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A GENERAL APPROACH TO MORSE THEORY

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The Morse theory of critical points was extended by Palais and Smale [10], [16] to a certain class of functions on Hilbert manifolds. However, there are many variational problems in a nonlinear setting which for technical reasons are posed not on Hilbert but on Banach manifolds of mappings. For example, the Plateau problem, the existence of harmonic mappings between finite dimensional Riemannian manifolds, and the fixed endpoint solution to the Euler equations of hydrodynamics to name a few. It would therefore be desirable to have an infinite dimensional Morse theory which applies to these problems. The purpose of this paper is to extend Morse theory to manifolds modelled on Banach spaces and to show how this theory applies to the problem of geodesics on finite dimensional Riemannian manifolds. Other applications will be given in future papers.

Such extensions have already been given by Uhlenbeck [22], [23] and we build upon her work to some extent. Our theory has the advantages (a) that the definition we give of nondegenerate critical point (§ 2) is intrinsic, that is, does not depend on the choice of a particular coordinate neighborhood, and (b) we abandon the condition (C) of Palais and Smale and replace it with a condition which works in a much more general setting (see the discussion at the end of § 1 and the beginning of § 3). In addition this new theory fits nicely with the authors [15], [20] generalization of vector field index theory to the Banach manifold category. Finally we assume that the mappings f which we consider are of class C^2. This is in the spirit of Smale's approach to Morse theory [6].

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For condition (C) to be satisfied Palais needed the manifold of L_1^2 maps of the interval into V. We show that in our theory we are free to choose *any* Sobolev manifold of maps functor L_k^p, $k > 0$. Condition (C) is then violated but not our conditions. The notion of nondegeneracy does not depend on the model space.

1. Preliminaries and a review of standard theory

Let M be a C^k, $k > 1$ Banach manifold and let TM denote its tangent bundle

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with $\pi: TM \to M$ the canonical projection. If TM is given a *Finsler structure* (e.g. see [11, p. 118]), M is called a C^k Finsler manifold. For a Finsler manifold there is a natural metric on the components of M induced by the Finsler structure on TM; namely if $p, q \in M$ and are in the same component we define

$$(1) \qquad \rho(p, q) = \inf \int_a^b \|\sigma'(t)\|_{\sigma(t)} dt ,$$

where the infinum is taken over all C^1 paths joining p and q. In [11] it is shown that ρ is a metric for each component of M which induces the given topology. M is said to be a *complete Finsler manifold* if the pair (M, ρ) is a complete metric space.

Definition. Let M be a C^1 Finsler manifold and $\sigma: (a, b) \to M$ a C^1 path on M. We define the length $l(\sigma)$ of σ by

$$l(\sigma) = \lim_{\substack{s \to a \\ t \to b}} \int_s^t \|\sigma'(u)\| \, du .$$

It is possible that $l(\sigma) = \infty$.

Proposition 1. *If M is a Finsler manifold, and $\sigma: (a, b) \to M$ is a C^1 curve of finite length, then the image of σ in M is totally bounded in the Finsler metric for M, and hence if M is complete the image of σ has compact closure in M.*

Proof. [11, § 9, Proposition 1].

A C^r, $r \geq 0$, $r \in Z$, vector field X is a C^r section of the tangent bundle TM. A vector field $X: M \to TM$ on a C^1 manifold M is C^{1-} if given a coordinate neighborhood $\mathcal{O}$, and a chart $\varphi: \mathcal{O} \to E$, the principle part $X_\varphi: \mathcal{O} \to E$ of the vector field X is locally Lipshitz. For $p \in M$ a solution curve of X with initial condition p is a C^1 map $\sigma_p: (a, b) \to M$, (a, b) an open interval about zero in R with $\sigma_p'(t) = X(\sigma_p(t))$ and $\sigma_p(0) = p$. The following results on solution curves of vector fields are standard [8].

Proposition 2. *Let M be a C^k manifold $\partial M = \emptyset$ and $X: M \to TM$ a C^r, $r \geq 1$-vector field on M. For each $p \in M$ there is a solution curve σ_p of X with initial condition p such that every solution curve of X with initial condition p is a restriction of σ_p.*

The solution curve above is called the maximal solution curve of X. Define $t^+: M \to (0, \infty]$ and $t^-: M \to [-\infty, 0)$ by the condition that domain $\sigma_p = (t^-(p), t^+(p))$.

Proposition 5. *Let X be a C^{1-} vector field on an open submanifold M^* of a complete C^2 Finsler manifold M and let $\sigma: (a, b) \to M^*$ be a maximal integral curve of X. If $b < \infty$ and $\int_0^b \|X(\sigma(t))\| \, dt < \infty$, then $\sigma(t)$ has a limit point in $M - M^*$ as $t \to b$. Similarly if $a > -\infty$ and $\int_a^0 \|X(\sigma(t))\| \, dt < \infty$, then $\sigma(t)$ has a limit point in $M - M^*$ as $t \to a$.*

Proof. [11, § 3, Theorem 9].

In order for Palais to do Lusternik-Schnirelman theory on Banach manifolds he needed the notion of a pseudo-gradient vector field which we present below.

Let M be a Finsler manifold and let $f\colon M \to R$ be differentiable at $p \in M$. Then $Y \in T_pM$ is called a *pseudo gradient* vector for f at p if

$$(2) \qquad \|Y\| \leq 2\,\|df_p\|\,,$$

$$(3) \qquad Y(f) = df_p(Y) \geq \|df_p\|^2\,,$$

$$\left(\|df_p\| = \sup_{\|v\|\leq 1} |df_p(v)|,\ v \in T_pM\right).$$

If f is differentiable at each point of $S \subseteq M$, and Y is a C^k pseudo-gradient field for f on S, then $Y(p)$ is a pseudo-gradient vector for f at each $p \in S$.

The following is the basic result of Palais' on the existence of such vector fields.

Proposition 6. *Let M be a C^k Finsler manifold with $\partial M = \emptyset$ and let $f\colon M \to R$ be C^1. Let M^* denote the open submanifold of M consisting of regular (i.e., noncritical) points of f. Then there is a C^{1-} pseudo-gradient vector field Y for f in M^*. If M admits C^k partitions of unity, we can choose Y to be C^k.*

Before reviewing the basic results of Morse theory on C^k-Riemannian manifolds we recall the now famous *sequential* version of the condition (C) of Palais and Smale.

Definition. Let M be a C^1 Finsler manifold, and $f\colon M \to R$ a C^1 map. We say that f satisfies condition (C) if given any sequence $\{s_n\}$ in M in which f is bounded but on which $\|df\|$ is not bounded away from zero there is a subsequence $\{s_{n_j}\}$ which converges.

This condition (C) is essentially a compactness condition on the function f. As a general rule in extending finite dimensional results in differential topology to infinite dimensions we transfer the compactness condition from the space M to the functions on M. Condition (C) is crucial to the Palais-Smale versions of Morse theory and to Schwartz's and Palais' version of Lusternik-Schnirelman theory.

