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CURVATURES OF COMPLEX SUBMANIFOLDS OF C^n

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0. Introduction

Complex submanifolds M^n of a complex N-space C^N from the viewpoint of hermitian geometry are distinguished by

(a) the existence of N holomorphic imbedding functions $f_1, f_2, \cdots, f_N$ so that the kähler form is of the form $i\partial\bar{\partial}(\sum |f_i|^2)$, and as a consequence

(b) the imbedding is minimal in the sense of riemannian geometry, and all the holomorphic sectional curvatures are nonpositive. In [2] Bochner demonstrated that the Poincaré metric of constant negative curvature on the unit disc cannot be holomorphically imbedded in C^N even locally. It seems therefore reasonable to pose the following

Question. Does there exist a complete complex submanifold M^n of C^N with holomorphic sectional curvature bounded away from zero?

In this paper we discuss partial results to this question. To begin with, we show in § 1 that a negative answer to this question would imply that there is no bounded complete complex submanifold of C^N. In § 2, utilizing an elementary observation on the Gauss map we answer the question in the negative for hypersurfaces, and in § 3 we show that it suffices to consider the question for holomorphic curves ($n = 1$).

In § 4 we recall the higher order curvature functions introduced by Calabi and show that two such functions are enough to determine a holomorphic curve uniquely up to a rigid motion in C^N, and thus providing a justification for a generalization of the theorem in § 2, in terms of the higher order curvature functions. In § 5, applying the method of extremal length we derive a criterion, which involves the curvature behavior at infinity of a simply connected metric riemann surface M for it to be conformally equivalent to the disc. It is subsequently used to sharpen the result in § 2.

The last section contains curvature estimate for a piece of curve in C^2 which is a graph over a domain in C.

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1. Boundedness of complex submanifolds in C^n

It is an open question whether a complete minimal submanifold of euclidean space can be realized in a bounded region. In the case of complex submanifolds we show that a negative answer to the question in the introduction would yield a negative answer here as well. This is accomplished by the following.

Proposition. *The unit ball B^N can be holomorphically imbedded in C^{2N} with the following properties: Let $F: B^N \to C^{2N}$ be the imbedding. Then*

(A) $|dF(v)| \geq |v|$ *for each* $v \in T_p(B^N)$, $p \in B^N$,

(B) *the holomorphic sectional curvatures of B^N are strongly negative, i.e., $K(v) \leq -c < 0$ for all $v \in T_p(B^N)$, $p \in B^N$.*

Proof. Consider the map $F: B^N \to C^{2N}$ given by $F(z_1, \cdots, z_N) = (z_1, \cdots, z_N, e^{z_1}, \cdots, e^{z_N})$. Relative to the coordinates $z_1, \cdots, z_N$, we have $g_{i\bar{j}} = \langle \partial F/\partial z_i, \overline{\partial F/\partial z_j}\rangle = \delta_{ij}(1 + |e^{z_i}|^2)$, so that $g^{i\bar{j}} = \delta_{ij}(1 + |e^{z_i}|^2)/G$, where $G = \det g_{i\bar{j}} = \prod_i (1 + |e^{z_i}|^2)$. Thus $K_{i\bar{j}k\bar{l}} = \delta_{ik}\delta_{jl}\delta_{ij}(|e^{z_i}|^2 - |e^{z_i}|^2(1 + |e^{z_i}|)^2/G)$, i.e., $K_{i\bar{i}i\bar{i}} = |e^{z_i}|^2(1 - (1 + |e^{z_i}|^2)/G)$, and all the other components vanish. Hence it is clear that the holomorphic sectional curvature of $F(B^N)$ at the point $F(z)$ in the direction $F_*(v) = F_*(v^i\partial/\partial z_i)$ satisfies

$$\begin{aligned} K(v) &= -(\textstyle\sum K_{i\bar{j}k\bar{l}}v^i\bar{v}^jv^k\bar{v}^l)/|\textstyle\sum g_{i\bar{j}}v_i\bar{v}_j|^4 \\ &= -\Big(\sum_i K_{i\bar{i}i\bar{i}}v^i\bar{v}^iv^i\bar{v}^i\Big)/|\textstyle\sum g_{i\bar{j}}v_i\bar{v}_{\bar{j}}|^4 \leq -c < 0 \end{aligned}$$

for some c. Condition (A) is satisfied, since $F(B^N)$ is a graph over B^N. q.e.d.

