

A CLASS OF COMPLEX ANALYTIC FOLIATE MANIFOLDS WITH RIGID STRUCTURE

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In 1957, R. Bott [1] proved that the complex projective spaces have a rigid complex structure. On the other hand, in 1961 Kodaira and Spencer [9] extended the deformation theory to general multifoliate complex structures and, particularly, to complex analytic foliations. But, so far as we know, no example of a rigid structure of this kind has been provided. It is our aim here to prove the rigidity of a class of complex foliate manifolds which generalizes the complex projective spaces. Our class contains as a particular case any product of two complex projective spaces.

The complex manifolds under consideration will be compact Kählerian, the result being obtained by the general method initiated by Bochner, which consists in studying the relations between curvature and cohomology. Namely, we shall go along the lines of Calabi-Vesentini's paper [3] to prove first a generalized Nakano inequality. In connection with our previous cohomology calculations of [13], [14], this will lead to the desired results.

Some other related remarks will also be made.

1. A complex analytic foliate (c.a.f.) structure \mathcal{F} of complex codimension n on a complex $(n + m)$ -dimensional manifold V is given by an atlas $\{U_\alpha; z_\alpha^a, z_\alpha^u\}$ ($a, b, \dots = 1, \dots, n; u, v, \dots = n + 1, \dots, n + m$), such that on $U_\alpha \cap U_\beta \neq \emptyset$ one has, besides analyticity,

$$(1.1) \quad \partial z_\beta^a / \partial z_\alpha^u = 0.$$

Then the maximal connected submanifolds which can be represented locally by $z_\alpha^a = \text{const.}$ are the *leaves* of \mathcal{F} , and the images $\varphi_\alpha(U_\alpha) \subset C^n$ of the submersions $\varphi_\alpha: U_\alpha \rightarrow C^n$ defined by $\varphi_\alpha(z_\alpha^a, z_\alpha^u) = (z_\alpha^a)$ (C is the complex line) are called the *local transverse manifolds*.

The tangent vectors of the leaves define the *structural subbundle* F of $T(V)$ with local bases $Z_u = \partial / \partial z_\alpha^u$ and transition functions $(\partial z_\beta^u / \partial z_\alpha^u)$. $T(V)/F = F'$ is the *transversal bundle* with the local bases defined by the equivalence classes $[\partial / \partial z_\alpha^a]$ and the transition functions $(\partial z_\beta^a / \partial z_\alpha^a)$.

Generally, we shall say that the elements depending only on the leaves are *foliate* and, particularly, c.a.f. For instance, $f: V \rightarrow C$ is foliate if $\partial f / \partial z_\alpha^u =$

$= \partial f / \partial \bar{z}_\alpha^u = 0$, and it is c.a.f. if, moreover, $\partial f / \partial \bar{z}_\alpha^a = 0$. A differential form is c.a.f. if it does not contain dz_α^u , $d\bar{z}_\alpha^u$ and has local c.a.f. coefficients. A vector bundle on V is c.a.f. if it has c.a.f. transition functions (for instance, the transversal bundle is such), etc.

Now suppose that V is hermitian with metric h . Then the orthogonal bundle F^\perp of F , which is differentially isomorphic to F' , has local bases of the form

$$(1.2) \quad Z_\alpha = \partial / \partial z^\alpha - t_\alpha^u (\partial / \partial z^u)$$

(the index α of the coordinate neighborhood will be omitted) and we shall use in the sequel the bases (Z_α, Z_u) to express different elements on V .

The corresponding dual cobases are

$$(1.3) \quad dz^\alpha, \quad \theta^u = dz^u + t_\alpha^u dz^\alpha,$$

and the metric can be expressed by

$$(1.4) \quad ds^2 = h_{\alpha\bar{b}} dz^\alpha d\bar{z}^{\bar{b}} + h_{uv} \theta^u \bar{\theta}^v.$$

These cobases allow us to speak of the *type* (p_1, p_2, q_1, q_2) of a differential form by counting in its expression the number of dz^α , $d\bar{z}^{\bar{\alpha}}$, θ^u , $\bar{\theta}^u$. One also introduces [13], [14] the *complex type* which is $(p_1 + q_1, p_2 + q_2)$ and the *mixed type* $(p_1, p_2 + q_1 + q_2)$.

