^q be a smooth q-dimensional manifold. An *N^q-cocycle* on the manifold M is a collection of triples $\{(U_\alpha, f_\alpha, g_{\alpha\beta})\}_{\alpha,\beta\in A}$ such that

(i) $\{U_\alpha\}_{\alpha\in A}$ is an open cover of M,

(ii) each f_α is a submersion of U_α onto an open subset of N^q,

(iii) $g_{\alpha\beta}$ is a local diffeomorphism of N^q such that for each $x \in U_\alpha \cap U_\beta$, $f_\alpha(x) = g_{\alpha\beta}(f_\beta(x))$.

We say the N^q-cocycle $\{(U_\alpha, f_\alpha, g_{\alpha\beta})\}$ represents the foliation $\mathscr{F}$ of M if the bundle E of tangents to the foliation is given locally by $E \mid U_\alpha = \ker f_{\alpha*} \subset T(U_\alpha)$. In this case, any additional structure supported by the manifold N^q, which is preserved by the local diffeomorphisms $g_{\alpha\beta}$, can be interpreted as a transverse structure for $\mathscr{F}$. For example, if we can find an N^q-cocycle representing $\mathscr{F}$ such that the $g_{\alpha\beta}$'s preserve some given Borel measure on N^q, we say that $\mathscr{F}$ admits a transverse measure [6].

From this point of view, Conlon's notion of a transverse H-structure for a foliation [1] may be defined as follows.

Definition. Let H be a Lie subgroup of $GL(q, \boldsymbol{R})$. A codimension-q foliation $\mathscr{F}$ is said to admit a *transverse H-structure* if an N^q-cocycle representing $\mathscr{F}$ can be chosen such that the $g_{\alpha\beta}$'s are local H-diffeomorphisms of some H-structure on the manifold N^q.

We study the extreme case where H is the trivial subgroup e. An e-structure for the manifold N^q is a framing of its tangent bundle (or an absolute parallelism on N^q).

Definition. An *e-foliation* is a foliation which admits a transverse e-structure.

In this case, we have a framing of the normal bundle $Q \subset T^*(M)$ of $\mathscr{F}$ by sections "parallel along the leaves of $\mathscr{F}$" [1] (a section $\omega \in \Gamma(Q)$ is parallel along the leaves of $\mathscr{F}$ if for each α, $\omega \mid U_\alpha = f_\alpha{}^*\eta$ for some 1-form η on N^q).

Via a choice of Riemannian metric on M, we may view Q as the subbundle of $T(M)$ orthogonal to the bundle of tangents of $\mathscr{F}$, $Q = E^\perp$. Then a section $X \in \Gamma(Q) \subset \mathfrak{X}(M)$ is parallel along the leaves of $\mathscr{F}$ if $f_{\alpha*}X$ is a well-defined vector field on $f_\alpha(U_\alpha) \subset N^q$ for each α. A useful equivalent formulation in terms of a Bott-basic connection gives: $X \in \Gamma(Q)$ is parallel along the leaves of $\mathscr{F}$ if for every $Y \in \Gamma(E)$, $[X, Y] \in \Gamma(E)$, [1].

The simplest example of an e-foliation is the foliation of the total space of a fibration $p: M \to N$ onto a parallelizable manifold by fibers. Indeed by [1, (4.4)] these are the only e-foliations of compact manifolds which admit a closed leaf.

As a final example of a transverse structure for a foliation, we ask that the transverse structure be modeled by a Lie group.

Definition. Let G be a Lie group. A foliation $\mathscr{F}$ is a *Lie foliation modeled on the Lie group* G if it can be represented by a G-cocycle $\{(U_\alpha, f_\alpha, g_{\alpha\beta})\}$ where the $g_{\alpha\beta}$'s are (restrictions of) left translation by elements of G.

Since the underlying manifold of G supports an e-structure given by left invariant vector fields, every Lie foliation is an e-foliation.

If $(\omega_1, \cdots, \omega_q)$ is an e-structure for $\mathscr{F}$ obtained by pulling back an e-structure for the manifold G given by left invariant 1-forms, we have

$$(*) \qquad d\omega_i = \sum_{1 \le j < k \le q} c^i_{jk} \omega_j \wedge \omega_k \,,$$

where c^i_{jk} are constants of structure for G.

Conversely, if $(\omega_1, \cdots, \omega_q)$ is a framing of the normal bundle of a foliation $\mathscr{F}$ and satisfies $(*)$, then $\mathscr{F}$ is a Lie foliation modeled on G [4, (5.1)].

The equivalent formulation for $Q \subset T(M)$ is: $\mathscr{F}$ is a Lie foliation modeled on G if and only if a framing $(X_1, \cdots, X_q)$ of Q can be chosen so that the Q-component of the brackets $[X_i, X_j]$ satisfies $[X_i, X_j]_Q = \sum_{k=1}^q C^k_{ij} X_k$ where C^k_{ij} are "dual" constants of structure for G.

2. The main theorem

The starting point of our analysis is the following.

