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PERTURBATION THEORY FOR CONDITION (C) IN THE CALCULUS OF VARIATIONS

JILL P. MESIROV

I. INTRODUCTION

1. Introduction

One classical method of solving the Dirichlet problem for $\Delta u = f$ on a bounded domain $\Omega \subset \boldsymbol{R}^n$, is to minimize the Dirichlet integral

$$J(u) = \int_\Omega \tfrac{1}{2}|\nabla u|^2 + f(x)u \ ,$$

over an appropriate class of functions u on Ω.

The *generalized variational Dirichlet problem* is to study the critical points of a functional

$$J(u) = \int \mathscr{L}(u) \ ,$$

where $\mathscr{L}$, the Lagrangian, is a nonlinear differential operator from sections, with prescribed boundary values, of a fiber bundle E over a compact manifold M with boundary, to sections of the trivial line bundle $\boldsymbol{R}_M$.

The key step in finding a critical point which is a minimum, for example, is to show that the functional actually achieves its minimum value. For this we need some sort of compactness condition; we use the Palais-Smale condition (*C*).

To state this precisely, we consider $L^p_k(E)$, a manifold modeled on the Sobolev space of sections whose distributional derivatives up through order k are in L^p, with norm $\| \ \|_{p,k}$. A functional $J: L^p_k(E) \to \boldsymbol{R}$ satisfies *condition* (*C*), if given any subset S of $L^p_k(E)$ on which $|J|$ is bounded but $\|DJ\|$ is not bounded away from zero, then there is a critical point of J in the closure of S. If J is C^2 and bounded below, and satisfies condition (*C*), then J assumes a minimum on each component of $L^p_k(E)$, [13], [16].

The main question we consider is, if J_0 satisfies the Palais-Smale condition (*C*), under what conditions on a perturbation $\mathscr{V}$ can we show that the per-

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turbed functional $J = J_0 - \mathscr{V}$ also satisfies condition (C)? With this in mind, we observe that a functional J having both of the following two properties satisfies condition (C):

(a) A functional J is *pseudo-proper* on L_k^p if $|J(S)| \leq a$ for some set $S \subset L_k^p$ implies $\|u\|_{p,k} \leq b$ for all $u \in S$.

(b) A functional J is *coercive* on L_k^p, if given any bounded sequence (u_i) in L_k^p such that

$$(DJ_{u_i} - DJ_{u_j})(u_i - u_j) \to 0 ,$$

then (u_i) has an L_k^p convergent subsequence. (See § 2 for a more precise definition.)

The condition we call pseudo-proper, is classically sometimes called the coercive condition, and is written: $\|u\|_{p,k} \to \infty$ implies $|J(u)| \to \infty$.

In practice this is how these conditions are used. The pseudo-proper condition on J provides a weakly compact set, from which we get a candidate for the minimum of J as a weak limit. The coercive condition is used to show the weak limit is in fact the strong limit of a convergent subsequence. In the literature, condition (C) is almost always verified by checking these or similar conditions. The pseudo-proper condition is also used in monotonicity methods of solving partial differential equations [7].

Our work is an investigation of the pseudo-proper and coercive conditions. If $J_0(u) = \int \mathscr{L}(u)$ is pseudo-proper (resp. coercive) on L_k^p, what conditions on $\mathscr{V}(u) = \int V(u)$ insure $J = J_0 - \mathscr{V}$ is still pseudo-proper (resp. coercive)? We treat the two conditions separately. This is more than a stability question. It is not difficult to show that if a perturbation $\mathscr{V}$ is "small" enough, then it preserves the two conditions. We ask, rather, how large $\mathscr{V}$ can be.

The contents of the paper are as follows: § 2 contains technical preliminaries and notation. (We suggest skipping this section, referring back to it as the need arises.)

In § 3 we discuss the motivating example of geodesics in the presence of a bounded potential as a perturbation problem. Boundedness is too restrictive a condition for most applications. The point of the remainder of the paper is to investigate pseudo-properness, coercivity, and condition (C) under weaker assumptions on perturbations.

§§ 4 through 7 contain the perturbation results for the pseudo-proper condition. We begin with an especially illuminating and useful special case. For $u \in L_1^2(M, \boldsymbol{R})$ with "zero boundary values," let

$$J_0(u) = \int |\nabla u|^2 .$$

J_0 is the square of the L_1^2 norm of u and thus pseudo-proper. Let $V: M \times \boldsymbol{R}$

$\to \boldsymbol{R}$ be continuous, and $\lambda_1 > 0$ be the first eigenvalue of the Laplacian. Finally, let $J(u) = J_0(u) - \int V(x, u)$. The essential content of Theorem 5.1 is

Theorem 5.1′. (a) *If*

$$V(x, s) \leq const. + Ks^2$$

for all $s \in \boldsymbol{R}$, *and* $K < \lambda_1$, *then* J *is bounded below and pseudo-proper.*

(b) *If*

$$V(x, s) \geq const. + \lambda_1 s^2 ,$$

then J *is not pseudo-proper.*

The theorem can be viewed as an asymptotic growth estimate on V, compare with [6, § 8]. Part (a) shows that if V grows at most quadratically, at a rate bounded by λ_1, then it preserves the pseudo-properness of $\int |\nabla u|^2$; part (b) shows the sharpness of the bound on the growth rate found in (a). As an outgrowth of Theorem 5.1, we give an example of a functional which satisfies condition (C), but is not pseudo-proper.

The remainder of §§ 5 through 7 extends Theorem 5.1 to more general functionals J_0, arising from kth order Lagrangians, and perturbations $\mathscr{V}$. In the second half of § 5 we extend part (a) to general perturbations of order zero, i.e., to those which only depend on u and not any derivatives of u. In this context we also discuss geodesics in the presence of possibly unbounded potentials. (The essential step in our discussion of geodesics is to find an analogue of λ_1 in a setting where there is no linear structure.) The extension of part (a) to general perturbations of order $k - 1$ is carried out in § 6, while in § 7 we extend part (b) to perturbations of order $k - 1$.

The coercive condition is dealt with in § 8. We consider coercive functionals $J_0(u) = \int \mathscr{L}(u)$, where $\mathscr{L}$ is a polynomial differential operator of order k and satisfies an auxiliary condition which insures $\mathscr{L}$ is smooth from $L_k^p(E)$ to $L_0^1(\boldsymbol{R}_M)$. The perturbations $\mathscr{V}(u) = \int V(u)$ are also polynomial, only depend on the $(k - 1)$-jet of u, and also satisfy the auxiliary condition. Under these hypotheses we get an optimal result, namely that all such perturbations preserve coercivity. We also show that we can relax the auxiliary conditions on V depending on the relation between p and the dimension of M. Related conditions are also given for the case where the functionals act on sections of a vector bundle ξ, in which case $L_k^p(\xi)$ is a Banach space.

Some of these results were announced in [9]. In a subsequent paper [11] we will continue our investigation of which functionals are pseudo-proper and coercive. There we show, for example, that if $\mathscr{L}(u)$ is a quadratic polynomial in u and its derivatives, then modulo $\int |u|^2$, $J(u) = \int \mathscr{L}(u)$ is pseudo-proper if and only if the bilinear form associated with $\mathscr{L}$ is uniformly strongly elliptic.

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2. Preliminaries and notation

M denotes a compact connected C^∞ Riemannian manifold of dimension $n \geq 1$ with or without smooth boundary, ∂M. ξ and η denote finite dimensional C^∞ vector bundles over M, and $C^k(\xi)$, $0 \leq k \leq \infty$, the linear space of C^k sections of ξ. $C_0^\infty(\xi)$ is the linear space of C^∞ sections with compact support in the interior of M. We define $C^\infty(E)$ for E a finite dimensional C^∞ fiber bundle over M in a similar manner. $\boldsymbol{R}_M$ is the trivial line bundle $M \times \boldsymbol{R}$ over M.

An RMC structure for a vector bundle ξ consists of a Riemannian metric for ξ, whose norm we write as $| \ |$ and inner product $\langle \ , \ \rangle$, together with a Riemannian connection [3]. We denote by ∇ the covariant derivative with respect to the connection, by ∇^j the jth covariant derivative, and by Δ the Laplacian with respect to the given metric. We use the sign convention giving $\Delta u = u_{xx} + u_{yy}$ for the standard metric on $\boldsymbol{R}^2$.

Choosing some RMC structure on TM and ξ, we define norms

$$\|u\|_{p,k} = \sum_{j=0}^{k} \left(\int_M |\nabla^j u|^p \right)^{1/p} ,$$

for nonnegative integers k, and real $p \geq 1$. We define $L_k^p(\xi)$ as the Sobolev space of sections whose covariant derivatives up through order k are in $L^p(\xi)$. $L_k^p(\xi)$ is a Banach space with respect to $\| \ \|_{p,k}$. If $p = 2$, $L_k^2(\xi)$ is a Hilbert space with inner product

$$(u, v)_k = \sum_{j=0}^{k} \int_M \langle \nabla^j u, \nabla^j v \rangle$$

and norm

$$\|u\|_{2,k} = (u, u)_k^{1/2} .$$

Different RMC structures will yield equivalent norms since M is compact. There are other norms equivalent to the one given above, we will use the one best suited to the problem at hand. We refer often to the standard Sobolev embedding and Rellich Theorems (see [4, pp. 22–23, 28, 31]).

If $pk > n$, we can give $L_k^p(E)$ the structure of a C^∞ infinite dimensional Finsler manifold, modeled on $L_k^p(\xi)$. For more detailed expositions of possible precedures see [19], [2], [3], and [12]. One step necessary in one procedure is the construction of vector bundle neighborhoods. As we will need this for a local argument in § 8, we give the definition here. For a proof of the existence of vector bundle neighborhoods see [12, § 12].

Definition 2.1. Let $u \in C^0(E)$. A *vector bundle neighborhood* of u in E is a vector bundle ξ over M such that ξ is an open subbundle of E and $u \in C^0(\xi)$.

We can now give a more precise definition of coercivity for functionals on $L_k^p(E)$.

Definition 2.2. Let E be a C^∞ fiber bundle over a compact connected n-dimensional C^∞ manifold M, and $pk > n$. A functional $J: L_k^p(E) \to \boldsymbol{R}$, is called *coercive* if on any vector bundle neighborhood $\xi \subset E$, if

$$[DJ_{s_i} - DJ_{s_j}](s_i - s_j) \to 0 ,$$

and $\|s_i\|_{L_k^p(\xi)}$ is bounded, then s_i has an $L_k^p(\xi)$ strongly convergent subsequence.

