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ISOMETRY TO SPHERES OF RIEMANNIAN MANIFOLDS ADMITTING A CONFORMAL TRANSFORMATION GROUP

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1. Introduction

Let M be an orientable smooth Riemannian manifold of dimension n with Riemannian metric g_{ij}. Let K_{hijk}, K_{ij} and K denote the Riemann curvature tensor, the Ricci tensor and the scalar curvature of M respectively. Let X be an infinitesimal conformal transformation of M so that

$$(1.1) \qquad (L_X g)_{ij} = 2\rho g_{ij} ,$$

where ρ is a function on M, and L_X denotes the Lie derivative with respect to X. Recently Yano and Hiramatu [3], [4] have obtained conditions for M to be isometric to a sphere without assuming any condition on the scalar curvature function. The purpose of the present paper is to extend the study of the above authors. Among the four lemmas which we shall prove, two (Lemmas 1.1 and 1.2) relate to some of the main results of [3] and [4]. Also Theorems 1.1 and 1.2 in this paper generalize some of the results of [3] and [4].

The tensor fields G, Z [2] and W [1] required in our study are given by

$$(1.2) \qquad G_{ij} = K_{ij} - \frac{K}{n} g_{ij} ,$$

$$(1.3) \qquad Z_{hijk} = K_{hijk} - \frac{K}{n(n-1)}(g_{hk}g_{ij} - g_{hj}g_{ik}) ,$$

$$(1.4) \qquad \begin{aligned} W_{hijk} = {} & aZ_{hijk} + b_1 g_{hk}G_{ij} - b_2 g_{hj}G_{ik} + b_3 g_{ij}G_{hk} \\ & - b_4 g_{ik}G_{hj} + b_5 g_{hi}G_{jk} - b_6 g_{jk}G_{hi} , \end{aligned}$$

where $a, b_1, \cdots, b_6$ are constants, and W was first introduced by Hsiung.

As usual ∇ denotes covariant differentiation on M. We denote $\nabla_i \rho$ by ρ_i and $g^{ij}\nabla_j \rho$ by ρ^i. $D\rho$ denotes the vector field on M associated with the differential 1-form $d\rho$. The Laplace-Beltrami operator on M is given by $\Delta = g^{ij}\nabla_i\nabla_j$.

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For the sake of easy reference we list some known formulas (for details see [1] and [2]):

$$L_XK = -2(n-1)\Delta\rho - 2K\rho \ , \tag{1.5}$$

$$[X, D\rho]K = L_XL_{D\rho}K - L_{D\rho}L_XK \ , \tag{1.6}$$

$$L_X(W_{hijk}W^{hijk}) = -4\rho W_{hijk}W^{hijk} - 2cG^{ij}\nabla_i\rho_j \ , \tag{1.7}$$

where $c \geq 0$ is given by

$$\begin{aligned} \frac{c-4a^2}{n-2} &= 2a\sum_{i=1}^{4} b_i + \left[\sum_{i=1}^{6}(-1)^{i-1}b_i\right]^2 \\ &\quad + (n-1)\sum_{i=1}^{6} b_i^2 - 2(b_1b_3 + b_2b_4 - b_5b_6) \ . \end{aligned} \tag{1.8}$$

We prove the following lemmas and theorems.

Lemma 1.1. *Let M be a compact orientable smooth Riemannian manifold of dimension $n \geq 2$ admitting an infinitesimal conformal transformation X satisfying* (1.1). *Then*

$$\int_M \rho K L_X K dV = (n-1)\int_M L_{D\rho}L_XKdV - \frac{1}{2}\int_M (L_XK)^2 dV \ . \tag{1.9}$$

Lemma 1.2 (*Yano and Hiramatu* [4]). *For a manifold M having the same properties as in Lemma* 1.1 *we have*

$$\begin{aligned} &\int_M K\rho^i\rho_i dV \\ &\quad = \frac{1}{4n(n-1)}\int_M [4(n-1)[X, D\rho]K + 2(n-1)(n+2)L_{D\rho}L_XK \\ &\qquad + 4nK^2\rho^2 - n(L_XK)^2]dV \ . \end{aligned} \tag{1.10}$$