(2.1) Theorem (*Conlon* [1]). *Let $\mathscr{F}$ be a codimension-two e-foliation of a closed manifold M. Then the universal covering space of M has the form $\hat{M} \cong \hat{A} \times \mathbf{R}^2$ where $\hat{A}$ is the universal covering space of a typical leaf A of $\mathscr{F}$. Furthermore, the lifted foliation $\hat{\mathscr{F}}$ of $\hat{M}$ is the foliation of $\hat{A} \times \mathbf{R}^2$ by leaves of the form $\hat{A} \times$ point.*

We remark that if we assume that $\mathscr{F}$ is a codimension-two Lie foliation, the above is a special case of (5.1) of [4].

(2.2) Theorem. *Every codimension-two e-foliation $\mathscr{F}$ of a closed manifold M is a Lie foliation.*

The proof will be a series of lemmas and observations. We choose a Riemannian metic on M invariant under the natural parallelism along the leaves of $\mathscr{F}$ (i.e., since $\mathscr{F}$ admits a transverse e-structure, it admits a transverse 0(2)-structure) and view Q as a subbundle of $T(M)$. Let (Y_1, Y_2) be an e-structure for $\mathscr{F}$ consisting of orthonormal vector fields $Y_i \in \Gamma(Q) \subset \mathfrak{X}(M)$, and let $(\hat{Y}_1, \hat{Y}_2)$ be a lifted e-structure for $\hat{\mathscr{F}}$, $\hat{Y}_i \in \Gamma(\hat{Q}) \subset \mathfrak{X}(\hat{M})$.

Conlon's analysis continues with the observation that the group of covering transformations $\pi_1(M)$ maps leaves of $\hat{\mathscr{F}}$ to leaves of $\hat{\mathscr{F}}$, and preserves the induced e-structure $(\hat{Y}_1, \hat{Y}_2)$ on $\hat{M}$. In particular, this defines a representation $\rho: \pi_1(M) \to \mathrm{Diff}^+(\boldsymbol{R}^2)$.

The crucial observation for the present results is that the transverse e-structure (Y_1, Y_2) induces a framing of the tangent bundle of $\boldsymbol{R}^2$ (i.e., an absolute parallelism or e-structure for the manifold $\boldsymbol{R}^2$) by the complete vector fields X_i, $i = 1, 2$, given by projecting the corresponding $\hat{Y}_i$ along the leaves of $\hat{\mathscr{F}}$. Thus for $\varphi \in \pi_1(M)$, $\rho(\varphi) \in \mathrm{Diff}^+(\boldsymbol{R}^2)$ is an automorphism of the absolute parallelism (X_1, X_2) on $\boldsymbol{R}^2$, i.e., we have a representation $\rho: \pi_1(M) \to \mathrm{Aut}(X_1, X_2) < \mathrm{Diff}^+(\boldsymbol{R}^2)$. The advantage of this is that the group $\mathrm{Aut}(X_1, X_2)$ is particularly amenable to study. Indeed we have the following:

(2.3) Theorem (*Kobayashi* [5]). *Let G be the group of automorphisms of an absolute parallelism on the manifold M. Then G is a Lie group. Furthermore G acts freely on the manifold M, and the orbits of this action are (regular) closed submanifolds of M. In particular,* $\dim G \leq \dim M$.

Following Conlon, we observe that the kernel of the representation ρ may be identified with the fundamental group of a typical leaf $A \in \mathscr{F}$. We designate the group $\ker(\rho)$ by $\pi_1(A)$, and notice that the quotient group $\pi_1(M)/\pi_1(A)$ may be identified with the relative homotopy set $\pi_1(M, A)$.

Let G be the group of automorphisms of the absolute parallelism (X_1, X_2) on $\boldsymbol{R}^2$ induced by the e-structure (Y_1, Y_2) for the foliation $\mathscr{F}$. Then $\dim G \leq 2$ and $\pi_1(M, A) \cong \rho(\pi_1(M)) < G$.

To prove our theorem, we must show that an e-structure (Y_1', Y_2') for $\mathscr{F}$ can be chosen to satisfy $[Y_1', Y_2']_Q = C_1 Y_1' + C_2 Y_2'$. We divide the argument into three cases corresponding to the possible dimensions of the group G associated to our original choice of e-structure.

(2.4) Lemma. *If* $\dim G = 2$, *then $\mathscr{F}$ is a Lie foliation modeled on the Lie group G.*

Proof. Since G acts freely on $\boldsymbol{R}^2$ and the orbits of this action are closed in $\boldsymbol{R}^2$, the mapping of G into $\boldsymbol{R}^2$ which takes $g \in G$ to $\varphi_g(0)(0 \in \boldsymbol{R}^2)$ is a diffeomorphism of the underlying manifold of G onto $\boldsymbol{R}^2$. Let (Z_1, Z_2) be a framing of the tangent bundle of $\boldsymbol{R}^2$ by vector fields invariant under the action of G. Then $[Z_1, Z_2] = C_1 Z_1 + C_2 Z_2$ where C_i are constants of structure for G, and since $\rho(\pi_1(M)) < G$ it follows that (Z_1, Z_2) is associated to an e-structure (Y_1', Y_2') for the foliation $\mathscr{F}$ such that $[Y_1', Y_2']_Q = C_1 Y_1' + C_2 Y_2'$.

