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SOME ALMOST HERMITIAN MANIFOLDS WITH CONSTANT HOLOMORPHIC SECTIONAL CURVATURE

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B. Smyth proved in [3]

Theorem A. *Let M be a complex hypersurface of a Kählerian manifold $\tilde{M}$ of constant holomorphic sectional curvature $\tilde{\mu}$. If M is of complex dimension ≥ 2, then the following statements are equivalent*:

(i) *M is totally geoddsic in $\tilde{M}$,*

(ii) *M is of constant holomorphic sectional curvature,*

(iii) *M is an Einstein manifold, and at one point of M all sectional curvatures of M are $\geq \frac{1}{4}\tilde{\mu}$ (resp. $\leq \frac{1}{4}\tilde{\mu}$) when $\tilde{\mu} \geq 0$ (resp. ≤ 0).*

Considering nearly Kähler manifolds, S. Sawaki and K. Sekigawa proved in [2] the following generalization of this theorem.

Theorem B. *Let M be a complex hypersurface of a nearly Kähler manifold $\tilde{M}$ with constant holomorphic sectional curvature $\tilde{\mu}$. If M is of complex dimension ≥ 2, then the following statements are equivalent*:

(i) *M is totally geodesic in $\tilde{M}$,*

(ii) *M is of constant holomorphic sectional curvature,*

(iii) *at every point $m \in M$ all the sectional curvatures of M satisfy*

$$K(x, y) \geq \tfrac{1}{4}\tilde{\mu}\{1 + 3g(x, Jy)^2\} ,$$

where x, y are any orthonormal vectors of $T_m(M)$.

An almost Hermitian manifold with J-invariant Riemann curvature tensor is called an RK-manifold [6]. RK-manifolds with pointwise constant type form a particularly nice class of almost Hermitian manifolds, and many properties for Kähler manifolds can be generalized to this class [4], [5], [6], [7]. An RK-manifold with pointwise constant type and pointwise constant holomorphic sectional curvature is an Einstein manifold. The main purpose of this paper is to generalize the theorem of Smyth to complex hypersurfaces of such manifolds satisfying an interesting condition. This is done in § 3 following the same arguments as in [2], [3].

In § 1 we give some generalizations of theorems for RK-manifolds [6] to almost Hermitian manifolds. In § 2 we state some differential-geometric pro-

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perties of a complex hypersurface of an almost Hermitian manifold satisfying a certain condition, and finally in § 4 we give some properties for the holomorphic bisectional curvature [1].

We remark that, if necessary, the complex hypersurface is supposed to be connected.

1. Let M be a C^∞ differentiable manifold which is *almost Hermitian*, that is, the tangent bundle has an almost complex structure J and a Riemannian metric g such that $g(JX, JY) = g(X, Y)$ for all $X, Y \in \chi(M)$ where $\chi(M)$ is the Lie algebra of C^∞ vector fields on M. We suppose that $\dim M = n = 2m$, and we denote by ∇ the Riemannian connection on M. Let R be the Riemann curvature tensor, S the Ricci tensor defined by

$$S(x, y) = \sum_{i=1}^{n} R(x, e_i, y, e_i) , \tag{1}$$

where $x, y \in T_m(M)$, $m \in M$ and $\{e_i\}$ is an orthonormal local frame field, and $K(x, y)$ the sectional curvature for a 2-plane spanned by x and y. We denote by $H(x)$ the holomorphic sectional curvature of the 2-plane spanned by x and Jx. The sectional curvature of the antiholomorphic plane spanned by x and y, where $g(x, y) = g(x, Jy) = 0$, is called the antiholomorphic sectional curvature.