Definition 2.3. A map $P: C^\infty(E) \to C^\infty(\xi)$ is called a *polynomial differential operator of order k and weight at most w*, if for each local representation of P, the jth component of $Pu(x)$ is

$$[Pu(x)]_j = F_j(x, u_1(x), \cdots, u_m(x), D^\alpha u_i(x)) , \qquad 1 \leq |\alpha| \leq k ,$$

where each of the functions F_j is a sum of terms of the form

$$\Phi(x, u_1(x), \cdots, u_m(x)) D^{\beta_1} u_{l_1}(x) \cdots D^{\beta_q} u_{l_q}(x)$$

with $1 \leq |\beta_1| \leq k$, and $|\beta_1| + \cdots + |\beta_q| \leq w$, [12, p. 69]. (An intrinsic definition may be found in [10, appendix II].)

Definition 2.4. A polynomial differential operator is said to be *strict*, if for some local representation of P the functions Φ are of the form

$$\Phi(x, u_1(x), \cdots, u_m(x)) = a(x) u_{i_1} \cdots u_{i_t} .$$

(Note that this notion is invariant in the base variable of E (in x), but not in the fiber variable (i.e., in u).)

In other words, a polynomial differential operator is a polynomial in the derivatives of u whose coefficients may depend on u, but not on the derivatives of u. For a strict polynomial, the coefficients are further restricted to be polynomials in u.

Examples. (1) $P(u) = a(x)e^u(u')^2$ is a polynomial differential operator of order 1 and weight 2.

(2) $P(u) = a(x)u^3(u')^2$ is a strict polynomial differential operator of order 1 and weight 2.

For the rest of the paper when considering L_k^p we assume $1 < p < \infty$, since then bounded sets are weakly compact. All integration is with respect to the Riemannian measure on M.

Notation. We designate an open set Ω with compact closure contained in M by $\Omega \subset\subset M$. D^n denotes the unit n-disk, i.e., $\{x \in \boldsymbol{R}^n \mid |x| \leq 1\}$. The tangent bundle of a manifold M is written TM. For the unit interval $[0, 1]$, we use I.

To designate the space of L_k^p maps from a manifold M into $\boldsymbol{R}$ we use either of the following: $L_k^p(\boldsymbol{R}_M)$ or $L_k^p(M, \boldsymbol{R})$. For an open set $\Omega \subset \boldsymbol{R}^n$ we use $L_k^p(\Omega)$ and $L_k^p(\Omega, \boldsymbol{R})$ interchangeably. $L_k^p(E)_{\partial f}$ (resp. $L_k^p(\xi)_{\partial f}$) denotes the closure in $L_k^p(E)$ (resp. $L_k^p(\xi)$) of the set of $g \in L_k^p(E)$ (resp. $L_k^p(\xi)$) which agree with f on some neighborhood of the boundary of M. The closure of $C_0^\infty(\xi)$ in $L_k^p(\xi)$ is $L_k^p(\xi)_0$, while $C_0^\infty(\xi|_\Omega)$ is the space of $u \in C_0^\infty(\xi)$ with compact support in Ω. The uniform norm is written $\| \ \|_\infty$.

$D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$ in the usual multi-index notation. To designate weak convergence we use $\rightharpoonup$. Finally, the k-jet of a section u is written $j_k(u)$.

3. An example for motivation

The motivation for our perturbation problem comes from classical mechanics. We are given the following data:

N: differentiable manifold = configuration space
K: Riemannian metric on TN = kinetic energy
V: $\boldsymbol{R}$-valued function on N = potential energy
(By abuse of notation we may think of V as acting on TN.)
L: $K - V$ = Lagrangian.

The *motion of the system* is an extremal of L at a point $(p, v) \in TN$, where p is a position vector and v a velocity vector.

We will show that the action integral

$$E(u) = \int L(u) = \int K(u, u') - V(u)dt$$

satisfies condition (C), assuming that (i) $K(u, u') = \|u'(t)\|^2_{u(t)}$ is the metric on configuration space N, (ii) the potential V is a smooth bounded function on N, and (iii) $u \in L_1^2(I, N)_{P,Q}$, i.e., L_1^2 paths from P to Q in N.

It is not difficult to show that the energy integral $E^0(u) = \int K(u, u')dt$, the "unperturbed functional", satisfies condition (C) by verifying pseudo-properness and coercivity [17]. Assuming this, we consider the perturbed functional E. As we observed in the introduction, it suffices to prove the following two claims to establish condition (C) for E.

Claim 3.1. *E is pseudo-proper on $L_1^2(I, N)_{P,Q}$.*

Proof. Since E^0 is pseudo-proper on $L_1^2(I, N)_{P,Q}$, it is enough to show that if E is bounded on $S \subset L_1^2(I, N)_{P,Q}$, then E^0 is also bounded on S. Now V is bounded, so for all $u \in S$

$$0 \le E^0(u) = E(u) + \int V(u)dt \le E(u) + (\text{const.}) \text{ length } (I) .$$

Claim 3.2. *E is coercive on $L_1^2(I, N)_{P,Q}$.*

Proof. We will show that if a sequence u_i is bounded in L_1^2, and $(DE_{u_i} - DE_{u_j})(u_i - u_j) \to 0$, then u_i has a strongly L_1^2 convergent subsequence. Since E^0 is coercive and

$$(DE_{u_i} - DE_{u_j})(u_i - u_j) = (DE^0_{u_i} - DE^0_{u_j})(u_i - u_j) - \int (DV_{u_i} - DV_{u_j})(u_i - u_j)dt ,$$

it is enough to show that, for a relabeled subsequence of u_i,

$$\int (DV_{u_i} - DV_{u_j})(u_i - u_j)dt \to 0 .$$

The inclusion of L_1^2 into C^0 is compact, so there is a relabeled subsequence u_i which converges uniformly. Since V is smooth this means DV_{u_i} converges, which, along with the uniform boundedness of u_i, gives the desired result. q.e.d.

Modifying the above, we get the following more general result.

Theorem 3.3. *Let $J_0 : L_k^p(E)_{\partial f} \to \boldsymbol{R}$ be a C^1 functional on sections of a trivial fiber bundle $\pi : E \to M$ over a compact manifold M with fiber N, and $pk > \dim M$. Let $V : N \to \boldsymbol{R}$ be a smooth uniformly bounded function on the fiber. If J_0 is pseudo-proper and coercive, then so is $J(u) = J_0(u) - \int V(u)$. Moreover, if J_0 is bounded below it is enough to assume V is bounded from above.*

II. PSEUDO-PROPERNESS

4. Almost J-boundedness

In § 3, we dealt with the pseudo-proper half of the perturbation problem by showing that for a uniformly bounded perturbation $\mathscr{V}$ (or one which is bounded above), if the perturbed functional $J = J_0 - \mathscr{V}$ is bounded on a set $S \subset L_k^p(E)_{\partial f}$, then the original functional J_0 must also be bounded on S. This assumption on $\mathscr{V}$ is too strong. It is enough for $\mathscr{V}$ to be dominated by J_0, which leads us to the concept of almost J-boundedness.

Throughout §§ 4 through 7, unless otherwise stated, we do *not* assume $pk > n$.

Definition 4.1. Let J and $\mathscr{V}$ be functionals. Then $\mathscr{V}$ is *almost J-bounded* if there exist constants $\theta < 1$ and K such that for all u,

$$\mathscr{V}(u) \leq \theta J(u) + K .$$

The following theorem is an obvious consequence of the definition.

Theorem 4.2. *Let $J_0\colon L_k^p(E)_{\partial f} \to \boldsymbol{R}$ be a functional which is bounded from below. If $\mathscr{V}\colon L_k^p(E)_{\partial f} \to \boldsymbol{R}$ is almost J-bounded, then the perturbed functional $J = J_0 - \mathscr{V}$ is bounded below and pseudo-proper.*

Using Theorem 4.2 and the pseudo-properness of the functional $J_0(u) = \|u\|_{p,k}^p$, in subsequent sections we will give various conditions on perturbations $\mathscr{V}$ which insure the pseudo-properness of the perturbed functional

$$J(u) = J_0(u) - \mathscr{V}(u) = \|u\|_{p,k}^p - \mathscr{V}(u) .$$

Since the results extend in a straightforward manner to more general functionals J_0 which dominate the norm $\|u\|_{p,k}^p$, i.e.,

$$\|u\|_{p,k}^p \leq cJ_0(u) + K , \qquad (c, K\colon \text{constants}) ,$$

we will omit the details of the extension.

5. 0-order perturbations

We begin with an especially illuminating and useful example.

Let M be a compact Riemannian manifold with nonempty boundary. For $u \in L_1^2(M, \boldsymbol{R})_0$, let

$$J_0(u) = \int |\nabla u|^2 = \|u\|_{2,1}^2 .$$

Let $V\colon M \times \boldsymbol{R} \to \boldsymbol{R}$, and define a functional $\mathscr{V}$ by

$$\mathscr{V}(u) = \int V(x, u) .$$

Note that $\mathscr{V}$ is 0-order since it depends on u, but not any derivatives of u. For $j = 1, 2, \cdots$ let $\lambda_j > 0$ be the jth eigenvalue of the Laplacian, with corresponding orthonormal eigenfunctions $\phi_j \in L_1^2(M, \boldsymbol{R})_0$, $-\Delta\phi_j = \lambda_j\phi_j$. Finally, let $J = J_0 - \mathscr{V}$.

Theorem 5.1. (a) *If*

$$V(x, s) \leq const. + Ks^2$$

for all $s \in \boldsymbol{R}$, and $K < \lambda_1$, then J is bounded below and pseudo-proper.

(b) *If*

$$V(x, s) = const. + Ks^2 ,$$

and $K \geq \lambda_1$, then J is not pseudo-proper. Moreover, if $K = \lambda_j$ for any $j = 1, 2, \cdots$, then J does not satisfy condition (C).

(c) *If V is continuous, and there are constants $\gamma \geq \lambda_1$ and c such that*

$$V(x, s) \geq c + \gamma s^2 , \tag{5.2}$$

for all $s \geq 0$ (*or* $s \leq 0$), *then* J *is not pseudo-proper. In fact* (5.2) *need only hold on some open set* $\Omega \subset M$.