(2.5) Lemma. *If* $\dim G = 0$, *then there is a smooth fibration $s: M \to T^2$, and $\mathscr{F}$ is the foliation of M by the fibers of s. In particular, $\mathscr{F}$ is a Lie foliation modeled on the Lie group $\boldsymbol{R}^2$.*

Proof. A foliation given by the fibers of a fibration onto a Lie group is always a Lie foliation modeled on the universal covering group. Since T^2 is the only compact parallelizable 2-manifold, by (4.4) of [1] it suffices to show $\mathscr{F}$ has a closed leaf.

Since the orbits of G and hence of $\rho(\pi_1(M))$ are closed regular 0-dimensional submanifolds of $\boldsymbol{R}^2$, the union of all leaves of $\hat{\mathscr{F}}$ covering a particular leaf A

of $\mathscr{F}$ is a closed regular submanifold of $\hat{M}$. Hence the leaf A is proper, and A is closed by [1, (4.2)]. q.e.d.

To complete the proof of (2.2) we may now assume that $\dim G = 1$. For this case, the proof will be accomplished by a series of lemmas.

Let G_0 be the connected component of the identity in G, and let $X \in \mathfrak{X}(\boldsymbol{R}^2)$ be the complete vector field that generates the action of G_0 on $\boldsymbol{R}^2$. Let $\langle\ ,\ \rangle$ be the Riemannian metric on $\boldsymbol{R}^2$ for which (X_1, X_2) is an orthonormal frame, and let $X^\perp \in \mathfrak{X}(\boldsymbol{R}^2)$ be the unit normal to X with respect to $\langle\ ,\ \rangle$ such that $(X_p^\perp, X_p)$ determines the same orientation as (X_{1_p}, X_{2_p}).

(2.6) Lemma. *$X^\perp$ is invariant under the action of G on $\boldsymbol{R}^2$.*

Proof. Since G_0 is a 1-dimensional normal subgroup of G and X is a basis of the Lie algebra of G_0, for $g \in G$ we have $\operatorname{ad}(g)X = f_g \cdot X$ for some $0 \neq f_g \in \boldsymbol{R}$. It follows that the diffeomorphism φ_g satisfies $\varphi_{g*p} X_p = f_g \cdot X_{\varphi g(p)}$.

Since φ_g preserves the fields X_1 and X_2, it preserves the orientation of $\boldsymbol{R}^2$ determined by (X_1, X_2). Since for $p \in \boldsymbol{R}^2$ at least one of (X_p, X_{1_p}) or (X_p, X_{2_p}) is a framing of $T_p(\boldsymbol{R}^2)$, it follows that $f_g > 0$ for all $g \in G$. Hence $(\varphi_{g*p} X_p^\perp, X_{\varphi g(p)})$ determines the same orientation as $(X_{\varphi g(p)}^\perp, X_{\varphi g(p)})$.

The lemma follows from the definition of $X^\perp$ and the observation that φ_g preserves the metric $\langle\ ,\ \rangle$.

(2.7) Corollary. *$X^\perp$ is complete as a vector field on $\boldsymbol{R}^2$.*

Proof. By definition, $X^\perp$ is a nowhere zero section of the normal bundle of the codimension-one foliation $\mathscr{F}_0$ of $\boldsymbol{R}^2$ given by the orbits of the action of G_0. Since $X^\perp$ is invariant under the action of G, it is invariant under the action of G_0, and hence is parallel along the leaves of $\mathscr{F}_0$.

Since $\rho(\pi_1(M))$ is a subgroup of G, it acts on $\boldsymbol{R}^2$ as a group of diffeomorphisms preserving the foliation $\mathscr{F}_0$ and the field $X^\perp$. Hence the codimension-one foliation $p^{-1}(\mathscr{F}_0)$ of $\hat{M}$ (where p is the projection $p\colon \hat{M} \to \boldsymbol{R}^2$) and the field $p^{-1}(X^\perp)$ (i.e., $p^{-1}(X^\perp) \in \Gamma(\hat{Q})$ and $p_*(p^{-1}(X^\perp)) = X^\perp$) are invariant under the action of $\pi_1(M)$ on $\hat{M}$. This defines a codimension-one e-foliation $\tilde{\mathscr{F}}(= \pi(p^{-1}(\mathscr{F}_0)))$ of M which is integral to the foliation $\mathscr{F}$ (i.e., the leaves of $\mathscr{F}$ are tangent to the leaves of $\tilde{\mathscr{F}}$) and whose e-structure $(\pi_*(p^{-1}(X^\perp)))$ is carried to the field $X^\perp$ via lifting to $\hat{M}$ and projection along the leaves of $\hat{\tilde{\mathscr{F}}}$. In particular, $X^\perp$ is a complete vector field on $\boldsymbol{R}^2$.

(2.8) Corollary. *There is a system of coordinates (x, y) on $\boldsymbol{R}^2$ such that $\partial/\partial x = X^\perp$ and $\partial/\partial y = X$.*

Proof. $X^\perp$ and X are complete everywhere linearly independent vector fields. By (2.6), $[X, X^\perp] = 0$, hence they are coordinate vector fields.

q.e.d.

Notice that with respect to this coordinate system, the action of G_0 on $\boldsymbol{R}^2$ is given by translation in the y-direction.