An almost Hermition manifold such that the Riemann curvature tensor R is J-invariant, that is,

$$R(JX, JY, JZ, JW) = R(X, Y, Z, W) , \qquad \forall X, Y, Z, W \in \chi(M) , \tag{2}$$

is said to be an *RK-manifold* [6]. For such a manifold we have

$$K(x, y) = K(Jx, Jy) , \qquad K(x, Jy) = K(Jx, y) , \tag{3}$$

$$S(x, y) = S(Jx, Jy) , \qquad S(x, Jy) + S(Jx, y) = 0 . \tag{4}$$

We say further that an almost Hermitian manifold is of *constant type* at $m \in M$ provided that for all $x \in T_m(M)$ we have

$$\lambda(x, y) = \lambda(x, z) \tag{5}$$

with

$$\lambda(x, y) = R(x, y, x, y) - R(x, y, Jx, Jy) , \tag{6}$$

whenever the planes defined by x, y and x, z are antiholomorphic and $g(y, y) = g(z, z)$. If this holds for all $m \in M$, we say that M has (*pointwise*) *constant type*. Finally, if $X, Y \in \chi(M)$ with $g(X, Y) = g(X, JY) = 0$, $\lambda(X, Y)$ is constant whenever $g(X, X) = g(Y, Y) = 1$, then M is said to have *global constant type*.

The following theorems are generalizations of theorems given in [6]. The proofs are easy verifications.

Theorem 1. *Let M be an almost Hermitian manifold and $x, y \in T_m(M)$. Then*

$$\begin{aligned} R(x, y, x, y) = &\frac{1}{32}\{3Q(x + Jy) + 3Q(x - Jy) - Q(x + y) \\ &\qquad - Q(x - y) - 4Q(x) - 4Q(y)\} \\ &+ \frac{1}{16}\{13\lambda(x, y) - 3\lambda(Jx, Jy)\} \\ &+ \frac{1}{16}\{\lambda(x, Jy) + \lambda(Jx, y\} , \end{aligned} \tag{7}$$

where $Q(x) = R(x, Jx, x, Jx)$.

Theorem 2. *Assume M is almost Hermitian, and let $x, y \in T_m(M)$ be such that $g(x, x) = g(y, y) = 1$ and $g(x, Jy) = \cos\theta \geq 0$. Then*

$$\begin{aligned} K(x, y) = &\frac{1}{8}\{3(1 + \cos\theta)^2 H(x + Jy) + 3(1 - \cos\theta)^2 H(x - Jy) \\ &\qquad - H(x + y) - H(x - y) - H(x) - H(y)\} \\ &+ \frac{1}{16}\{13\lambda(x, y) - 3\lambda(Jx, Jy)\} + \frac{1}{16}\{\lambda(x, Jy) + \lambda(Jx, y)\} , \end{aligned} \tag{8}$$

if $g(x, y) = 0$.

Theorem 3. *Suppose M has constant holomorphic sectional curvature μ at a point $m \in M$, and let $x, y \in T_m(M)$ with $g(x, x) = g(y, y) = 1$ and $g(x, y) = 0$. Then*

$$\begin{aligned} K(x, y) = &\frac{\mu}{4}\{1 + 3g(x, Jy)^2\} + \frac{1}{16}\{13\lambda(x, y) - 3\lambda(Jx, Jy)\} \\ &+ \frac{1}{16}\{\lambda(x, Jy) + \lambda(Jx, y)\} . \end{aligned} \tag{9}$$

Theorem 4. *Let M be an almost Hermitian manifold with pointwise constant holomorphic sectional curvature μ and pointwise constant type α. Then M is an Einstein manifold with*

$$2S(x, x) = (m + 1)\mu + 3(m - 1)\alpha \tag{10}$$

for $g(x, x) = 1$, and M is a space of constant holomorphic sectional curvature if and only if M has global constant type α.

The definition of α in the theorem is given by

$$\lambda(x, y) = \alpha , \tag{11}$$

if $g(x, x) = g(y, y) = 1$ where x and y span an antiholomorphic plane.

The following theorem is proved in [6].

Theorem 5. *Assume M is an RK-manifold. Then M has (pointwise) constant type if and only if there exists a C^∞-function α such that*

$$\lambda(X, Y) = \alpha\{g(X, X)g(Y, Y) - g(X, Y)^2 - g(X, JY)^2\} \tag{12}$$

for all $X, Y \in \chi(M)$. Furthermore, M has global constant type if and only if (13) holds with a constant function α.

2. For our purpose we need some considerations on complex hypersurfaces of an almost Hermitian manifold. We follow the notation of [2] and refer to that paper for the proofs of the given properties. See also [3].