The fundamental form of h is

$$(1.5) \quad \omega = \omega' + \omega'',$$

where

$$(1.6) \quad \omega' = \frac{1}{2} i h_{\alpha\bar{b}} dz^\alpha \wedge d\bar{z}^{\bar{b}}, \quad \omega'' = \frac{1}{2} i h_{uv} \theta^u \wedge \bar{\theta}^v,$$

and it follows that h is a Kähler metric ($d\omega = 0$) if and only if [14]

$$(1.7) \quad \begin{array}{ll} \text{(a)} \quad Z_c h_{\alpha\bar{b}} - Z_\alpha h_{c\bar{b}} = 0, & \text{(d)} \quad Z_u h_{v\bar{w}} - Z_v h_{u\bar{w}} = 0, \\ \text{(b)} \quad Z_u h_{\alpha\bar{b}} - h_{uv} Z_\alpha \bar{t}_b^v = 0, & \text{(e)} \quad Z_c t_\alpha^u - Z_\alpha t_c^u = 0, \\ \text{(c)} \quad Z_\alpha h_{uv} - h_{uv} Z_\alpha t_a^v = 0, & \text{(f)} \quad h_{uv} \bar{Z}_w t_a^u - h_{uw} \bar{Z}_v t_a^u = 0. \end{array}$$

2. On the manifolds V above, we can consider the classical scalar product and the operators $*$, d , δ , L , A , C [4], and it is important here to get decompositions of these operators with respect to the mixed type.

In order to avoid considerations on the supports of the forms, we shall assume hereafter that the manifold V is compact.

Thus from (1.5) we have $L = L' + L''$, where L' denotes the left exterior multiplication by ω' and has the mixed type $(1, 1)$, and similarly L'' has the mixed type $(0, 2)$.

The operator d has an obvious decomposition [13], [14] into three parts of the respective mixed types $(1, 0)$, $(0, 1)$, $(2, -1)$:

$$(2.1) \quad d = \mu + \lambda + \nu .$$

It is important to remark that in the Kählerian case the condition (1.7)(e) implies $\nu = 0$, whence *the differential forms of a foliate Kähler manifold are organized by mixed types as a double cochain complex.*

Next, because $*$ is not homogeneous with respect to mixed types, we shall introduce the operator $\#$ defined by the composition of $*$ with the complex conjugation. $\#$ sends forms of the mixed type (p, q) to forms of the mixed type $(n - p, n + 2m - q)$ and it allows to write the scalar products as

$$(2.2) \quad (\alpha, \beta) = \int_V \alpha \wedge \# \beta .$$

As in the classical theory [4] it follows:

$$(2.3) \quad \#^{-1} \alpha = (-1)^{\deg \alpha} \# \alpha ,$$

$$(2.4) \quad \delta = -\# d \# ;$$

hence

$$(2.5) \quad \begin{aligned} \delta &= \mu^* + \lambda^* + \nu^* , & \mu^* &= -\# \mu \# , \\ \lambda^* &= -\# \lambda \# , & \nu^* &= -\# \nu \# , \end{aligned}$$

where the terms have the mixed types $(-1, 0)$, $(0, -1)$, $(-2, 1)$, and in the Kählerian case $\nu^* = 0$.

It also follows

$$(2.6) \quad A = A' + A'' , \quad A' = \#^{-1} L' \# , \quad A'' = \#^{-1} L'' \# ,$$

where the terms have the mixed types $(-1, -1)$ and $(0, -2)$.

Finally, in order to handle with C we write down a form φ of mixed type (p, q) as

$$(2.7) \quad \varphi = \sum_{h=0}^q \varphi_{p+h, q-h} ,$$

where the indices denote the complex type of the respective terms. This gives

$$(2.8) \quad C\varphi = i^{p-q} \sum_{h=0}^q (-1)^h \varphi_{p+h, q-h} ,$$

which shows that C preserves the mixed type and that

$$(2.9) \quad C^{-1} = (-1)^{p-q} C .$$

3. Let us proceed now to the derivation of the announced generalized Nakano inequality. We suppose here that (V, h) is a compact Kähler manifold.