Let M be a C^k, $k \geq 3$ complete Riemannian manifold modelled on a seperable Hilbert space H with $\langle\ ,\ \rangle_p\colon T_pM \times T_pM \to R$ a complete inner product on T_pM for all $p \in M$ (the Riemannian structure). The Riemannian structure induces a Finsler structure on TM in the standard way: if $u \in T_pM$, then $\|u\|_p = \sqrt{\langle u, u\rangle_p}$. Let $f\colon M \to R$ be a C^3 function. Then $df_p\colon T_pM \to R$ is a linear functional on T_pM. Therefore by the Riesz representation theorem there exists a unique element $\nabla f(p) \in T_pM$ so that $df_p(u) = \langle \nabla f(p), u\rangle$ for all $u \in T_pM$ and with $\|df_p\| = \|\nabla f(p)\|$. $\nabla f\colon M \to TM$ is a C^2 vector field on M called the gradient of f at p and it is also a pseudogradient field for f on M.

Palais and Smale originally phrased condition (C) in terms of the gradient of f.

We now proceed to define the notion of nondegenerate critical point. Let E be a Banach space. A continuous symmetric bilinear form $B: E \times E \to R$ is said to be nondegenerate if the induced map $B_\#: E \to E^*$ (E^* the dual space of E) given by $B_\#(u) = B(u, \cdot)$ is an isomorphism of E with E^*; otherwise B is said to be degenerate. A critical point p of f is said to be nondegenerate if the Hessian $H_p(f): T_pM \times T_pM \to R$ of f at p defined by $H_p(f)(u, v) = d^2f_p(u, v)$ is a nondegenerate bilinear form. Unfortunately this notion of nondegeneracy requires that E be isomorphic to E^* which rarely occurs in practice. For example the Sobolev space L_k^p is isomorphic to $(L_k^p)^* = L_k^q$ if and only if $p = q = 2$.

By the index of a bilinear form B we mean the dimension of the maximal subspace on which it is negative definite. Recall that B is negative on a subspace E_0 if $(B(u, u) < 0$ for all $u \in E_0$, $u \neq 0$, and is negative definite if $B(u, u) \leq -c\,\|u\|^2$, $c > 0$ some constant. The index of B may be infinite. Also a maximal subspace on which B is negative may not be unique, but its dimension is unique.

We may then define the index of a nondegenerate critical point p of f to be the index of $H_p(f)$, the Hessian of f at p. The following is the basic result of the Morse theory on Riemannian manifolds as developed by Palais and Smale.

Proposition 7. *Let $f: M \to R$ ($\partial M = \emptyset$) be C^{k+3}, $k \geq 0$ satisfy condition* (C) *and have only nondegenerate critical points.*

(i) *For any closed interval $[a, b] \subset \boldsymbol{R}$ there are only finitely many critical points of f in $f^{-1}[a, b]$.*

(ii) *Suppose f^{-1}(a) and f^{-1}(b) contain no critical points. Let $p_1, \cdots, p_n$ be the critical points of f in $f^{-1}[a, b]$ of index $k_1, \cdots, k_n$ respectively ($k_i = \infty$ is possible). Then $M^b = \{x \mid f(x) \leq b\}$ has the homotopy type of M^a with n cells of dimensions $k_1, \cdots, k_n$ attached.* (*Palais actually showed that M^b has the diffeomorphism type of M^a with n-handles attached.*)

(iii) *In* (ii) *if $p_1, \cdots, p_m$, $m \leq n$ are of infinite index, then M^b has the homotopy type of M^a with $n - m$ handles attached each of dimensions $k_{m+1}, \cdots, k_n$.* (*The critical points of infinite index are homotopically invisible.*)

From (i) and (ii) it is possible to prove a version of the classical Morse inequalities (see [9], [10]).

Morse theory on Hilbert manifolds has been applied by Palais [10] to give an intrinsic development of the existence theory of geodesics on finite dimensional closed Riemannian manifolds, by Gromoll and Meyer [5], [6] to the existence of infinitely many distinct periodic geodesics, by Palais [14], [12] and Smale [14], [16] to a nonlinear generalization of the Dirichlet problem, and finally by Uhlenbeck [24] and Eliasson [2] to the existence of harmonic mappings.

Up to the present time the principle stumbling block to the development of a Morse theory on Banach manifolds has been a proper definition of nondegenerate critical point in the Banach space setting. The Hilbert space definition does not work because it implies that the model space E is isomorphic to its adjoint space E^*. This is one of the factors which led Palais to speculate that the natural setting for Morse theory was Hilbert manifolds.

This prompted Smale in 1968 to conjecture that weak nondegeneracy might be the answer. By weak nondegeneracy he meant that the Hessian $B = H_p(f)$ induces only an injective map $B_\#: E \to E^*$. It is not hard to see that such a definition of nondegeneracy does not work; in fact, weakly nondegenerate critical points need not be isolated. For example let $M = l_2$ be a seperable Hilbert space. Each $x \in l_2$ is an infinite sequence $\{x_i\}$ with $\sum x_i^2 < \infty$.

Define $f: H \to R$ by $f(x) = -\sum_i (\cos ix_i)/i^4$. Then f is C^2 and $0 \in H$ is a critical point for f. Moreover $H_0(f)(u, v) = \sum_{i=1}^{\infty} u_i v_i / i^2$ and so 0 is weakly nondegenerate. But it is clear that any neighborhood of 0 has infinitely many critical points.

Also crucial to the Palais version of the Morse theory was the Palais-Morse lemma (see [10], [13]) which says that if $f: \mathcal{O} \to R$ is C^3, $\mathcal{O} \subset H$ open, $p \in \mathcal{O}$ a nondegenerate critical point, then there is a change of variables $\phi: \mathcal{U} \to \mathcal{O}$, $\mathcal{U}$ a neighborhood of p, so that

$$f\phi(q) = \tfrac{1}{2}H_p(f)(q, q) + f(p);$$

that is, f could be "linearized" in a neighborhood of its nondegenerate critical point.

In particular the Morse lemma explicitly shows that nondegenerate critical points must be isolated. When the author first considered the problem of generalizing the Morse theory to Banach manifolds he attempted to find a definition of nondegeneracy in Banach spaces which would give a Morse lemma. He succeeded in doing this (e.g., see [17], [18]). Unfortunately his definition of nondegeneracy was not intrinsic, and to make matters worse a Morse lemma in the Banach space category is incompatible with condition (C) in the case that E is not isomorphic to E^*, and E reflexive. Recently the author found a nondegeneracy condition which was intrinsic and implied a Morse lemma [21].