Let $\tilde{M}$ be an almost Hermitian manifold of complex dimension $m + 1$, and denote the almost complex structure and the Hermitian metric of $\tilde{M}$ by J and g respectively. Moreover, let M be a complex hypersurface of $\tilde{M}$ i.e., suppose that there exists a complex analytic mapping $f: M \to \tilde{M}$. Then for each $m \in M$ we identify the tangent space $T_m(M)$ with $f_*(T_m(M)) \subset T_{f(x)}(\tilde{M})$ by means of f_*. Since $f^* \circ g = g'$ and $J \circ f_* = f_* \circ J'$ where g' and J' are the Hermitian metric and the almost complex structure of M respectively, g' and J' are respectively identified with the restrictions of the structures g and J to the subspace $f_*(T_m(M))$.

As is known, we can choose the following special neighborhood $\mathscr{U}(m)$ of m for a neighborhood $\tilde{\mathscr{U}}(f(m))$ of $f(m)$. Let $\{\tilde{\mathscr{U}}; \tilde{m}^i\}$ $(i = 1, 2, \cdots, 2m + 2)$ be a system of coordinate neighborhoods of $\tilde{M}$. Then $\{\mathscr{U}; m^i\}$ is a system of coordinate neighborhoods of M such that $m^{2m+1} = m^{2m+2} = 0$ where $m^i = \tilde{m}^i \circ f$.

By $\tilde{\nabla}$ we always mean the Riemannian covariant differentiation on $\tilde{M}$, and by N a differentiable unit vector field normal to M at each point of $\mathscr{U}(m)$.

If X and Y are vector fields on the neighborhood $\mathscr{U}(m)$, we have

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)N + k(X, Y)JN , \tag{12}$$

where $\nabla_X Y$ denote the component of $\tilde{\nabla}_X Y$ tangent to M, ∇ is the covariant differentiation of the almost complex Hermitian manifold M, and h and k are symmetric covariant tensor fields of degree 2 on $\mathscr{U}(m)$. We have further

$$\tilde{\nabla}_X N = -AX + s(X)JN , \tag{13}$$

$$\tilde{\nabla}_X(JN) = -BX + t(X)N , \tag{14}$$

where AX and BX are tangent to M. A, B, s and t are tensor fields on $\mathscr{U}(m)$ of type (1,1) and (0,1) respectively, and A and B are symmetric with respect to g and satisfy

$$h(X, Y) = g(AX, Y) , \tag{15}$$

$$k(X, Y) = g(BX, Y) . \tag{16}$$

Now let M be a complex hypersurface satisfying the condition

$$h(X, Y) = k(X, JY) \tag{17}$$

for any vector fields X and Y on $\mathscr{U}(m)$ at every point $m \in M$. It is easy to verify that this condition is independent of the choice of N. For such a hypersurface we have

$$JA = -AJ \,, \qquad JB = -BJ \,, \tag{18}$$

where JA and JB are symmetric with respect to g. Condition (17) is equivalent to

$$B = JA \,. \tag{19}$$

Moreover we have

Lemma 6 [2]. *In a complex hypersurface M of $\tilde{M}$ satisfying* (17), *at any point $p \in \mathscr{U}(m)$ there exists an orthonormal basis $\{e_i, Je_i\}$ $(i = 1, 2, \cdots, m)$ of $T_p(M)$ with respect to which the matrix A is diagonal of the form*

$$\begin{bmatrix} \lambda_1 & & & & & & & \\ & \lambda_2 & & & & & & \\ & & \ddots & & & & & \\ & & & \lambda_m & & & & \\ & & & & -\lambda_1 & & & \\ & & & & & -\lambda_2 & & \\ & & & & & & \ddots & \\ & & & & & & & -\lambda_m \end{bmatrix}$$

where $Ae_i = \lambda_i e_i$ and $AJe_i = -\lambda_i Je_i$.

