J. DIFFERENTIAL GEOMETRY
12 (1977) 101-117

ESTIMATES OF THE LENGTH OF A CURVE

B. V. DEKSTER

In this article we establish some upper bounds for the length of a curve γ lying in a convex region T of an n-dimensional Riemannian space. The results obtained here have the character of a comparison theorem of the following type. Let k_s, κ be respectively the minimum values of the sectional curvature in T and of the normal curvature of the boundary of T. Under the condition that $k_s > -\kappa^2$, one can assign to the region T a circle T_0 in a k_s-plane (a two-dimensional sphere, plane or hyperbolic plane of curvature k_s) whose boundary has the geodesic curvature κ. Then, if the maximum curvature ξ of γ is less than κ, the length of γ does not exceed the length of the longest arc contained in T_0, having constant curvature ξ. (See the corollary of Theorem 1 of § 1.)

The question on estimates of the length of a curve a in a region on a two-dimensional surface was explored by A. D. Aleksandrov and V. V. Strel'cov in 1953 (see [1]). The estimates obtained in [1] contain some integral characteristics of the curve and the region. Their estimates and ours (when $n = 2$) do not follow from one another.

The plan of the proof of inequality (1.1) and Lemma 4 was discussed with J. D. Burago who reported to the author a convenient version of the Rauch theorem connected with Γ-Jacobi field, where Γ is a submanifold. The author thanks J. D. Burago for his attention and help.

1. The basic construction and the results

In n-dimensional Riemannian space M, $n \geq 2$ (of regularity class C^4) we consider a connected region which has a compact closure T and is bounded by a nonempty, possibly disconnected regular hypersurface Γ (of class C^4). The surface Γ divides a sufficiently small ball neighborhood of any of its points into two components; we suppose that only one of them belongs to T. (Instead of this we could suppose that T is the image under an immersion of some connected compact n-dimensional manifold with a smooth edge into M.) Let the boundary Γ of the region T be strictly convex in the following sense: all the normal curvatures of Γ on the side of the interior normal are not less than some positive κ. Finally, let us suppose that in the compact region T the

Received March 26, 1975, and, in revised form, October 29, 1975. This research was supported by NRC Grant A-2338 (awarded to Professor H. S. M. Coxeter).

sectional curvature $\geq k_s > -\kappa^2$. Such a compact region T is said to be normal.

Let us denote the distance between subsets of T (in the metric induced by deleting T from the space M) by $\rho(\cdot, \cdot)$.

The basic construction is as follows. We assign to the normal region T a circle T_0 (on the k_s-plane) whose circumference has geodesic curvature κ. This circle exists because of the condition $k_s > -\kappa^2$. It will be proved in § 3 that the radius R of the ball inscribed in T ($R \overset{\text{def}}{=} \max_{X \in T} \rho(X, \Gamma)$) does not exceed the radius R_0 of the circle T_0, i.e.,

$$
(1.1) \qquad R \leq R_0 = R_0(\kappa, k_s) = \begin{cases} \dfrac{1}{k} \cot^{-1} \dfrac{\kappa}{k}, & \text{if } k_s > 0, \\ \dfrac{1}{\kappa}, & \text{if } k_s = 0, \\ \dfrac{1}{k} \coth^{-1} \dfrac{\kappa}{k}, & \text{if } k_s < 0, \end{cases}
$$

where $k = \sqrt{|k_s|}$.

Let γ: $[0, L] \to T$ be a normal curve (i.e., parametrized by arc-length) of class C^2, whose curvature does not exceed χ: $0 \leq \chi < \kappa$; $X = \gamma(0) \in \operatorname{int} T$. Furthermore let a shortest path $XY \subset T$ satisfy the conditions: $Y \in \Gamma$, $\rho(X, Y) = \rho(X, \Gamma)$, and the angle ϕ between the curves XY and γ does not exceed $\frac{1}{2}\pi$.

In T_0 we construct points Y_0, X_0 and a direction μ at the point X_0 in such a way that $Y_0 \in \Gamma_0$, the shortest path $Y_0X_0 \perp \Gamma$, $X_0Y_0 = XY$, and the angle α between the shortest path X_0Y_0 and the direction μ satisfies $\phi \leq \alpha \leq \frac{1}{2}\pi$ (see Fig. 1). We assign to the curve γ the normal arc of the circumference

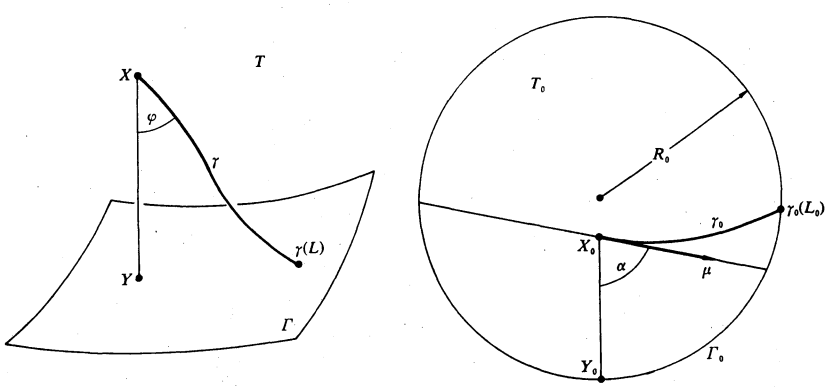

Fig. 1

$$
\gamma_0 \colon [0, L_0)] \longrightarrow T_0, \quad \gamma_0(0) = X_0, \quad \gamma_0(L_0) \in \Gamma_0,
$$

which starts from X_0 in the direction μ, has geodesic curvature χ and (if $\chi \neq 0$)

is situated on that side of the geodesic going through X_0 in the direction μ which does not contain the point Y_0. (This circumference intersects Γ_0 because of the condition $\chi < \kappa$.)

Let us put $r(\lambda) = \rho(\gamma(\lambda), \Gamma)$, $\lambda \in [0, L]$; $\tau(\lambda) = \rho_0(\gamma_0(\lambda), \Gamma_0)$, $\lambda \in [0, L_0]$, where $\rho_0(\cdot, \cdot)$ is the distance in the circle T_0.

The following theorem is obtained in this paper.

Theorem 1. *For a normal region T the following inequalities hold:*

(1) $L \leq L_0$,

(2) $r(\lambda) \leq \tau(\lambda)$, *when* $\lambda \in [0, L]$,

(3) *if* $\lambda_1 \in [0, L]$ *and* $\lambda_2 \in [0, L]$ *are such that* $r(\lambda_1) = \tau(\lambda_2) = a > 0$, *and if*

$$c_1\colon [0, a] \longrightarrow T \quad \text{and} \quad c_2\colon [0, a] \longrightarrow T_0$$

are normal geodesics such that $c_1(0) = \gamma(\lambda_1)$, $c_1(a) \in \Gamma$, $c_2(0) = \gamma_0(\lambda_2)$, $c_2(a) \in \Gamma_0$, *then*

$$\langle \dot\gamma(\lambda_1), \dot c_1(0)\rangle \geq \langle \dot\gamma_0(\lambda_2), \dot c_2(0)\rangle \; . \tag{1.2}$$

Corollary. *The length of any curve (of class C^2), which has the maximum curvature χ satisfying $0 \leq \chi < \kappa$ and is contained in the normal region T, does not exceed the length of an arc of a circumference in the circle T_0 which has geodesic curvature χ, and whose ends are opposite points of Γ_0.*

The theorem and its corollary will be proved in § 4.

Remarks. 1. By the corollary, the length of any geodesic in T is not more than $2R_0$. Therefore neither closed nor infinitely long geodesics exist in T, and the diameter D of the region T satisfies the inequality

$$D \leq 2R_0 \; . \tag{1.3}$$

By (1.1), $R_0(\kappa, k_s)$ is a strictly decreasing function of both arguments. Therefore it is more convenient to take as k_s and κ the corresponding exact lower bounds.

The global behavior of geodesics in a complete Riemannian space was also explored in [5] where there is a detailed bibliography and history of the question.