(2.9) Lemma. *$G/G_0 \cong \boldsymbol{Z}$.*

Proof. G/G_0 acts freely on $\boldsymbol{R}^1(= G_0/\boldsymbol{R}^2)$ as a group of automorphisms of the Lie group structure on $\boldsymbol{R}^1$ given by projecting the vector field $X^\perp$ along the leaves of $\mathscr{F}_0$. Hence G/G_0 is realized as a subgroup of the Lie group $\boldsymbol{R}$.

Since the orbits of G are closed in $\boldsymbol{R}^2$, it follows that the orbits of G/G_0 are closed in $\boldsymbol{R}^1$ and hence that $G/G_0 \cong \boldsymbol{Z}^r$, $r = 0$ or 1.

Since M is compact and $p: \hat{M} \to \boldsymbol{R}^2$ induces a continuous map $\bar{p}: M = \pi_1(M)\backslash\hat{M} \to \rho(\pi_1(M))\backslash\boldsymbol{R}^2$, we conclude that $\rho(\pi_1(M))\backslash\boldsymbol{R}^2$ is compact. The inclusion of $\rho(\pi_1(M))$ into G induces a continuous map $\rho(\pi_1(M))\backslash\boldsymbol{R}^2 \to G\backslash\boldsymbol{R}^2$. Hence $G\backslash\boldsymbol{R}^2$ is compact.

Since $G\backslash\boldsymbol{R}^2 \cong (G/G_0)\backslash(G_0\backslash\boldsymbol{R}^2) \cong (G/G_0)\backslash\boldsymbol{R}^1$, G/G_0 is nontrivial. Hence $G/G_0 \cong \boldsymbol{Z}$. q.e.d.

By replacing $X^\perp$ by some constant multiple $cX^\perp$ we can assume that the action of a generator $\bar{g} \in G/G_0$ on $\boldsymbol{R}^1 \cong G_0\backslash\boldsymbol{R}^2$ is given by $\bar{g}(x) = x + 1$.

(2.10) Lemma. *G is a semi-direct product of $\boldsymbol{R}$ by $\boldsymbol{Z}$($G \cong \boldsymbol{R} \times_\varphi \boldsymbol{Z}$). More precisely, $G \cong \{(t, n) \mid t \in \boldsymbol{R}, n \in \boldsymbol{Z}\}$ with group operation given by $(t_1, n_1) \circ (t_2, n_2) = (t_1 + a^{n_1}t_2, n_1 + n_2)$ where $a \neq 0$ is a real number.*

Proof. The lemma follows easily from the exact sequence of Lie groups: $0 \longrightarrow \boldsymbol{R}(\cong G_0) \stackrel{i}{\longrightarrow} G \stackrel{j}{\longrightarrow} \boldsymbol{Z}(\cong G/G_0) \longrightarrow 0$.

Let $\bar{g} \in G/G_0$ be a generator and choose $g \in G$ such that $j(g) = \bar{g}$. Then $\lambda: \boldsymbol{Z} \to G$ given by $\lambda(n) = g^n$ splits the exact sequence and is a Lie group homomorphism. Since $\boldsymbol{R} \cong G_0$ is a normal subgroup of G, we can define a homomorphism $\varphi: \boldsymbol{Z} \to \operatorname{Aut}(\boldsymbol{R})$ by $\varphi(n): t \mapsto g^n t g^{-n}$. By identifying $\operatorname{Aut}(\boldsymbol{R})$ with the multiplicative group of real numbers, we find some $a \neq 0$ such that $\varphi(1)$ is multiplication by a.

It follows that $\psi: \boldsymbol{R} \times_\varphi \boldsymbol{Z} \to G$ defined by $\psi(t, n) = tg^n$ is a Lie group isomorphism.

(2.11) Lemma. *For $(t, n) \in G \cong \boldsymbol{R} \times_\varphi \boldsymbol{Z}$ and (x, y) the coordinate system of (2.8), $\varphi_{(t,n)}(x, y) = (x + n, a^n y + t + c_n)$ where c_n is a constant depending on n and the above choice of g.*

Proof. We have already noted that $\varphi_{(t,0)}(x, y) = (x, y + t)$. Let $c(x)$ be the function defined by $\varphi_{(0,1)}(x, 0) = (\bar{g}(x), c(x)) = (x + 1, c(x))$. Since $\partial/\partial x = X^\perp$ is invariant under the action of G, it follows that $c(x)$ is the constant function c_1. Then

$$\varphi_{(0,1)}(x, y) = \varphi_{(0,1)}\varphi_{(y,0)}(x, 0) = \varphi_{(ay,0)}\varphi_{(0,1)}(x, 0) = (x + 1, ay + c_1) ,$$

$$\begin{aligned} \varphi_{(0,n)}(x, y) = \varphi_{(0,n-1)}(x + 1, ay + c_1) &= \cdots \\ &= (x + n, a^n y + a^{n-1}c_1 + a^{n-2}c_1 + \cdots + c_1) \\ &\equiv (x + n, a^n y + c_n) . \end{aligned}$$

The formula follows from writing $\varphi_{(t,n)} = \varphi_{(t,0)} \circ \varphi_{(0,n)}$. q.e.d.

We can now compute $\varphi_{(t,n)*(x,y)}(\partial/\partial y_{(x,y)}) = a^n(\partial/\partial y)_{\varphi_{(t,n)}((x,y))}$. Recall that from (2.6) we had written $\varphi_{g*p}X_p = f_g X_{\varphi_g(p)}$ where $f_g > 0$, hence $a > 0$ and we can define the field $a^x(\partial/\partial y)$.