We start with the following fundamental formula of Kählerian geometry [4]:

$$(3.1) \quad \Lambda d - d\Lambda = -C^{-1}\delta C .$$

Using the previous decompositions and identifying the different (mixed) homogeneous parts of this formula we get

Proposition 1. *On a compact Kähler foliate manifold the following relations hold:*

$$\begin{aligned} \Lambda'\mu - \mu\Lambda' + \Lambda''\lambda - \lambda\Lambda'' &= -C^{-1}\lambda^*C , \\ \Lambda''\mu - \mu\Lambda'' &= 0 , \\ \Lambda'\lambda - \lambda\Lambda' &= -C^{-1}\mu^*C . \end{aligned}$$

Our main interest will be in the last of these formulas. If we apply it to an homogeneous form of the mixed type (p, q) and use (2.8) and (2.9), we see that the formula becomes

$$(3.2) \quad \Lambda'\lambda - \lambda\Lambda' = -i\mu^* ,$$

which is the relation to be used here.

Consider now a c.a.f. vector bundle S on V . Then $\#$ and the operators of (3.2) make sense on S -valued forms by componentwise application. If S is given a hermitian metric $a = (a_{\alpha\beta})$, $(\alpha, \beta, \dots = 1, \dots, \text{the dimension of the fiber of } S)$, then the product of the S -valued forms $\varphi = (\varphi^\alpha)$ and $\psi = (\psi^\alpha)$ is given by

$$(3.3) \quad (\varphi, \psi) = \int_V a_{\alpha\beta} \varphi^\alpha \wedge \# \psi^\beta ,$$

and we shall denote by $+$ the adjointness with respect to this product.

For the following calculations, the operator μ^{*+} will be needed. To obtain it, we consider S -valued forms φ and ψ of the mixed type $(p-1, q)$ and (p, q) . After putting

$$(3.4) \quad \omega_\beta^\alpha = a^{r\alpha} \mu_{\beta r} , \quad (a_{\alpha r} a^{r\beta} = \delta_\alpha^\beta) ,$$

$$(3.5) \quad \tilde{D}\varphi^\alpha = \mu\varphi^\alpha + \omega_\beta^\alpha \wedge \varphi^\beta ,$$

one gets

$$d(a_{\alpha\beta} \varphi^\alpha \wedge \# \psi^\beta) = a_{\alpha\beta} \tilde{D}\varphi^\alpha \wedge \# \psi^\beta - (-1)^{p+q} a_{\alpha\beta} \varphi^\alpha \wedge \# \#^{-1} \mu \# \psi^\beta ,$$

whence by integrating along V we get

$$(3.6) \quad \mu^{*+} = \tilde{D} .$$

The geometrical interpretation of \tilde{D} is analogous to that of the classical case [4]. It is easy to see that ω_{β}^{α} defines a connection on S which is uniquely determined by the following conditions: 1°. the connection forms are of the mixed type $(1, 0)$; 2°. the metric a is invariant by parallel translations "transverse" to the leaves. (Generally, this is not the metric connection of S). Next, the covariant exterior derivative D with respect to this connection is just

$$(3.7) \quad D = \tilde{D} + \lambda ,$$

which gives the interpretation looked for: \tilde{D} is the term of the mixed type $(1, 0)$ of D .

From known properties of connections [4], we can write

$$(3.8) \quad D^2\varphi^{\alpha} = \Omega_{\beta}^{\alpha} \wedge \varphi^{\beta} ,$$

where

$$(3.9) \quad \Omega_{\beta}^{\alpha} = d\omega_{\beta}^{\alpha} - \omega_{\beta}^{\gamma} \wedge \omega_{\gamma}^{\alpha} = \lambda\omega_{\beta}^{\alpha}$$

are the curvature forms of the previous connection and have the mixed type $(1, 1)$. This implies $\tilde{D}^2 = 0$ and

$$(3.10) \quad \tilde{D}\lambda + \lambda\tilde{D} = e(\Omega) ,$$

where $e(\Omega)$ is the operator defined by the right-hand side of formula (3.8).