2. In the estimates of the theorem and the corollary, equalities hold if $T = T_0$.

3. If $\kappa \leq 0$ or $k_s \leq -\kappa^2$, then in such a region infinitely long geodesic can exist.

4. The requirement of regularity of class C^4 is explained by the references in [4]. It could be reduced to regularity of class C^2 by means of a suitable approximation. Later on we will remark only on the fact that certain objects are regular without mentioning the class.

5. L_0 is a strictly increasing function of the arguments α and χ and a strictly decreasing function of the arguments κ and k_s in the region

$$\kappa > 0,\ k_s > -\kappa^2,\ 0 \leq \chi < \kappa,\ 0 < \alpha \leq \tfrac{1}{2}\pi,\ 0 < XY < R_0(\kappa, k_s) \; .$$

The dependence of L_0 on XY is not monotonic. In § 4, as an example, the strict decrease of L_0 as a function of k_s will be proved. Other assertions can be proved in the same way or reduced to simple considerations in the k_s-plane.

6. In the notation of Remark 5, $L_0 = L_0(\alpha) \leq L_0(\frac{1}{2}\pi)$. Obviously, $L_0(\frac{1}{2}\pi)$, as a function of $XY \in (0, R_0]$, has the maximum value $\bar{L}_0(\frac{1}{2}\pi)$ when the points Y_0 and $\gamma_0(L_0(\frac{1}{2}\pi))$ are on orthogonal radii of the circle T_0. Accordingly

$$L \leq L_0 \leq \bar{L}_0(\tfrac{1}{2}\pi) . \tag{1.4}$$

7. The function $r(\lambda)$, $\lambda \in [0, L]$, is strictly decreasing so that $\gamma[0, L) \subset \operatorname{int} T$. Let us put $r(L) \leq r_1 < r_2 \leq XY$. Then the following inequality holds

$$r^{-1}(r_1) - r^{-1}(r_2) \leq \tau^{-1}(r_1) - \tau^{-1}(r_2) . \tag{1.5}$$

This remark will be explained in § 4.

8. At $a = 0$ (see the assertion (3) of Theorem 1) inequality (1.2) is retained if one understands $\dot{c}_1(0)$ and $\dot{c}_2(0)$ as the exterior normals of Γ and Γ_0. (According to Remark 7, equality $a = 0$ is possible only if $\lambda_1 = L$, $\lambda_2 = L_0$.)

The plan of the proof of Theorem 1 is as follows.

The function $\tau(\lambda)$ satisfies the differential equation (4.1), where κ_τ is some function of τ and the numbers κ and k_s (see § 4, § 3, formulas (4.1) and (3.1)). We shall show that in the case of a general disposition of the curve γ, the function $r(\lambda)$ is regular almost everywhere (see § 3) and satisfies the analogous differential inequality (3.7). Moreover, at the points where the regularity fails the left derivative r'_- of the function r is not less than the right one r'_+, (Theorem 2, § 3).

Because of this, as we will see in § 4, the solution $\tau(\lambda)$ of (4.1) under the corresponding initial conditions majorises the function $r(\lambda)$. Thus $r(\lambda)$ vanishes earlier than $r(\lambda)$. This circumstance gives the desired estimate.

We should say that Theorem 1 proved in the last paragraph is a direct analytical consequence of Theorem 2, § 3, which establishes inequalities (3.7) and $r'_- \geq r'_+$. We find it difficult to state the complete text of Theorem 2 just now; let us remark only that Theorem 2 is a generalization of Theorem 1.10 in [3] which states that the distance from a point on a geodesic in a totally convex set to the boundary of the set is a convex function of the position of the point on the geodesic. A detailed remark about this is made at the end of § 3.

The proof of the inequality $r'_- \geq r'_+$ is based on the results of § 2 where the Toponogov lemma on the limit angle (see [6, § 6.4, Lemma 2]) is generalized a little (see the remark at the end of § 4 and Lemmas 2, 3). In particular, Lemma 3 makes it possible to obtain a one-sided estimate of the speed of convergence of angles to the limit angle.

2. A few general remarks

Let Q be a hypersurface in the considered Riemannian space M and a point $q \in Q$. We denote by ν some fixed unit normal for Q at the point q. Let $h\colon [-\lambda_0, \lambda_0] \to M$ be a normal curve and $h(0) = q$. Supposing that $\lambda_0 > 0$ is sufficiently small we denote by $\rho(\lambda)$ the distance between $h(\lambda)$ and Q in the space M taken with the sign "$-$" if the point $h(\lambda)$ is situated on that side of the surface Q which corresponds to the normal ν, and with sign "$+$" otherwise. When $\lambda_0 > 0$ is sufficiently small, the functions $\rho(\lambda)$ are regular.

We denote by K the normal curvature of the surface Q on the side of the normal $-\nu$ at the point q in the direction of the component of the vector $\dot{h}(0)$ which is tangent to Q if this component (i.e., vector $\tau = \dot{h}(0) - \nu\langle\nu, \dot{h}(0)\rangle$) is not zero.

Lemma 1. *The following equality takes place*:

$$(2.1) \qquad \rho''(0) = \begin{cases} -\langle\nu, \ddot{h}(0)\rangle - K(1 - \langle\nu, \dot{h}(0)\rangle^2)\,, & \text{if } |\langle\nu, \dot{h}(0)\rangle| \neq 1\,, \\ 0\,, \quad \text{if } |\langle\nu, \dot{h}(0)\rangle| = 1\,. \end{cases}$$

Corollary. *Let ϕ be the angle between ν and $\dot{h}(0)$, ψ the angle between ν and $\ddot{h}(0)$ (if $\ddot{h}(0) \neq 0$), $k(\lambda)$ the curvature of the curve h (i.e., $k(\lambda) = |\ddot{h}(\lambda)|$), and K_q the minimum of the normal curvatures of the surface Q at the point q. Then*

$$(2.2) \qquad \rho''(0) \leq k(0)\sin\phi - K\sin^2\phi\,.$$

Actually, a simple reasoning based on orthogonality of the vectors $\dot{h}(0)$ and $\ddot{h}(0)$ shows that $-\cos\psi \leq \sin\phi$. Now (2.2) *follows from* (2.1). *Since $\rho'(0) = -\cos\phi$, we have*

$$(2.3) \qquad \rho''(0) \leq k(0)\sqrt{1 - \rho'^2(0)} - K(1 - \rho'^2(0)\,,$$

where K can be replaced by K_q.

Remarks. 1. Equality (2.1) can be rewritten in the following way:

$$(2.4) \qquad \rho''(0) = -k(0)\cos\psi - K\sin^2\phi\,,$$

assuming that $k(0)\cos\psi = 0$ if ψ is not defined, and $K\sin^2\phi = 0$ if K is not defined.

2. Let M be a two-dimensional manifold and the angle $\psi \geq \frac{1}{2}\pi$. Then $-\cos\psi = \sin\phi$, $K = K_q$, and equality takes place in (2.2) and (2.3).

3. Equality in (2.2) and (2.3) also takes place when $\dot{h}(0) = \pm\nu$ (then $\rho'' = 0$).

Proof of Lemma 1. The second equality in (2.1) is obvious. So, let $\dot{h}(0) \neq \pm\nu$.

Let $g(\lambda) \in Q$ be such that the geodesic passing through the points $g(\lambda), h(\lambda)$ is orthogonal to Q (when $\lambda = 0$ we assume $g(0) = h(0)$ and the corresponding geodesic is orthogonal to Q). Denote by N the 2-dimensional surface formed by these geodesics (in a neighborhood of q). Obviously, $\rho(\lambda)$ is the length of the segment $g(\lambda)h(\lambda)$. At the same time, $\rho(\lambda)$ is the distance on the surface N from the point $h(\lambda)$ to the curve g. Obviously, the geodesic curvature of the curve g on the surface N is equal to K.