Remark. Part (a) of Theorem 5.1 says that if V grows at most quadratically with a rate of growth bounded by λ_1, then J will be pseudo-proper. We can rewrite this condition on V in terms of an asymptotic growth estimate, i.e.,

$$\limsup_{|s| \to \infty} \frac{V(x, s)}{|s|^2} \leq K < \lambda_1 .$$

In part (b) we see that if V grows quadratically, then the λ_1 "growth constant" is a sharp bound for the pseudo-properness of J.

Part (c) shows the sharpness of the restriction in part (a). In particular, if V grows faster than quadratically even on an open set $\Omega \subset M$, e.g., $V(x, s) \geq c_1 + c_2 |s|^t$ where $t > 2$, then J will not be pseudo-proper. As a special case,

$$V(x, s) = f(x) \pm s^3$$

satisfies (5.2) for $s \geq 0$ (for $+$), or $s \leq 0$ (for $-$).

There are similar phenomena in the more general situation.

Proof. (a) By Theorem 4.2, it is enough to show that V is almost J_0-bounded. First we recall that

$$\lambda_1 = \inf_{u \in L_1^2(M, \mathbf{R})_0} \frac{\int |\nabla u|^2}{\int |u|^2} > 0 .$$

Now

$$\int V(x, u) \leq \int (\text{const.} + Ku^2) \leq (\text{const.}) \operatorname{vol}(M) + \frac{K}{\lambda_1} \int |\nabla u|^2 .$$

But $K < \lambda_1$, so we see V is almost $\int |\nabla u|^2$-bounded.

(b) First observe that

$$J(u) = \int |\nabla u|^2 - Ku^2 = - \int u(\Delta u + Ku) ,$$

and

$$DJ_u(v) = 2 \int \nabla u \cdot \nabla v - Kuv = 2 \int u(-\Delta v) - Kuv .$$

Now writing u in an eigenfunction expansion, $u = \sum a_i \phi_i$, we see $\|u\|_{2,1}^2 = \sum \lambda_i a_i^2$, $J(u) = \sum (\lambda_i - K) a_i^2$, and

$$\|\nabla J_u\|_{2,1}^2 = \sum \lambda_l\left(1 - \frac{K}{\lambda_l}\right)^2 a_l^{\,2} .$$

Say $K \geq \lambda_1$, since $\lambda_j \to \infty$, $\lambda_N \leq K < \lambda_{N+1}$ for some $N \geq 1$. Let

$$u = \phi_1 + \left(\frac{\lambda_1 - K}{K - \lambda_{N+1}}\right)^{1/2} \phi_{N+1} .$$

If we define $u_j = ju$, then $J(u_j) = 0$ while $\|u_j\|_{2,1} \to \infty$, and hence J is not pseudo-proper. Note that if $K = \lambda_1$, we can let $u = \phi_1$. This establishes the first part of (b).

If $K = \lambda_N$, let $u_j = j\phi_N + (1/j)\phi_{N+1}$. Then we see that $|J(u_j)| \leq \lambda_{N+1} - K$, $\|\nabla J_{u_j}\| \to 0$, but $\|u_j\|_{2,1} \to \infty$. Hence we see J does not satisfy condition (C).

(c) Replacing u by $-u$ if necessary, we need only consider the case $s \geq 0$. Pick $z \perp \phi_1$. Then for any $\alpha \in \boldsymbol{R}$,

$$\|\alpha\phi_1 + z\|_{2,1}^2 = \int |\nabla(\alpha\phi_1 + z)|^2 = \alpha^2\lambda_1 + \int |\nabla z|^2 . \tag{5.3}$$

Also by (5.2)

$$J(\alpha\phi_1 + z) \leq \alpha^2\lambda_1 + \int |\nabla z|^2 - \gamma \int (\alpha\phi_1 + z)^2 - c_1 .$$

Consequently,

$$J(\alpha\phi_1 + z) \leq (\lambda_1 - \gamma)\alpha^2 + \int |\nabla z|^2 - c_1 . \tag{5.4}$$

Thus for fixed $z \perp \phi_1$, since $\gamma \geq \lambda_1$,

$$J(\alpha\phi_1 + z) \leq \text{const.} \qquad \text{for all } \alpha \in \boldsymbol{R} . \tag{5.5}$$

There are essentially two cases: as $\alpha \to \infty$, either (i) $\lim J(\alpha\phi_1 + z) \geq$ const. or (ii) $\lim J(\alpha\phi_1 + z) = -\infty$. The first corresponds to $V(x, s) = \lambda_1 s^2$, when $J(\alpha\phi_1) = 0$, and the second to $\gamma > \lambda_1$ as can be seen from (5.4).

Case (*i*). Say there is a $z \perp \phi_1$ such that $\limsup J(\alpha\phi_1 + z) \geq$ const. as $\alpha \to \infty$. Then there are $\alpha_j \in \boldsymbol{R}$, $\alpha_j \to \infty$,s uch that $J(\alpha_j\phi_1 + z) \geq$ const.. If $u_j = \alpha_j\phi_1 + z$, then by (5.3) and (5.5) we see that $\|u_j\|_{2,1} \to \infty$ but $|J(u_j)| \leq$ const.. Thus J is not pseudo-proper.

Case (*ii*). Assume that for all $z \perp \phi_1$, we have $\limsup J(\alpha\phi_1 + z) = -\infty$ as $\alpha \to \infty$. We can pick $z_j \perp \phi_1$ such that $\|z_j\|_\infty \leq 1$ and $\int |\nabla z_j|^2 \to \infty$. Let $F_j(\alpha) = J(\alpha\phi_1 + z_j)$. Since V is continuous and $\|z_j\|_\infty \leq 1$, $V(x, z_j)$ is bounded. Thus $\int |\nabla z_j|^2 \to \infty$ implies $F_j(0) \to \infty$ as $j \to 0$. Consequently for j sufficiently large, $F_j(0) \geq \mu$, for some constant μ. Moreover, $F_j(\alpha) \to -\infty$ as $\alpha \to \infty$. Therefore, since F_j is continuous, for each j sufficiently large there is an α_j such that $F_j(\alpha_j\phi_1 + z_j) = \mu$. Now let $u_j = \alpha_j\phi_1 + z_j$. Then $J(u_j) = \mu$ but $\|u_j\|_{2,1} \to \infty$, thus J is not pseudo-proper.

It is enough for (5.2) to hold only on some open set, since if $\Omega \subset M$ and J is not pseudo-proper on $L_1^2(\Omega)_0$, then J is not pseudo-proper on $L_1^2(M)_0$. This follows from the fact that if $u \in L_1^2(\Omega)_0$, then it can be extended to $\bar{u} \in L_1^2(M)_0$ by letting $\bar{u} \equiv u$ on Ω and $\bar{u} \equiv 0$ on $M - \Omega$.

Remark. In the case of $V(x, s) = Ks^2$ we can say more than part (b) of Theorem 5.1. In fact we can completely analyze $J(u) = \int |\nabla u|^2 - Ku^2$ in terms of K. J satisfies condition (C) if and only if $K \neq \lambda_j$ for $j = 1, 2, \cdots$. Furthermore, if $K < \lambda_1$ the critical points of J are minima; while if $K > \lambda_1$ and J satisfies condition (C) (so $K \neq \lambda_j$), the critical points of J are "saddle points." We will show in Theorem 8.13 that J is coercive for all K, although this fact is not needed here.

Proof of Remark. First we show that if J is bounded on a sequence u_j, and $\|\nabla J_{u_j}\|_{2,1} \to 0$, then the u_j's are converging strongly in $L_1^2(M, \boldsymbol{R})_0$. In fact, we show $\|u_j\|_{2,1}^2 \to 0$. If $u_j = \sum a_l\phi_l$, then

$$\|\nabla J_{u_j}\|_{2,1}^2 = \sum \lambda_l\left(1 - \frac{K}{\lambda_l}\right)^2 a_l^2 \geq \min_\nu\left(1 - \frac{K}{\lambda_\nu}\right)^2 \sum \lambda_l a_l^2 ,$$

since each term on the right hand side is positive. Now because $\lambda_l \to \infty$, $1 - K/\lambda_l \to 1$ as $l \to \infty$. Since K is not equal to any eigenvalue, $1 - K/\lambda_l \neq 0$ for any l, and therefore there is a $\gamma > 0$ such that $|1 - K/\lambda_l| \geq \gamma$ for every $l = 1, 2, \cdots$. Hence we see

$$\|\nabla J_{u_j}\|_{2,1}^2 \geq \gamma^2 \sum \lambda_l a_l^2 = \gamma^2 \|u\|_{2,1}^2 .$$

To study the nature of the critical points of J, we examine the second variation

$$D^2J_u(v, v) = 2\int \nabla v \cdot \nabla v - Kv^2 = 2\int v(-\Delta v - Kv) .$$

Thus, if $v = \sum b_l\phi_l$, then

$$D^2J_u(v, v) = \sum (\lambda_l - K)b_l^2 .$$

When $K < \lambda_1$, if $v \not\equiv 0$ then $D^2J_u(v, v) > 0$, because $\lambda_1 \leq \lambda_2 \leq \cdots$. Hence, if u is a critical point then it must be a minimum. On the other hand, if $\lambda_j < K < \lambda_{j+1}$, then we can choose v to make $D^2J_u(v, v)$ positive or negative.

q.e.d.

With this insight we are ready to answer the question of which zero order perturbations are almost J_0-bounded for $J_0(u) = \|u\|_{p,k}^p = \int |\nabla^k u|^p$, $u \in (L_k^p)_0$. If M has no boundary we must modify J_0 to $J_0(u) = \int (|\nabla^k u|^p + |u|^p)$, but the theorems go through with little change.

Let ξ be a finite dimensional C^∞ vector bundle over a compact connected C^∞ finite dimensional manifold M with boundary. Choose an *RMC* structure for TM and ξ. Let $J_0 : L_k^p(\xi)_0 \to \boldsymbol{R}$ be a functional defined as follows

$$J_0(u) = \int_M |\nabla^k u|^p \,.$$

Define constants $\lambda(p, k)$ by

$$\lambda(p, k) = \inf_{u \in L_k^p(\xi)_0} \frac{\int |\nabla^k u|^p}{\int |u|^p} \,.$$

Note that $\lambda(p, k)$ is the reciprocal of the norm of the continuous linear inclusion of $L_k^p(\xi)_0$ into $L_0^p(\xi)_0$, and thus is positive.