Now consider also the operator λ^+ . The space $\mathcal{H}(S) = \ker \lambda \cap \ker \lambda^+$ is called the space of λ -harmonic forms, and the desired inequality is just for such forms. Thus for a λ -harmonic S -valued form φ of the mixed type (p, q) we can repeat, in our situation, the calculations of [3], which gives

$$\begin{aligned} 0 \leq (\tilde{D}\varphi, \tilde{D}\varphi) &= (\mu^{*+}\tilde{D}\varphi, \varphi) = i([A'\lambda - \lambda A']\tilde{D}\varphi, \varphi) \\ &= i(A'[\lambda\tilde{D} + \tilde{D}\lambda]\varphi, \varphi) - i(A'\tilde{D}\varphi, \lambda^+\varphi) = i(A'e(\Omega)\varphi, \varphi) . \end{aligned}$$

Hence using also the adjointness of A' and L' we have

Proposition 2 (*The Nakano inequality for the foliate case*). *If V is a compact c.a.f. Kähler manifold and S is a c.a.f. vector bundle on V , then for any S -valued λ -harmonic form φ one has*

$$(3.11) \quad i(e(\Omega)\varphi, L'\varphi) \geq 0 ,$$

equality holding if and only if $\tilde{D}\varphi = 0$.

In fact, this is a generalization of the inequalities of [3], which are obtained if $m=0$.

4. We go over now to the rigidity problem. In [9] Kodaira and Spencer showed that the infinitesimal deformations of a c.a.f. structure \mathcal{F} are the elements of the cohomology space $H^1(V, \Theta)$, where Θ is the sheaf of germs of vector fields on V such that the corresponding infinitesimal transformations preserve the structure \mathcal{F} .

Such a vector field can be expressed as

$$(4.1) \quad \xi = \xi^a \frac{\partial}{\partial z^a} + \bar{\xi}^a \frac{\partial}{\partial \bar{z}^a} + \eta^u \frac{\partial}{\partial z^u} + \bar{\eta}^u \frac{\partial}{\partial \bar{z}^u},$$

and must satisfy the conditions

$$(4.2) \quad \left[\xi, \frac{\partial}{\partial z^u} \right] = A^v_u \frac{\partial}{\partial z^v}, \quad \left[\xi, \frac{\partial}{\partial \bar{z}^a} \right] = B^c_a \frac{\partial}{\partial z^c} + B^u_a \frac{\partial}{\partial \bar{z}^u}$$

for some functions A and B, or equivalently

- (a) the functions η^u are analytical ($\partial \eta^u / \partial \bar{z}^a = \partial \eta^u / \partial \bar{z}^v = 0$),
- (b) the functions ξ^a are c.a.f. ($\partial \xi^a / \partial \bar{z}^b = \partial \xi^a / \partial \bar{z}^u = \partial \xi^a / \partial z^u = 0$).

In particular, if we take $\xi^a = 0$ and η^u analytical, the previous conditions are satisfied so that, denoting by Φ the sheaf of germs of analytic sections of the structural bundle F , Φ is a subsheaf of Θ .

On the other hand, let Ψ be the sheaf of germs of c.a.f. sections of the transverse bundle F' . Then we have a natural projection $p: \Theta \rightarrow \Psi$ which sends the germ defined by (4.1) to the germ defined by $\xi^a [\partial / \partial z^a]$, where the bracket denotes classes in F' .

It follows quite straightforward [13], [14]:

Proposition 3. *For any c.a.f. structure \mathcal{F} there is an associated exact sequence of sheaves:*

$$(4.3) \quad 0 \longrightarrow \Phi \xrightarrow{\subseteq} \Theta \xrightarrow{p} \Psi \longrightarrow 0.$$

Corollary. *If $H^1(V, \Phi) = H^2(V, \Phi) = 0$, then $H^1(V, \Theta) = H^1(V, \Psi)$.*

It is just this corollary which we plan to use. Namely we shall give conditions for the hypotheses of the corollary to hold and at the same time for the vanishing of $H^1(V, \Psi)$. This last condition will be implied by Proposition 2 above, and the first conditions will be deduced from Griffiths' generalization of the Kodaira vanishing theorem [6]. We arrive at our class of c.a.f. structures by examining the mentioned theorems and their possible application for the complex projective spaces.

5. First, from the just mentioned Griffiths' theorem [6, Theorem G'] it follows that if

$$(5.1) \quad Q(\xi, \eta) = (m+1) R_{uvij} \xi^u \bar{\xi}^v \eta^i \bar{\eta}^j - (R_u{}^u{}_{ij} - \rho_{ij}) \eta^i \bar{\eta}^j |\xi|^2,$$

is a positive definite form in both ξ and η , where $i, j, \dots = 1, \dots, n + m$, ρ is the Ricci tensor of (V, h) , R is the curvature tensor of the metric connection of some hermitian metric of the structural bundle F , and $|\xi|$ is the norm of ξ with respect to this metric, then $H^q(V, \Phi) = 0$ for $q > 0$.