Lemma 7 [2]. *If $\tilde{R}$ and R are the Riemannian curvature tensors of $\tilde{M}$ and a complex hypersurface M of $\tilde{M}$ satisfying* (17) *respectively, then for any vector fields X, Y, Z, W on $\mathscr{U}(m)$ we have the following Gauss equation*:

$$\begin{aligned} \tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) & \\ - \{g(AX, Z)g(AY, W) - g(AX, W)g(AY, Z)\} & \\ - \{g(JAX, Z)g(JAY, W) - g(JAX, W)g(JAY, Z)\} \,. & \end{aligned} \tag{20}$$

Lemma 8 [2]. *Let M be a complex hypersurface of $\tilde{M}$ and satisfy condition* (17).

(i) *If $\{x, y\}$ is a 2-plane tangent to M at a point of $\mathscr{U}(m)$, then*

$$\begin{aligned} \tilde{K}(x, y) = K(x, y) - \{g(Ax, x)g(Ay, y) - g(Ax, y)^2\} & \\ - \{g(JAx, x)g(JAy, y) - g(JAx, y)^2\} \,, & \end{aligned} \tag{21}$$

where x, y form an orthonormal basis of the 2-plane.

(ii) *If x is a unit vector tangent to M at a point of $\mathscr{U}(m)$, then*

$$\tilde{H}(x) = H(x) + 2\{g(Ax, x)^2 + g(JAx, x)^2\} . \tag{22}$$

Proposition 9 [2]. *Let M be a complex hypersurface of $\tilde{M}$ of (pointwise) constant holomorphic sectional curvature $\tilde{\mu}$. If M is of complex dimension ≥ 2 and satisfies condition (17), then at each point of M there exists a holomorphic plane whose sectional curvature in M is $\tilde{\mu}$, and therefore if M is of (pointwise) constant holomorphic sectional curvature μ, then $\mu = \tilde{\mu}$.*

Finally this proposition gives

Theorem 10 [2]. *Let M be a complex hypersurface of $\tilde{M}$ of constant holomorphic sectional curvature. If M is of complex dimension ≥ 2 and satisfies condition (17), then the following statements are equivalent:*

(i) *M is totally geodesic in $\tilde{M}$,*

(ii) *M is of constant holomorphic sectional curvature.*

3. Let $\tilde{M}$ be an almost Hermitian manifold, and M a complex hypersurface of $\tilde{M}$ satisfying condition (17). It follows at once from (20), (19) and (18) that

$$\begin{aligned} &\tilde{R}(JX, JY, JZ, JW) - \tilde{R}(X, Y, Z, W) \\ &\qquad = R(JX, JY, JZ, JW) - R(X, Y, Z, W) \end{aligned} \tag{23}$$

for any vector fields X, Y, Z, W on $\mathscr{U}(m)$. Hence

Theorem 11. *Let $\tilde{M}$ be an almost Hermitian manifold, and M a complex hypersurface of $\tilde{M}$ satisfying condition (17). If $\tilde{M}$ is an RK-manifold, then M is also an RK-manifold.*

Further we have also

$$\begin{aligned} &\tilde{R}(X, Y, Z, W) - \tilde{R}(X, Y, JZ, JW) \\ &\qquad = R(X, Y, Z, W) - R(X, Y, JZ, JW) \end{aligned} \tag{24}$$

for any vector fields X, Y, Z, W on $\mathscr{U}(m)$. Hence

$$\tilde{\lambda}(X, Y) = \lambda(X, Y) , \tag{25}$$

and from (25) and Theorems 5, 11 we obtain

Theorem 12. *Let $\tilde{M}$ be an RK-manifold of (pointwise) constant type α, and M a complex hypersurface satifying condition (17). Then M has (pointwise) constant type α.*

We need only this theorem for *RK*-manifolds, but it is easy to prove that this is still valid for a general almost Hermitian manifold.

With the help of Theorem 4 we obtain an equivalent version of Theorem 10 for manifolds with (pointwise) constant type.

Theorem 13. *Let $\tilde{M}$ be an almost Hermitian manifold of (pointwise) constant type, and M a complex hypersurface of complex dimension ≥ 2 satisfy-*

ing condition (17). *Then the following statements are equivalent*:

(i) *M is totally geodesic in $\tilde{M}$*,

(ii) *M has global constant type and pointwise constant holomorphic sectional curvature.*

The following theorem is an immediate consequence of (9) and (12).