Denote by $\ddot{h}_N(0)$ the covariant derivative of the field $\dot{h}$ using parallel transfer on the surface N. It is known that $\ddot{h}_N(0)$ is the orthogonal projection of the vector $\ddot{h}(0)$ onto 2-dimensional direction of the surface N. Therefore $\langle \nu, \ddot{h}(0)\rangle = \langle \nu, \ddot{h}_N(0)\rangle$.

So, to prove (2.1) it is sufficient to establish it for the curve h on the 2-dimensional surface N. But for the case $n = 2$, equality (2.1) can be obtained easily by direct calculation based on the formula for geodesic curvature of a curve.

Lemma 2. *Let a sequence of points q_ν converge to an interior point p of a normal region T in such a way that $q_\nu \neq p$ and the directions of the shortest paths pq_ν converge to some direction μ. Let α be the angle at the point p between μ and some shortest path pp', $p' \in \Gamma$, of the length $\rho(p, \Gamma)$. Denote by ξ_ν the angle between the direction of the shortest path pq_ν (from p to q_ν) at the point q_ν and some shortest path $q_\nu q'_\nu$, $q'_\nu \in \Gamma$, of the length $\rho(q_\nu, \Gamma)$. Then the sequence ξ_ν converges and the limit*

$$\bar{\alpha} = \lim_{\nu\to\infty} \xi_\nu \leq \alpha \ . \tag{2.5}$$

The proof of this lemma is based on the following.

Lemma 3. *Let $p, q \in T$, $p \neq q$, $\rho(p, \Gamma) > \bar{\rho} > 0$, $\rho(q, \Gamma) > \bar{\rho}$; $p', q' \in \Gamma$, $\rho(p, p') = \rho(p, \Gamma)$, $\rho(q, q') = \rho(q, \Gamma)$. Let α be the angle between the shortest paths pp' and pq, and β the angle at the point q between qq' and the direction of the shortest path pq (from p to q). Then there exist, depending only on the surface Γ and $\bar{\rho}$, numbers $\varepsilon > 0$ and C such that*

$$\frac{\cos\alpha - \cos\beta}{\rho(p, q)} \leq C \ , \tag{2.6}$$

when $\rho(p, q) < \varepsilon$.

Proof of Lemma 3. Let $\varepsilon_1 > 0$ be such that the closed neighborhood T_{ε_1} of the normal region T is still a compact set in the space M. We put $\Delta = \rho(p, q)$, $l = \rho(p, \Gamma)$.

Let us extend the shortest path $p'p$ as a geodesic within the region T to the point p'' such that the length λ of the geodesic $p'p''$ is equal to the diameter $\mathscr{D}$ of the region T, if it is possible, or, if it is not possible, let $p'' \in \Gamma$. The geodesic $p'p''$ is orthogonal to Γ at the point p' and, by strict convexity of Γ, intersects Γ at a nonzero angle at the point p'' (if $p'' \in \Gamma$). It follows easily from this that λ is a continuous function $\lambda(\rho')$ of the point $p' \in \Gamma$. Obviously, $l \in [\bar{\rho}, \lambda(p')]$.

Let $c\colon [0, l] \to T$ be the normal shortest path $p'p$, $c(0) = p'$, $c(l) = p$. Denote by v the unit parallel field along c such that the vector $v(l)$ has the direction d of the shortest path pq. Let $C_\Delta\colon [0, l] \to T$ be the curve given by the formula

$$c_\Delta(x) = \exp\left(\frac{x}{l} v(x)\right),$$

so that $c_\Delta(0) = p'$, $c_\Delta(l) = q$. Obviously, when $\Delta < \varepsilon_1$ the curve c_Δ exists and $c_\Delta([0, l]) \subset T_{\varepsilon_1}$.

Let σ be the sphere of directions at the point p. Then the curve c_Δ is determined identically by representation of a set $X \overset{\text{def}}{=} (p', l, d, \Delta) \in \Gamma \times [\bar{\rho}, \lambda(p')] \times \sigma \times [0, \varepsilon_1]$. Since the triplet (p', l, d) varies in the compact region $\Gamma \times [\bar{\rho}, \lambda(p')] \times \sigma$, it is easy to see that a positive number $\varepsilon < \varepsilon_1$ can be chosen such that the curve c_Δ is regular for any set $X \in \Omega \overset{\text{def}}{=} \Gamma \times [\bar{\rho}, \lambda(p')] \times \sigma \times [0, \varepsilon]$. Obviously, ε depends only on Γ and $\bar{\rho}$.

Using the first variation formula and Hadamard's lemma we can easily see that the length l_Δ of the curve c_Δ satisfies

$$l_\Delta = l - \cos\alpha \cdot \Delta + f(X) \cdot \Delta^2, \tag{2.7}$$

where $f(X)$ is a regular function in the compact set Ω. (It is important here that $\bar{\rho} > 0$.) So, if $\Delta < \varepsilon$, then

$$l_\Delta \leq l - \cos\alpha \cdot \Delta + C\Delta^2, \tag{2.8}$$

where the constant C depends only on Γ and the numbers $\bar{\rho}$ and $\varepsilon = \varepsilon(\Gamma, \bar{\rho})$. Since $\rho(q, p') \leq l_\Delta$ we have

$$\rho(q, q') \leq \rho(q, p') \leq \rho(p, p') - \cos\alpha \cdot \Delta + C \cdot \Delta^2. \tag{2.9}$$

Similarly, changing the order of the points in the pair p, q, we have

$$\rho(p, p') \leq \rho(p, q') \leq \rho(q, q') - \cos(\pi - \beta) \cdot \Delta + C\Delta^2, \tag{2.10}$$

if $\Delta < \varepsilon$.

Now (2.6) follows by combining (2.9) and (2.10).

Proof of Lemma 2. Following word for word the exposition in [6, § 6.4, Lemma 2] we remark that in order to prove our Lemma 2 it is sufficient to establish inequality (2.5) under the assumption that the sequence ξ_ν converges.

Let us put $\rho(p, \Gamma) = 2\bar{\rho} > 0$. Then, for a sufficiently large ν, $\rho(q_\nu, \Gamma) > \bar{\rho}$. By Lemma 3

$$\cos\alpha - \cos\xi_\nu \leq C\cdot\rho(p, q_\nu)\ ,$$

when $\rho(p, q_\nu) < \varepsilon$. Now we obtain (2.5) passing to the limit when $\nu \to \infty$.

Remark. 3. Originally, Lemmas 2 and 3 were stated in much more generality, but their proof was very long. The idea of the present short proof was suggested by the referee whom the author thanks very much. This proof can still be easily generalized for the case when Γ is an arbitrary compact set in a complete space and $p, q \notin \Gamma$. If Γ is a point, and the points q_ν lie on a geodesic, then Lemma 2 turns into Toponogov's lemma (see [6, § 6.4, Lemma 2]).

3. The distance to the boundary of the normal region as a function of a point on a curve

Let F be the cut locus of the region T from its boundary Γ, i.e., the union of the ends Y of geodesics $XY \subset T$ with $X \in \Gamma$, which are orthogonal to Γ and have maximum length XY subject to $XY = \rho(Y, \Gamma)$. A more detailed description of the cut locus is given, for example, in [4, § 4] (there it is denoted by $F(\partial'\Omega)$). It is shown in [4] that F is closed and its n-dimensional measure is zero. (See Lemma 8.)

We denote by $\Gamma(t)$ the set of points of the normal region T whose distance from Γ is t where $0 \leq t < R = \max_{X\in T} \rho(X, \Gamma)$. It follows from [4, § 4] that every component of the set $\Gamma(t)\backslash F$ is a hypersurface which is parallel to Γ. Let us remark that if $q \in \Gamma(t)\backslash F$, $p \in \Gamma$, $pq = t$, then there are no focal points of the surface Γ on pq (see [6, the end of § 4.3]).