Next we recall the Gauss-Codazzi equation for computing the curvature of a submanifold $\varphi: M \to \tilde{M}$:

$$\begin{aligned} K(v) = R(v, Jv, v, Jv) = \tilde{R}(v, Jv, v, Jv) &- \langle B(v, Jv), B(Jv, v)\rangle \\ &+ \langle B(Jv, Jv), B(v, v)\rangle\ , \end{aligned} \tag{1}$$

where R = curvature tensor of submanifold M, $\tilde{R}$ = curvature tensor of manifold $\tilde{M}$, B = second fundamental form of M, K = holomorphic sectional curvature of M, $\tilde{K}$ = holomorphic sectional curvature of $\tilde{M}$. Recalling that the second fundamental form is complex linear ($B(Jx, Y) = B(x, Jy) = JB(x, y)$), we may rewrite (1) as

$$K(v) = \tilde{K}(v) - 2|B(v, v)|^2\ ,$$

which expresses the curvature decreasing phenomenon of a complex submanifold of a kähler manifold.

Thus we are in a position to conclude the argument. Suppose $\varphi: M^n \to B^N$ is a complete holomorphic immersion. Then composing φ with the map F constructed in the proposition, we see clearly that $F \circ \varphi$ is still a complete immersion (a consequence of (A)). Since $F\varphi(M)$ is a submanifold of $F(B)$, the curvature decreasing property implies that $F \circ \varphi(M)$ has strongly negative holomorphic sectional curvature.

2. Hypersurfaces

In this section we study the Gauss map of a complex hypersurface in C^{n+1}. An elementary computation shows that the kähler form of the metric induced by the Gauss map is the negative of the ricci form of M itself. It will follow that no complete hypersurface of C^{n+1} can have strongly negative holomorphic sectional curvature.

To fix notation, let $M^n \overset{\varphi}{\hookrightarrow} C^{n+1}$ be a complex hypersurface. The Gauss map is defined analogously as the classical gauss map of a surface in R^3. Let ξ_p be a unit normal to M at p. ξ_p is determined up to a multiplication by $e^{i\theta}$, so ξ_p determines a point $[\xi_p]$ in P_nC. (Actually we shall use the fact that $p \mapsto \xi_p \mapsto [\xi_p]$ goes through the Hopf fibration $M \xrightarrow{\xi} S^{2n+1} \xrightarrow{\pi} P_nC$, and the Fubini-Study metric comes from the standard metric on the sphere S^{2n+1}.) The Gauss map is simply $G(p) = [\xi_p]$. If we represent M locally by the zero set of a holomorphic function f, then $G(p) = [\overline{(\partial f/\partial z_1)(p)}, \cdots, \overline{(\partial f/\partial z_{n+1})(p)}]$ is a local representation of G, hence G is conjugate holomorphic.

Let $\omega =$ kähler form of Fubini-Study metric in P_nC. Then $\pi^*\omega = \langle dz, dz \rangle$. Let S denote the ricci form of M, i.e., the (1,1) form corresponding to the ricci tensor. The claim is

$$\pi^*\omega = -S . \tag{2}$$

Let $1 \leq \alpha, \beta, \gamma \leq n+1$; $1 \leq i, j, h \leq n$. Let $e_\alpha(x)$ be a field of unitary frames with $e_1, \cdots, e_n$ tangent to M, and e_{n+1} normal to M. Its dual coframe field consists of $n+1$ complex valued linear differential forms θ_α of type $(1,0)$, and the kähler metric for C^{n+1} is written as

$$ds^2 = \sum \theta_\alpha \bar{\theta}_\alpha .$$

The connection forms $\theta_{\alpha\beta}$ satisfy

$$\theta_{\alpha\beta} + \bar{\theta}_{\beta\alpha} = 0 , \qquad d\theta_\alpha = \sum \theta_\beta \wedge \theta_{\beta\alpha} .$$

The curvature forms $\Theta_{\alpha\beta}$ are given by

$$\begin{aligned} d\theta_{\alpha\beta} &= \sum \theta_{\alpha\gamma} \wedge \theta_{\gamma\beta} + \Theta_{\alpha\beta} , \\ \Theta_{\alpha\beta} &= -\bar{\Theta}_{\beta\alpha} = \sum R_{\alpha\beta\gamma\delta} \theta_\gamma \wedge \bar{\theta}_\delta \ (\equiv 0 \text{ in this case}) . \end{aligned}$$

Restricting everything to the hypersurface we have from

$$\theta_{n+1} = 0$$

that

$$0 = d\theta_{n+1} = \sum \theta_i \wedge \theta_{i,n+1}$$

so that by Cartan's lemma

$$\theta_{i,n+1} = \sum a_{ik}\theta_k \,, \qquad \text{where } a_{ik} = a_{ki} \,,$$
$$d\theta_{ij} = \sum \theta_{ik} \wedge \theta_{kj} - \theta_{i,n+1} \wedge \theta_{j,n+1} \,.$$