Let $Z_1 = X^\perp = \partial/\partial x$ and $Z_2 = a^x(\partial/\partial y)$. Then Z_1 and Z_2 are everywhere linearly independent complete vector fields on $\boldsymbol{R}^2$, which are invariant under the action of G. Since $\rho(\pi_1(M)) < G$, the absolute parallelism (Z_1, Z_2) on $\boldsymbol{R}^2$ is associated to an e-structure (Y_1', Y_2') for $\mathscr{F}$. Finally since $[Z_1, Z_2] = \log(a)Z_1$, it follows that $[Y_1', Y_2']_Q = \log(a)Y_1'$. This completes the proof of (2.2).

(2.12) Corollary. *Let $\mathscr{F}$ be a codimension-two e-foliation of a closed manifold M. Then $\mathscr{F}$ is defined by a 2-form $\omega = \omega_1 \wedge \omega_2$ such that either $d\omega_1 = 0 = d\omega_2$ or $d\omega_1 = \omega_1 \wedge \omega_2$ and $d\omega_2 = 0$.*

Proof. There are only two simply connected 2-dimensional Lie groups, $\boldsymbol{R}^2$ and the 2-dimensional affine group $\mathfrak{A}$.

3. An example of a Lie foliation modeled on the affine group $\mathfrak{A}$

In [1], Conlon had remarked that a codimension-two e-foliation which admits a closed leaf is a Lie foliation modeled on the Lie group $\boldsymbol{R}^2$ (in Conlon's terminology, a foliation with a "strong transverse e-structure"). In addition, assuming $\pi_1(M, A)$ was abelian, he showed that a codimension-two e-foliation of a closed manifold M always admits a C^0 strong transverse e-structure (i.e., there exists an $\boldsymbol{R}^2$-cocycle of class C^0 representing the foliation such that the $g_{\alpha\beta}$'s are restrictions of translations in $\boldsymbol{R}^2$).

Notice that for a codimension-two Lie foliation of a closed manifold M, modeled on the Lie group G, we have realized $\pi_1(M, A)$ as a subgroup of G such that the space $\pi_1(M, A)\backslash G$ is compact. Since G is either $\boldsymbol{R}^2$ or the affine group $\mathfrak{A}$, and since $\mathfrak{A}$ does not contain an abelian subgroup which compactifies it, we have the following corollary of (2.2).

(3.1) Corollary. *A codimension-two e-foliation of a closed manifold M is a Lie foliation modeled in the Lie group $\boldsymbol{R}^2$ if and only if $\pi_1(M, A)$ is abelian.*

The loss of differentiability in Conlon's approach resulted from applications of a theorem of Sacksteder where one must introduce a possibly new differentiable structure on the manifold.

Conlon also conjectured that every codimension-two e-foliation of a closed manifold M admits a strong transverse e-structure and noted that a codimension-two e-foliation for which the group $\pi_1(M, A)$ was not abelian would give a counterexample to this conjecture. By our present results, such a foliation would be a Lie foliation modeled on the Lie group $\mathfrak{A}$. We have the following example.

Let $\varphi\colon T^2 \to T^2$ be the diffeomorphism induced by the linear mapping $\hat{\varphi}\colon \boldsymbol{R}^2 \to \boldsymbol{R}^2$ given by multiplication by the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, and let M^3 be the manifold $T^2 \times [0, 1]/(p, 0) \sim (\varphi(p), 1)$. Then $\boldsymbol{R}^3 = \hat{M}^3$ and $\pi_1(M^3)$ is the group of diffeomorphisms generated by

$$\varphi_1\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+1 \\ y \\ z \end{pmatrix}, \qquad \varphi_2\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y+1 \\ z \end{pmatrix}$$

and

$$\varphi_3\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y \\ x + 2y \\ z - 1 \end{pmatrix} .$$

Let $\hat{X}, \hat{Y}, \hat{Z} \in \mathfrak{X}(\boldsymbol{R}^3)$ be given by

$$\hat{X} = \left(\frac{3 + \sqrt{5}}{2}\right)^{-z}\left(\frac{\sqrt{5} - 1}{2}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right),$$
$$\hat{Y} = \left(\frac{3 - \sqrt{5}}{2}\right)^{-z}\left(\frac{\partial}{\partial x} + \left(\frac{1 - \sqrt{5}}{2}\right)\frac{\partial}{\partial y}\right),$$
$$\hat{Z} = \frac{\partial}{\partial z} .$$

Then the fields $\hat{X}$, $\hat{Y}$ and $\hat{Z}$ are everywhere linearly independent and are invariant under the action of $\pi_1(M)$. Hence the fields $X = \pi_* \hat{X}$, $Y = \pi_* \hat{Y}$ and $Z = \pi_* \hat{Z} \in \mathfrak{X}(M)$, (where $\pi: \boldsymbol{R}^3 \to M^3$ is the projection), are well defined.