Lemma 4. *The normal curvatures of the surface $\Gamma(t)/F$ are not less than κ_t, where*

$$\kappa_t = \kappa_t(\kappa, k_s) = \begin{cases} k\dfrac{\kappa + k\tan kt}{k - \kappa\tan kt} & \text{when } k_s > 0\ (k = \sqrt{|k_s|})\ , \\[2ex] \dfrac{\kappa}{1-\kappa t} & \text{when } k_s = 0\ , \\[2ex] k\dfrac{\kappa - k\tanh kt}{k - \kappa\tanh kt} & \text{when } k_s < 0\ . \end{cases} \tag{3.1}$$

Remark. κ_t is the geodesic curvature of a circumference of radius $R_0 - t$ on a k_s-plane, where R_0 is defined by formula (1.1). Calculations show that $\kappa_t(\kappa, k_s)$ is a strictly increasing function of the arguments t, κ and k_s. If T is a ball in a space of constant curvature k_s, then the normal curvature of the sphere $\Gamma(t)$ is exactly κ_t.

For the proof of this lemma we shall need the following version of the Rauch Theorem.

Rauch comparison theorem. *Let M and $\tilde{M}$ be Riemannian manifolds of the same dimension (≥ 2), Γ and $\tilde{\Gamma}$ hypersurfaces in M and $\tilde{M}$, σ: $[0, t] \to$*

M and $\tilde{\sigma}\colon [0, t] \to \tilde{M}$ normal geodesics, and suppose that $\sigma(0) \in \Gamma$, $\tilde{\sigma}(0) \in \tilde{\Gamma}$, $\dot{\sigma}(0) \perp \Gamma$, $\dot{\tilde{\sigma}}(0) \perp \tilde{\Gamma}$. Assume the following conditions:

(i) *there are no focal points of the submanifold Γ on $\sigma_{[0,t)}$,*

(ii) *the maximum of the normal curvatures of the surface $\tilde{\Gamma}$ at the point $\tilde{\sigma}(0)$ on the side of the normal $\dot{\tilde{\sigma}}(0)$ does not exceed minimum of the normal curvatures of Γ at the point $\sigma(0)$ on the side of the normal $\dot{\sigma}(0)$ and*

(iii) *for any $\tau \in [0, t]$ the sectional curvatures K and $\tilde{K}$ at the points $\sigma(\tau)$ and $\tilde{\sigma}(\tau)$ satisfy the condition $K(P) \geq \tilde{K}(\tilde{P})$ for any pair of two-dimensional directions P and $\tilde{P}$ such that P is tangent to σ and $\tilde{P}$ is tangent to $\tilde{\sigma}$. Finally, let V and $\tilde{V}$ be Γ- and $\tilde{\Gamma}$-Jacobi fields[1] along σ and $\tilde{\sigma}$ and $|V(0)| = |\tilde{V}(0)|$. Then*

$$|V(\tau)| \leq |\tilde{V}(\tau)|\,, \qquad \tau \in [0, t]\,. \tag{3.2}$$

(Thus on $\tilde{\sigma}_{[0,t)}$ there are no focal points of the surface $\tilde{\Gamma}$.)

The proof of this statement is entirely similar to the proof of the Rauch theorem given, for example, in [2] (see [2, § 11.9]; intermediate statement (a) there does not need a proof in our case). The proof shows also that if $V(0) \neq 0$, and $\tilde{V}(0) \neq 0$, then

$$\frac{\langle V, V\rangle'}{\langle V, V\rangle} \leq \frac{\langle \tilde{V}, \tilde{V}\rangle'}{\langle \tilde{V}, \tilde{V}\rangle} \qquad \text{on } [0, t]\,. \tag{3.3}$$

Moreover, the inequality (3.3) holds even if $|V(0)| \neq |\tilde{V}(0)|$.

Let us prove the inequality (1.1).

We consider a normal shortest path $\sigma\colon [0, R] \to M$ which connects the boundary Γ and the point $\sigma(R) \in T$ most distant from Γ. Such a shortest path is orthogonal to Γ and does not contain any focal point of the surface Γ inside itself. Let $\tilde{\Gamma}$ be an $(n-1)$-dimensional sphere with the normal curvature κ in an n-dimensional space $\tilde{M}$ which is S^n, R^n or the hyperbolic space of curvature k_s according as $k_s > 0$, $= 0$, or < 0. The radius of the sphere $\tilde{\Gamma}$ is equal to R_0.

Let $\tilde{\sigma}\colon [0, R] \to \tilde{M}$ be a normal geodesic issuing from the point $\tilde{\sigma}(0) \in \tilde{\Gamma}$ along the radius of the sphere $\tilde{\Gamma}$. Let us take arbitrary Γ- and $\tilde{\Gamma}$-Jacobi fields V and $\tilde{V}$ along σ and $\tilde{\sigma}$ in such a way that $|V(0)| = |\tilde{V}(0)| \neq 0$. According to (3.2), $|\tilde{V}(\tau)| \geq |V(\tau)| > 0$ when $\tau \in [0, R)$. Therefore the center of the sphere $\tilde{\Gamma}$ does not lie on $\tilde{\sigma}[0, R)$, i.e., $R \leq R_0$.

Proof of Lemma 4. Let X be a unit vector at a point $q \in \Gamma(t)\backslash F$ tangent to the surface $\Gamma(t)\backslash F$. We consider the normal shortest path $\sigma\colon [0, t] \to T \subset M$, $\sigma(t) = q$, $\sigma(0) \in \Gamma$. The shortest path σ is orthogonal to the parallel surfaces Γ and $\Gamma(t)\backslash F$. It is easy to construct a geodesic variation of the shortest path σ such that (i) its longitudinal lines are shortest paths of length t, orthogonal to

[1] I.e., fields associated with a geodesic variation whose longitudinal lines are orthogonal to Γ and $\tilde{\Gamma}$; see [2, § 11.2, Theorem 2].

Γ and $\Gamma(t)\backslash F$, (ii) its transversal lines lie on the surfaces $\Gamma(\tau)\backslash F$, $\tau \in [0, t]$, and (iii) the point $\sigma(t)$ moves with speed X. The field V associated with this variation is a Γ- and $\Gamma(t)\backslash F$-Jacobi field, and moreover $V(t) = X$. Therefore the normal curvature of the surface $\Gamma(t)\backslash F$ in the direction of the vector X

$$k(X) = -\langle V', V\rangle\Big|_t = -\frac{1}{2}\frac{\langle V, V\rangle'}{\langle V, V\rangle}\Big|_t . \tag{3.4}$$

Let a sphere $\tilde{\Gamma} \subset \tilde{M}$ and $\tilde{\sigma}: [0, t] \to \tilde{M}$ be defined as above, and $\tilde{\Gamma}(t)$ be the sphere of radius $R_0 - t$ concentric for the sphere $\tilde{\Gamma}$, $\tilde{V} \neq 0$ an arbitrary $\tilde{\Gamma}$- and $\tilde{\Gamma}(t)$-Jacobi field along σ. Similarly we have

$$\kappa_t = -\frac{1}{2}\frac{\langle \tilde{V}, \tilde{V}\rangle'}{\langle \tilde{V}, \tilde{V}\rangle}\Big|_t . \tag{3.5}$$

Assuming that σ contains no focal points of the surface Γ we can easily show that $V(0) \neq 0$. Then on the basis of (3.3) it follows from (3.4) and (3.5) that

$$k(X) \geq \kappa_t . \tag{3.6}$$

Theorem 2. *Let $\gamma: [0, L] \to T$ be a normal curve of class C^2 whose curvature satisfies $\chi(\lambda) \leq \bar{\chi}$. Suppose that the set $\Phi = \{\lambda: \lambda \in [0, L], \gamma(\lambda) \in F\}$ has linear measure zero and $\rho(\gamma[0, L], \Gamma) > \bar{\rho} > 0$. Let $r(\lambda) \overset{\text{def}}{=} \rho(\gamma(\lambda), \Gamma)$, $\lambda \in [0, L]$. Then*

(1) *$|r(\lambda_1) - r(\lambda_2)| \leq |\lambda_1 - \lambda_2|$ if $\lambda_1, \lambda_2 \in [0, L]$ (thus $r(\lambda)$ is absolutely continuous).*

(2) *The function $r(\lambda)$ is regular on the set $[0, L]\backslash\Phi$, and on these points we have*

$$r'' \leq \sqrt{1 - r'^2}\cdot\bar{\chi} - (1 - r'^2)\kappa_r , \tag{3.7}$$

where $\kappa_r = \kappa_{r(\lambda)}$ is defined by formula (3.1).