Therefore $\tilde{\Theta}_{ij} = \Theta_{ij} - \theta_{i,n+1} \wedge \bar{\theta}_{j,n+1} = -\theta_{i,n+1} \wedge \bar{\theta}_{j,n+1}$. The Ricci form is given by

$$(3) \qquad \sum S_{jk}\theta_j \wedge \bar{\theta}_k = -\sum_{j,k,i} a_{ij}\bar{a}_{ik}\theta_j \wedge \bar{\theta}_k \,.$$

Since the Gauss map is given by $G(p) = e_{n+1}(p)$, the induced kähler form is

$$(4) \qquad \begin{aligned} G^*\omega &= de_{n+1} \wedge \overline{de}_{n+1} = \sum \theta_{n+1,i} \wedge \bar{\theta}_{n+1,i} \\ &= \sum \theta_{i,n+1} \wedge \bar{\theta}_{i,n+1} = \sum a_{ij}\bar{a}_{ik}\theta_j \wedge \bar{\theta}_k \,. \end{aligned}$$

Comparing (3) with (4) yields the claim.

Theorem. *Let $p: M^n \to C^{n+1}$ be a complete complex hypersurface. Then (M^n, φ) cannot have strongly negative holomorphic sectional curvature.*

Proof. Suppose not; say in fact $K(v) \leq -c < 0$, therefore the Ricci curvature is $\leq -(n-1)c$. In view of (2) this means that the Gauss map is distance-increasing by a factor greater than $(n-1)c$, hence (M, G) is complete with respect to the induced metric $G^*\omega$. Since the Gauss map is equidimensional and antiholomorphic, the induced metric must have constant holomorphic sectional curvature $+4$, which would imply that $M \approx P_nC$, i.e., M is a compact manifold, a clear contradiction.

Remarks. (1) It is easy to obtain quantitative version of this theorem, for example, using the same argument one can show that if $\varphi: M^n \to C^{n+1}$ is a complete complex hypersurface then for any point $p \in M^n$, $c > 0$, there exist a sequence of points p_i and vectors $v_i \in T_{p_i}M^n$ such that $K(v_i) > -c/\text{dist}\,(p, p_i)^2$, where $\text{dist}\,(p, p_i)$ can be either geodesic distance on M^n or the euclidean distance $|p - p_i|$. It is easy to see in either case that if the conclusion is false, then the induced metric $G^*\omega$ is complete.

(2) The theorem goes through for a minimal hypersurface of R^{N+1} in a completely analogous manner.

3. Reduction to holomorphic curves

The general case of arbitrary codimension can be reduced to the consideration of holomorphic curves, as shown in the following proposition.

Proposition. *Suppose M^n is a complete complex submanifold of C^N with either of the following properties:*

(1) *the holomorphic sectional curvature K satisfies* $K(v) \leq -c < 0$,
(2) $\varphi: M^n \to C^N$ *is bounded.*

Then there exists an affine subspace L^{N-n+1} *of* C^N *such that* $L^{N-n+1} \cap M^n$ *is a nonsingular complete holomorphic curve with the same property* (1) *or* (2).

Proof. For each nonnegative integer $i \geq 0$, let B_i denote the set $\{z \in C^N \,|\, |z| \leq i\}$. The set of affine subspaces L of C^N of dimension $N - n + 1$, whose intersection with $B_i \cap M^n$ is a nonsingular curve, is clearly open and dense in the space of all affine linear subspaces. L and S can be made into a complete metric space. A category argument then yields the existence of an L^{N-n+1} such that $L^{N-n+1} \cap M^n$ is a nonsingular curve, which being a closed subset of M^n is cleary complete. Property (1) is satisfied due to the curvature-decreasing property. Property (2) is trivial.

Remark. The properties (1) and (2) and completeness are inherited when one passes to the universal covering manifold. Therefore for the questions under consideration, it suffices, in view of the uniformization theorem, to take the unit disc as the underlying Riemann surface.

4. Holomorphic curves

Consider a holomorphic curve $\varphi: M^1 \to C^n$. In terms of local coordinates $z = x + iy$, the hermitian metric induced from C^n may be written as $ds^2 = F\,|dz|^2$, where $F = \sum_i |f_i'|^2$ and $\varphi = (f_1, f_2, \cdots, f_n)$. As the metric is conformal, the Laplace operator can be expressed simply as $\Delta = (4/F)(d^2/dzd\bar{z})$, and the Gauss curvature is found to be $K = (-2/F)(d^2/dzd\bar{z}) \log F = -\frac{1}{2}\Delta \log F$.