To see that condition (C) is incompatible with the Morse lemma in the case $E \not\cong E^*$, with E reflexive suppose $f: E \to R$ is already in linearized form $f(x) = \frac{1}{2}B(x, x)$ where $B: E \times E \to R$ is continuous bilinear and symmetric and $B_\#: E \to E^*$ is injective. Then $df_x(h) = B(x, h)$, the range of $B_\#$ is dense in E^*, and $\|df_x\| = \|B_\#(x)\|$. Since $B_\#$ is not invertible there exists a sequence $x_n \in E$, $\|x_n\| = 1$ with $\|B_\#(x_n)\| \to 0$. Since B is continuous, $\{f(x_n)\}$ is a bounded sequence and moreover $\|df_{x_n}\| \to 0$. But 0 is the only critical point of f, and therefore there cannot be a critical point in $\bar{S}$, $S = U_n x_n$, which contradicts condition (C).

Now in our quest for a Banach manifold Morse theory we find ourselves at a fork in the road. It seems that we can either find an alternate version of concition (C) and an alternate intrinsic notion of nondegeneracy which gives us a Morse lemma or clutch onto condition (C) and find a nondegeneracy condition which is strong enough for a Morse theory yet to weak to imply a Morse lemma. We shall do neither.

We shall change our point of view somewhat and develop a theory which we believe is general enough to include these two directions. That is to say we shall in § 6 give examples where one of the above approaches will work and the other will not; yet our theory will work in both cases (e.g., see the concluding remarks of this paper).

Our point of view will be to consider real valued maps $f: M \to R$, M a complete Finsler manifold, along with an associated "globally defined" vector field X on M satisfying certain compatibility conditions with f. As a special case we will obtain a Morse theory for maps f satisfying condition (C) and having nondegenerate critical points in a new sense.

In § 6 we study some examples to see how the theory applies to variational problems. Other applications will be published in separate papers. This paper was partly motivated by the author's work on the index theory of vector fields on Banach manifolds [20].

2. Nondegenerate critical points

In the remainder of the paper we shall assume that M is at least a C^2 paracompact Banach manifold without boundary modelled on a real Banach space E with an equivalent C^1 norm and hence M admits C^1 partitions of unity.

By a C^1 norm $\| \quad \|$, we mean that $\| \quad \|: E - \{0\} \to R$ is C^1 $\{C^1$ away from $0\}$. We shall assume that the Frechét derivative $\| \quad \|_*: \{E - \{0\}\} \to \mathscr{L}(E, R)$, where $\mathscr{L}(E, R)$ are the continuous linear maps from E to R, is bounded in a neighborhood of 0. That is there are a neighborhood W of 0 and a constant $\bar{N}$ so that $\|(\|q\|_*)\| < \bar{N}$ for all $q \in W - \{0\}$.

This certainly holds for the Sobolev spaces L_k^{2m}, $m \geq 1$.

Definition. Let $f: M \to R$ be C^2. A critical point $p \in M$ is said to be B-nondegenerate if there exist a neighborhood $\mathcal{O}$ of p and a C^1 vector field $V: \mathcal{O} \to TM \mid \mathcal{O}$ with

(i) $V_q(f) = df_q(V(q)) > 0$ for $q \in \mathcal{O}$, $q \neq p$,

(ii) $V(p) = 0$ and $\mathscr{D}V_p: T_pM \to T_pM$, the Frechét derivative of V at p, is symmetric with respect to the Hessian $H_p(f)$, i.e.,

$$H_p(f)(\mathscr{D}V_p(u), v) = H_p(f)(u, \mathscr{D}V_p(v))$$

for all $u, v \in T_pM$,

(iii) $\mathscr{D}V_p: T_pM \to T_pM$ is an isomorphism with spectrum off the imaginary axis,

(iv) $H_p(f)(\mathscr{D}V_p(u), u) > 0$ if $u \neq 0$.

Remark 1. Since $V: M \to TM$, $\mathcal{D}V_p: T_pM \to T_{v(p)}(TM)$. However in the case where p is a zero for V we can interpret $\mathcal{D}V_p$ as a linear map of T_pM into itself.

Remark 2. For the purpose of Morse theory it may be possible that condition (ii) can be weakened.

The following two results are immediate consequences of the above definition.

Theorem 1. *B-nondegenerate critical points are isolated.*

Theorem 2. *B-nondegeneracy is intrinsic.*

Theorem 3. *Suppose $f: M \to R$ is C^2 with a M Riemannian Hilbert manifold. If $p \in M$ is a nondegenerate critical point, then p is B-nondegenerate.*

Proof. Let $V(q) = \nabla f(q)$. Then

$$df q(\nabla f(q)) = \|\nabla f(q)\|_q^2 = \langle \nabla f(q), \nabla f(q)\rangle_q.$$

Consequently (i) is satisfied and $\nabla f(p) = 0$. For notational convenience let us denote the Frechét derivative of ∇f at p by $\nabla f_*(p): T_pM \to T_pM$. From the definition of the gradient it follows that for $u, v \in T_pM$

$$H_p(f)(u, v) = d^2 f_p(u, v) = \langle \nabla f_*(p)u, v\rangle_p .$$

The symmetry of the Hessian guarantees the symmetry of $\nabla f_*(p)$ as an operator on T_pM. Therefore from standard Hilbert space theory we can conclude that $\nabla f_*(p)$ has only real spectrum. The nondegeneracy condition implies that $\nabla f_*(p)$ is an isomorphism. Thus 0 is not in the spectrum, and the spectrum is disjoint from the imaginary axis.

In addition

$$H_p(f)(\nabla f_*(p)u, v) = \langle \nabla f_*(p)u, \nabla f_*(p)v\rangle_p ,$$

whence $H_p(f)(\nabla f_*(p)u, u) = \|\nabla f_*(p)u\|_p^2 > 0$ if $u \neq 0$. Thus nondegenerate points are B-nondegenerate.

To see that B-nondegenerate points are not in general nondegenerate *in the sense that the Hessian induces an isomorphism between T_pM and T_pM^**, consider the following example:

Let $M = L^4[0, 1] = E$, $J: M \to R$ given by $J(g) = \frac{1}{4}\int_0^1 |g|^4 + \frac{1}{2}\int_0^1 |g|^2$. One easily checks that J satisfies condition (C). The only critical point for J is $g \equiv 0$, and

$$H_0(J)(u, v) = \int_0^1 uv = B_\#(u)(v) ,$$

$B_\#: E \to E^*$. Now $E^* = T_0M^* \cong L^{4/3}[0, 1]$ where $\cong$ denotes isometric isomorphism. Making the identification of E^* with $L^{4/3}[0, 1]$ we see that $B_\#(u) = u$ or $B_\#$ is the natural inclusion of L^4 into $L^{4/3}$. This clearly cannot be an isomorphism and so 0 is not nondegenerate. On the other hand define the vector field $V(g) = g$. It is immediate that V satisfies conditions (i)–(iv). Consequently B-nondegeneracy is weaker than nondegeneracy.

In § 5 we shall study how such vector fields arise in variational problems.

The reason we required $\nabla f_*(p): T_pM \to T_pM$ to have spectrum disjoint from the imaginary axis was so we could apply the following fundamental fact.