Theorem 5.6. *Let V be a 0-order differential operator from ξ to $\boldsymbol{R}_M$, i.e., $V : \xi \to \boldsymbol{R}_M$ is a fiber bundle morphism, such that there exist constants A and α such that*

$$\int V(u) \le A + \alpha \int |u|^p \qquad \text{for all } u \in L_k^p(\xi)_0 \,. \tag{5.7}$$

If $\alpha < \lambda(p, k)$, then

$$J(u) = J_0(u) - \int V(u)$$

is bounded below and preudo-proper on $L_k^p(\xi)_0$.

Proof. The natural modification of the proof of Theorem 5.1 part (a).

Theorem 5.8. *In the notation of Theorem 5.6, let $J_0 : L_k^p(\xi)_{\partial f} \to \boldsymbol{R}$ and V satisfy (5.7) for all $u \in L_k^p(\xi)_{\partial f}$. Then there is a constant $\gamma > 0$, such that if $\alpha < \gamma$, then*

$$J(u) = J_0(u) - \int V(u)$$

is bounded below and pseudo-proper on $L_k^p(\xi)_{\partial f}$. Moreover, $\gamma \ge \lambda(p, k)/2^p$.

Proof. It suffices to show there exist positive constants B, γ such that

$$\int |u|^p \le B + \frac{1}{\gamma} \int |\nabla^k u|^p \,. \tag{5.9}$$

Then the argument is again similar to the proof of Theorem 5.1 part (a). Note this γ and the γ in the statement of the theorem are identical.

In turn, (5.9) follows from the inequality

$$\|u\|_{p,0} \leq B_1 + \frac{1}{\gamma_1}\|u\|_{p,k} \ . \tag{5.10}$$

(To obtain (5.9) from (5.10) use $(a+b)^p \leq 2^p(a^p + b^p)$ for any $a, b > 0$.) Then one finds $B = (2B_1)^p$, and $\gamma = (\gamma_1/2)^p$. To prove (5.10), fix $\phi \in L_k^p(\xi)_{\partial f}$. If $u \in L_k^p(\xi)_{\partial f}$, then $v \equiv u - \phi \in L_k^p(\xi)_0$. Now use the triangle inequality and the definition of $\lambda(p, k)$ applied to $\|v\|_{p,0}$. The constant B_1 depends only on ϕ, and $\gamma_1^p \geq \lambda(p, k)$. Thus $\gamma = (\gamma_1/2)^p \geq \lambda(p, k)/2^p$.

Remarks. 1. As we remarked before for the case of $\partial M = \emptyset$, $\int |\nabla^k u|^p$ is no longer a norm, so we must add $\int |u|^p$ to $J_0(u)$. The same general theorems are true in this case.

2. The value of the constants $\lambda(p, k)$ and γ will crucially depend on the choice of norms and the *RMC* structure. Their existence is of course independent of such choices.

3. In the case of $M =$ bounded domain in $\boldsymbol{R}^n$, we can verify condition (5.7) on V by checking for the following pointwise conditions

$$V(x, s) \leq A + \alpha|s|^p \ , \tag{a}$$

or

$$\limsup_{|s|\to\infty} \frac{V(x, s)}{|s|^p} < \alpha \ . \tag{b}$$

An appropriate version of the above should extend to the general fiber bundle setting. We now present some results in that direction. We consider, as in § 3, the classical mechanics case, i.e., geodesics in the presence of a potential.

For the rest of this section, let N be a complete noncompact Riemannian manifold, and ρ the distance function induced by the metric (if N is compact all smooth potentials are bounded). Let $I = [0, 1]$, and let

$$J_0(u) = \int_0^1 |u'(t)|^2 dt$$

be a function defined for $u \in L_1^2(I, N)$, where $u(0) = P$, $u(1) = Q$ for fixed $P, Q \in N$, i.e., L_1^2 paths from P to Q, written $L_1^2(I, N)_{P,Q}$. We seek a perturbation theorem of the nature of Theorems 5.1, 5.6 and 5.8. That is, we want to find asymptotic growth conditions on a potential function $V: N \to \boldsymbol{R}$ such that

$$J(u) = J_0(u) - \int_0^1 V(u(t))dt$$

is pseudo-proper.

Recall the condition from Theorem 5.1 was

$$\limsup_{|s| \to \infty} \frac{V(x, s)}{s^2} \leq K < \lambda_1 ,$$

where λ_1 is the first eigenvalue of the Laplacian, and is the "best" constant possible in the inequality

$$\int |u|^2 \leq \frac{1}{\lambda_1} \int |\nabla u|^2 , \qquad u \in L_1^2(M, \boldsymbol{R})_0 .$$

Thus in order to handle the case of geodesics, we will find a function on N with which to compare V corresponding to $f(s) = s^2$ on $\boldsymbol{R}$, and an invariant constant corresponding to λ_1.

For a fixed $x_0 \in N$, define a function $\rho_{x_0}: N \to \boldsymbol{R}$ by $\rho_{x_0}(x) = \rho(x_0, x)$. Let S_0 be the set of $\gamma \in \boldsymbol{R}$ such that for some $B_\gamma \in \boldsymbol{R}$,

$$\int_0^1 \rho(x_0, u(t))^2 dt \leq \frac{1}{\gamma} J_0(u) + B_\gamma$$

for all $u \in L_1^2(I, N)_{P,Q}$. We will show that the constant $\gamma_0 = \sup(S_0)$ depends only on the metric on N. This constant γ_0 plays the role of λ_1 in the perturbation theorem. One can show that $\gamma_0 < \infty$, but since we do not use this fact the proof is omitted.

In order to be able to state the asymptotic growth conditions, we must introduce the notion of asymptotic equivalence. Let X be a connected topological space.

Definition 5.11. For $f: X \to \boldsymbol{R}$, we say

$$\limsup_{x \to \infty} f(x) = \alpha ,$$

if for every $\varepsilon > 0$ there is a compact set $K_\varepsilon \subset X$ such that $f(x) \leq \alpha + \varepsilon$ for $x \in X - K_\varepsilon$, and no smaller α will do.

Definition 5.12. Let V, W be continuous functions on X. V is *asymptotically dominated* by W if

$$\limsup_{x \to \infty} \left(\frac{V(x)}{W(x)} \right) \leq 1 .$$

V and W are *asymptotically equivalent* if each one asymptotically dominates the other.

Claim 5.13. *If V is dominated by W, then for all $\varepsilon > 0$ there is a constant $K_\varepsilon > 0$ such that*

$$V(x) \leq K_\varepsilon + (1 + \varepsilon)W(x)$$

for all $x \in X$.

Proof. Definitions (5.11) and (5.12). q.e.d.

Asymptotic equivalence relates to the notion of almost J-boundedness as follows.

Theorem 5.14. *Let M, N be Riemannian manifolds, N complete noncompact and connected, and M compact. Let $J: L_k^p(M, N)_{\partial f} \to \boldsymbol{R}$, where $pk > n$. If $W: N \to \boldsymbol{R}$ is almost J-bounded, and $V: N \to \boldsymbol{R}$ is dominated by W, then V is almost J-bounded.*

Proof. We know

$$\int W(u) \leq K + \theta J(u) , \qquad \text{where } \theta < 1 .$$

Since W dominates V, pick $\varepsilon > 0$ such that $(1 + \varepsilon) < 1/\theta$. Then by (5.13) there is a $K_\varepsilon > 0$ such that

$$\int V(u) \leq K_\varepsilon + (1 + \varepsilon) \int W(u) \leq K_\varepsilon + K + \theta(1 + \varepsilon)J(u)$$

for all $u \in L_k^p(M, N)_{\partial f}$. But $\theta(1 + \varepsilon) < 1$, therefore V is almost J-bounded. q.e.d.

The function with which we will compare V, corresponding to $f(s) = s^2$ in the linear case, is $\rho(x_0, x)^2$. We first show that the "comparison" is independent of the specific point $x_0 \in N$ which we might pick.

Theorem 5.15. *Given any points x_0 and x_1 in N, then ρ_{x_0} and ρ_{x_1} are asymptotically equivalent.*

Proof. Given any $\varepsilon > 0$, let $K_\varepsilon = \{x \in N \,|\, \rho(x_1, x) \leq \rho(x_1, x_0)/\varepsilon\}$. Then K_ε is compact, and for all $x \in N - K_\varepsilon$

$$\frac{\rho(x_0, x)}{\rho(x_1, x)} \leq 1 + \varepsilon .$$

The argument is symmetric in x_1 and x_0. q.e.d.

We now give a perturbation theorem corresponding to Theorem 5.1 part (a), for the space of paths $L_1^2(I, N)_{P,Q}$. Recall that we seek conditions on $V: N \to \boldsymbol{R}$ such that $\mathscr{V}(u) = \int_0^1 V(u(t))dt$ is almost J_0-bounded where

$$J_0(u) = \int_0^1 |u'(t)|^2 dt , \qquad u \in L_1^2(I, N)_{P,Q} .$$

Before we state and prove the theorem, we prove two lemmas. The first establishes an inequality similar to (5.9), and the second establishes the existence of the invariant γ_0 which corresponds to the first eigenvalue λ_1.

Lemma 5.16. *If $x_0 \in N$, then there are constants $B, \gamma > 0$ such that*

$$\int_0^1 \rho(x_0, u(t))^2 dt \leq \frac{1}{\gamma} \int_0^1 |u'(t)|^2 dt + B \tag{5.17}$$

for all $u \in L_1^2(I, N)_{P,Q}$. Moreover, $\gamma \geq 1/2$.

Proof. For any path $u \in L_1^2(I, N)_{P,Q}$,

$$\rho(u(0), u(t)) \leq \int_0^1 |u'(s)| ds \leq \left(\int_0^1 |u'(s)|^2 ds\right)^{1/2} . \tag{5.18}$$

Therefore,

$$\rho(x_0, u(t)) \leq \rho(x_0, u(0)) + \rho(u(0), u(t)) \leq \rho(x_0, P) + \left(\int_0^1 |u'(t)|^2 dt\right)^{1/2} .$$

Consequently,

$$\int_0^1 \rho(x_0, u(t))^2 dt \leq 2\rho(x_0, P)^2 + 2\int_0^1 |u'(t)|^2 dt .$$

Lemma 5.19. *Let S_0 be the set of $\gamma \in \boldsymbol{R}$ satisfying* (5.17) *for some $B_\gamma \in \boldsymbol{R}$. Then each of the following holds:*

(a) *$S_0 \neq \emptyset$.*

(b) *If $\gamma_0 = \sup(S_0)$, then $\gamma_0 > 0$ and is independent of x_0.*

(c) *If $x_0 = P$, then $\gamma_0 \geq 1$ and we can let $B = 0$.*

Proof. (a) This is immediate from Lemma 5.16.