According to Calabi, there is a sequence of inductively defined nonnegative real analytic functions on M:

Lemma (*Calabi* [3]). *Suppose that the image* $\varphi(M)$ *lies in no hyperplane. Then we may define a sequence of functions* $\{F_n\}$ *as follows:*

$$F_0 = 1\,, \qquad F_1 = F\,,$$
$$F_{k+1} = \frac{F_k^2}{F_{k-1}}\left(\frac{d^2}{dzd\bar{z}} \log F_k\right), \qquad \text{for } 1 \leq k \leq n\,. \tag{5}$$

F_k *is nonnegative and vanishes only at isolated points. The succeeding function is defined by* (5) *away from these points, but extend to a real analytic function on all of* M. $F_k \equiv 0$ *for* $k \geq n + 1$.

Theorem (*Calabi* [3]). *Let* $ds^2 = F\,|dz|^2$ *be a real analytic hermitian metric on* M, *and suppose that a sequence of functions* F_k *satisfying* (5) *can be defined with the same properties as in the above lemma. Then there exists a unique holomorphic isometric immersion of* (M, ds^2) *into* C^n *up to a motion of* C^n.

Simple computations give the following explicit formulas for these functions in terms of the imbedding functions $(f_1, \cdots, f_n)$:

$$F_1 = \sum_i |f_i'|^2\,, \qquad F_2 = \sum_{i>j} |f_i''f_j' - f_i'f_j''|^2\,,$$

$$F_3 = \sum_{i>j>k} \left| \det \begin{pmatrix} f_i & f_j & f_k \\ f_i' & f_j' & f_k' \\ f_i'' & f_j'' & f_k'' \end{pmatrix} \right|^2 ,$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$F_n = |\det (f_j^{(i)})|^2 .$$

From these intrinsically defined functions it is possible to define higher order curvature functions $K_0 = 0$, $K_k = (F_{k+1}F_{k-1})/(FF_k^2)$ for $k \geq 1$. A simple computation then gives $K_k = \frac{1}{2}\Delta \log F_k$. These intrinsic curvature functions have geometric meaning: they are the squared norms of higher order fundamental forms (Lawson [7]). In particular, $K_1 = -\frac{1}{2}K$ where K is the Gauss curvature of ds^2 as usual, and $K_k = 0$ if $k \geq n$.

The curvature functions satisfy certain recurrence relations; the ones of interest to us are

$$(6) \qquad \tfrac{1}{2}\Delta \log K_k = K_{k+1} - 2K_k + K_{k-1} - K_1 , \qquad k = 1, \cdots, n-1 ,$$

and, as a consequence,

$$(7) \qquad \tfrac{1}{4} \log K_1K_2 \cdots K_{n-1} = -2K_{n-1} - (n-1)K_1 .$$

In terms of these curvature functions it is possible to formulate the following rigidity condition.

Proposition. *If $\varphi, \varphi' : M' \to C^N$ are two holomorphic immersions satisfying $K_1 = K_1'$, $K_2 = K_2'$, then $F = F'$, and consequently φ and φ' are congruent.*

Proof. If K_1 (and hence K_1') vanishes identically, then Calabi's theorem (Lemma 1) implies that $\Phi(M)$ and $\Phi'(M)$ lie in a 1-dimensional linear subvariety of C^n. In this case checking the congruence of Φ and Φ' is trivial.

Let us then assume that neither K_1 nor K_1' vanishes identically. To prove Φ and Φ' are congruent, it suffices to prove $F = F'$ in some open set because of the analyticity of F and F'. Take a coordinate neighborhood on which all the F_k and F_k' are nowhere zero, and denote this neighborhood by U. On U we have

$$F(K_2 - 3K_1) = \frac{d^2}{dzd\bar{z}} \log K_1 = \frac{d^2}{dzd\bar{z}} \log K_1' = F'(K_2' - 3K_1') .$$

We know $K_2 - 3K_1 = K_2' - 3K_1'$. We will show in the following that $K_2 - 3K_1$ is not identically zero in U. Hence we may cancel the common factor to get $F = F'$, as desired.