Lemma 1. *Let $A: E \to E$ be a linear endomorphism of a Banach space E with spectrum disjoint from the imaginary axis. Then the space E is the direct sum of two subspaces $E_- \oplus E_+$ both invariant under A and with the property that $A_- = A \mid E_-$ has spectrum to the left of the imaginary axis and $A = A \mid E_+$ has spectrum to the right of the imaginary axis. E_+ and E_- are called the positive and negative invariant subspaces of A.*

In addition there exist projection operators $P_+: E \to E_+$, $P_-: E \to E_-$ with $P_+^2 = P_+$, $P_-^2 = P_-$, $P_-P_+ = P_+P_- = 0$, $P_+ + P_- = I$, and moreover P_+ and P_- are expressible as a limit of power series in A.

Proof. The proof is essentially contained in [15, p. 421–423] after one passes to the complexification of E, $E \otimes C$, and the complexification of A.

Using Lemma 1 it is now easy to give a characterization of the index of a B-nondegenerate critical point. Recall that in the last section we defined the index of a nondegenerate critial point to be the dimension of the maximal subspace in which the Hessian is negative definite.

Theorem 4. *Let $f: M \to R$ with $p \in M$ a B-nondegenerate critical point of f. Let V denote the associated local vector field and set $A = \mathscr{D}V_p$. Then $A: T_pM \to T_pM$, and the index of f at p is the dimension of the space T_pM_-. Therefore p is of finite index if and only if* $\dim T_pM_- < \infty$.

Proof. Straightforward.

3. The general setting for Morse theory on Banach manifolds

In our approach to abstract variational calculus we switch emphasis away from the real valued map $f: M \to R$ (for which we are trying to describe the relation between the critical points and the geometry of certain level sets) to an associated vector field X. In the case where M is a Riemannian manifold, such a "nice" associated vector field X will exist (by nice we mean that its zeros will be precisely the critical points of f, and $df_{(p)}(X(p)) \geq 0$), namely the gradient of f. In the case where M is a Banach manifold, there is no Riemannian structure and hence apparently no "natural" way to produce such an associated vector field. In [21] the author introduced the notion of "almost-Riemannian" structure on a Banach manifold. Such structures generally exist on Sobolev manifolds of mappings. For such manifolds there is a nice "gradient" defined. It is the authors' belief that in most variational problems which arise in practice there is a natural globally defined nice vector field associated to the variational mapping of $f: M \to R$. We shall not attempt to justify this statement here nor attempt even to give a full justification in this paper. Examples are given in § 6 and [20].

We shall start by giving a definition paralleling condition (C) for smooth vector fields $X: M \to TM$. As in the rest of this paper M is a complete C^2 paracompact Finsler manifold without boundary modelled on a real Banach space E with an equivalent C^1 norm.

Definition. A set $S \subset M$ is bounded if $\sup_{p,q \in S} \rho(p, q) < \infty$ where ρ is the distance function induced by the Finsler on M (see § 1).

Definition. A C^1 vector field $X: M \to TM$ satisfies condition (CV) if whenever $\{p_i\}$ is a bounded sequence in M and $\|X(p_i)\| \to 0$ then there is a subsequence $\{p_{i_j}\}$ which converges.

We have an immediate consequence of this definition, namely,

Proposition 1. *Let X be a vector field on M satisfying condition (CV), and $S \subset M$ any bounded set. Then, if* zer (X) *denotes the zeros of X, we have that* zer $(X) \cap \bar{S}$ *is a compact set. Hence, if the zeros of X in any closed set C are isolated, then C contains at most finitely many of these zeros.*

We wish now to define what it means for a vector field to behave like a gradient with respect to some scalar function. Let $t \to \sigma_p(t)$ denote the trajectory of X with initial condition p. Further let $f: M \to R$ be a C^2 function.

Definition. We say that a C^1 vector field X is *gradient like* for f if

(G0) X satisfies (CV),

(G1) $X_p(f) = df_p(X_p) \geq 0$ and equals zero only if p is simultaneously a critical point of f and a zero of X.

This condition implies that f increases along the trajectories of X.

(G2) Let $p \in M$. The trajectory σ_p of X through p has a maximal domain $(\alpha, \beta) \subset \boldsymbol{R}$. Then as $t \to \beta$ either

(i) $f(\sigma_p(t)) \to +\infty$ or

(ii) $\|X(\sigma_p(t))\| \to 0$ and $\sigma_p[0, \beta)$ is bounded.

Similarly as $t \to \alpha$ either

(iii) $f(\sigma_p(t)) \to -\infty$ or

(iv) $\|X(\sigma_p(t))\| \to 0$ and $\sigma_p(\alpha, 0]$ is bounded.

(G3) (Regularity condition). Let $K(a, b)$ denote the zeros of X in $f^{-1}[a, b]$, $-\infty < a \leq b < \infty$. Then $K(a, b)$ is bounded. From condition (G0) and Proposition 1 it follows that $K(a, b)$ is also compact.

The following proposition is crucial to the development of Morse theory.

Proposition 2. *In axiom* (G2) *if, as $t \to \beta$, $\|X(\sigma_p(t))\| \to 0$ and $\sigma_p[0, \beta)$ is bounded, then $\beta = +\infty$ and $\sigma_p(t)$ has a critical point as a limit point as $t \to \infty$.*

Similarly if, as $t \to \alpha$, $\|X(\sigma_p(t))\| \to 0$ and $\sigma_p(\alpha, 0]$ is bounded, then $\alpha = -\infty$ and $\sigma_p(t)$ has a critical point as a limit point as $t \to -\infty$.

Proof. Condition (G0) implies that if, as $t \to \beta$, $\|X(\sigma_p(t))\| \to 0$ with $\sigma_p[0, \beta)$ bounded, then $\sigma_p(t)$ has a limit point in M as $t \to \beta$. By Proposition 5 of § 1 this is impossible unless $\beta = \infty$. Since $\|X(\sigma_p(t)\| \to 0$ as $t \to \beta$, this limit point must be a zero of X and hence a critical point of f.

The proof for $t \to \alpha$ is exactly the same.

Remark. Of course not every real valued smooth map has a gradient like vector field (e.g., set $f =$ constant). In § 5 we shall state formally that if f satisfies condition (C), is bounded below, bounded on bounded sets, and has B-nondegenerate critical points in the sense of § 2, then there exists a gradient like vector field for f.

Proposition 3. *Let $f: M \to R$, and X be gradient like for f. Let $b = f(p)$, and $\sigma: (\alpha, \beta) \to M$ be a maximal integral curve of X with initial condition p. Suppose $\lim_{t \to \alpha} f(\sigma_p(t)) = a > -\infty$. By the last proposition $\alpha = -\infty$. Then as $t \to -\infty$, $\sigma_p(t)$ converges to $K(a, b)$. Similarly if $\lim_{t \to \beta} f(\sigma_p(t)) = c < \infty$, then $\beta = \infty$, and as $t \to \infty$, $\sigma_p(t)$ converges to $K(c, b)$.*

Proof (by contradiction). Suppose that $\sigma_p(t) \not\to K(a, b)$ as $t \to -\infty$. Then there are a neighborhood $\mathscr{U}$ of $K(a, b)$ and a sequence of $t_n \to -\infty$ with $\sigma_p(t_n) \notin \mathscr{U}$. Since $\|X(\sigma_p(t_n))\| \to 0$ and $\sigma_p(\infty, 0]$ is bounded, condition (CV) implies that there is a subsequence $\sigma_p(t_{n_j})$ which converges to a point in $K(a, b)$, a contradiction. The case for $t \to \beta$ follows exactly as above.