(b) Again by Lemma 5.16 we know $\gamma_0 > 0$. It remains to show γ_0 is independent of x_0. Pick $x_1 \in M$, $x_1 \neq x_0$, with corresponding S_1 and γ_1. Note that if $\gamma \in S_1$ (resp. S_0), then $\beta < \gamma$ implies $\beta \in S_1$ (resp. S_0).

If $\gamma_1 \neq \gamma_0$, say $\gamma_0 < \gamma_1$, we will obtain a contradiction. Pick $\varepsilon_1 > 0$ such that $\gamma_0 + \varepsilon_1 < \gamma_1$. Now let $\varepsilon = \varepsilon_1/\gamma_0 > 0$, so $\varepsilon > 0$ and $\gamma_0 + \gamma_0\varepsilon < \gamma_1$, i.e., $\gamma_0 < \gamma_1/(1 + \varepsilon)$. With ε thus chosen, since $\rho(x_0, \cdot)^2$ and $\rho(x_1, \cdot)^2$ are asymptotically equivalent, there is a $K_\varepsilon > 0$ such that

$$\int_0^1 \rho(x_0, u(t))^2 \leq \int_0^1 K_\varepsilon + (1 + \varepsilon)\int_0^1 \rho(x_1, u(t))^2 . \tag{5.20}$$

Since $\gamma_0 < \gamma_1/(1 + \varepsilon)$, there is a $\delta > 0$ such that $\gamma_0 < (\gamma_1 - \delta)/(1 + \varepsilon)$. Now $\gamma_1 = \sup(S_1)$, so there is a γ in S_1 such that $\gamma + \delta > \gamma_1$, that is, $\gamma > \gamma_1 - \delta$ which implies $\gamma_1 - \delta \in S_1$ by an above note. Combining this with (5.20) we see

$$\int_0^1 \rho(x_0, u(t))^2 \leq K_\varepsilon + B(1 + \varepsilon) + \frac{(1 + \varepsilon)}{\gamma_1 - \delta} J(u) . \tag{5.21}$$

Now (5.21) implies $(\gamma_1 - \delta)/(1 + \varepsilon) \in S_0$, but $\gamma_0 < (\gamma_1 - \delta)/(1 + \varepsilon)$ and $\gamma_0 = \sup (S_0)$. This is a contradiction.

(c) This follows immediately from (5.18).

Theorem 5.22. *If $V: N \to \boldsymbol{R}$ is asymptotically dominated by $\alpha\rho(x, \cdot)^2$ for any $x \in N$, $\alpha < \gamma_0$, then the functional*

$$J(u) = \int_0^1 (|u'(t)|^2 - V(u(t)))dt \ , \qquad u \in L_1^2(I, N)_{P,Q}$$

is bounded below and pseudo-proper.

Proof. By Theorem 5.14 and 5.15 it is enough to show that $\alpha\rho(x, \cdot)^2$ is almost J_0-bounded for some $x \in N$, where

$$J_0(u) = \int_0^1 |u'(t)|^2 dt \ .$$

This follows from Lemma 5.19.

6. $(k - 1)$-order perturbations

In this section we investigate conditions on a perturbation V, which imply almost J_0-boundedness for $J_0(u) = \|u\|_{p,k}^p = \int |\nabla^k u|^p$ on the space $L_k^p(\xi)_0$. The results in this section generalize those in § 5, especially part (a) of Theorem 5.1. They apply to the case of arbitrary boundary values with an appropriate change of the constants involved, using the technique of Theorem 5.8.

We analyze almost $\int |\nabla^k u|^p$-boundedness, for the case of perturbations V which are dominated by some *strict polynomial differential* operator P of order at most $k - 1$, and homogeneous of degree at most p, i.e.,

$$V(u) \leq \text{constant} + P(u) \ .$$

Locally one can express this condition as

$$V(x, u, D^\alpha u) \leq \text{constant} + \sum_a A_a(x) D^{a_1}u \cdots D^{a_r}u \ ,$$

where $a = (a_1, \cdots, a_r)$ is an r-tuple of multi-indices, $0 \leq |a_i| \leq k - 1$, and $r \leq p$.

One should understand the above condition as an asymptotic growth condition on V. For example:

1. In the case $k = 1$ and $p = 2$, where V is a function of the zero jet and

$$\limsup_{|s| \to \infty} \frac{V(x, s)}{|s|^2} \leq a(x) \ ,$$

we have $V(x, u) \leq b + a(x)u^2$ for some constant b.

2. In particular, there is no growth requirement in the "negative direction" such as the $-e^u$ term in

$$V(x, u, u') = -e^u + a(\sin u)u^2 + b(u')^2 \leq au^2 + b(u')^2 .$$

One might ask how necessary is the above restriction on V. We have shown the necessity in the case $k = 1$, $p = 2$ in Theorem 5.1. In Theorem 7.1 we will discuss the necessity of the growth condition for arbitrary p and k.

Given the above growth condition on V, we wish to show that V is almost $\int |\nabla^k u|^p$-bounded, that is,

$$\int V(u) \leq \theta \int |\nabla^k u|^p + \text{const} .$$

for some $\theta < 1$. Clearly it is enough for us to consider the dominating polynomials $P(u)$.

Lemma 6.1. *Let P be a strict polynomial differential operator from ξ to $\boldsymbol{R}_M$, of order at most k, and homogeneous of degree $r \leq p$. Then P extends to a C^∞ map of $L_k^p(\xi)$ into $L_0^1(\boldsymbol{R}_M)$.*

Proof. This is a local question, so we may assume $M = D^n$, $\xi = D^n \times \boldsymbol{R}^m$, and $\boldsymbol{R}_M = D^n \times \boldsymbol{R}$. If $u = (u_1, \cdots, u_m)$, then $P(u)$ is a sum of terms of the form

$$A(x)D^{\alpha_1}u_{a_1} \cdots D^{\alpha_r}u_{a_r} ,$$

where $A \in C^\infty(M, \boldsymbol{R})$, $0 \leq |\alpha_i| \leq k$. The map $u \to D^{\alpha_i}u_{a_i}$ is a linear differential operator of order $|\alpha_i|$ from ξ to $\boldsymbol{R}_M$, and thus extends to a continuous linear map of $L_k^p(\xi)$ into $L_{k-|\alpha_i|}^p(M, \boldsymbol{R})$. Therefore it suffices to show that multiplication is a continuous multilinear map of $\bigoplus_{i=1}^r L_{k-|\alpha_i|}^p(M, \boldsymbol{R})$ into $L_0^1(M, \boldsymbol{R})$. The proof of this, given $r \leq p$, is a straightforward application of the Sobolev theorems and Hölder's inequality, see, for example, [12, Theorem 9.4].

Theorem 6.2. *Let P be a strict polynomial differential operator of order k from ξ to $\boldsymbol{R}_M$, homogeneous of degree at most p. Then there is a constant $\gamma > 0$ such that for all $u \in L_k^p(\xi)_0$*

$$\int P(u) \leq \gamma \int |\nabla^k u|^p . \tag{6.3}$$

Moreover, if γ_0 is the greatest lower bound of the set of $\gamma > 0$ satisfying (6.3), *and*

$$\gamma_0 < 1 ,$$

then $\int P(u)$ is almost $\int |\nabla^k u|^p$-bounded for $u \in L_k^p(\xi)_0$.

Proof. By Lemma 6.1, P extends to a continuous map from $L_k^p(\xi)_0$ into $L_0^1(\boldsymbol{R}_M)$. Since P is homogeneous of degree p, there is a $\gamma > 0$ such that

$$\|Pu\|_{1,0} \leq \gamma \|u\|_{p,k}^p = \gamma \int |\nabla^k u|^p .$$

(If not, there are $u_j \in L_k^p(\xi)_0$ with $\|u_j\|_{p,k}^p \to 0$ and $\|P(u_j)\|_{1,0} = 1$. By continuity, if $u_j \to 0$ in L_k^p then $P(u_j) \to P(0) = 0$ in $L_0^1(\boldsymbol{R}_M)$.) q.e.d.

For Theorem 6.2 to yield an effective procedure, one must find the constant γ_0 more explicitly. We carry this out in the case where P acts on scalar valued functions on a bounded open set Ω in Euclidean space.

Let

$$\|u\|_{p,l}^p \equiv \sum_{|\alpha|=l} \int |D^\alpha u|^p$$

be the norm on $L_l^p(\Omega, \boldsymbol{R})_0$, for $0 \leq l \leq k$. Define constants $\lambda(p, k; l)$ for $0 \leq l \leq k$ by

$$\lambda(p, k; l) \equiv \inf_{u \in L_k^p(\Omega)_0} \frac{\|u\|_{p,k}^p}{\|u\|_{p,l}^p} > 0 ,$$

so $\lambda(p, k; l)$ is the reciprocal of the norm of the continuons linear inclusion of $L_k^p(\Omega)_0$ into $L_l^p(\Omega)_0$.

Theorem 6.4. *Let $V(u) = A(x)(D^{\alpha_1}u)^{a_1} \cdots (D^{\alpha_N}u)^{a_N}$, where A is smooth, $0 \leq |\alpha_i| \leq k - 1$, and $\sum a_i \leq p$. Then each of the following holds:*

(a) *There exist constants $p_1, \cdots, p_N$ such that $\sum 1/p_i = 1$, and*

$$\limsup_{|s| \to \infty} \frac{|s|^{a_i p_i}}{s^p} = \beta_i < \infty$$

for $i = 1, 2, \cdots, N$. (In fact β_i is 0 or 1.)

(b) *There is a smallest constant K such that for all $\gamma > K$,*

$$\int V(u) \leq \gamma \|u\|_{p,k}^p + \text{const} . \tag{6.5}$$

for all $u \in L_k^p(\Omega)_0$. Moreover, if $K < 1$, then $V(u)$ is almost $\|u\|_{p,k}^p$-bounded.