(3) *The function $r'(\lambda)$, defined on $[0, L]\backslash\Phi$, has a limit from the left $r'_-(\lambda)$ ($\lambda \neq 0$) and a limit from the right $r'_+(\lambda)$ ($\lambda \neq L$) at every point $\lambda \in [0, L]$. Let $X = \gamma(\lambda)$, $\lambda \in [0, L]$; let $\{XY_j\}$ be the set of the shortest paths of length $r(\lambda)$, $Y_j \in \Gamma$; let ϕ_j be the angle between the shortest path XY_j and $\dot{\gamma}(\lambda)$ at the point X. Then $r'_-(\lambda) = -\cos\max_j \phi_j$ ($\lambda \neq 0$) and $r'_+(\lambda) = -\cos\min_j \phi_j$ ($\lambda \neq L$), so $r'_-(\lambda) \geq r'_+(\lambda)$ when $\lambda \in (0, L)$.*

(4) *The left and the right derivatives of the function $r(\lambda)$ exist and are equal to $r'_-(\lambda)$ and $r'_+(\lambda)$ respectively.*

(5) *There exist constants $\mu > 0$ and C depending only on the region T and the numbers $\bar{\rho}$ and $\bar{\chi}$ such that for $0 < \lambda_1 < \lambda_2 < L$ and $\lambda_2 - \lambda_1 < \mu$ the following inequality holds:*

$$\tilde{r}'(\lambda_2) - \tilde{r}'(\lambda_1) < C(\lambda_2 - \lambda_1) , \tag{3.8}$$

where $\tilde{r}'(\lambda) \in [r'_+(\lambda), r'_-(\lambda)]$. *(If* $\lambda \notin \Phi$, *then* $\tilde{r}' = r'_- = r'_+ = r'$.)

Proof. (1) is an obvious consequence of the triangle inequality.

Let $\lambda \in [0, L]\backslash\Phi$, i.e., $\gamma(\lambda) \notin F$. Then in a neighborhood of $\gamma(\lambda)$ the set $\Gamma(r(\lambda))$ is a regular surface parallel to Γ, and the function $r(\lambda + \Delta\lambda)$ is regular when $|\Delta\lambda|$ is small. According to (1), $|r'(\lambda)| \leq 1$. Let us put $\rho(\Delta\lambda) = r(\lambda + \Delta\lambda) - r(\lambda)$. Obviously, $|\rho(\Delta\lambda)|$ is the distance from the point $\gamma(\lambda + \Delta\lambda)$ to the surface $\Gamma(r(\lambda))$, $r'(\lambda) = \rho'(0)$ and $r''(\lambda) = \rho''(0)$. Now from (2.3) and the relation $k(0) = \chi(\lambda)$, it follows

$$r'' \leq \sqrt{1 - r'^2}\cdot\chi(\lambda) - (1 - r'^2)K \,, \tag{3.9}$$

and (3.7) holds since $\chi(\lambda) \leq \bar{\chi}$ and $K \geq K_q \geq \kappa_r$ (see Lemma 4).

Let us prove (3), for example, for r'_+. Let $\lambda_i \to \lambda$, $\lambda_i > \lambda$, $\lambda_i \in [0, L]\backslash\Phi$. We introduce the following notation:

ξ_i is the angle between the shortest path $\gamma(\lambda_i)Z_i$ of length $r(\lambda_i)$, $Z_i \in \Gamma$, and the (directed) shortest path $X\gamma(\lambda_i)$ at the point $\gamma(\lambda_i)$.

η_i is the angle between $\gamma(\lambda_i)Z_i$ and the vector $\dot{\gamma}(\lambda_i)$.

ζ_i is the angle between $\dot{\gamma}(\lambda_i)$ and the (directed) shortest path $X\gamma(\lambda_i)$.

Since the directions of the shortest paths $X\gamma(\lambda_i)$ converge to the direction of the vector $\dot{\gamma}(\lambda)$, by Lemma 2 there exists a limit $\phi_0 = \lim_{i\to\infty}\xi_i$ and $\phi_0 \leq \phi_j$. We can suppose that the shortest paths $\gamma(\lambda_i)Z_i$ converge to some shortest path $XY \in \{XY_j\}$ forming the angle ϕ_0 with $\dot{\gamma}(\lambda)$, so that $\phi_0 \in \{\phi_j\}$.

Since $|\xi_i - \eta_i| \leq \zeta_i \to 0$, there exists a limit $\lim_{i\to\infty}\eta_i = \phi_0 \leq \phi_j$. But $-\cos\eta_i = r'(\lambda_i)$. Therefore there exists

$$r'_+(\lambda) \overset{\text{def}}{=} \lim_{i\to\infty} r'(\lambda_i) = -\cos\phi_0 \leq -\cos\phi_j \,.$$

Since the last inequality applies for any j and $\phi_0 \in \{\phi_j\}$, we have

$$r'_+(\lambda) = -\cos\left(\min_j \phi_j\right) \,.$$

Let us prove (4) for r'_+. Since $r(\lambda)$ is absolutely continuous,

$$\Delta r \overset{\text{def}}{=} r(\lambda + \Delta\lambda) - r(\lambda) = \int_\lambda^{\lambda+\Delta\lambda} r'(t)dt \,.$$

According to (3) for any $\varepsilon > 0$ there is a number $\delta > 0$ such that $|r'_+(\lambda) - r'(t)| < \varepsilon$ when $0 < t - \lambda < \Delta\lambda < \delta$, $t \notin \Phi$. Then

$$\int_\lambda^{\lambda+\Delta\lambda} (r'_+(\lambda) - \varepsilon)dt < \Delta r < \int_\lambda^{\lambda+\Delta\lambda} (r'_+(\lambda) + \varepsilon)dt \,,$$

whence $|r'_+(\lambda) - \Delta r/\Delta\lambda| < \varepsilon$ when $\Delta\lambda < \delta$.

Now we prove (5). Let us put $p = \gamma(\lambda_1)$, $q = \gamma(\lambda_2)$. Then

$$\tilde{r}'(\lambda_2) - \tilde{r}'(\lambda_1) \leq r'_-(\lambda_2) - r'_+(\lambda_1) = -\cos\phi^2 + \cos\phi^1 ,$$

where ϕ^2 (resp. ϕ^1) is the maximum (resp. minimum) angle between the vector $\dot{\gamma}(\lambda_2)$ (resp. $\dot{\gamma}(\lambda_1)$) and the shortest paths from the set set $\{q\Gamma\}$ (resp. $\{p\Gamma\}$). Let this maximum (resp. minimum) take place for a shortest path $q\Gamma \in \{q\Gamma\}$ (resp. $p\Gamma \in \{p\Gamma\}$). We denote by β (resp. α) the angle between $q\Gamma$ (resp. $p\Gamma$) and the directed shortest path pq at the point q (resp. p).

Using the compactness argument it is easy to prove the existence of a constant $C' > 0$ depending only on the region T and the number $\bar{\chi}$ such that the angle between (any) shortest path pq and the vector $\dot{\gamma}(\lambda_1)$ is less than $C'\Delta$ where $\Delta = \lambda_2 - \lambda_1$. Then $|\phi^2 - \beta| < C'\Delta$, $|\phi^1 - \alpha| < C'\Delta$, and if Δ is sufficiently small

$$\begin{aligned}\tilde{r}'(\lambda_2) - \tilde{r}'(\lambda_1) &\leq -\cos(\beta + C'\Delta) + \cos(\alpha - C'\Delta)\\ &\leq (\cos\alpha - \cos\beta)\cos C'\Delta + 2C'\Delta .\end{aligned}$$

We take as μ the constant m from Lemma 3. Then $pq \leq \Delta < m$ and by Lemma 3, $\cos\alpha - \cos\beta < C \cdot pq \leq C \cdot \Delta$. Consequently

$$\tilde{r}'(\lambda_2) - \tilde{r}'(\lambda_1) < C\Delta + 2C'\Delta = (C + 2C')\Delta .$$

The number $(C + 2C')$ depends only on $T, \bar{\rho}$, and $\bar{\chi}$.