Thus, if F has a hermitian metric $g_{uv}\xi^u\bar{\xi}^v$ which induces on the leaves Kähler metrics of constant nonnegative holomorphic sectional curvature (h.s.c.) and if the curvature forms of the corresponding metric connection of F have the type $(0, 0, 1, 1)$, one has

$$(5.2) \quad R_{uvab} = R_{uva\bar{v}} = 0,$$

and (just as for the Kähler manifolds of constant h.s.c. [5])

$$(5.3) \quad R_{uvst} = \frac{1}{2}k(g_{uv}g_{st} + g_{ut}g_{sv}), \quad (k \geq 0).$$

This gives

$$(5.4) \quad Q(\xi, \eta) = \frac{1}{2}(m + 1)k(g_{ut}\xi^u\bar{\eta}^t)(\overline{g_{vs}\xi^v\bar{\eta}^s}) + \rho_{ij}\eta^i\bar{\eta}^j|\xi|^2.$$

Hence, if we also ask that (V, h) has positive definite Ricci tensor, Q will be positive definite and the corresponding cohomology conclusions hold.

Next, let us look for conditions implying $H^1(V, \Psi) = 0$.

Since the transverse bundle F' is foliate, we have from our previous calculations of the cohomology spaces of a c.a.f. manifold [13], [14] that $H^1(V, \Psi) = \mathcal{H}^{01}(F')$ so that we must look for conditions under which every λ -harmonic $(0, 1)$ -form vanishes.

Let $g' = (g'_{ab})$ be a hermitian metric on F' and consider the connection ω_a^b associated with g' by (3.4), for which we use the notation of § 3.

Now let us first suppose that the corresponding curvature forms Ω_a^b have the type $(1, 1, 0, 0)$, i.e., the curvature coefficients satisfy the conditions

$$(5.5) \quad R_a^b{}_{cu} = R_a^b{}_{uv} = 0.$$

From (3.9) it follows easily that this is equivalent to asking ω_a^b to be foliate forms (i.e., to have foliate, not necessarily analytic, coefficients), which means that ω is a *projectable connection* [10], i.e., ω induces connections on the local transverse manifold of the foliation.

Next, keeping in mind the complex projective space, we are led to ask that the connection induced by ω on any local transverse manifold be a *projective euclidian connection* (which means that it has vanishing Weyl's projective curvature tensor [5]). By calculations similar to those of [5, pp. 206–207] one then gets

$$(5.6) \quad R_a^b{}_{cd} = \frac{1}{n + 1}(R_{ad}\delta_c^b + R_{cd}\delta_a^b)$$

on the local transverse manifold, R_{ad} being the corresponding Ricci tensor. Since (5.6) is clearly also valid on V , we can lower the index b by the help of g' which, as in [5], gives

$$(5.7) \quad R_{abcd} = \frac{R}{2n(n+1)}(g'_{ad}g'_{cb} + g'_{ab}g'_{cd}),$$

where

$$(5.8) \quad R = \frac{1}{2}g'^{ab}R_{ab}.$$

Suppose next that the metric g' can be extended to a Kähler metric h' on V and that we use this new metric for the definition of local bases on V . Then it is clear that formulas (5.5), (5.7) remain valid and we shall use this h' in (3.11).

Now let

$$(5.9) \quad \varphi = \varphi'_b d\bar{z}^b + \psi'_u \theta'^u + \chi'_u \bar{\theta}'^u,$$

where θ' are the forms playing the role of θ with respect to h' , be a λ -harmonic F' -valued $(0, 1)$ -form. Put $L' = \frac{1}{2}ie(\tilde{L})$ where

$$(5.10) \quad \tilde{L} = g'_{ab} dz^a \wedge d\bar{z}^b,$$

and e denotes the left exterior multiplication. (5.10) together with (5.5), (5.7) gives

$$(5.11) \quad \Omega'_b = \frac{R}{2n(n+1)}(\delta'_b \tilde{L} + M),$$

where

$$(5.12) \quad M = \delta'_c g'_{ba} dz^c \wedge d\bar{z}^a.$$

Suppose that $R > 0$, which is equivalent to the fact that R_{ab} is a positive definite tensor. Then (3.11) becomes

$$(5.13) \quad (e(\tilde{L})\varphi, e(\tilde{L})\varphi) + (e(M)\varphi, e(\tilde{L})\varphi) \leq 0.$$