Theorem 14. *Let $\tilde{M}$ be an RK-manifold with* (*pointwise*) *constant holomorphic sectional curvature $\tilde{\mu}$ and* (*pointwise*) *constant type α. If $x, y \in T_m(\tilde{M})$, $m \in \tilde{M}$ and $g(x,x) = g(y,y) = 1$, $g(x,y) = 0$, then*

$$K(x,y) = \tfrac{1}{4}\mu\{1 + 3g(x,Jy)\}^2 + \tfrac{3}{4}\alpha\{1 - g(x,Jy)^2\} . \tag{26}$$

We prove now the main theorem of this paper.

Theorem 15. *Let M be a complex hypersurface of an RK-manifold $\tilde{M}$ with constant holomorphic sectional curvature $\tilde{\mu}$ and constant type α. If M is of complex dimension ≥ 2 and satisfies condition* (17), *then the following statements are equivalent*:

(i) *M is totally geodesic in $\tilde{M}$*,

(ii) *M is of constant holomorphic sectional curvature* (*or equivalently, M has global constant type and pointwise constant holomorphic sectional curvature*),

(iii) *at every point $m \in M$, all the sectional curvatures of M satisfy*

$$K(x,y) \geq \tfrac{1}{4}\tilde{\mu}\{1 + 3g(x,Jy)^2\} + \tfrac{3}{4}\alpha , \qquad \text{if } \alpha \geq 0 , \tag{27}$$

or

$$K(x,y) \leq \tfrac{1}{4}\tilde{\mu}\{1 + 3g(x,Jy)^2\} + \tfrac{3}{4}\alpha , \qquad \text{if } \alpha \leq 0 , \tag{28}$$

where x, y are orthonormal vectors spanning the 2-plane of $T_m(M)$.

Proof. (i) is equivalent to (ii) by Theorem 10 and Theorem 13.

Next, if M is of constant holomorphic sectional curvature μ, then $\mu = \tilde{\mu}$ by Proposition 9, and therefore we have (26) which implies (27) and (28).

Finally, we prove that (iii) implies (i). Therefore consider an orthonormal basis as in Lemma 6, and set

$$x = \frac{1}{\sqrt{2}}(e_i + Je_i) , \qquad y = \frac{1}{\sqrt{2}}(e_i - Je_i) . \tag{29}$$

Then from (21) follows

$$\tilde{K}(x,y) = K(x,y) + 2\lambda_i^2 . \tag{30}$$

In the case $\alpha \geq 0$, from (27) and the expression (26) for $\tilde{M}$ we obtain

$$-\tfrac{3}{4}\alpha g(x,Jy)^2 \geq 2\lambda_i^2 ,$$

which implies $\lambda_i = 0$ $(i = 1, \cdots, m)$. It follows then from Lemma 6 that A is identically zero at each point of M, so that M is totally geodesic in $\tilde{M}$. In the same way we can treat the case $\alpha \leq 0$.

Following the same arguments we obtain

Theorem 16. *Let M be a complex hypersurface of an RK-manifold $\tilde{M}$ with pointwise constant holomorphic sectional curvature $\tilde{\mu}$ vnd vanishing constant type. If M is of complex dimension ≥ 2 and satisfies condition* (17), *then the following statements are equivalent*:

(i) *M is totally geodesic in $\tilde{M}$,*

(ii) *M has pointwise constant holomorphic sectional curvature,*

(iii) *at every point $m \in M$, all the sectional curvatures of M satisfy*

$$K(x, y) \geq \tfrac{1}{4}\tilde{\mu}\{1 + 3g(x, Jy)^2\} , \tag{31}$$

where x, y are orthonormal vectors which span the 2-plane of $T_m(M)$.