Remark. It follows from the proof that inequality (3.7) of Theorem 2 can be replaced by (3.9). On the other hand, if γ is a geodesic ($\chi = \bar{\chi} = 0$), then we can replace inequality (3.7) with the stronger inequality $r'' \leq 0$. Thus it can be proved that Theorem 2 means that for a geodesic "of general position" (i.e., when measure of the set Φ is zero) the function $r(\lambda)$ is convex. Since one can approximate an arbitrary geodesic with geodesics "of general position", $r(\lambda)$ is a convex function for any geodesic. This fact, proved under the additional condition that $k_s \geq 0$ and M is a noncompact complete manifold (but with admission of degenerate region T), is a statement of Theorem 1.10 in [3].

4. Proof of Theorem 1

It is enough to consider only the case when $XY < R_0$. If $XY = R_0$, then the theorem can be proved by varying the curve γ to include it in a regular family of curves $\gamma_\sigma\colon [0, L] \to T$ such that $\gamma_\sigma(0) \in XY \setminus X$ and $\gamma_\sigma(\lambda) \to \gamma(\lambda)$ as $\sigma \to 0$, $\lambda \in [0, L]$. Applying the theorem to γ_σ and passing to limit as $\sigma \to 0$ we obtain the theorem for γ.

Similarly we can consider only the case when $\gamma[0, L] \in \operatorname{int} T$, $0 < \phi < \alpha \leq \frac{1}{2}\pi$ and the curvature of the curve γ is strictly less than $\chi > 0$ (one can take $\chi + \varepsilon$, $\varepsilon > 0$ instead of χ, apply the theorem and pass to limit as $\varepsilon \to 0$).

Finally, we can suppose that the set $\Phi = \{\lambda\colon \lambda \in [0, L],\ \gamma(\lambda) \in$ the cut locus F from the boundary $\Gamma\}$ has measure zero. In fact, otherwise we can include the curve γ in an $(n-1)$-parameter family of curves which make up some nonzero volume in the manifold M. Among these curves there is a curve γ' for which $\gamma' \cap F$ has linear measure zero or else by the Fubini theorem F has nonzero measure, and this is contrary to Lemma 8 of [4]. Applying the theorem to γ' and passing to the limit as $\gamma' \to \gamma$ we obtain the theorem for γ.

The proof is based on the integration of the differential inequality (3.7). We consider the differential equation

$$t'' = \sqrt{1-t'^2}\chi - (1-t'^2)\kappa_t\,, \tag{4.1}$$

where κ_t is given by formula (3.1) so that the right side of (4.1) is defined when $0 \le t < R_0(\kappa, k_s)$ and $|t'| \le 1$. We study the solution of (4.1) noncontinuable in the open region Ω: $\lambda \in (-\infty, \infty)$, $t \in (0, R_0)$, $|t'| < 1$ with the initial data: $t(0) = XY$, $t'(0) = -\cos\alpha$. The point $a = (0, t(0), t'(0)) \in \Omega$ so that this solution exists and is unique.

The function $\tau(\lambda)$ (see § 1) is a solution of (4.1) with the initial data: $\tau(0) = XY$, $\tau'(0) = -\cos\alpha$. This can be checked by calculation or can be seen from the remark after Lemma 4, § 3 and Remark 2 after Lemma 1, § 2. (When $T = T_0$ and $\gamma = \gamma_0$, equality holds in (2.3) and in the estimate of Lemma 4, from which follows (3.7)).

(A) Let us show that $\tau'(\lambda) < 0$ when $\lambda > 0$. It is obvious for sufficiently small λ. In fact, if $\tau'(0) < 0$ then $\tau'(\lambda) < 0$ by continuity, and if $\tau'(0) = 0$ then $\tau'(\lambda) < 0$ because

$$\tau''(0) = \chi - \kappa_{\tau(0)} < \chi - \kappa_0 = \chi - \kappa < 0\,,$$

(see (4.1) and the remark after Lemma 4). Let us suppose to the contrary that $\tau'(\lambda_1) \ge 0$ for some $\lambda_1 > 0$. Let λ_2 be the minimum root of the equation $\tau'(\lambda) = 0$ on the set $(0, \lambda_1]$. Then $\tau''(\lambda_2) \ge 0$ which is impossible since $\tau''(\lambda_2) = \chi - \kappa_{\tau(\lambda_2)} \le \chi - \kappa < 0$.

(B) It is easy to see that $|\tau'(\lambda)| < 1$ for $\lambda \in [0, L_0]$. The case $|\tau'(\lambda)| > 1$ is impossible by a metric reasoning. The case $|\tau'(\lambda)| = 1$ means that the curve γ_0 is tangent to a radius at the point $\gamma_0(\lambda)$ and that is also impossible by the geometry of the k_s-plane (remember that $\alpha > 0$).

(C) Let us consider the curve $\tilde{\gamma}$: $t = \tau(\lambda)$, $t' = \tau'(\lambda)$, $\lambda \in [0, L_0]$ in the space of the parameters λ, t, t'. It follows from (A) that $\tau(\lambda) \le \tau(0) = XY$. According to (B), $|\tau'| < 1$. Therefore $\tilde{\gamma}|_{[0,L_0)}$ lies in the region Ω. Since $\lim_{\lambda \to L_0} \tau(\lambda) = \tau(L_0) = 0$, the curve $\tilde{\gamma}$ reaches the boundary of the region Ω.

Let us return now to the given curve γ: $[0, L] \to T$, and consider on the plane of the variables λ and t the graphs $\Xi(\tau)$ and $\Xi(r)$ of the functions $t = \tau(\lambda)$, $\lambda \in [0, L_0]$ and $t = r(\lambda)$, $\lambda \in [0, L]$. We remark that $r(0) = \tau(0) = XY$ and, according to Theorem 2 (4),

$$r'_+(0) = -\cos\left(\min_j \phi_j\right) \leq -\cos\phi < -\cos\alpha = \tau'(0) \ ,$$

so that if $\lambda > 0$ is sufficiently small, then

$$r(\lambda) < \tau(\lambda) \tag{4.2}$$

(see Fig. 2).

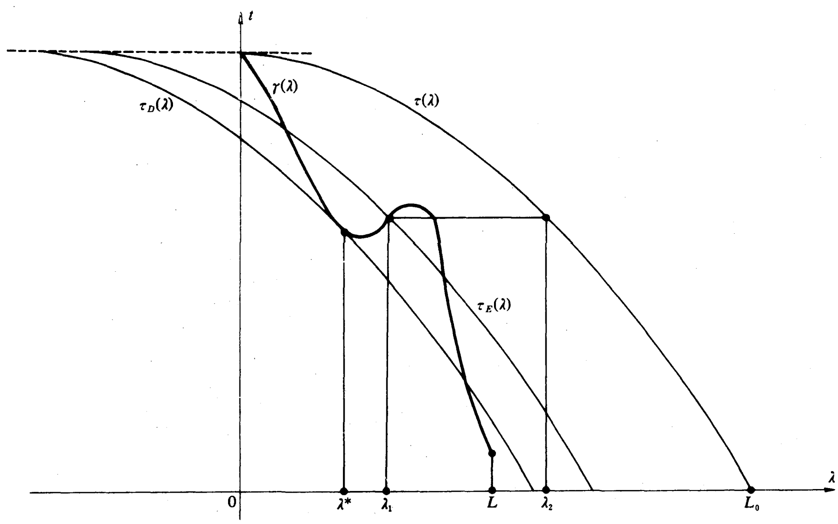

Figure 2

We remark also that the function $\tau_c(\lambda) = \tau(\lambda - C)$ for an arbitrary C is a solution of (4.1) with the initial data: $\tau_c(C) = XY$, $\tau'_c(C) = -\cos\alpha$. (The graph $\Xi(\tau_c)$ is a result of translating $\Xi(\tau)$ a distance C along the λ-axis.)