Now suppose $K_2 - 3K_1 = 0$ in U, and we will deduce a contradiction. This condition implies both $\Delta \log K_1 = 0$ and $K_2 = 3K_1$. Hence $\Delta \log K_2 = \Delta \log 3K_1 = 0$. Since $K_3 = \frac{1}{2}\Delta \log K_2 + 2K_2$, we see that $K_3 = 6K_1$. Now suppose at the kth stage, we have $K_k = \frac{1}{2}k(k+1)K_1$. With the help of the recurrence relation,

$$K_{k+1} = \tfrac{1}{2}\Delta \log K_k + 2K_k - K_{k-1} + K_1 , \qquad 1 \leq k \leq n-1 ,$$

we get

$$\begin{aligned} K_{k+1} &= \tfrac{1}{2}\Delta \log (\tfrac{1}{2}k(k+1)K_1) + k(k+1)K_1 - k(k-1)K_1 + K_1 \\ &= \tfrac{1}{2}(k+1)(k+2)K_1 . \end{aligned}$$

In particular, $0 = K_n = \frac{1}{2}n(n+1)K_1$, which contradicts our assumption that K_1 is not identically zero. q.e.d.

The previous proposition indicates that perhaps one should consider higher order osculations in our study of the general question. In this direction it is easy to prove

Theorem. *If $\Phi: D \to C^N$ is a complete holomorphic curve, then either* $\inf K_1 = 0$ *or* $\inf K_1 \cdots K_{n-1} = 0$.

Proof. Suppose not; then there is a curve $\Phi: D \to C^N$ with the property that $K_1 \geq c > 0$, $K_1, \cdots, K_{n-1} \geq c > 0$ for some c. Consider the metric $\widetilde{ds}^2 = K_1 \cdots K_{n-1} ds^2$, which is a complete hermitian metric by the assumptions. Further, the relation $\Delta \log K_1 \cdots K_{n-1} = -2K_{n-1} - (n-1)K_1$ shows that $\widetilde{ds}^2$ has strictly positive curvature. This is a contradiction to the fact that unit disc cannot carry such a metric (see Greene-Wu [4]).

5. A condition for hyperbolicity

In Greene-Wu [4], it is stated that if M is a simply connected Riemann surface with hermitian metric g. Suppose there exist $p_0 \in M$ and $c > 0$ such that $-c/d(p_0, p)^{2+\varepsilon} \leq K(p) \leq 0$ for some positive c. Then M is conformally equivalent to C. A complementary result is the following.

Theorem. *If M is a simply connected Riemann surface, and g a hermitian metric on M with nonpositive curvature satisfying: there exist $p_0 \in M$ and a compact set $G \subset M$, such that $K(p) \leq -c/d(p_0, p)^2$ for all $p \notin G$. Then M is conformally equivalent to a disc.*

Remark. This result has been extended by Greene and Wu [5] to a condition for a complete manifold M of arbitrary dimension to be complete hyperbolic in the sense of Kobayashi.

The proof employs the method of extremal length. We give a short convenient formulation.

Definition. Let Ω be a region in the plane, and Γ a set of rectifiable 1-chains in Ω. Consider the family of all conformal metrics $ds = \rho|dz|$, where ρ is required to be Borel-measurable. Let

$$L(\gamma, \rho) = \int_\gamma \rho |dz| \qquad \text{for each } \gamma \in \Gamma ,$$

$$A(\Omega, \rho) = \int_\Omega \rho^2 dxdy , \qquad L(\Gamma, \rho) = \inf_{\gamma \in \Gamma} L(\gamma, \rho) .$$

Define the extremal length of the family Γ (relative to Ω) to be

$$\wedge_\Omega(\Gamma) = \sup_\rho L(\Gamma, \rho)^2 / A(\Omega, \rho) \ , \qquad \text{where } \rho \text{ satisfies } 0 < A(\Omega, \rho) < \infty \ .$$

The following properties of extremal length are well-known; see for instance Ahlfors and Sario [1] or Otsuka [8].

(I) $\wedge_\Omega(\Gamma)$ is a conformal invariant. More precisely, if φ is a conformal map sending Ω to Ω', Γ to Γ', then $\wedge_\Omega(\Gamma) = \wedge_{\Omega'}(\Gamma')$.

(II) Let D be a domain in the plane such that its complement with respect to the extended plane consists of mutually disjoint nonempty closed subsets F, F'. Let D_n be a sequence of domains increasing to D such that the complement of each D_n with respect to the extended plane consists of closed sets F_n, F'_n which decrease to F and F' respectively. Let Γ_n be the set of rectifiable arcs in D_n which join F_n to F'_n, and Γ be the set of rectifiable arcs in D which join F to F'. Then $\wedge_{D_n}(\Gamma_n)$ increases to $\wedge_D(\Gamma)$.