Here the first term is obviously nonnegative. Let us prove that the same is true of the second term. Since the scalar product of two forms can be expressed by integrating the punctual scalar product of the respective tensors [4] we see from (5.9), (5.10) and (5.12) that the second term of (5.13) is given by integrating along V the quantity

$$(5.14) \quad g'_{cb}g'^{ad}\varphi'_a\bar{\varphi}'_d - \varphi'_b\bar{\varphi}'_a + g'_{ab}g'^{vu}\psi'_u\bar{\psi}'_v + g'_{ab}g'^{vu}\chi'_u\bar{\chi}'_v,$$

where g'^{vu} are the components defined by the metric h' .

The last two terms of (5.14) are obviously nonnegative. As for the others, we see that $\varphi_b^a \bar{\varphi}_a^b = \overline{\varphi_b^a \bar{\varphi}_a^b}$, which is therefore a real quantity, and that if it is nonpositive, the whole expression (5.14) is nonnegative. If, on the contrary, $\varphi_b^a \bar{\varphi}_a^b \geq 0$, we consider unitary frames, which allow us to get, for the first two terms of (5.14),

$$\sum_{a,c} \varphi_a^c \bar{\varphi}_a^c - \sum_{a,b} \varphi_b^a \bar{\varphi}_a^b = \left(\sum_{a,c} \varphi_a^c \bar{\varphi}_a^c \right)^{1/2} \left(\sum_{b,d} \bar{\varphi}_b^d \bar{\varphi}_b^d \right)^{1/2} - \sum_{a,b} \varphi_b^a \bar{\varphi}_b^a,$$

where we denote $\bar{\varphi}_b^a = \varphi_a^b$. The above quantity is again nonnegative in view of the well-known Schwartz inequality.

Hence (5.14) is nonnegative at every point of V , which implies

$$(e(M)\varphi, e(\tilde{L})\varphi) \geq 0,$$

and by combining this argument with (5.13) we get

$$(5.15) \quad (e(\tilde{L})\varphi, e(\tilde{L})\varphi) = 0, \quad (e(M)\varphi, e(\tilde{L})\varphi) = 0.$$

Thus from the second equation of (5.15) expressed by integrating (5.14) we get $\psi_u^a = \chi_u^a = 0$, and from the first equation of (5.15), which is given by the integral of

$$(n - 1)g'_{ab}g'^{cd}\varphi_c^a \bar{\varphi}_a^b,$$

we get $\varphi_c^a = 0$ if $n \neq 1$. Hence under the mentioned conditions, there is no nonzero harmonic F' -valued $(0, 1)$ -form and $H^1(V, \Psi) = 0$.

So, using the corollary of Proposition 3 and summing up the previous discussion we have

Proposition 4. *Let \mathcal{F} be a c.a.f. structure of complex codimension different from 1 on a compact manifold V such that the following conditions are satisfied: (a) V admits a Kähler metric h of positive definite Ricci curvature tensor, (b) V admits a Kähler metric h' which induces in the transverse bundle F' a hermitian metric whose connection (3.4) defines on the local transverse manifolds a projectively euclidean connection with a positive definite Ricci curvature tensor, (c) the structural bundle F admits a hermitian metric, whose curvature forms have the type $(0, 0, 1, 1)$ (with respect to h) and which induces on the leaves Kähler metrics of constant nonnegative h.s.c. Then this c.a.f. structure has no nonzero infinitesimal deformation.*

6. We shall see now that one can obtain a nicer result if all the conditions of Proposition 4 are imposed on a single Kähler metric on V . In fact, if (V, h) is a hermitian manifold, and \mathcal{F} a c.a.f. structure on it, we introduced in [14] a canonically associated connection, called the *second connection* of V , which

is characterized by several geometric properties and is given, in the notation of § 1, by

$$(6.1) \quad \begin{aligned} \omega_a^b &= (h^{ab} Z_c h_{ad}) dz^c, & \omega_u^v &= (Z_u t_a^v) dz^a + (h^{sv} Z_w h_{us}) \theta^w, \\ \omega_a^u &= \omega_u^a = 0. \end{aligned}$$