Let us suppose now that Theorem 1 is not true, i.e., one of the following possibilites occurs:

(a) $L > L_0$.

(b) $L \leq L_0$ but $r(\lambda) > \tau(\lambda)$ for some $\lambda \in [0, L]$.

(c) $L \leq L_0$, $r(\lambda) \leq \tau(\lambda)$ for all $\lambda \in [0, L]$ but there exist (described in Theorem 1 (3)) λ_1, λ_2 and c_1, c_2 such that $(\widehat{\dot\gamma(\lambda_1), \dot c_1(0)}) > (\widehat{\dot\gamma_0(\lambda_2), \dot c_2(0)})$ where $(\widehat{\cdot\,,\,\cdot})$ denotes an angle.

It is easy to see that in any of these three cases there exists $D \in (-\infty, \infty)$ and $\lambda_* \in (0, L)$ such that $r(\lambda_*) = \tau_D(\lambda_*)$ and

$$\tau_D(\lambda) \leq r(\lambda) \tag{4.3}$$

in some neighborhood $|\lambda - \lambda_*| < \delta$ of the point λ_*. (In other words during the translation of $\Xi(\tau)$ along λ-axis there is a moment D such that the translated graph $\Xi(\tau_D)$ is "tangent from the left" to the graph $\Xi(r)$ at its interior point $(\lambda_*, r(\lambda_*))$.)

We consider, for example, the case (c) (see Fig. 2). Let $\bar{r}(\lambda)$ be the restriction of the function $r(\lambda)$ to the set $[0, \lambda_1] \subset [0, L]$. Let us show that one can take as D the minimum number C for which $\Xi(\tau_C) \cap \Xi(\bar{r}) \neq \varnothing$ and can take as λ_* a root of the equation $\bar{r}(\lambda_*) = \tau_D(\lambda_*)$.

On the strength of (4.2) and the fact that $\tau(\lambda)$ is strictly decreasing (see (A)), the number $D < 0$ and $\lambda_* \neq 0$. Let a number E be such that $\Xi(\tau_E) \ni (\lambda_1, \bar{r}(\lambda_1))$ (see Fig. 2). Then

$$\begin{aligned} r'_-(\lambda_1) &= -\cos(\max \phi_j) \geq -\cos(\widehat{\dot{\gamma}(\lambda_1), \dot{c}_1(0)}) > -\cos(\widehat{\dot{\gamma}_0(\lambda_2), \dot{c}_2(0)}) \\ &= \tau'(\lambda_2) = \tau'_E(\lambda_1) \ . \end{aligned}$$

From this and the fact that $\tau_E(\lambda)$ is strictly decreasing (see (A)) the number $D < E$ and $\lambda_* \neq \lambda_1$. Thus $\lambda_* \in (0, \lambda_1) \subset (0, L)$. Now (4.3) follows from strict decrease of $\tau_D(\lambda)$ and the minimality of D.

In view of the minimality of D we have $r'_- \leq \tau'_D$ and $r'_+ \geq \tau'_D$ at the point λ_*. According to Theorem 2 (4), $r'_+ \leq r'_-$ and therefore

$$r'_- = r'_+ = \tau'_D \qquad (\text{when } \lambda = \lambda_*) \ . \tag{4.4}$$

In order to get a contradiction it will be enough to establish that for some $\delta > 0$, $r' < \tau'_D$ on the set $(\lambda_*, \lambda_* + \delta]\backslash\Phi$. In this case, the absolute continuity of the function $r(\lambda)$ (see Theorem 2 (1)) yields

$$r(\lambda) = r(\lambda_*) + \int_{\lambda_*}^{\lambda} r'(u)du < \tau_D(\lambda_*) + \int_{\lambda_*}^{\lambda} \tau'_D(u)du = \tau_D(\lambda) \ , \qquad \lambda \in (\lambda_*, \lambda_* + \delta] \ ,$$

and this is contrary to (4.3).

Let us denote by $F_\chi(\lambda, t, t')$ the right side of (4.1). The function $F_\chi(\lambda, t, t')$ is defined for any χ (and, in reality, does not depend on λ). Let $\bar{\chi}$ be the maximum curvature of the curve γ, so that $\bar{\chi} < \chi$. Since $|\tau'_D| < 1$, the point $a_* = (\lambda_*, \tau_D(\lambda_*), \tau'_D(\lambda_*)) \in \Omega$ and $F_{\bar{\chi}} = F_\chi - \xi$, $\xi > 0$, at this point. If points $a_1, a_2 \in \Omega$ are sufficiently close to a_*, then $F_{\bar{\chi}}(a_1) < F_\chi(a_2) - \frac{1}{2}\xi$. Obviously, $(\lambda, \tau_D(\lambda), \tau'_D(\lambda)) \to a_*$ as $\lambda \to \lambda_*$. In view of the continuity of the function $r(\lambda)$ and also in consequence of Theorem 2 (3) and equality (4.4) we have $(\lambda, r(\lambda), r'(\lambda)) \to a_*$ as $\lambda \to \lambda_*$, $\lambda \notin \Phi$. Now, for the values of $\lambda \notin \Phi$ such that $|\lambda - \lambda_*| \leq \delta$, (3.7) implies

$$r''(\lambda) \leq F_{\bar{\chi}}(\lambda, r(\lambda), r'(\lambda)) < F_\chi(\lambda, \tau_D(\lambda), \tau'_D(\lambda)) - \tfrac{1}{2}\xi = \tau''_D(\lambda) - \tfrac{1}{2}\xi \ . \tag{4.5}$$

Let us put $I = [\lambda_*, \lambda_* + \delta]$, $\Psi = I \cap \Phi$. Since the cut locus F is closed (see [4, Lemma 8]), Φ and Ψ are also closed. Set $\eta(\lambda) = \tau'_D(\lambda) - r'_+(\lambda)$, $\lambda \in I$. Since $\tau'_D(\lambda)$ is regular, the function $\eta(\lambda)$ is subject to the following conditions:

1. $\eta(\lambda)$ is regular on $I \backslash \Psi$, where Ψ is closed and has measure zero.
2. $\eta(\lambda_*) = 0$ on the basis of (4.4).
3. On the set $I \backslash \Psi$, according to (4.5), $\eta'(\lambda) > \frac{1}{2}\xi > 0$.

According to (3.8), when $\lambda_1, \lambda_2 \in I$, $\lambda_2 > \lambda_1$, are sufficiently small we have $\eta(\lambda_2) - \eta(\lambda_1) > C(\lambda_2 - \lambda_1)$. (Maybe $C < 0$.) Let us show that, under this condition, $\eta(\lambda) > 0$ when $\lambda \in I \backslash \lambda_*$. The set $\Psi \cap [\lambda_*, \lambda]$ can be covered by a finite number of segments whose total tength is smaller than any $\varepsilon > 0$. On these segments the function η can decrease but not by more than $|C|\,\varepsilon$. On the other part of the segment $[\lambda_*, \lambda]$ it increases not by less than $\frac{1}{2}(\lambda_* - \lambda - \varepsilon)\xi$. Taking ε sufficiently small we see that $\eta(\lambda) > 0$. So, $r' = r'_+ < \tau'_D$ when $\lambda \in I \backslash (\Psi \cup \lambda_*)$.

Proof of the corollary. Let $b : [0, \Lambda] \to T$ be a normal curve mentioned in the corollary. As in the proof of Theorem 1 we can suppose that $b[0, \Lambda] \subset \operatorname{int} T$. According to Remark 6 after Theorem 1, it is enough to prove that $\Lambda \leq 2\bar{L}_0(\frac{1}{2}\pi)$.