(III) For a doubly connected region D in the plane bounded by simple closed curves γ_1 and γ_2 as below, consider the two families of curves in D:

$$\Gamma_1 = \{\text{closed curves which separate } \gamma_1 \text{ and } \gamma_2\} \ ,$$
$$\Gamma_2 = \{\text{"radial arcs" which join } \gamma_1 \text{ to } \gamma_2\} \ .$$

Then we have $\wedge_D(\Gamma_1) \wedge_D(\Gamma_2) = 1$.

(IV) For a doubly connected region D in the plane bounded by a simple closed curve γ and the component of D^c containing ∞, $\wedge_D(\Gamma_2) < \infty$ implies that $F \cup D$ is conformally a disc.

Proof of the Theorem. Since M is simply connected, we may consider M as a subset of $\boldsymbol{C}$. Let 0 correspond to p_0. Then the exponential map $\exp_{p_0}$: $T_0(M) \to M$ is a diffeomorphism by the curvature assumption, and we have well-defined geodesic polar coordinates (r, θ) on M, (which is distinct from the usual $z = re^{i\theta}$) with the metric $ds^2 = dr^2 + G(r, \theta)d\theta^2$. Let $K(p) \leq -c/d(0, p)^2 = -c/r(p)^2$ for $r(p) \geq r_0 > 0$. Let

$$D_R = \{p \in M : r_0 < r(p) < R\} \quad \text{and} \quad \gamma_0 = \{p \in M : r_0 = r(p)\} \ ,$$
$$D = \{p \in M : r_0 < r(p)\} \qquad \text{and} \quad \gamma_R = \{p \in M : R = r(p)\} \ ,$$
$$\Gamma_1^R = \{\text{simple closed curves in } D_R \text{ which separate } \gamma_0 \text{ and } \gamma_R\} \ ,$$
$$\Gamma_2^R = \{\text{arcs in } D_R \text{ which join } \gamma_0 \text{ to } \gamma_R\} \ ,$$
$$\Gamma_2 = \{\text{arcs in } D \text{ which join } \gamma_0 \text{ to } \partial M\} \ .$$

An upper estimate for the extremal length $\wedge_{D_R}(\Gamma_2^R)$ will be derived. We remark that $dr^2 + G(r, \theta)d\theta^2$ is a conformal metric on the underlying Riemann surface, and therefore so is $\rho_0 |dz|^2 = (dr^2 + G(r, \theta)d^2)/G(r, \theta)$. It follows immediately from (III) that

$$\bigwedge_{D_R}(\Gamma_2^R) = \frac{1}{\bigwedge_{D_R}(\Gamma_1^R)} = \inf_\rho \frac{A(D_R, \rho)}{L(\Gamma_1^R, \rho)^2} \leq \frac{A(D_R, \rho_0)}{\inf_{\gamma \in \Gamma_1^R} L(\gamma, \rho_0)^2} .$$

It is clear that $L(\gamma, \rho_0) \geq 2\pi$ for all $\gamma \in \Gamma_1^R$ so that

$$\bigwedge_{D_R}(\Gamma_2^R) \leq \tfrac{1}{2} A(D_R, \rho_0)/\pi .$$

To estimate $A(D_R, \rho_0)$ we compare $G(r, \theta)$ against the solution of the Jacobi equation

$$(\sqrt{G})_{rr} + K\sqrt{G} = 0 .$$

For $r \geq r_0$,

$$(8) \qquad -K = (\sqrt{G})_{rr}/\sqrt{G} \geq c/r^2 .$$

Let $\sqrt{G}_0 = \min_\theta \sqrt{G}(r_0, \theta)$, $(\sqrt{G}_r)_0 = \min_\theta \sqrt{G}_r(r_0, \theta) > 0$. The solution of the ordinary differential equation

$$y_{rr} = c/r^2 , \quad y(r_0) = \sqrt{G}_0 , \quad y_r(r_0) = (\sqrt{G}_r)_0$$

is found to be

$$y(r) = \frac{(\sqrt{G}_r)_0 r_0 + \sqrt{G}_0 \cdot \frac{1}{2}(\sqrt{1+4c} - 1)}{\sqrt{1+4c} r_0 \cdot \frac{1}{2}(\sqrt{1+4c})} r \cdot \tfrac{1}{2}(1 + \sqrt{1+4c})$$
$$+ \frac{-(\sqrt{G}_r)_0 r_0 + \sqrt{G}_0 \cdot \frac{1}{2}(\sqrt{1+4c} + 1)}{\sqrt{1+4c}_{r_0} \cdot \frac{1}{2}(1 - \sqrt{1+4c})} r \cdot \tfrac{1}{2}(1 - \sqrt{1+4c}) .$$