This second connection satisfies the following metric conditions :

$$(6.2) \quad \begin{aligned} dh_{ab} - h_{cb} \omega_a^c - h_{ac} \bar{\omega}_b^c &= 0 \quad (\text{mod } \theta^u = \bar{\theta}^u = 0), \\ dh_{uv} - h_{wv} \omega_u^w - h_{uw} \bar{\omega}_v^w &= 0 \quad (\text{mod } dz^a = d\bar{z}^a = 0). \end{aligned}$$

Now from (3.4) and the first formula of (6.1) we see that h induces on the transverse bundle (which can be differentially identified with the orthogonal bundle F^\perp) a metric whose connection (3.4) is just ω_a^b of (6.1).

Also, if h is a Kähler metric it follows from (1.7) and the second equation of (6.2) that ω_u^v of (6.1) is just the metric connection of the hermitian metric induced by h on the structural bundle F .

Hence by Proposition 4 we get the following desired result.

Theorem. *Let \mathcal{F} be a c.a.f. structure of complex codimension $n \neq 1$ on a compact manifold V . Suppose that V has a Kähler metric h of positive definite Ricci curvature tensor, which induces on the leaves of \mathcal{F} Kähler metrics of constant positive holomorphic sectional curvature and is such that for the second connection of V with respect to (h, \mathcal{F}) the following conditions hold: (a) the structural part of this connection has curvature forms of type $(0, 0, 1, 1)$, (b) the transversal part of this connection induces on the local transverse manifolds projective euclidean connections of positive definite Ricci tensor. Then \mathcal{F} has no nonzero infinitesimal deformations.*

From known results about Kähler manifolds [8], it follows that for $m = 0$, V is just a complex projective space so that we have a generalization of Bott's result which has been mentioned in the introduction, and we shall see that this is a real generalization since it covers other cases too.

Remark first that by a result of Kobayashi [7], the manifold V of the theorem must be simply connected since it is compact Kähler and has positive definite Ricci curvature tensor. Also, by a theorem of Bott [2] and Molino [10], some of the Chern classes of the transverse bundle F' must vanish. (Namely $\text{Chern}^n(F') = 0$ for $h > n$).

A second remark is that condition (b) of the theorem is implied by the condition that h_{ab} induces on the local transverse manifolds Kähler metrics of constant positive h.s.c., in which case ω_a^b is (following the first equation of (6.2)) the corresponding metric connection. The fact that h_{ab} induces metrics on the local transverse manifolds is obviously equivalent to

$$(6.3) \quad Z_u h_{ab} = 0,$$

which means that h is a Reinhart (bundle-like) metric [11], [14]. Hence, in view of (1.7) (b) and (e), F^\perp is integrable in the sense of the complex Frobenius theorem of Nirenberg [9], [14], and this implies that F^\perp defines on V a differentiable foliation with complex analytic n -dimensional leaves.

If, as a stronger condition, F^\perp is analytic, which means that t_a^u are analytic functions, then (1.7) (b) implies (6.3) and F^\perp defines a c.a.f. structure \mathcal{F}^\perp which is complementary to \mathcal{F} . It is simple to derive that the Kählerian character of the metric together with the analyticity of F^\perp also implies that the second connection is the Levi-Civita connection of (V, h) and induces on the transverse bundle the metric connection of the induced metric. In this case, V has a complex local product structure (almost product complex analytic integrable structure) and, by a change of the complex coordinates, we can consider $t_a^u = 0$. Then, by (1.7), V is a decomposable Kähler manifold [8] and, by the corresponding de Rham decomposition theorem, V (which is compact and simply connected) is the product of two complete Kähler manifolds of constant positive h.s.c., i.e., V is the product of two complex projective spaces, [8].