Consider again an almost Hermitian manifold $\tilde{M}$ of constant holomorphic sectional curvature $\tilde{\mu}$ and (pointwise) constant type α. We know from Theorem 4 that $\tilde{M}$ is an Einstein manifold with

$$\tilde{S} = \tilde{\rho} g , \qquad 2\tilde{\rho} = (m + 1)\tilde{\mu} + 3(m - 1)\alpha . \tag{32}$$

Now let M be a complex hypersurface of $\tilde{M}$ which satisfy condition (17), and consider further the basis $\{e_i, Je_i\}$ of Lemma 6. Then it follows with the help of (9), (21) and (22) that

$$H(e_i) = \tilde{\mu} - 2\lambda_i^2 , \tag{33}$$

$$S(e_i, e_i) = \tfrac{1}{2}(m + 1)\tilde{\mu} + \tfrac{3}{2}(m - 1)\alpha - 2\lambda_i^2 . \tag{34}$$

If M is an Einstein manifold, then we have

$$S = \rho g , \qquad \rho = \tilde{\rho} - 2\lambda_i^2 , \tag{35}$$

$$A^2 = \lambda^2 I , \tag{36}$$

where

$$4\lambda^2 = 4\lambda_i^2 = (m + 1)\tilde{\mu} + 3(m - 1)\alpha - 2\rho . \tag{37}$$

Moreover

$$H(e_i) = \rho - \frac{m - 1}{2}(\tilde{\mu} + 3\alpha) = \rho - 2(m - 1)\tilde{\nu} , \tag{38}$$

$\tilde{\nu}$ denoting the antiholomorphic sectional curvature. Hence

Theorem 17. *Let $\tilde{M}$ be an almost Hermitian manifold with constant holomorphic sectional curvature $\tilde{\mu}$ and* (*pointwise*) *constant type α, and let M be* *a complex Einstein hypersurface satisfying condition* (17). *If ρ is the Ricci curvature of M, then*

(i) $\rho \leq \frac{1}{2}(m+1)\tilde{\mu} + \frac{3}{2}(m-1)\alpha$,

(ii) *there exists a basis $\{e_i, Je_i\}$ as in Lemma 6 such that $A^2 = \lambda^2 I$ where*

$$4\lambda^2 = (m+1)\tilde{\mu} + 3(m-1)\alpha - 2\rho ,$$

(iii) *at each point of M there exists a holomorphic plane whose sectional curvature is $\rho - 2(m-1)\tilde{\nu}$ where $\tilde{\nu} = \tilde{\mu} + 3\alpha$.*

It follows further from (37) that if $\lambda_i = 0$ at one point, then $\lambda_i = 0$ on the whole manifold M (supposed to be connected). Hence

Theorem 18. *Theorem* 15 (iii) *may be replaced by*

(iii) *M is an Einstein manifold, and at one point of M all sectional curvatures of M satisfy* (27) *or* (28).

Theorem 16 (iii) *may be replaced by*

(iii) *M is an Einstein manifold, and at one point of M all sectional curvatures of M satisfy* (31).

In relation with Theorem 15 and formula (32) it is easy to verify

Theorem 19. *Let M be a complex hypersurface of an RK-manifold $\tilde{M}$ with constant holomorphic sectional curvature $\tilde{\mu}$ and constant type. If M is of complex dimension ≥ 2 satisfies condition* (17) *and is totally geodesic in $\tilde{M}$, then the following statements are equivalent*:

(i) *the antiholomorphic sectional curvature of M (or of $\tilde{M}$ on M) is zero,*

(ii) $\tilde{k} = k$ *on M,*

(iii) $k(x,x) = \tilde{\mu}$ *for* $g(x,x) = 1$,

where $\tilde{k}$ (resp. k) denotes the Ricci tensor of $\tilde{M}$ (resp. M).

4. Let σ (resp. σ') be a holomorphic 2-plane defined by the unit vector x (resp. y). Then the holomorphic bisectional curvature $H(\sigma, \sigma')$ is defined by [1]

$$H(\sigma, \sigma') = R(x, Jx, y, Jy) . \tag{39}$$

It is easy to verify that $H(\sigma, \sigma')$ depends only on σ and σ'. Using (6) we obtain

$$H(\sigma, \sigma') = R(x,y,x,y) + R(Jx,y,Jx,y) - \lambda(x,y) - \lambda(Jx,y) , \tag{40}$$

which together with (7) gives

Theorem 20. *Let M be an almost Hermitian manifold, and σ (resp. σ') a holomorphic 2-plane in $m \in M$ defined by a unit vector x (resp. y). Then*

$$\begin{aligned} H(\sigma, \sigma') = \frac{1}{16}\{Q(x+Jy) + Q(x-Jy) + Q(x+y) \\ + Q(x-y) - 4Q(x) - 4Q(y)\} \\ - \frac{1}{8}\{\lambda(x,y) + \lambda(Jx,Jy) + \lambda(Jx,y) + \lambda(x,Jy)\} . \end{aligned} \tag{41}$$