It is easy to see that $y(r) \geq A(r_0, c, \sqrt{G}_0, (\sqrt{G}_r)_0) r \cdot \frac{1}{2}(1 + \sqrt{1+2c})$, so that $G(r, \theta) \geq A(r_0, c, \sqrt{G}_0, (\sqrt{G}_r)_0) r \cdot \frac{1}{2}(1 + \sqrt{1+2c})$. Hence

$$A(D_R, \rho_0) = 2\pi \int_{r_0}^R (r/G(r, \theta)) dr \leq \frac{2\pi}{A} \int_{r_0}^R \frac{dr}{1 + \sqrt{1+2c}} < M < \infty ,$$

which implies $\bigwedge_{D_R}(\Gamma_2^R) < \frac{1}{2}M/\pi$. It follows from (II) that $\bigwedge_D(\Gamma_2) < \infty$, and M is conformally equivalent to a disc by (IV).

Corollary. *Let $\Phi: M \to C^n$ be a simply connected complete immersed curve, with $\Phi(p_0) = 0$. If $K(p) \leq -c/|\Phi(p)|^2$ for large $|\Phi(p)|$, then M is conformally a disc.*

Proof. It is clear that $|\Phi(p)| \leq d(p_0, p)$.

We now apply this criterion to an immersed holomorphic curve in C^2.

Corollary. *If $\varphi: M \to C^2$ is a simply connected complete holomorphic curve, then there is no $p_0 \in M$ such that, for some constant $c > 0$ and compact set $G \subset M$, $p_0 \in G$ and*

$$K(p) \leq -c/d(p_0, p)^2 \qquad \textit{for } p \notin G .$$

Proof. Suppose that on the contrary, such p_0, c and G do exist. It follows from the theorem that M is conformally a disc. Let $\{z_1, \cdots, z_n\}$ be the finite set of points in G where K vanishes. Then near each z_i, K can be written as $K(z) = -|z - z_i|^{2n_i}h(z)$, where h is a positive function. Consider the metric $(\prod_i |z - z_i|^{-2n_i})(-K)ds^2$. Since the first factor is bounded away from zero outside the compact set G, it follows as before that it is a complete metric. But the curvature of the new metric is $\prod_i |z - z_i|^{2n_i}$, hence it stays bounded away from zero outside G. Therefore an application of comparison theorem shows that M is again compact, which is a contradiction. q.e.d.

The method of construction in the proof of the above corollary may also be used to obtain the following.

Theorem. *Let M be obtained from a compact Riemann surface $\tilde{M}$ by deleting a finite number of closed connected subsets each of which has nonempty interior, and let $\varphi: M \to \boldsymbol{C}^2$ be a complete holomorphic curve. Then there is no p_0 in M such that, for some constant $c > 0$ and compact $G \subset M$, $K(p) \leq -c/d(p_0, p)^2$ for $p \notin G$.*

Proof. Suppose on the contrary, such p_0, c and G do exist. We choose f to be a meromorphic function on M satisfying:

(a) the zeroes of $|f|^2$ in M are precisely those of K, counted according to order of multiplicity,

(b) the poles if any are contained in the interior of $\tilde{M} - M$,

(c) $\min_{z \in \partial M} |f(z)|^2 > 0$.

Since the zeroes of K are finite in number, and ∂M is compact, such function f can easily be found. Then we can prove as in the corollary above that $-|f|^2 K ds^2$ is a complete metric of curvature $|f|^2$. Thus by a previous argument M would be compact, a contradiction.

6. A curvature estimate

While the previous discussions center around global restrictions on the curvature of a complete immersed curve, there is, as in the case of minimal surfaces in $\boldsymbol{R}^3$, a semilocal restriction on the curvature.

Theorem. *Suppose a complex holomorphic curve in $\boldsymbol{C}^2$ is parametrized as a graph $(z, f(z))$ over a disc $|z| < R$, then we have the estimate*

$$R \leq 4\sqrt{2}(\alpha)^{-1/2} , \qquad \textit{where } \alpha = \min |K| .$$

Proof. The idea, due to E. Heinz [6], is simply to estimate the area of the surface over $\{|z| < r\}$. To proceed, we work with polar coordinates $z = re^{i\theta}$, let $D_\rho = \{(z, f(z)) : r < \rho\}$, and we have

$$\text{Area } (D_\rho) = 2 \iint_{|z|<\rho} (1 + f'\bar{f}')dx \wedge dy .$$

Observe that

$$d^c = \sqrt{-1}(d'' - d') = r\frac{\partial}{\partial r}d\theta - \frac{1}{r}\frac{\partial}{\partial\theta}dr ,$$

an application of Stoke's theorem yields

$$\rho\frac{d}{d\rho}\int_0^{2\pi} f'(\rho e^{i\theta})\overline{f'(\rho e^{i\theta})}d\theta = \int_0^{2\pi}\rho\frac{d}{d\rho}(1 + f'(\rho e^{i\theta})\overline{f'(\rho e^{i\theta})})d\theta$$
$$= \int_{|z|=\rho} d^c(1 + f'\bar{f}') = \iint_{|z|<\rho} dd^c(1 + f'\bar{f}') .$$