Moreover, if on the product of two complex projective spaces we consider the sheaf Θ' of germs of the infinitesimal transformations which preserve the complex local product structure, then the germs of Θ' can be represented by (4.1) where one also has $\partial\eta^u/\partial z^a = 0$, and it follows that Θ' is the direct sum of two sheaves Φ' and Ψ' which both behave like the sheaf Ψ of the previous sections. Hence by the same proof we shall get $H^1(V, \Theta') = 0$, which means that the considered complex local product structure has no nonzero infinitesimal deformations. (The author was not able to find a proof of this fact which would be essentially simpler than the proof of the previous sections). Thus we have

Proposition 5. *Let $V = CP_m \times CP_n$ ($m, n \neq 1$) be a product of two complex projective spaces, J be the complex structure of V , \mathcal{F}_n be the natural foliate structure of codimension n , \mathcal{F}_m be the natural foliate structure of codimension m , and \mathcal{F}' be the natural complex local product structure of V . Then the structures $J, \mathcal{F}_n, \mathcal{F}_m$, and \mathcal{F}' have no nonzero infinitesimal deformations.*

In fact the result was proved for $\mathcal{F}_n, \mathcal{F}_m$ and \mathcal{F}' , and for J it can be seen to be a consequence of Lemma 4 of [3] or it can be obtained by remarking that the cohomology of the sheaf of germs (4.1) with analytic coefficients ξ^a, η^u is equal (by the classical Dolbeault-Serre theorem) to the cohomology of the elliptic complex

$$(K_{(m)} \otimes \Omega_{(n)}) \oplus (\Omega_{(m)} \otimes K_{(n)}),$$

where $K_{(m)}$ is the complex of vector valued $(0, \cdot)$ -forms, $\Omega_{(m)}$ is the complex of scalar $(0, \cdot)$ -forms on CP_m , and $K_{(n)}, \Omega_{(n)}$ are the similar complexes for CP_n . Then we get the desired result using the Künneth formula for elliptic complexes (see for instance [12]), the rigidity of the complex structure of CP_m and known results regarding the vanishing of the cohomology spaces of $\Omega_{(m)}$ for CP_m [7].

7. From the previous section, we see that it is important to know whether a c.a. foliation admits an analytic complementary distribution. Obviously, this happens if and only if the exact sequence of vector bundles

$$(7.1) \quad 0 \longrightarrow F \longrightarrow T(V) \longrightarrow F' \longrightarrow 0$$

admits an analytic splitting (then the sequence (4.3) also has an analytic splitting), it is known that such a splitting exists if and only if some cohomological obstruction vanishes. This obstruction has been calculated for the general case of an analytic subbundle of a vector bundle on V (see for instance [6]), but we want here to express it, in our case, in a simpler manner.

Consider a c.a.f. manifold V with the notation of § 1. Even without the introduction of a metric, one sees that a complementary subbundle F^\perp of F in $T(V)$ can be described using local bases of the form of Z_a given by (1.2), where t_a^u are locally defined functions. By technical calculations, one derives that on an intersection $U_\alpha \cap U_\beta \neq \emptyset$ of coordinate neighborhoods one has

$$(7.2) \quad t_\beta^u = \frac{\partial z_\alpha^v}{\partial z_\beta^a} \frac{\partial z_\beta^u}{\partial z_\alpha^v} + t_\alpha^v \frac{\partial z_\alpha^b}{\partial z_\beta^a} \frac{\partial z_\beta^u}{\partial z_\alpha^b}.$$

Then it follows that any other complementary bundle \tilde{F} of F is generated by local vector fields

$$(7.3) \quad \tilde{Z}_a = \partial/\partial z^a - (t_a^u + q_a^u)(\partial/\partial z^u),$$

where q_a^u define a global section of the bundle $\text{Hom}(F', F)$.

Now we see that a complementary analytic distribution of F exists if and only if there are functions q_a^u such that

$$(7.4) \quad d_z t_a^u = -d_z q_a^u.$$

But from (7.2) it follows that

$$(7.5) \quad \omega_a^u = d_z t_a^u$$

defines a global 1-form ω on V with values in $\text{Hom}(F', F)$, which is d_z -closed, hence it gives a cohomology class $w \in H^1(V, \mathcal{O}(\text{Hom}(F', F)))$ (in view of the Dolbeault-Serre theorem [4]).

By (7.4) we now see that the obstruction looked for is just w . Hence we have

Proposition 6. *The foliation \mathcal{F} admits a complementary analytic distribution if and only if $w = 0$ (or ω is a d_z -exact form).*

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