Corollary 1. *Let f, X, p, a, b, c be as above. If $a > -\infty$ and $K(a, b)$ are isolated points (and hence finite many), $\sigma_p(t)$ converges to a critical point $q \in K(a, b)$ as $t \to -\infty$. Similarly if $c < \infty$, $\sigma_p(t)$ converges to a critical point $q \in K(c, b)$.*

Proof. Obvious.

Corollary 2. *Suppose $q \in f^{-1}(a, b)$ is the only critical point of f in $f^{-1}[a, b]$. Let $p \in f^{-1}[a, b]$ be arbitrary. If $\sigma_p: (\alpha, \beta) \to M$ is the maximal integral curve of X with initial condition p, then either $\sigma_p(t)$ converges to q as $t \to \alpha$ or $\sigma_p(t)$ drops below the level $f^{-1}(a)$; i.e., there exists a $t_0 > \alpha$ so that for all $t \leq t_0$, $f(\sigma_p(t)) \leq a$.*

Proof. By Proposition 3 either $\lim_{t \to \alpha} f(\sigma_p(t)) = -\infty$ or else $\alpha = -\infty$ and $\sigma_p(t)$ has a critical point as a limit point as $t \to \infty$. If the former we are clearly done. If the latter then $\alpha = -\infty$ and either q is a limit point of $\sigma_p(t)$ as $t \to -\infty$ or it is not. If not then, since q is the only critical point of f in $f^{-1}[a, b]$, $\sigma_p(t)$ must drop below the level surface $f^{-1}(a)$ after time t_0 and hence for all time $t \leq t_0$. If q is a limit point of $\sigma_p(t)$, then $f(q) = a_1 = \lim_{t \to -\infty} f(\sigma_p(t))$. Applying Corollary 1 finishes the proof.

Corollary 3. *Suppose $K(a, b) = \emptyset$. Again let $p \in f^{-1}[a, b]$ be arbitrary. If $\sigma_p: (\alpha, \beta) \to M$ is the maximal integral curve of X through p, then after some finite time $\sigma_p(t)$ drops below the level $f^{-1}(a)$.*

The following theorem permits us to deform a manifold M along a gradient like vector field X. It is one of the two basic results used in the handle body decomposition theorem in the next section.

Theorem 1. *Let $M^b = \{x \in M \mid f(x) \leq b\}$ with M^a defined analogously. If*

$K(a, b) = \emptyset$, then M^a is homotopically equivalent to M^b.

Proof. Condition (G1) and the assumptions of the theorem guarantee that $df(p)(X(p)) > 0$ for all $p \in f^{-1}[a, b]$. Thus the vector field X is transverse to the level surfaces $f^{-1}(c)$, $c \in [a, b]$. From Corollary 3 it follows that for each $p \in M^b$ there is a first time $\gamma(p)$ so that $\sigma_p(\gamma(p)) \in M^a$. The transversality of X to the level surfaces of f insures that $p \to \gamma(p)$ is continuous (in fact smooth if f and X are smooth).

Define $H: I \times M^b \to M^b$, I the unit interval by $H(t, p) = \sigma_p(t\gamma(p))$. H is the desired homotopy equivalence.

In the next section we shall again study a pair (f, X) where $f: M \to R$ is a C^2 real valued map, M a C^2 paracompact Banach manifold without boundary, and X a gradient like vector field. It is for these pairs that we shall complete the development of the Morse theory of critical points.

Before we conclude this section we shall give the definition of nondegenerate critical point for the pair (f, X).

Definition. Let $f: M \to R$ be C^2 with X a C^1 gradient like vector field for f. A critical point p of f is B-nondegenerate with respect to X if

(a) $DX(p): T_pM \to T_pM$, the Frechet derivative of X at p is symmetric with respect to the Hessian $H_p(f)$,

(b) $DX(p)$ is an isomorphism with spectrum of the imaginary axis,

(c) $H_p(f)(DX(p)u, u) > 0$ if $u \neq 0$.

Compare these with (i)–(iv) of the first part of § 2.

4. The handle-body theorem

The major part of Morse theory is the analysis of the behavior of the trajectories of a vector field in the neighborhood of a critical point. In order to study this we shall need a sequence of results the first of which is due to Karen Uhlenbeck [22].

Proposition 1. *Let $A: E \to E$ be a linear isomorphism with real spectrum and with E_+ and E_- the positive and negative invariant subspaces of E. Then there exist a norm $|\ \ |$ for E and a $\rho > 0$ such that for $v = v_+ + v_-$*

(i) $|v_+ + v_-| = |v_+| + |v_-|$,

(ii) $|e^{tA}v_+| > (1 + \rho t)|v_+|$ *for all* $t > 0$,

(iii) $|e^{tA}v_-| > (1 - \rho t)|v_-|$ *for all* $t < 0$.

Moreover the norm $|\ \ |$ has the same differentiability properties as the given norm for E.

Proof. Since $E = E_+ \oplus E_-$ once we have defined $|\ \ |$ on E_+ and E_-, we can define $|v_+ + v_-| = |v_+| + |v_-|$. We define $|\ \ |$ only on E_+. e^A is expanding on E_+ so for any norm $\|\ \ \|$ on E_+ there exist an $\varepsilon > 0$ and a $k > 1$ so that $\|e^{NA}v_+\| > \varepsilon k^N \|v_+\|$ for all $v_+ \in E_+$ and all integers N. Choose N large enough so that $\varepsilon k^N > 1$ and then define

$$|v_+| = \int_0^N \|e^{\lambda A}v_+\|\, d\lambda \,.$$

This is a norm on E_+ with the same smoothness properties as $\| \quad \|$ on E_+. Now

$$|e^{tA}v_+| = \int_0^N \|e^{(\lambda+t)A}v_+\|\, d\lambda \,.$$

Making a change of variables we find this is equal to

$$\begin{aligned}
&\int_t^{N+t} \|e^{\lambda A}v_+\|\, d\lambda \\
&\quad = \int_0^N \|e^{\lambda A}v_+\|\, d\lambda + \int_N^{t+N} \|e^{\lambda A}v_+\|\, d\lambda - \int_0^t \|e^{\lambda A}v_+\|\, d\lambda \\
(4) \qquad &\quad = |v_+| + \int_0^t \|e^{(\lambda+N)A}v_+\|\, d\lambda - \int_0^t \|e^{\lambda A}v_+\|\, d\lambda \\
&\quad = |v_+| + \int_0^t \{\|e^{NA}(e^{\lambda A}v_+)\| - \|e^{\lambda A}v_+\|\}\, d\lambda \\
&\quad \geq |v_+| + (\varepsilon k^N - 1)\int_0^t \|e^{\lambda A}v_+\|\, d\lambda \,.
\end{aligned}$$