(c) $$K \leq \|A\|_\infty \prod_{j=1}^N \left(\frac{\beta_j}{\lambda(p, k; |\alpha_j|)}\right)^{1/p_j} .$$

(d) *If $\sum a_i < p$, then we can make K arbitrarily close to 0, but the constant in (6.5) may go to $+\infty$.*

Proof. (a) We first observe that if $\sum a_i = p$, then we let $p_i = p/a_i$ and $\beta_i = 1$. If $\sum a_i < p$, then $\sum a_i/p < 1$, hence we can pick $1/p_i > a_i/p$ such that $\sum 1/p_i = 1$, and

$$\limsup_{|s|\to\infty} \frac{|s|^{a_i p_i}}{|s|^p} = \beta_i = 0 .$$

(b) This follows immediately from Theorem 6.2, but is proved again in (c).
(c) We observe that

$$\begin{aligned} \int V(u) &= \int A(x)(D^{\alpha_1}u)^{a_1} \cdots (D^{\alpha_N}u)^{a_N} \\ &\leq \|A\|_\infty \int |(D^{\alpha_1}u)^{a_1} \cdots (D^{\alpha_N}u)^{a_N}| . \end{aligned} \tag{6.6}$$

Now pick p_i's as in (a), and constants μ_i, which we will choose later, such that $\prod_{i=1}^N \mu_i = 1$. Since $\sum 1/p_i = 1$, we know that for any $x_1, \cdots, x_N$,

$$x_1 \cdots x_N = \mu_1 x_1 \cdots \mu_N x_N \leq (\mu_1 x_1)^{p_1}/p_1 + \cdots + (\mu_N x_N)^{p_N}/p_N .$$

Combining this with (a) and (6.6) we find

$$\begin{aligned} \int V(u) &\leq \|A\|_\infty \left[\frac{\beta_1 \mu_1^{p_1}}{p_1} \int |D^{\alpha_1}u|^p + \cdots + \frac{\beta_N \mu_N^{p_N}}{p_N} \int |D^{\alpha_N}u|^p \right] + \text{const} . \\ &\leq \|A\|_\infty \left[\frac{\beta_1 \mu_1^{p_1}}{p_1 \lambda(p, k; |\alpha_1|)} + \cdots + \frac{\beta_N \mu_N^{p_N}}{p_N \lambda(p, k; |\alpha_N|)} \right] \|u\|_{p,k}^p + \text{const} . \end{aligned} \tag{6.7}$$

So our first approximation of K is

$$K \leq \|A\|_\infty \left[\frac{\beta_1 \mu_1^{p_1}}{p_1 \lambda(p, k; |\alpha_1|)} + \cdots + \frac{\beta_N \mu_N^{p_N}}{p_N \lambda(p, k; |\alpha_N|)} \right] . \tag{6.8}$$

Now we will pick the μ_i's to minimize the right hand side of (6.7). Using standard techniques we get

$$\mu_i^{p_i} = \left(\prod_{j=1}^N c_j \right) \Big/ c_i^{p_i} ,$$

where $c_i = \beta_i / \lambda(p, k; |\alpha_i|)$. Substituting back into (6.8) and using the fact that $\sum 1/p_i = 1$, we get

$$K \leq \|A\|_\infty \prod_{j=1}^N \left(\frac{\beta_j}{\lambda(p, k; |\alpha_j|)} \right)^{1/p_j} .$$

Note the constants in (6.7) depend on $\|A\|_\infty$, and arise from the lim sup statement of (a).

(d) is clear from the above and the fact that if $\sum a_i < p$ then $\beta_i = 0$ for $i = 1, \cdots, N$.

Remark. Condition (b) is stated as it is because K might be $-\infty$ in which case we cannot use it in (6.5).

It is straightforward to extend Theorem 6.4 to V's which are a sum

$$V(u) = \sum A_{\alpha,a}(x)(D^{\alpha_1}u)^{a_1} \cdots (D^{\alpha_{N(\alpha,a)}}u)^{a_{N(\alpha,a)}} , \tag{6.9}$$

where the $A_{\alpha,a}$ are smooth, each α is an $N(\alpha, a)$-tuple of multi-indices with $0 \leq |\alpha_i| \leq k - 1$, and each a is an $N(\alpha, a)$-tuple such that $\sum a_i \leq p$. The result of this extension is

Corollary 6.10. *With V as in* (6.9) *let $K_{\alpha,a}$ be the K of Theorem* 6.4 *corresponding to the $A_{\alpha,a}$ monomial. If $\sum K_{\alpha,a} < 1$, then V is almost $\|u\|^p_{p,k}$-bounded.*

Remark. It is clear how one extends Theorems 6.2, 6.4, 6.10, to kth order perturbations.

7. $(k-1)$-order perturbations: Growth restrictions

The results of this section continue the generalization of Theorem 5.1. First we focus on part (c), and consider the question of the necessity of restricting ourselves to perturbations dominated by polynomials which are homogeneous of degree at *most* p. The idea is to show that if the perturbation V grows faster than a polynomial in the derivatives, homogeneous of degree p, then

$$J(u) = \|u\|^p_{p,k} - \int V(u)$$

is not pseudo-proper. Next we generalize part (b) of Theorem 5.1 by showing that if V is homogeneous of degree p, then there is a constant K such that if $K\eta > 1$, then

$$J_\eta(u) = \|u\|^p_{p,k} - \eta \int V(u)$$

is not pseudo-proper.

Let

$$J_0(u) = \|u\|^p_{p,k} \equiv \int |\nabla^k u|^p$$

on $L^p_k(\xi)_0$, and

$$\mathscr{V}(u) = \int V(u) ,$$

where V is a $(k-1)$-order differential operator from ξ to $\boldsymbol{R}_M$, continuous on $L^p_k(\xi)_0$.

Theorem 7.1. *If there is a continuons functional $\mathscr{V}_1(u)$ on $L^p_k(\xi)_0$, an open set $\Omega \subset\subset M$, and $\psi \in C^\infty_0(\xi|_\Omega)$ such that*

$$\mathscr{V}(\lambda\psi) \geq c_1 + \mathscr{V}_1(\lambda\psi) \qquad \textit{for all } \lambda > 0 , \tag{7.2}$$

and

$$(7.3) \qquad \lim_{\lambda\to\infty} \frac{\mathscr{V}_1(\lambda\psi)}{\lambda^p} = +\infty \ ,$$

then $J(u) = J_0(u) - \mathscr{V}(u)$ *is not pseudo-proper on* $L^p_k(\xi)_0$.

Proof. Pick an open set $\Omega_0 \subset\subset M$, with $\Omega \cap \Omega_0 = \emptyset$, and $\phi_j \in C_0^\infty(\xi|_{\Omega_0})$ such that

$$(7.4) \qquad \|\phi_j\|_{\infty,k-1} \le 1 \ ,$$

$$(7.5) \qquad J_0(\phi_j) = \|\phi_j\|^p_{p,k} \to \infty \ .$$

We will show there are constants α_j, such that $J(u_j) = 0$ and $\|u_j\|^p_{p,k} = J_0(u_j) \to \infty$, where $u_j = \phi_j + \alpha_j\psi$, thereby establishing that J is not pseudo-proper.

Observe that

$$(7.6) \quad J_0(u_j) = \int_{\Omega_0} |\nabla^k\phi_j|^p + |\alpha_j|^p \int_\Omega |\nabla^k\psi|^p = J_0(\phi_j) + |\alpha_j|^p J_0(\psi) \ge J_0(\phi_j) \ ,$$

so $J_0(u_j) \to \infty$ as $j \to \infty$.

Also if 0 represents the 0-section,

$$(7.7) \qquad \begin{aligned} \mathscr{V}(u_j) &= \int_{\Omega_0} V(\phi_j) + \int_\Omega V(\alpha_j\psi) + \int_{M-\Omega_0-\Omega} V(0) \\ &= \mathscr{V}(\phi_j) + \mathscr{V}(\alpha_j\psi) + c \ , \end{aligned}$$

where

$$c = -\int_{M-\Omega_0} V(0) - \int_{M-\Omega} V(0) + \int_{M-\Omega-\Omega_0} V(0)$$

is a constant independent of j.

Since V is continuous, and $\|\phi_j\|^p_{\infty,k-1} \le 1$, we know

$$(7.8) \qquad \left|\int V(\phi_j)\right| \le K \qquad \text{for all } j = 1, 2, \cdots \ .$$

Hence assuming $\alpha_j > 0$, we see by (7.7), (7.8), and (7.2) that

$$(7.9) \qquad -K + c + c_1 + \mathscr{V}_1(\alpha_j\psi) \le \mathscr{V}(u_j) \le K + \mathscr{V}(\alpha_j\psi) + c \ .$$

Using a continuity argument, we now show there are $\alpha_j > 0$ such that $J(u_j) = 0$, i.e., $\mathscr{V}(u_j)/J_0(u_j) = 1$. By (7.6), (7.7), and (7.9), we have

$$(7.10) \qquad \frac{-K + c + c_1 + \mathscr{V}_1(\alpha_j\psi)}{J_0(\phi_j) + |\alpha_j|^p J_0(\psi)} \le \frac{\mathscr{V}(u_j)}{J_0(u_j)} \le \frac{K + \mathscr{V}(\alpha_j\psi) + c}{J_0(\phi_j)} \ .$$

Since $J_0(\phi_j) \to \infty$, pick j so large that $J_0(\phi_j) > K + c + \mathscr{V}(0)$, where 0 denotes the 0-section. Then letting $\alpha_j = 0$, we see

$$\frac{K + c + \mathscr{V}(0)}{J_0(\phi_j)} < 1 ,$$

and hence for $\alpha_j = 0$,

$$\mathscr{V}(u_j)/J_0(u_j) \leq 1 .$$

For a fixed j, by condition (7.3) on $\mathscr{V}_1$ we see that

$$\frac{-K + c + c_1 + \mathscr{V}_1(\alpha_j\psi)}{J_0(\phi_j) + |\alpha_j|^p J_0(\psi)} \to \infty$$

as $\alpha_j \to \infty$. Therefore, there is an $\alpha_j > 0$ such that

$$1 \leq \mathscr{V}(u_j)/J_0(u_j) .$$

Since $\mathscr{V}(u_j)/J_0(u_j)$ is a continuous function of α_j, for j sufficiently large there are $\alpha_j > 0$ such that

$$\mathscr{V}(u_j)/J_0(u_j) = 1 ,$$

i.e., $J(u_j) = 0$. q.e.d.

What sorts of functionals $\mathscr{V}$ satisfy the conditions of Theorem 7.1? To answer this we give several examples.