Lemma 1.3. *For a manifold M having the same properties as in Lemma* 1.1 *we have*

$$\begin{aligned} &\int_M \left[K_{ij}\rho^i\rho^j - \frac{1}{4n(n-1)}(2K\rho + L_XK)^2\right]dV \\ &\quad = \frac{2}{c}\int_M \rho^2 W_{hijk}W^{hijk}dV - \frac{1}{2nc}\int_M L_XL_X(W_{hijk}W^{hijk})dV \\ &\qquad + \frac{1}{2}\int_M \left[K\rho_i\rho^i - \frac{1}{2n(n-1)}\{2nK^2\rho^2 + (n+2)K\rho L_XK \right. \\ &\qquad\qquad \left. + (L_XK)^2\}\right]dV \ , \end{aligned} \tag{1.11}$$

where c is given by (1.8) *and is assumed to be positive.*

Lemma 1.4. *For a manifold M having the same properties as in Lemma* 1.1 *we have*

$$\begin{aligned}(1.12)\quad &\int_M \left[K_{ij}\rho^i\rho^j - \frac{1}{4n(n-1)}(2K\rho + L_XK)^2\right]dV \\ &= \frac{2}{c}\int_M \rho^2 W_{hijk}W^{hijk}dV - \frac{1}{2nc}\int_M L_XL_X(W_{hijk}W^{hijk})dV \\ &\quad + \frac{1}{2n}\int_M [X, D\rho]KdV ,\end{aligned}$$

where c is given by (1.8) *and is assumed to be positive.*

Theorem 1.1. *If a compact orientable smooth Riemannian manifold M of dimension $n > 2$ admits an infinitesimal nonhomothetic conformal transformation X satisfying* (1.1) *such that*

$$\begin{aligned}(1.13)\quad &\int_M L_XL_X(W_{hijk}W^{hijk})dV \\ &- nc\int_M \left[K\rho_i\rho^i - \frac{1}{2n(n-1)}\{2nK^2\rho^2 + (n+2)K\rho L_XK \right. \\ &\left. \qquad + (L_XK)^2\}\right]dV \leq 0 ,\end{aligned}$$

where $c > 0$, then M is isometric to a sphere.

Theorem 1.2. *For a manifold M having the same properties as in Lemma* 1.1 *with $c > 0$ we have*

$$(1.14)\quad \int_M [L_XL_X(W_{hijk}W^{hijk}) - c[X, D\rho]K]dV \geq 0 \qquad (c > 0) ,$$

where the equality holds if and only if M is isometric to a sphere.

Remark. Theorems 1.1 and 1.2 are equivalent and generalize [3, Proposition 12] and [4, Proposition 3] respectively.

We need the following known lemmas and theorem.

Lemma A (*Yano and Sawaki* [5]). *If a compact orientable smooth Riemannian manifold M of dimension n admits an infinitesimal conformal transformation X satisfying* (1.1), *then for any smooth function f on M we have*

$$\int_M \rho f dV = -\frac{1}{n}\int_M L_X f dV .$$

Lemma B (*Yano and Hiramatu* [4]). *For a manifold M having the same properties as in Lemma A we have*

$$-n\int_M \rho\rho^i\nabla_i K dV = \frac{n}{2}\int_M \rho^2 \Delta K dV = \int_M L_X L_{D\rho} K dV , \tag{1.15}$$

$$-\int_M (\Delta\rho) L_X K dV = \int_M L_{D\rho} L_X K dV . \tag{1.16}$$

Lemma C (*Yano and Hiramatu* [4]). *For a manifold M having the same properties as in Lemma A we have*

$$-\int_M (\Delta\rho)^2 dV = \int_M \rho^i \nabla_i (\Delta\rho) dV . \tag{1.17}$$

Theorem A (*Yano and Hiramatu* [3]). *If a compact orientable Riemannian manifold M of dimension $n > 2$ admits an infinitesimal nonhomothetic conformal transformation X satisfying* (1.1), *then*

$$\int_M K_{ij}\rho^i\rho^j dV \leq \frac{1}{4n(n-1)}\int_M (2K\rho + L_X K)^2 dV , \tag{1.18}$$

equality holding if and only if M is isometric to a sphere.

2. Proofs of lemmas and theorems

Proof of Lemma 1.1. Multiplying (1.5) by $L_X K$, integrating over M and using (1.16) we obtain (1.9).