Again since e^A is expanding on E_+, $\inf_{t\geq 0}\|e^{tA}v_+\| \geq \varepsilon' \|v_+\|$ for all $v_+ \in E_+$. But $|v_+| = \int_0^N \|e^{\lambda A}v_+\|$ is an equivalent norm for E_+, and so

$$\inf_{t\geq 0}\|e^{tA}v_+\| \geq \varepsilon'' |v_+| = \varepsilon'' \int_0^N \|e^{\lambda A}v_+\|\, d\lambda \,.$$

Consider the functions

$$t \xrightarrow{g_1} \varepsilon'' t \int_0^N \|e^{\lambda A}v_+\|\, d\lambda \,, \qquad t \xrightarrow{g_2} \int_0^t \|e^{\lambda A}v_+\|\, d\lambda \,.$$

Both take the value 0 at 0. Moreover

$$g_1'(t) \leq g_2'(t) = \|e^{tA}v_+\| \,,$$

which implies that $g_1(t) \leq g_2(t)$ for all t, or

$$\varepsilon'' t \int_0^N \|e^{\lambda A}v_+\|\, d\lambda \leq \int_0^t \|e^{\lambda A}v_+\|\, d\lambda \,.$$

Putting this into (4) we find that

$$|e^{tA}v_+| \geq |v_+| + (\varepsilon k^N - 1)\varepsilon'' t \int_0^N \|e^{\lambda A}v_+\|\, d\lambda = |v_+| + \rho t |v_+| \,,$$

where $\rho = (\varepsilon k^N - 1)\varepsilon''$. A similar argument works for (iii). We shall call the norm $|\quad|$ the *norm induced by* A.

Continuing we have

Proposition 2. *Let* $f: M \to R$ *be* C^2 *with* $p \in M$ *a B-nondegenerate critical point. Let* V *be the associated local vector field about* p *and let* $A = DV_p: T_pM \to T_pM$. *For ease of exposition identify* E *with* T_pM. *Let* $E = E_+ \oplus E_-$ *be the decomposition of* E *induced by* A *with projections* P_+ *and* P_- *onto* E_+ *and* E_- *respectively* (*see Lemma* 1, § 2) *and* $H_p(f): E \times E \to R$ *the Hessian of* f *at* p. *Then* $H_p(f)$ *is positive on* E_+ *and negative on* E_-; *that is,* $H_p(f)(u, u) > 0$ *if* $u \in E_+$, $u \neq 0$, *and* $H_p(f)(u, u) < 0$ *if* $u \in E_-$, $u \neq 0$. *Moreover, if* $\dim E_- < \infty$, *then* $H_p(f)$ *is negative definite on* E_- *which means that there is a positive constant* $v > 0$ *with* $H_p(f)(u, u) \leq -v\,\|u\|^2$ *for all* $u \in E_-$.

Proof. Since the spectrums of $-A_-$ and A_+ are both entirely to the right of the imaginary axis we can, using the functional calculus (e.g., see [15]) define square roots S_- and S_+ to $-A_-$ and A_+ which are expressable in power series in A_- and A_+. Since A_+ and A_- are symmetric with respect to $H_p(f)$ so will S_- and S_+. Consequently $S_-^2 = -A_-$, $S_+^2 = A_+$ and S_- and S_+ are isomorphisms of E_- to E_- and E_+ to E_+. If $u \in E_-$, then for some v

$$\begin{aligned} H_p(f)(u, u) &= H_p(f)(S_-v, S_-v) = H_p(f)(S_-^2 v, v) \\ &= H_p(f)(-A_-v, v) = -H_p(f)(A_-v, v) < 0\,. \end{aligned}$$

Similarly we get that $H_p(f)$ is positive on E_+. If $\dim E_- < \infty$, any negative form on a E_- will be negative definite.

Proposition 3. *Suppose* $p \in M$ *is a critical point of finite index, and B-nondegenerate. Let* $E_-^\perp$ *be the* $H_p(f)$ *orthogonal complement of* E_-. *So* $E_-^\perp = \{v \,|\, H_p(f)(u, v) = 0$ *for all* $u \in E_-\}$. *Then* $E_+ = E_-^\perp$.

Proof. First let us show that $E = E_- \oplus E_-^\perp$. On E_- define the bilinear form $Q(u, v) = -H_p(f)(u, v)$. Since $\dim E_- < \infty$, by the last theorem there is a $\nu > 0$ with $Q(u, u) \geq \nu\,\|u\|^2$ for all $u \in E_-$. Consequently Q gives a Riemannian structure to E_-. Let $w \in E$ be arbitrary. Then w induces a linear functional on E_- by the rule $w_\#(u) = -H_p(f)(u, w)$. The Riesz representation theorem says that there must be a unique $u_0 \in E_-$ with $w_\#(u) = Q(u, u_0)$. Therefore

$$Q(u, u_0) = -H_p(f)(u, u_0) = -H_p(f)(w, u)$$

for all $u \in E_-$, or $H_p(f)(w - u_0, u) = 0$ for all $u \in E_-$. Thus $w - u_0 \in E_-^\perp$, $u_0 \in E_-$, and $w = (w - u_0) + u_0$ which shows that $E = E_- \oplus E_-^\perp$. We also know that $E = E_- \oplus E_+$ so that to show that $E_+ = E_-^\perp$ it suffices to show that $E_+ \subset E_-^\perp$, and then the finite dimensionality of E_- will imply the result.

By Lemma 1 of § 2 the projection operator $P_-: E \to E_-$ associated to A_- is the limit of a sequence of power series in A and therefore symmetric with respect to $H_p(f)$. Let $v \in E_+$ and $u \in E_-$. Then

$$H_p(f)(u, v) = H_p(f)(P_-u, v) = H_p(f)(u, P_-v) = 0 ,$$

since $P_-v = P_-P_+v = 0$. Thus $E_+ \subset E_-^{\perp}$ and Proposition 3 is established.

Now back to a local result. Let U be a coordinate neighborhood of the B-nondegenerate critical point p. Identify this again with an open neighborhood of 0 in E. Give E the C^1 norm $|\quad|$ of Proposition 1 of § 4. Let $|v|_* \in \mathscr{L}(E, R)$ denote the Frechét derivative of $|\quad|$ at v. So for $h \in E$, $|v|_*(h) \in R$.

Proposition 4. *There is a $\rho > 0$ so that:*

(i) *if $v_- \in E_-$, then*

$$|v_-|_* (Av_-) \leq -\rho |v_-| ,$$

or A is negative definite on E_-,

(ii) *if $v_+ \in E_+$, then*

$$|v_+|_* (Av_+) \geq \rho |v_+| ,$$

or A is positive definite on E_+,

(iii) $|v|_* (Av) = |v_+|_* (Av_+) + |v_-|_* (Av_-)$.