Example 1. Let M be a bounded open domain in $\boldsymbol{R}^n$, and $\xi = M \times \boldsymbol{R}$ so we are considering real-valued functions on M. Say $\mathscr{V}(u) = \int V(u)$. If there exist constants $c_\alpha \geq 0$ and not all zero, and $q_\alpha > p$, such that on some open set $\Omega \subset M$,

$$V(u) \geq \text{constant} + \sum_{|\alpha| \leq k-1} c_\alpha (D^\alpha u)^{q_\alpha} ,$$

then $\int |\nabla^k u|^p - \mathscr{V}(u)$ is not pseudo-proper on $L_k^p(M, \boldsymbol{R})_0$. This follows immediately from Theorem 7.1, with

$$\mathscr{V}_1(u) = \int \sum_{|\alpha| \leq k-1} c_\alpha (D^\alpha u)^{q_\alpha} .$$

Example 2. Since $\mathscr{V}(u) = \int V(u)$ is only determined up to integration by parts, we need to assume more than $V(u) \geq \text{constant} + \sum P_j(u)$ where the P_j are strict polynomial differential operators homogeneous of degree greater than p. This can be seen clearly in the following:

(a) Say $J_0(u) = \int_0^1 (u'')^2$ on $L_2^2([0, 1], \boldsymbol{R})_0$. Say $\mathscr{V}(u) = \int V(u)$ for $V(x, u, u') = 3u^2 u'$. Then

$$\mathscr{V}(u) = 3\int_0^1 u^2u' = \int_0^1 (u^3)' = 0 .$$

Hence $J_0 - \mathscr{V}$ is pseudo-proper, even though $\mathscr{V}$ is a cubic polynomial in u and u'.

(b) Say $J_0(u) = \int (u''')^2$ on $L_3^2([0,1], \boldsymbol{R})_0$. Let $\mathscr{V}(u) = \int V(u)$ where $V(x, u, u', u'') = u^3u''$. Then

$$\mathscr{V}(u) = \int u^3u'' = -3\int u^2(u')^2 \leq 0 .$$

Hence $J_0 - \mathscr{V}$ is pseudo-proper, despite the fact that V is quartic.

Example 3. Here we show the additional assumption needed on $\mathscr{V}_1$ to eliminate the difficulties exhibited in Example 2.

Say $J_0(u) = \int |\nabla^k u|^p$ as in Theorem 7.1, $\mathscr{V}(u) = \int V(u)$, and on an open set $\Omega \subset M$,

$$V(u) \geq \text{constant} + P(u) ,$$

where P is a strict polynomial differential operator homogeneous of degree $q > p$.

If there is a section $\psi \in C_0^\infty(\xi|_\Omega)$ such that

$$P(\psi) > 0 , \tag{7.11}$$

then $\int |\nabla^k u|^p - \mathscr{V}(u)$ is *not* pseudo-proper on $L_k^p(\xi)_0$.

This is immediate from Theorem 7.1. One cannot satisfy (7.11) in Examples 2a and 2b.

Now we consider the special case of the above, where the perturbation $\mathscr{V}$ arises from a Lagrangian V which is a strict polynomial operator, homogeneous in u of degree p. We investigate the pseudo-properness of functionals of the form

$$J_\eta(u) = J_0(u) - \eta\mathscr{V}(u) = \|u\|_{p,k}^p - \eta\int V(u) , \qquad \eta \geq 0 .$$

First, we may as well assume that $\mathscr{V}(u)$ is not bounded above, because in this case $J_\eta(u)$ is pseudo-proper for all nonnegative η. Thus for some $u \in L_k^p(\xi)_0$, $\mathscr{V}(u) > 0$. Let

$$K = \sup \frac{\mathscr{V}(u)}{J_0(u)} ,$$

where the sup is taken over all $u \in L_k^p(\xi)_0$ for which $\mathscr{V}(u) > 0$. Then $K > 0$, and $K < \infty$ since by Theorem 6.4, (6.7) and the homogeneity of $\mathscr{V}$, there is a constant $c < \infty$ such that $\mathscr{V}(u) \leq cJ_0(u)$. K corresponds to the reciprocal of the first eigenvalue of the Laplacian, which plays a crucial role in Theorem 5.1.

Theorem 7.12. *If $\eta > 1/K$, then $J_\eta(u)$ is not pseudo-proper. Moreover, this holds for $\eta = 1/K$ if K is attained for some u.*

Proof. Since $1/\eta < K$, and $\mathscr{V}$ is homogeneous, there is a $u_\eta \not\equiv 0$ in $L_k^p(\xi)_0$ such that $1/\eta < \mathscr{V}(u_\eta)/J_0(u_\eta)$. Thus we see

$$J_\eta(u_\eta) = J_0(u_\eta) - \eta\mathscr{V}(u_\eta) < 0 \ . \tag{7.13}$$

The argument is now similar to Theorem 7.1.

It is possible to construct a smooth one-parameter family $\phi_t \in L_k^p(\xi)_0$ such that $\phi_0 \equiv 0$, and

(a) $\|\phi_t\|_{\infty,k-1} \leq 1$,

(b) $\|\phi_t\|_{p,k} \to \infty$ as $t \to \infty$,

(c) $u_\eta + \phi_t \not\equiv 0$.

Consider the function $f(t) = J_\eta(u_\eta + \phi_t)$, which is a continuous function of t since the ϕ_t's vary smoothly. Since $\phi_0 \equiv 0$, we have $f(0) = J_\eta(u_\eta) < 0$. As $t \to \infty$, $f(t) \to \infty$ because the continuity of V and (a) imply $\mathscr{V}(u_\eta + \phi_t)$ is bounded, while (b) implies $J_0(u_\eta + \phi_t) \to \infty$. Thus there is a τ such that $f(\tau) = 0$, i.e., a ϕ_τ with $\|u_\eta + \phi_\tau\|_{p,k} \neq 0$ and $J_\eta(u_\eta + \phi_\tau) = 0$.

Using the homogeneity of $\mathscr{V}$ in the usual way, we get a sequence $v_j = j(u_\eta + \phi_\tau)$ showing J_η is not pseudo-proper.

Theorem 7.14. *If $K = \sup \mathscr{V}(u)/J_0(u)$ taken over all nonzero $u \in L_k^p(\xi)_0$ is nonnegative, then $J_\eta(u)$ is pseudo-proper for $0 \leq \eta < 1/K$.*

Proof. Since $\mathscr{V}(u) \leq KJ_0(u)$ for all $u \in L_k^p(\xi)_0$, we see $\eta\mathscr{V}(u) \leq \eta KJ_0(u)$ for $\eta \geq 0$. If $\eta K < 1$, this implies $\eta\mathscr{V}(u)$ is almost J_0-bounded. q.e.d.

It is likely that a more precise version of the above two theorems is true which includes the case when K might be negative.

Note that the above theorems generalize almost immediately to the case where V is an in homogeneous polynomial of degree p, since the terms of lower degree homogeneity always preserve the pseudo-proper condition (see Theorem 6.4 (d)).

III. COERCIVITY

8. Perturbing coercive functionals

We consider the following problem. Say a functional $J_0(u) = \int \mathscr{L}(u)$ is coercive on $L_k^p(E)$, where $\mathscr{L}$ is a differential operator from E to $\boldsymbol{R}_M$ of order k. If V is an operator of order $k - 1$, what conditions on a perturbation $\mathscr{V}(u) = \int V(u)$ insure that $J(u) = J_0(u) - \mathscr{V}(u)$ is also coercive on $L_k^p(E)$? We will only consider the case where $\mathscr{L}$ is a polynomial differential operator of weight pk, and V is also a polynomial (not necessarily strict) differential operator. The weight restriction on $\mathscr{L}$ is placed to insure that $\mathscr{L}$ extends to a C^∞ map from $L_k^p(E)$ into $L_0^1(\boldsymbol{R}_M)$ (see [12, pp. 69–77]). In this case Theorem 8.6 gives the optimal result: as long as we stay within the weight restriction imposed by the original Lagrangian $\mathscr{L}$, *all* lower order polynomial perturbations preserve coercivity. Since we are working in a fiber bundle setting, we must assume $pk > \dim M$ throughout this part of the section.

In the case of a vector bundle ξ, the spaces $L_k^p(\xi)$ are Banach spaces, and thus the $pk > \dim M$ assumption is not required. In this situation, if we assume the perturbation V is polynomial in u, i.e., V is a strict polynomial differential operator of order at most $k-1$ and degree at most p, then we can get explicit algebraic conditions on V to preserve the coercivity condition (see Theorem 8.13).

From Definition 2.2 of coercivity we can reduce our considerations to vector bundles ξ over compact M. Further, if a perturbation $\mathscr{V}(u) = \int V(u)$ satisfies the following condition:

$$\text{(8.1)} \qquad \text{if } s_j \rightharpoonup s \text{ in } L_k^p(\xi), \text{ then } (D\mathscr{V}_{s_i} - D\mathscr{V}_{s_j})(s_i - s_j) \to 0\ ,$$

and J_0 is a coercive functional, then it follows that $J = J_0 - \mathscr{V}$ is also coercive. In fact, using a partition of unity argument, it is not difficult to show that it is enough to check condition (8.1) locally. Thus we can reduce the perturbation problem of coercivity to the verification of (8.1), on vector bundles $\xi = \Omega \times \boldsymbol{R}^m$ where $\Omega \subset \boldsymbol{R}^n$ is a bounded open domain. Indeed, by composing V with coordinate functions we reduce further to the case $\xi = \Omega \times \boldsymbol{R}$. Therefore we will state and prove our theorems in this setting with the understanding that they hold in the general fiber bundle case.

Consider a functional $J_0\colon L_k^p(\Omega, \boldsymbol{R}) \to \boldsymbol{R}$ which is coercive, and perturbations of the form $\mathscr{V}(u) = \int V(j_r(u))$, where $pk > n$, V is say C^1, and $0 \le r \le k-1$. We are looking for conditions on V, so that it satisfies (8.1), where, in coordinates, $D\mathscr{V}$ is of the following form:

$$\text{(8.2)} \qquad D\mathscr{V}_{s_i}(\eta) = \int \sum_{0 \le |\gamma| \le r} \frac{\partial V}{\partial u^{(\gamma)}}(s_i, D^\alpha s_i) \cdot (D^\gamma \eta)$$

and $1 \le |\alpha| \le r$.

Theorem 8.3. *If $k - r > n/p$, and V is any C^1 function of $j_r(u)$, then condition* (8.1) *holds, and therefore $J = J_0 - \mathscr{V}$ is coercive.*

Proof. If $k - r > n/p$, then L_k^p is compactly contained in C^r. Hence, if $s_i \rightharpoonup s$ in L_k^p, then $D^\alpha s_i \to D^\alpha s$ uniformly for $0 \le \alpha \le r$. This together with the continuity of DV gives condition (8.1).