Proof of Lemma 1.2. Using (1.5) and (1.6) we have

$$[X, D\rho]K = L_X L_{D\rho} K + 2(n-1)\rho^i\nabla_i(\Delta\rho) + 2\rho\rho^i\nabla_i K + 2K\rho_i\rho^i .$$

Integrating over M and using (1.15) and (1.17) we get

$$\int_M K\rho_i\rho^i dV = \frac{1}{2}\int_M [X, D\rho]K dV - \frac{n-2}{2n}\int_M L_X L_{D\rho} K dV + (n-1)\int_M (\Delta\rho)^2 dV ,$$

which in view of (1.5) and (1.6) takes the form

$$\int_M K\rho_i\rho^i dV = \frac{1}{n}\int_M [X, D\rho]K dV - \frac{n-2}{2n}\int_M L_{D\rho} L_X K dV + \frac{1}{4(n-1)}\int_M (2K\rho + L_X K)^2 dV . \tag{2.1}$$

Now by Lemma 1.1 we have

$$(2.2)\qquad \int_M (2K\rho + L_X K)^2 dV = \int_M [4K^2\rho^2 + 4(n-1)L_{D\rho}L_X K - (L_X K)^2]dV .$$

Substituting (2.2) in (2.1) we obtain (1.10).

Proof of Lemma 1.3. From (1.7) it follows that

$$(2.3)\qquad K_{ij}\nabla^i\rho^j = -\frac{2}{c}\rho W_{kjih}W^{kjih} - \frac{1}{2c}L_X(W_{kjih}W^{kjih}) + \frac{K}{n}\Delta\rho ,$$

On the other hand, using $\nabla^j K_{ji} = \frac{1}{2}\nabla_i K$ we have

$$(2.4)\qquad \nabla^j(K_{ij}\rho\rho^i) = \tfrac{1}{2}(\nabla_i K)\rho\rho^i + K_{ij}\rho^i\rho^j + \rho K_{ij}\nabla^i\rho^j .$$

Also

$$(2.5)\qquad \nabla_i(K\rho\rho^i) = (\nabla_i K)\rho\rho^i + K\rho_i\rho^i + K\rho\Delta\rho .$$

Eliminating $K_{ij}\nabla^i\rho^j$ and $(\nabla_i K)\rho\rho^i$ from (2.3), (2.4) and (2.5), integrating over M and using (1.5) and Lemma A we obtain

$$(2.6)\qquad \begin{aligned} \int_M K_{ij}\rho^i\rho^j dV &= \frac{2}{c}\int_M \rho^2 W_{kjih}W^{kjih}dV \\ &\quad - \frac{1}{2nc}\int_M L_X L_X(W_{kjih}W^{kjih})dV + \frac{1}{2}\int_M K\rho_i\rho^i dV \\ &\quad - \frac{n-2}{4n(n-1)}\int_M K\rho(2K\rho + L_X K)dV . \end{aligned}$$

Subtracting $\dfrac{1}{4n(n-1)}\displaystyle\int_M (2K\rho + L_X K)^2 dV$ from both sides of (2.6) we obtain (1.11).

Proof of Lemma 1.4. Eliminating $\int_M K\rho_i\rho^i dV$ from (1.10) and (1.11) and using (1.9) we obtain (1.12).

Proof of Theorem 1.1. Assumption (1.13) of the theorem and Lemma 1.3 lead to the inequality

$$\int_M \left[K_{ij}\rho^i\rho^j - \frac{1}{4n(n-1)}(2K\rho + L_X K)^2\right]dV \geq 0 ,$$

which by Theorem A implies that M is isometric to a sphere.

Proof of Theorem 1.2. From (1.12) we have

$$(2.7)\qquad \begin{aligned} &\frac{2}{c}\int_M \rho^2 W_{hijk}W^{hijk}dV + \int_M \left[\frac{1}{4n(n-1)}(2K\rho + L_X K)^2 - K_{ij}\rho^i\rho^j\right]dV \\ &= \frac{1}{2nc}\left[\int_M \{L_X L_X(W_{hijk}W^{hijk}) - c[X, D\rho]K\}dV\right] . \end{aligned}$$

Theorem 1.2 follows from (2.7), Theorem A and the assumption that $c > 0$.

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