Consider the following function $F(r)$, defined for $0 < r < R$,

$$F(r) = \int_0^r \rho d\rho \int_0^{2\pi} (1 + f'(\rho e^{i\theta})\overline{f'(\rho e^{i\theta})})d\theta \ (= \text{Area of } (D_r)) .$$

Let $K = -f''\bar{f}''(1 + f'\bar{f}')^{-3} \leq -\alpha < 0$. Differentiating $F(r)$, we have

$$F'(r) = r\int_0^{2\pi} (1 + f'\bar{f}')d\theta ,$$
$$F''(r) = \int_0^{2\pi} (1 + f'\bar{f}')\Big|_{|z|=r} d\theta + r\frac{d}{dr}\int_0^{2\pi}(1 + f'\bar{f}')d\theta ,$$

so that

$$F''(r) \geq \int_{|z|<r} dd^c(1 + f'\bar{f}')$$
$$= \int_{|z|<r} f''\bar{f}''dx \wedge dy \geq \int_{|z|<r} \alpha(1 + f'\bar{f}')^3 dx \wedge dy .$$

Since, by Hölder's inequality,

$$\pi r^2 \leq F(r) \leq (\pi r^2)^{2/3}\left(\int_{|z|<r} (1 + f'\bar{f}')^3 dx \wedge dy\right)^{1/3} ,$$

we obtain

$$F''(r) \geq \alpha\iint_{|z|<r}(1 + f'\bar{f}')^3 dxdy \geq \frac{\alpha(F(r))^3}{(\pi r^2)^2} = \frac{\alpha}{\pi^2 r^4}(F(r))^3 ,$$

from which we deduce the differential inequality

$$\frac{d}{d\sigma}(F'(\rho))^2 = 2F'(\rho)F''(\rho) \geq 2F'(\rho)\frac{\alpha}{\pi^2\rho^4}(F(\rho))^3$$
$$\geq \frac{\alpha}{2\pi^2\rho^4}\frac{d}{d\rho}(F(\rho))^4 \qquad \text{for all } 0 < \rho < R .$$

Therefore

$$\frac{d}{d\rho}(F'(\rho))^2 \geq \frac{\alpha}{2\pi^2 r^4} F(\rho))^4$$

or

$$\frac{(F'(r))^2}{(F(r))^4} \geq \frac{\alpha}{2\pi^2 r^4} \quad \text{or} \quad \frac{F'(r)}{(F(r))^2} \geq \left(\frac{\alpha}{2}\right)^{1/2} \frac{1}{\pi r^2} ,$$

which implies

$$(9) \qquad -\frac{d}{dr}(F(r)^{-1}) \geq \left(\frac{\alpha}{2}\right)^{1/2} \frac{1}{\pi r^2} \qquad \text{for } 0 < r < R .$$

Now let $0 < R_1 < R_2 < R$. Integrating the inequality (9) gives

$$-\int_{R1}^{R2} \frac{d}{dr}\left(\frac{1}{F(r)}\right) dr \geq \left(\frac{\alpha}{2}\right)^{1/2} \frac{1}{\pi} \int_{R_1}^{R_2} \frac{1}{r^2} dr ,$$

$$\frac{1}{\pi R_1^2} \geq \frac{1}{F(R_1)} \geq \frac{1}{F(R_1)} - \frac{1}{F(R_2)} \geq \left(\frac{\alpha}{2}\right)^{1/2} \frac{1}{\pi}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) .$$

Letting $R_2 \to R$ we obtain

$$\frac{1}{R_1^2} \geq \left(\frac{\alpha}{2}\right)^{1/2}\left(\frac{1}{R_1} - \frac{1}{R}\right) .$$

Hence

$$\frac{1}{R} \geq \frac{1}{R_1} - \left(\frac{\alpha}{2}\right)^{-1/2}\left(\frac{1}{R_1^2}\right) \quad \text{or} \quad R \leq \left(\frac{(\alpha/2)^{1/2}R_1 - 1}{(\alpha/2)^{1/2}R_1^2}\right)^{-1} .$$

Choose $R_1 = 2(2/\alpha)^{1/2}$, we obtain $R \leq 4(\alpha/2)^{-1/2}$ as asserted.

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