If $g(x, Jy) = \cos\theta$ *and* $g(x, y) = \cos\phi$, *then*

$$\begin{aligned} H(\sigma, \sigma') = \tfrac{1}{4}\{&(1 + \cos\theta)^2 H(x + Jy) + (1 - \cos\theta)^2 H(x - Jy) \\ &+ (1 + \cos\phi)^2 H(x + y) + (1 - \cos\phi)^2 H(x - y) \\ &- H(x) - H(y)\} \\ &- \tfrac{1}{8}\{\lambda(x, y) + \lambda(Jx, Jy) + \lambda(Jx, y) + \lambda(x, Jy)\} . \end{aligned} \tag{42}$$

Using (12) we obtain

Theorem 21. *Let M be an RK-manifold with pointwise constant holomorphic sectional curvature* μ *and pointwise constant type* α. *Then*

$$H(\sigma, \sigma') = \tfrac{1}{2}(\mu - \alpha) + \tfrac{1}{2}(\mu + \alpha)(\cos^2\theta + \cos^2\phi) , \tag{43}$$

$$\tfrac{1}{2}(\mu - \alpha) \leq H(\sigma, \sigma') \leq \mu , \qquad \text{if } \mu + \alpha \geq 0 , \tag{44}$$

$$\tfrac{1}{2}(\mu - \alpha) \geq H(\sigma, \sigma') \geq \mu , \qquad \text{if } \mu + \alpha \leq 0 . \tag{45}$$

The value $\frac{1}{2}(\mu - \alpha)$ *is attained when* σ *is perpendicular to* σ', *whereas the value* μ *is attained when* $\sigma = \sigma'$.

In [7] we proved the following proposition.

Proposition 22. *Let M be an RK-manifold with pointwise constant type* α. *Then M is a space of constant curvature if and only if the holomorphic sectional curvature is equal to* α.

From this and Theorem 21 we obtain

Theorem 23. *Let M be an RK-manifold with pointwise constant type and pointwise constant holomorphic sectional curvature. Then M is a space of constant curvature if and only if the holomorphic bisectional curvature* $H(\sigma, \sigma')$ *vanishes when* σ *is perpendicular to* σ'.

From (43) it follows that $H(\sigma, \sigma')$ is constant at a point m if $\mu + \alpha = 0$, and we see in consequence of (9) and (12) that this means that the antiholomorphic sectional curvature is equal to $-\frac{1}{2}\mu$. Hence

Theorem 24. *Let M be an RK-manifold as in Theorem* 23. *Then the holomorphic bisectional curvature is constant at a point m of M if and only if the antiholomorphic sectional curvature is* $-\frac{1}{2}\mu$ *where* μ *is the holomorphic sectional curvature at m* (*or if and only if the holomorphic bisectional curvature is equal to the holomorphic sectional curvature*).

Finally, consider again an almost Hermitian manifold $\tilde{M}$, and let M be a complex hypersurface which satisfies condition (17). Then we obtain, from (20) and (18),

$$\tilde{R}(x, Jx, y, Jy) = R(x, Jx, y, Jy) + 2g(Ax, y)^2 + 2g(Ax, Jy)^2 , \tag{46}$$

which proves

Theorem 25. *Let* $\tilde{M}$ *be an almost Hermitian manifold, and M a complex hypersurface satisfying condition* (17). *Then the holomorphic bisectional cur-*

vature of M does not exceed that of $\tilde{M}$. In particular, if $\tilde{M}$ is a complex Euclidean space, then the holomorphic bisectional curvature of M is nonpositive.

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