Proof. We shall prove only (ii) and (iii). Recall from Proposition 1 that there is a $\rho > 0$ with $|e^{tA}v_+| \geq (1 + \rho t)|v_+|$ for all $t \geq 0$. Thus

$$\frac{1}{t}(|e^{tA}v_+| - |v_+|) \geq \rho |v_+| .$$

Since $|\quad|$ is C^1, the limit on the left exists as $t \to 0$ and equals $|v_+|_* (Av_+)$ by the chain rule. This shows (i). To demonstrate (iii) assume we have (i) and (ii). Then

$$|v| = |v_+| + |v_-| , \qquad |e^{tA}v| = |e^{tA}v_+| + |e^{tA}v_-| ,$$

so

$$\frac{1}{t}(|e^{tA}v| - |v|) = \frac{1}{t}(|e^{tA}v_+| - |v_+|) + \frac{1}{t}(|e^{tA}v_-| - |v_-|) .$$

Taking the limit as $t \to 0$ yields (iii). We are now ready to prove the major step in the handle-body decomposition theorem.

Theorem 1. *Let $f: M \to R$ be C^2 on a complete C^2 Finsler manifold M with X a gradient like vector field for f. Let $p \in f^{-1}(a, b)$, $f(p) = \lambda$ be the only critical point of f in $M_a^b = f^{-1}[a, b]$ with p B-nondegenerate with respect to X and of finite index. Then there exists a differentiable embedding $\psi: D_\eta \times D_\xi$ of radius η and ξ onto a neighborhood of p such that*

(i) $\psi(0, 0) = p$, $\dim D_\eta =$ *index of f at p,*

(ii) *X is transverse to $D_\eta \times \partial D_\xi$ (we write $X \pitchfork D_\eta \times \partial D_\xi$),*

(iii) *there is some $\varepsilon > 0$ with $f(\partial D_\eta \times D_\xi) \leq \lambda - \varepsilon$, $f^{-1}(\lambda - \varepsilon)$ transverse to $D_\eta \times \{0\}$ and also transverse to X.*

Proof. Let U be a neighborhood of p identified as usual via a coordinate mapping $\phi: U \to E$ ($\phi(p) = 0$) with an open subset U of E. Let S_R, $S_{R/2}$ be the balls of radius R and $R/2$ about 0 in E, where E is again assumed to have the norm $|\quad|$ of Proposition 1. Then $E = E_+ \oplus E_-$.

Let D_η be the disc of radius η on E_- with center 0, and let D_ξ be the disc of radius ξ in E_+ with center 0. We shall eventually pick ξ and η small enough so that $D_\eta \times D_\xi \subset S^\circ_{R/2}$. Set $\xi = \mu\eta$ where μ is also to be picked to guarantee (ii) and (iii). Once we have picked the appropriate η and ξ, (i) will be automatically satisfied since we just take the embedding ψ to be ϕ^{-1} restricted to $D_\eta \times D_\xi$.

The proof involves keeping track of lots of constants. Let us start listing them. Since $|\quad|_*: U \to \mathscr{L}(E, R)$ is locally bounded about $0 \in E$, it follows that if m is small enough, then there is a constant $\bar{N}$ with

$$\||q|_*\| \leq \bar{N} \qquad \text{for all } q \in S_m - \{0\} ,$$

S_m the ball of radius m about 0.

From Proposition 2 there is a $1 > \nu > 0$ with $H_p(f)(u, u) \leq -\nu |u|^2$ for all $u \in E_-$. Since $H_p(f): E \times E \to R$ is continuous, there is a constant $K_1 > 1$ with $|H_p(f)(u, v)| \leq K_1 |u| \cdot |v|$ for all $u, v \in E$.

Since X is C^1, we can write

$$X(q) = A(q) + R(q) ,$$

where $|R(q)| \leq w(q)|q|$ with $w(q) \to 0$ as $q \to 0$. Also $q \to df_q$ is C^1 so locally around p ($p = 0$) we have

$$df_q = d^2f_p(q, \cdot) + R_1(q) ,$$

where $|R_1(q)| \leq w_1(q)|q|$, $w_1(q) \to 0$ as $q \to 0$.

In addition since f is C^2 we have from Taylors' formula

$$f(q) = H_p(f)(q, q) + R_2(q) + \lambda ,$$

where $\lambda = f(p)$ and $|R_2(q)| \leq w_2(q)|q|^2$ with $w_2(q) \to 0$ as $|q| \to 0$.

Finally from Proposition 4 there is a $\rho > 0$ so that $|v_-|_* (Av_-) \leq -\rho |v_-|$ and $|v_+|_* (Av_+) \geq \rho |v_+|$. Choose $S_m \subset S_{R/2}$ to be a ball about p with $0 < m < \frac{1}{2}$ small enough to insure that for $q \in S_m$

$$w_2(q) < \tfrac{1}{8}\nu , \tag{5}$$

$$w_1(q) < \tfrac{1}{2}\nu , \tag{6}$$

$$w(q) < \min\left(\frac{\rho}{8\bar{N}\|P_+\|}, \frac{\rho\mu}{8\bar{N}\|P_+\|}\right) , \tag{7}$$

where $P_+ : E \to E_+$ is the projection into E_+ of Lemma 1, § 2.

Pick ξ and η to be fixed numbers less than $\frac{1}{2}m$ and with $\xi = \mu\eta$ where μ is some number

$$\mu < \sqrt{\tfrac{1}{8}\nu/K_1} < \sqrt{\tfrac{1}{8}\nu} < 1/\sqrt{8}\,.$$

Then $D_\xi \times D_\xi \subset S^\circ_{m/2}$.

Let us begin now by considering (ii), and show X is transverse to $D_\eta \times \partial D_\xi$. Let $(q_-, q_+) \in D_\eta \times \partial D_\xi$. The tangent space to $D_\eta \times \partial D_\xi$ at (q_+, q_-) is $(\text{Kernel}\,|q_+|_*) \times E_-$. Writing X in terms of components we have $X(q) = X(q)_+ + X(q)_-$. To show that $X \pitchfork D_\eta \times \partial D_\xi$ it therefore suffices to show that $|q_+|_* X(q)_+ > 0$. But

$$|q_+|_* X(q)_+ = |q_+|_* (Aq_+) + |q_+|_* [P_+(R(q))]\ .$$

The first term on the left is $\geq \rho|q_+| = \rho\xi$ and second is $\leq \bar{N}\|P_+\|\cdot w(q)\cdot|q| = w(q)\bar{N}\|P_+\|\{|q_-| + |q_+|\} \leq w(q)\bar{N}\|P_+\|\{\xi/\mu + \xi\}$ and from choice (7) of $w(q)$ this is bounded by $\frac{1}{4}\rho\xi$.

Consequently

$$|q_+|_* V(q)_+ \geq \frac{\rho\xi}{4} > 0\ ,$$

if $(q_-, q_+) \in D_\eta \times \partial D_\xi$. This shows (ii).

(iii) has to be done in three parts.