Corollary 8.4. *If V only depends on the* 0*-jet of u, i.e., $V: \boldsymbol{R} \to \boldsymbol{R}$, and is C^1, then $J_0 - \mathscr{V}$ is coercive.*

Corollary 8.5. *If V depends on the $(k-1)$-jet of u, $\Omega \subset \boldsymbol{R}^1$ (i.e., $n = 1$), and V is C^1, then $J_0 - \mathscr{V}$ is coercive.*

We now restrict to the case of polynomial perturbations $V(j_r(u))$, $0 \leq r \leq k-1$, on $u \in L_k^p(\Omega, \boldsymbol{R})$ where $\Omega \subset \boldsymbol{R}^n$ and $pk > n$. As we said before our result is the best possible since we want to perturb polynomial Lagrangians of weight pk on L_k^p by lower order polynomial Lagrangians keeping $J_0 - \mathscr{V}$ smooth from $L_k^p(E)$ into $L_0^1(\boldsymbol{R}_M)$.

Theorem 8.6. *Let V be a polynomial differential operator of order at most $k-1$ on $L_k^p(\Omega, \boldsymbol{R})$, where $\Omega \subset \boldsymbol{R}^n$ and $pk > n$. If the* weight $(V) \leq pk$, *then $\mathscr{V}(u) = \int V(u)$ satisfies condition* (8.1). *Thus $J = J_0 - \mathscr{V}$ is coercive.*

Proof. It is enough to consider V of the form

$$V(u) = f(x, u) D^{\alpha_1} u \cdots D^{\alpha_\tau} u ,$$

where f is smooth, and the α_i are multi-indices, not necessarily distinct, $1 \leq |\alpha_i| \leq k-1$, and weight $(V) = \sum_{i=1}^{\tau} |\alpha_i| \leq pk$, since V is a sum of such terms. In fact, for ease in exposition we will do the case of only two distinct multi-indices since there is no great difference in the proof for the more general case. Thus, let

$$V(u) = f(x, u)(D^\gamma u)^s (D^\delta u)^t$$

$1 \leq |\gamma|, |\delta| \leq k-1$ and $|\gamma| s + |\delta| t \leq pk$. Let $u_i \rightharpoonup u$ in L_k^p, and let $h_i = D^\gamma u_i$, $g_i = D^\delta u_i$. We note that for $l < k$, since $pk > n$ by the Rellich theorems, L_k^p is compactly contained in $L_l^{pk/l}$. Hence we see that

$$h_i \to h \text{ in } L_0^{pk/|\gamma|} , \tag{8.7}$$

and

$$g_i \to g \text{ in } L_0^{pk/|\delta|} .$$

Writing $\partial f(x, u)/\partial u = f_2(x, u)$, we have

$$\begin{aligned}
&(D\mathscr{V}_{u_i} - D\mathscr{V}_{u_j})(u_i - u_j) \\
&\quad = \int (f_2(x, u_i) h_i{}^s g_i{}^t - f_2(x, u_j) h_j{}^s g_j{}^t) \cdot (u_i - u_j) \\
&\qquad + s \int [f(x, u_i) h_i^{s-1} g_i{}^t - f(x, u_j) h_j^{s-1} g_j{}^t] \cdot (h_i - h_j) \\
&\qquad + t \int [f(x, u_i) h_i{}^s g_i^{t-1} - f(x, u_j) h_j{}^s g_j^{t-1}] \cdot (g_i - g_j) .
\end{aligned} \tag{8.8}$$

We will show that each of the three terms of (8.8) tends to 0.

For the first term we observe

$$\left|\int (f_2(x, u_i)h_i^s g_i^t - f_2(x, u_j)h_j^s g_j^t)(u_i - u_j)\right| \\ \leq \|f_2(x, u_i)h_i^s g_i^t - f_2(x, u_j)h_j^s g_j^t\|_{1,0} \|u_i - u_j\|_\infty .$$

But $u_i \rightharpoonup u$ in L_k^p and $pk > n$, so by the Sobolev and Rellich theorems, the $u_i \in C^0$, and $u_i \to u$ uniformly; therefore $\|u_i - u_j\|_\infty \to 0$. Thus it is enough to show that $\|f_2(x, u_i)h_i^s g_i^t\|_{1,0}$ is bounded. Since f is smooth and the u_i's are uniformly bounded, there is a constant $K > 0$ such that

$$\int |f_2(x, u_i)h_i^s g_i^t| \leq K \int |h_i^s g_i^t| . \tag{8.9}$$

By (8.7) we know that the h_i are bounded in $L_0^{pk/|\gamma|}$ and the g_i are bounded in $L_0^{pk/|\delta|}$. Since

$$\int |h_i^s g_i^t| \leq \left(\int |h_i|^{sc}\right)^{1/c} \left(\int |g_i|^{td}\right)^{1/d} ,$$

where $1/c + 1/d = 1$, expression (8.9) is bounded as long as $sc \leq pk/|\gamma|$ and $td \leq pk/|\delta|$, i.e., as long as

$$1 = \frac{1}{c} + \frac{1}{d} \geq \frac{s|\gamma| + t|\delta|}{pk} ,$$

which is true since weight $(V) = s|\gamma| + t|\delta| \leq pk$.

We show the second term tends to 0 in a similar way

$$\left|\int [f(x, u_i)h_i^{s-1} g_i^t - f(x, u_j)h_j^{s-1} g_j^t](h_i - h_j)\bigg)\right| \\ \leq \|f(x, u_i)h_i^{s-1} g_i^t - f(x, u_j)h_j^{s-1} g_j^t\|_{r,0} \|h_i - h_j\|_{q,0} , \tag{8.10}$$

where $1/r + 1/q = 1$. By (8.7) we know that $\|h_i - h_j\|_{q,0} \to 0$ as long as $q \leq pk/|\gamma|$. It remains to find conditions on r such that $\|f(x, u_i)h_i^{s-1} g_i^t\|_{r,0}$ is bounded. Since f is smooth and the u_i's uniformly bounded, there is a constant $K_r > 0$ such that

$$\int |f(x, u_i)h_i^{s-1} g_i^t|^r \leq K_r \int |h_i^{s-1} g_i^t|^r . \tag{8.11}$$

Since

$$\int |h_i^{s-1} g_i^t|^r \leq \left(\int |h_i^{(s-1)rc}\right)^{1/c} \left(\int |g_i|^{trd}\right)^{1/d}$$

for $1/c + 1/d = 1$, using (8.7) we see expression (8.11) is bounded as long as $(s-1)rc \le pk/|\gamma|$ and $trd \le pk/|\delta|$, that is, as long as

$$\frac{1}{r} = \frac{1}{cr} + \frac{1}{dr} \ge \frac{(s-1)|\gamma|}{pk} + \frac{t|\delta|}{pk}\,.$$

Combining this with the previous condition on q, we see expression (8.10) tends to 0 if

$$1 = \frac{1}{q} + \frac{1}{r} \ge \frac{|\gamma|}{pk} + \frac{(s-1)|\gamma|}{pk} + \frac{t|\delta|}{pk}\,,$$

that is, if $pk \ge s|\gamma| + t|\delta| = \text{weight}\,(V)$.

The third term tends to zero by the same argument used for the second term. q.e.d.

Theorem 8.6 tells us that as long as we remain within the weight restriction imposed by the original functional, all lower order polynomial perturbations preserve coercivity. However, by "milking" the Sobolev theorems we can obtain more precise conditions on perturbations V which insure the preservation of coercivity. For example, if $p > n$, from Theorem 8.3 we know that any C^1 function V of $j_{k-1}(u)$ will preserve coercivity. We now briefly explain how one gets the stronger results.

First observe that using the full power of the Sobolev and Rellich theorems we can replace (8.7) by:

> If $u_i \rightharpoonup u$ in L_k^p, $|\gamma| \le k-1$, and $p(k-|\gamma|) > n$, then $D^\gamma u_i \to D^\gamma u$ uniformly, and hence in L_0^a for all a. If, however, $p(k-|\gamma|) \le n$, then $D^\gamma u_i \to D^\gamma u$ in L_0^a for all $a < pn/[n-p(k-|\gamma|)]$.

Using this fact, and the methods used in the proof of Theorem 8.6 we can prove the following theorem, from which Theorem 8.6 follows as a corollary.

Let $V(u) = f(x,u)(D^{\gamma_1}u)^{s_1}\cdots(D^{\gamma_N}u)^{s_N}$ on $L_k^p(\Omega, \boldsymbol{R})$ where $\Omega \subset \boldsymbol{R}^n$, $pk > n$, and the γ_i are distinct multi-indices $1 \le |\gamma_i| \le k-1$. Let $a_i = pn/[n - p(k-|\gamma_i|)]$.

Theorem 8.12. *If $\sum (s_i/a_i) < 1$, where the sum is taken only over those i for which $p(k-|\gamma_i|) \le n$, then $\mathscr{V}(u) = \int V(u)$ satisfies condition* (8.1) *and therefore preserves coercivity.*

Remark. One can interpret Theorem 8.12 as saying that as long as $\mathscr{V}(u)$ is well defined, i.e., as long as $V(u) \in L_0^1(\Omega, \boldsymbol{R})$, then $\mathscr{V}(u)$ satisfies condition (8.1).

What about the case where V is a strict polynomial operator, i.e., polynomial in u? In this setting we *do not* have to assume $pk > n$, and using the same method as in Theorems 8.6 and 8.12 we obtain the following result which extends to the spaces $L_k^p(\xi)$ for vector bundles ξ.

Let Ω, a_i, and γ_i be as for Theorem 8.12, except that now $0 \leq |\gamma_i| \leq k - 1$. Let

$$V(u) = f(x)(D^{\gamma_1}u)^{s_1} \cdots (D^{\gamma_N}u)^{s_N}, \qquad u \in L_k^p(\Omega, \boldsymbol{R}).$$

Theorem 8.13. *If $\sum (s_i/a_i) < 1$, where the sum is taken only over those i for which $p(k - |\gamma_i|) \leq n$, then $\mathscr{V}(u) = \int V(u)$ satisfies condition* (8.1) *and thus preserves coercivity.*

Theorem 8.13 has many implications for functionals arising from strict polynomial Lagrangians, acting on real or vector valued functions on a compact manifold. For example, for the special case which we considered in Theorem 5.1,

$$J(u) = \int |\nabla u|^2 - V(x, u),$$

Theorem 8.13 implies that for any quadratic strict polynomial perturbation V, J is a coercive functional regardless of the dimension of the manifold M.

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