Let $\mathscr{F}$ be the codimension-two foliation of M^3 given by integral curves to the vector field X. We compute $[X, Y] = [\pi_* \hat{X}, \pi_* \hat{Y}] = \pi_*[\hat{X}, \hat{Y}] = 0$ and

$$[X, Z] = \log\left(\frac{3 + \sqrt{5}}{2}\right)X ,$$

and conclude that (Y, Z) is an e-structure for $\mathscr{F}$. Furthermore since

$$[Y, Z] = \log\left(\frac{3 - \sqrt{5}}{2}\right)Y ,$$

$\mathscr{F}$ is a Lie foliation modeled on $\mathfrak{A}$.

For the coordinates

$$x' = \left(\frac{\sqrt{5} - 1}{2}x + y\right), \quad y' = \left(x + \frac{\sqrt{5} - 1}{2}y\right), \quad z' = z$$

on $\boldsymbol{R}^3 (= \hat{M}^3)$, $\hat{\mathscr{F}}$ is the foliation of $\boldsymbol{R}^3$ by lines parallel to the x'-axis (${}^3\hat{M} \cong \boldsymbol{R}^1_{(x')} \times \boldsymbol{R}^2_{(y',z')}$ as in (2.1)), and the induced action of $\pi_1(M^3)$ on $\boldsymbol{R}^2_{(y',z')}$ is given by

$$\rho(\varphi_1)\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} y' + 1 \\ z' \end{pmatrix}, \qquad \rho(\varphi_2)\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} y' + \dfrac{1 - \sqrt{5}}{2} \\ z' \end{pmatrix}$$

and

$$\rho(\varphi_3)\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} \dfrac{3-\sqrt{5}}{2}y' \\ z'-1 \end{pmatrix}.$$

That is, $\pi_1(M) \cong \pi_1(M, A) \cong \rho(\pi_1(M)) \cong \boldsymbol{Z}^2 \times_\psi \boldsymbol{Z}^1$ where $\psi(1)$ is multiplication by $((3-\sqrt{5})/2)$.

Since $Y \in \Gamma(Q)$ is parallel along the leaves of $\mathscr{F}$, the bundle $E \oplus \text{span}\,(Y)$ is the tangent bundle of a codimension-one foliation $\mathscr{F}_Y$ of M.

It is interesting to note that in the example the codimension-one foliation $\mathscr{F}_Y$ is an e-foliation of M^3 (it is the foliation of M^3 by the fibers of a bundle $T^2 \hookrightarrow M^3 \to S^1$). However the codimension-one foliation $\mathscr{F}_Z$ admits a transverse H-structure where H is a discrete subgroup of $GL(1)$, but does not admit a transverse e-structure. Indeed E. Fedida [3] points out that this foliation has every leaf dense in M^3, but not all leaves are diffeomorphic—some leaves are diffeomorphic to $\boldsymbol{R}^2$ and some to $S^1 \times \boldsymbol{R}^1$.

4. A structure theorem for codimension-two e-foliations

Let $\mathscr{F}$ be a codimension-two e-foliation of the closed manifold M. Then the lifted foliation $\hat{\mathscr{F}}$ of $\hat{M}$ is the foliation of $\hat{M}$ by the fibers of a fibration $p: \hat{M} \to G$ ($G = \boldsymbol{R}^2$ or $\mathfrak{A}$). To study a leaf $A \in \mathscr{F}$, we study the orbit of a leaf $\hat{A} \in \hat{\mathscr{F}}$ covering A under the action of $\pi_1(M)$. This is just the inverse image under p of the orbit of a point $g \in G$ under left translation by elements of the subgroup $\pi_1(M, A) < G$. In particular, the closure $\bar{A}$ in M of a leaf $A \in \mathscr{F}$ corresponds to an orbit of the closed subgroup $\overline{\pi_1(M, A)} < G$.

If the Lie group $\overline{\pi_1(M, A)}$ is 2-dimensional, then $\overline{\pi_1(M, A)} = G$ and the orbits of $\pi_1(M, A)$ are dense in G, and it follows that the leaves of $\mathscr{F}$ are dense in M.

If $\overline{\pi_1(M, A)}$ is 0-dimensional, then $\overline{\pi_1(M, A)} = \pi_1(M, A)$ and it follows as in (2.5) that every leaf of $\mathscr{F}$ is closed, and $\mathscr{F}$ is the foliation of M by fibers of a smooth fibration $s: M \to T^2$. In this case $\pi_1(M, A) \cong \boldsymbol{Z}^2$.

If $\overline{\pi_1(M, A)}$ is 1-dimensional, then the arguments of (2.10) show that $\overline{\pi_1(M, A)} \cong \boldsymbol{R} \times_\varphi \boldsymbol{Z}$. Furthermore, each leaf L of the codimension-one e-foliation $\bar{\mathscr{F}}$ of M in the proof of (2.7) is closed in M and is itself e-foliated (in codimension one) by the leaves of $\mathscr{F}$. Then by Reeb's theorem $\bar{\mathscr{F}}$ is the foliation of M by fibers of a smooth fibration $s: M \to S^1$ and the group $\pi_1(M, L) \cong \boldsymbol{Z}^1$. Each leaf $A \in \mathscr{F}$ is dense in a leaf $L \in \bar{\mathscr{F}}$ and $\pi_1(L, A) \cong \boldsymbol{Z}^k$, $k \geq 2$. It follows that $\pi_1(M, A) \cong \boldsymbol{Z}^k \times_\varphi \boldsymbol{Z}^1$.