Let $r(\lambda) = \rho(b(\lambda), \Gamma)$ (> 0), $\lambda \in [0, \Lambda]$. Let $a^j_\lambda : [0, r(\lambda)] \to T$ be a normal shortest path: $a^j_\lambda(0) = b(\lambda)$, $a^j_\lambda(r(\lambda)) \in \Gamma$. (Index j belongs to the index set J_λ of such shortest paths when λ is fixed.) We denote by $\phi^j(\lambda)$ the angle between the vectors $\dot{a}^j_\lambda(0)$ and $\dot{b}(\lambda)$. Obviously, there exist indices "+" and "−" $\in J_\lambda$ such that $\phi^+(\lambda) = \max_{j \in J_\lambda} \phi^j(\lambda)$ and $\phi^-(\lambda) = \min_{j \in J_\lambda} \phi^j(\lambda)$. (Possibly, J_λ consists of only one index; then $a^+_\lambda = a^-_\lambda$, $\phi^+(\lambda) = \phi^-(\lambda)$.)

If $\phi^-(0) \leq \frac{1}{2}\pi$, then the situation is described by the conditions of Theorem 1 (with shortest path a^-_λ instead of XY). According to (1.4), $\Lambda \leq \bar{L}_0(\frac{1}{2}\pi)$ and the corollary is proved.

Let $\phi^-(0) > \frac{1}{2}\pi$, and put $E = \{\lambda : \lambda \in [0, \Lambda],\ \phi^-(\lambda) > \frac{1}{2}\pi\}$, $\lambda_* = \sup E$. Let us show that $\phi^+(\lambda_*) \geq \frac{1}{2}\pi$. If $\lambda_* \in E$, then it is true because $\phi^+(\lambda_*) \geq \phi^-(\lambda_*) > \frac{1}{2}\pi$. If $\lambda_* \notin E$, then $\lambda_* \neq 0$ and there exists a sequence $\lambda_i \to \lambda_*$ such that $\lambda_i < \lambda_*$, $\phi^-(\lambda_i) > \frac{1}{2}\pi$. We can suppose that the shortest paths $a^-_{\lambda_i}$ converge to some shortest path $a^j_{\lambda_*}$, $j \in J_{\lambda_*}$. Then $\phi^+(\lambda_*) \geq \phi^j(\lambda_*) \geq \frac{1}{2}\pi$.

If $\lambda_* = \Lambda$, then $\phi^+(\Lambda) \geq \frac{1}{2}\pi$, and changing the direction of the curve b we again get the case considered above: $\phi^-(0) \leq \frac{1}{2}\pi$. Let $\lambda_* \in [0, \Lambda)$. Then there exists a sequence $\lambda_i \to \lambda_*$ such that $\lambda_i > \lambda_*$, $\phi^-(\lambda_i) \leq \frac{1}{2}\pi$. We can suppose that the shortest paths $a^-_{\lambda_i}$ converge to some shortest path $a^j_{\lambda_*}$, $j \in J_{\lambda_*}$. Then $\phi^-(\lambda_*) \leq \phi^j(\lambda_*) \leq \frac{1}{2}\pi$.

By assumption, $\phi^-(0) > \frac{1}{2}\pi$ so that $\lambda_* \neq 0$. Thus $\lambda_* \in (0, \Lambda)$ and $\phi^-(\lambda_*) \leq \frac{1}{2}\pi \leq \phi^+(\lambda_*)$. Therefore one can apply Theorem 1 to the curves $b_{[0,\lambda_*]}$ and $b_{[\lambda_*,\Lambda_*]}$. According to (1.4), $\lambda_* \leq \bar{L}_0(\frac{1}{2}\pi)$, $\Lambda - \lambda_* \leq \bar{L}_0(\frac{1}{2}\pi)$ whence $\Lambda \leq 2\bar{L}_0(\frac{1}{2}\pi)$.

Proof of Remarks 5 *and* 7. Let us prove that $L_0 = L_0(k_s)$ is strictly decreasing. $L_0(k_s)$ is the first positive root of the equation $\tau(\lambda) = 0$, where $\tau(\lambda)$ is the solution of (4.1) with initial data $\tau(0) = XY$, $\tau'(0) = -\cos\alpha$. We denote by $\theta(\lambda)$ the solution of (4.1) with the same initial data under the condition that the argument k_s of the function $\kappa_t(\kappa, k_s)$ is replaced by $k_s + \Delta$, $\Delta > 0$. We should prove that the first positive root $L_0(k_s + \Delta)$ of the equation $\theta(\lambda) = 0$ is less than $L_0(k_s)$.

Let us assume the contrary so that $L_0(k_s + \Delta) \geq L_0(k_s)$. Since κ_t is striclty increasing in k_s (see remark after Lemma 4) and $-1 < -\cos\alpha = \tau'(0) = \theta'(0) \leq 0$, it follows from (4.1) that $\theta''(0) < \tau''(0)$. Thus $\theta(\lambda) < \tau(\lambda)$ when $\lambda > 0$ is sufficiently small. Let $\tilde{\lambda} \in (0, L_0(k_s)]$ be the minimum number for which $\theta(\tilde{\lambda}) = \tau(\tilde{\lambda})$.

According to (A) of § 4, $\tau'(\lambda) < 0$ and $\theta'(\lambda) < 0$ when $\lambda > 0$. Let us put $\psi(t) = \tau^{-1}(t) - \theta^{-1}(t)$, $t \in [\tau(\tilde{\lambda}), XY]$. Then $\psi(\tau(\tilde{\lambda})) = \psi(XY) = 0$ and $\psi(t) > 0$ when $t \in (\tau(\tilde{\lambda}), XY)$. At the point $t_* \in (\tau(\tilde{\lambda}), XY)$ where $\psi(t)$ has a maximum

$$\psi'(t_*) = \frac{d\tau^{-1}}{dt}(t_*) - \frac{d\theta^{-1}}{dt}(t_*) = 0 , \tag{4.6}$$

$$\psi''(t_*) = \frac{d^2\tau^{-1}}{dt^2}(t_*) - \frac{d^2\theta^{-1}}{dt^2}(t_*) \leq 0 . \tag{4.7}$$

We define λ_1 and λ_2 by the condition:

$$\theta(\lambda_1) = \tau(\lambda_2) = t_* . \tag{4.8}$$

Using the formula for the derivative of an invese function and taking into consideration (A) and (B) of § 4 we get, from (4.6),

$$-1 < \theta'(\lambda_1) = \tau'(\lambda_2) < 0 . \tag{4.9}$$

Similarly, it follows from (4.7) and (4.9) that

$$\theta''(\lambda_1) \geq \tau''(\lambda_2) .$$

But this is impossible since, on the basis of (4.1) and (4.9),

$$\tau''(\lambda_2) - \theta''(\lambda_1) = (1 - \tau'^2(\lambda_2)) \cdot (\kappa_t(\kappa, k_s + \Delta) - \kappa_t(\kappa, k_s)) > 0 .$$

Remark 7 for a curve γ with the set $\gamma[0, L] \cap F$ of measure zero follows easily enough from Theorem 1 (3) and from Theorem 2. In the general case Remark 7 can be proved by a limit argument.

Bibliography

[1] A. D. Aleksandrov & V. V. Strel'cov, *Isoperimetric problem and estimates of the length of a curve on a surface,* Proc. Steklov Inst. Math. **76** (1965) 81–99.
[2] R. L. Bishop & R. J. Crittenden, *Geometry of manifolds,* Academic Press, New York, 1964.
[3] J. Cheeger & D. Gromoll, *On the structure of complete manifolds of nonnegative curvature,* Ann. of Math. **96** (1972) 413–443.
[4] B. V. Dekster, *An inequality of the isoperimetric type for a domain in a Riemannian space,* Math. USSR-Sb. **19** (1973) 257–274.
[5] D. Gromoll & W. Meyer, *Periodic geodesics on compact Riemannian manifolds,* J. Differential Geometry **3** (1969) 493–510.
[6] D. Gromoll, W. Klingenberg & W. Meyer, *Riemannische Geometrie im Grossen,* Lecture Notes in Math. Vol. 55, Springer, Berlin, 1968.

UNIVERSITY OF TORONTO