We have the following theorem.

(4.1) Theorem. *Let $\mathscr{F}$ be a codimension-two e-foliation of a closed manifold M. Then one of the following holds:*

(i) *every leaf is closed and is the fiber of a smooth fibration $s: M \to T^2$ and the group $\pi_1(M, A) \cong \boldsymbol{Z}^2$,*

(ii) *the closure of every leaf is a fiber of a smooth fibration* $s: M \to S^1$ *and the group* $\pi_1(M, A) \cong Z^k \times_\varphi Z^1$, $k \geq 2$,

(iii) *every leaf is dense.*

Remark. If $\mathscr{F}$ is a Lie foliation modeled on R^2, then $\pi_1(M, A) = Z^k$, $k \geq 2$. If $\mathscr{F}$ is a Lie foliation modeled on $\mathfrak{A}$, then we can choose an e-structure $(Y_1, Y_2) \in \Gamma(Q) \subset \mathfrak{X}(M)$ such that $[Y_1, Y_2]_Q = Y_1$. In particular $\mathscr{F}_{Y_1}$ is a codimension-one e-folation of M, and for $L \in \mathscr{F}_{Y_1}$ we have $\pi_1(M, A) = \pi_1(L, A) \times_\varphi \pi_1(M, L)$. For case (iii) of (4.1), we have $\pi_1(M, A) = \pi_1(L, A) \times_\varphi Z^k$, $k \geq 2$ where $\pi_1(L, A)$ is a torsion free abelian group. We do not know if $\pi_1(L, A)$ is necessarily finitely generated, or indeed if there exists a Lie foliation modeled on $\mathfrak{A}$ of a closed manifold with every leaf dense.

We also remark that Conlon had proven a C^0 version of (4.1) under the assumption that $\pi_1(M, A)$ was abelian, [1, (4.5)], and that (4.1) is a special case of a theorem of Fedida on the structure of Lie foliations [3].

Added in proof. K. M. de Cesare [*On transversely parallelizable, codimension-two foliations*, preprint] has announced that a Lie foliation modelled on the Lie group of a closed manifold cannot admit a dense leaf and has stated a refinement of Theorem (4.1). We also remark that P. Molino [*Étude des feuilletages transversalement complets et applications*, Ann. Sci. École Norm. Sup., to appear] has stated a similar structure theorem for codimension-two e-foliations as a particular case of a structure theorem for transversally complete foliations.

5. Tischler's theorem and some corollaries

In this section, we use (2.12) to draw some immediate but pleasant corollaries of the celebrated theorem of D. Tischler.

(5.1) Theorem (*Tischler* [8]). *Let M be a closed manifold. Suppose M admits m linearly independent nonvanishing closed* 1*-forms. Then M is a fiber bundle over* T^m.

(5.2) Corollary. *If M supports a codimension-two e-foliation, then M is a fiber bundle over* S^1. *In particular* $\dim_R H^1(M, R) \geq 1$.

(5.3) Corollary. *M supports a Lie foliation modeled on* R^2 *if and only if M is a fiber bundle over* T^2. *In particular* $\dim_R H^1(M, R) \geq 2$.

Tischler also shows that a codimension-one e-foliation of a compact manifold M with $\dim_R H^1(M, R) = 1$ must have every leaf closed [8, Theorem 2].

Since a codimension-two e-foliation is always tangent to a codimension-one e-foliation, we have

(5.4) Corollary. *If M supports a codimension-two e-foliatiotion* $\mathscr{F}$ *and* $\dim_R H^1(M, R) = 1$, *then* $\mathscr{F}$ *is a Lie foliation modeled on* $\mathfrak{A}$, *and the closure of every leaf of* $\mathscr{F}$ *is a fiber of a smooth fibration* $s: M \to S^1$.

Notice also in this case, the fiber F of the above fibration is itself e-foliated in codimension one, hence by (5.1) there is a fibration $p: F \to S^1$ but the manifold M does not fiber over T^2.

6. Codimension-two e-foliations of 3-manifolds

Let M^3 be a closed oriented 3-dimensional manifold, and $\mathscr{F}$ a codimension-two e-foliation of M^3.

(6.1) Proposition. *If no leaf of $\mathscr{F}$ is closed, then M^3 is a T^2-bundle over S^1, and $\pi_1(M^3)$ is the semi-direct product of $\boldsymbol{Z}^2$ by $\boldsymbol{Z}^1$.*

Proof. Since no leaf of $\mathscr{F}$ is closed, every leaf is diffeomorphic to $\boldsymbol{R}^1$ and $\pi_1(M^3) \cong \pi_1(M^3, A)$. In particular $\pi_1(M^3)$ is a subgroup of either $\boldsymbol{R}^2$ or $\mathfrak{A}$.

By (5.1) there is a fibration $p: M^3 \to S^1$. The fiber F is a closed oriented surface hence is determined by its genus. From the exact homotopy sequence for fibrations we have $0 \to \pi_1(F) \to \pi_1(M^3) \to \pi_1(S^1) \to 0$. Hence $\pi_1(F)$ is isomorphic to a subgroup of either $\boldsymbol{R}^2$ or $\mathfrak{A}$. In particular the commutator subgroup of $\pi_1(F)$ is abelian. From the classification of the fundamental groups of surfaces, it follows that F is a surface of genus ≤ 1. Since $\hat{M}^3 = \boldsymbol{R}^3$, $F \neq S^2$. Hence $F = T^2$ and we have the exact sequence

$$0 \to \boldsymbol{Z}^2(= \pi_1(T^2)) \to \pi_1(M^3) \to \boldsymbol{Z}(= \pi_1(S^1)) \to 0 . \quad \text{q.e.d.}$$

The following is a special case of [2, Corollary 4].

(6.2) Corollary. *If $\pi_1(M^3)$ is abelian and no leaf of $\mathscr{F}$ is closed, then $M^3 = T^3$.*

Proof. Since $\mathscr{F}$ is a Lie foliation modeled on $\boldsymbol{R}^2$, by (5.3) we have a fibration $S^1 \to M^3 \to T^2$. By a standard spectral sequence argument we conclude that this bundle is nontrivial if and only if rank $H^1(M^3, \boldsymbol{Z}) = 2$. Since by (6.1), $\pi_1(M^3) = \boldsymbol{Z}^3 = H^1(M^3, \boldsymbol{Z})$ it follows that $M^3 = T^3$.

(6.3) Corollary. *If $\pi_1(M^3)$ is nonabelian and no leaf of $\mathscr{F}$ is closed, then rank $H^1(M^3, \boldsymbol{Z}) = 1$.*

Proof. We have $0 \longrightarrow \boldsymbol{Z}^2 \xrightarrow{i} \pi_1(M^3) \xrightarrow{j} \boldsymbol{Z} \longrightarrow 0$ and $\pi_1(M^3) < \mathfrak{A}$. Write $\mathfrak{A} = \{(s, t) \in \boldsymbol{R}^2 \mid (s_1, t_1) \circ (s_2, t_2) = (s_1 + a^{t_1}s_2, t_1 + t_2), 1 \neq a > 0\}$. The commutator subgroup C of $\pi_1(M^3)$ is nontrivial and consists of elements of the form $(s, 0)$. Since $C < i(\boldsymbol{Z}^2)$ and $i(\boldsymbol{Z}^2)$ is abelian, it follows that $i(\boldsymbol{Z}^2)$ is generated by elements $(s_1, 0)$, $(s_2, 0)$ for s_1, s_2 rationally independent real numbers.

Let $(\bar{s}, \bar{t}) \in \pi_1(M^3)$ be such that $j(\bar{s}, \bar{t})$ is a generator of $\boldsymbol{Z}$, $(\bar{t} \neq 0)$. Then $\pi_1(M^3)$ is generated by $(s_1, 0)$, $(s_2, 0)$, $(\bar{s}, \bar{t})$. Computing the commutators of the generator we get $(s_1(1 - a^{\bar{t}}), 0)$ and $(s_2(1 - a^{\bar{t}}), 0)$ are rationally independent elements of the commutator subgroup of $\pi_1(M^3)$ and it follows that rank $H^1(M^3, \boldsymbol{Z}) \leq 1$. q.e.d.

If $\mathscr{F}$ admits a closed leaf, then by [1, (4.4)] $\mathscr{F}$ is the foliation of M^3 by fibers of a fibration $S^1 \hookrightarrow M^3 \to T^2$. In summary we have

(6.4) Proposition. *Let M^3 be an oriented 3-manifold, and $\mathscr{F}$ a codimen-*

sion-two e-foliation of M^3. *If* $\beta_1 = \dim_R H^1(M^3, R)$, *then* $1 \leq \beta_1 \leq 3$ *and each of the following holds.*

(i) $\beta_1 = 1$ *if and only if* $\mathscr{F}$ *is a Lie foliation modeled on* $\mathfrak{A}$ (5.4). *In this case the closure of every leaf of* $\mathscr{F}$ *is a fiber of a smooth fibration* $T^2 \hookrightarrow M^3 \to S^1$.

(ii) $\beta_1 = 2$ *if and only if* $\mathscr{F}$ *is the foliation of* M^3 *by fibers of a nontrivial boundle* $S^1 \hookrightarrow M^3 \to T^2$.

(iii) $\beta_1 = 3$ *if and only if* $M^3 = T^3$.

Remark. In case (i) above, we can choose an e-structure (Y_1, Y_2) for $\mathscr{F}$ satisfying $[Y_1, Y_2]_Q = Y_1$. Then $\mathscr{F}_{Y_1}$ is the foliation of M^3 by fibers of a fibration $T^2 \hookrightarrow M^3 \to S^1$, and the foliation $\mathscr{F}_{Y_2}$ has every leaf dense in M^3.

A theorem of J. Plante's [6] asserts that for M^3 as above, if $\beta_1(M^3) \leq 1$ and the group $\pi_1(M^3)$ has non-exponential growth, then every transversely oriented foliation of M^3 has a compact leaf. In our case, since $\mathscr{F}_{Y_2}$ has no closed leaf, it follows that $\pi_1(M^3)$ must have exponential growth.

Plante also remarks that construction of § 3 of this paper with the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ replaced by an integer matrix of determinant ± 1 whose eigenvalues are on the unit circle but different from one yields a 3-manifold satisfying the hypotheses of his theorem. In particular, these are T^2 bundles over S^1 which do not admit codimension-two e-foliations.

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