Part 1. There is some positive $\varepsilon > 0$ with $f(\partial D_\xi \times D_\xi) \leq \lambda - \varepsilon$. From Taylors's theorem we have for q close to p

$$\begin{aligned} f(q) &= H_p(f)(q, q) + R_2(q) + \lambda \\ &= H_p(f)(q_-, q_-) + H_p(f)(q_+, q_+) + R_2(q) + \lambda \\ &\leq -\nu|q_-|^2 + K_1|q_+|^2 + w_2(q)(|q_-| + |q_+|)^2 + \lambda \\ &\leq -\nu\eta^2 + K_1\xi^2 + w_2(q)\{\eta^2 + 2\xi\eta + \xi^2\} + \lambda \\ &= -\nu\eta^2 + K_1\mu^2\eta^2 + w_2(q)\{1 + 2\mu + \mu^2\}\eta^2 + \lambda \end{aligned}$$

From the choice of μ it follows that this is bounded by

$$\leq -\nu\eta^2 + \tfrac{1}{8}\nu\eta^2 + w_2(q)\{1 + \tfrac{1}{8}\nu + 2\sqrt{\tfrac{1}{8}\nu}\}\eta^2 + \lambda$$

and since $w_2(q) \leq \frac{1}{8}\nu$ and $\nu < 1$ we have this

$$\leq -\nu\eta^2 + \tfrac{1}{8}\nu\eta^2 + \tfrac{1}{2}\nu\eta^2 + \lambda \leq -\tfrac{1}{4}\nu\eta^2 + \lambda\ .$$

Setting $\varepsilon = \frac{1}{4}\nu\eta^2$ finishes part 1 of Case (iii).

Part 2. We must show that $f^{-1}(\lambda - \varepsilon)$ is transverse to $D_\eta \times \{0\}$. Let $q \in$

$D_\eta \times \{0\}$. Then q_- is in the tangent space to $D_\eta \times \{0\}$ at q. If we can show that $df_q(q_-) \neq 0$, then, since the codimension of the tangent space to $f^{-1}(\lambda - \varepsilon)$ at q is one, we will have shown that $f^{-1}(\lambda - \varepsilon)$ is transverse to $D_\eta \times \{0\}$. Again since f is C^2,

$$\begin{aligned} df_q(q_-) &= H_p(f)(q_-, q_-) + R_1(q)(q_-) \\ &\leq -\nu|q_-|^2 + w_1(q)|q_-|\{|q|\} \\ &= -\nu|q_-|^2 + w_1(q)|q_-|^2 \leq -\tfrac{1}{2}\nu|q_-|^2 . \end{aligned}$$

This concludes part 2.

Part 3 of (iii) is trivial. The fact that $f^{-1}(\lambda - \varepsilon)$ is transverse to X follows immediately from the fact that for $q \in M_a^b$, $q \neq p$, $df_q(X(q)) > 0$. This concludes the proof of Theorem 1.

We are now prepared to prove the main theorem of this paper and the principle result of the Morse theory on Banach manifolds.

Theorem 2 (*Morse handle-decomposition theorem*). *Let $f: M \to R$ be a C^2 function with X a gradient like vector field for f where M is a complete Finsler manifold modelled on a Banach space E with an equivalent C^1 norm with locally bounded differential about* 0. *Suppose that f has a single B-nondegenerate critical point $p \in f^{-1}[a, b] = M_a^b$ of finite index k with $a < f(p) < b$. Then $M^b = f^{-1}(-\infty, b]$ has the homotopy type of M^a with a cell of dimension k attached.*

Proof. Let $D_\eta \times D_\xi$ be the embedded disc product given by Theorem 1. First from Theorem 1 of § 3 it follows that if $f(p) = \lambda$ then for all $\varepsilon > 0$ sufficiently small $M^{\lambda-\varepsilon}$ has the homotopy type of M^a. We shall show that for the $\varepsilon > 0$ given by Theorem 1, $M^{\lambda-\varepsilon} \cup (D_\eta \times D_\xi)$ has the homotopy type of M^b.

Let $\sigma_q: (\alpha, \beta) \to M$ be a maximal integral curve for the vector field X with initial condition $q \in M_{\lambda-\varepsilon}^b$. By Corollary 2 of Proposition 3 of § 3 as $t \to \alpha$, $\sigma_q(t)$ either converges to the critical point q or drops below the level $f^{-1}(\lambda - \varepsilon)$ after some finite time. Thus after some finite time $\sigma_q(t)$ must enter $M^{\lambda-\varepsilon} \cup (D_\eta \times D_\xi)$. Define the map $H_t: M^b \to M^b$ by $H_t(q) = \sigma_q(t\gamma(q))$ where $\gamma(q)$ is the first time that $\sigma_q(t) \in M^{\lambda-\varepsilon} \cup (D_\eta \times D_\xi)$. The transversality conditions (ii) and (iii) of Theorem 1 guarantee that γ and hence H are continuous. Thus M^b has the homotopy type of M^a with a handle $D_\eta \times D_\xi$ attached. But the fact that $f^{-1}(\lambda - \varepsilon)$ is transverse to $D_\eta \times \{0\}$ coupled with the fact that $\dim D_\eta = k < \infty$ implies that we can actually force $M^{\lambda-\varepsilon} \cup D_\eta \times \{0\}$ to be a deformation retract of $M^{\lambda-\varepsilon} \cup (D_\eta \times D_\xi)$ (of course this might involve choosing a somewhat smaller ξ and η than in Theorem 1). So composing all deformations we get that M^a has the homotropy type of M^b with a cell D_η of dimension k attached.

Remark 1. An easy modification of Theorem 2 shows that if there are n B-nondegenerate critical points $\{p_i\}$, $1 \leq i \leq n$, each of index k_i in $f^{-1}(a, b)$, then M^b has the homotopy type of M^a with n-cells $\{e_i\}$, $1 \leq i \leq n$, $\dim e_i = k_i$, attached.

Remark 2. If $f: M \to R$ has a gradient like vector field X and has only B-nondegenerate critical points, then there only a finite number of critical points in M_a^b. This follows immediately from Proposition 1 and axioms (G0) and (G3) of § 3, since B-nondegenerate critical points are isolated (cf. Theorem 1, § 2).

Theorem 2 also implies that we have the Morse inequalities for C^2 functions f having gradient like vector fields and B-nondegenerate critical points. The proof of the Morse inequalities in this context is exactly the same as in [10]; however for completeness we shall state them without proof.

First we give a few definitions. Let Q denote the rational field, and H_* the singular homology functor. A pair of spaces X and Y is called admissable if $H_*(X, Y)$ is of finite type, that is to say that $\dim H_k(X, Y) < \infty$ for all k and $H_k(X, Y) = 0$ if k is sufficiently large. If (X, Y) is admissable, the Euler characteristic $\chi(X, Y)$ of the pair (X, Y) is defined by

$$\chi(X, Y) = \sum_{i=0}^{\infty} (-1)^i \dim H_i(X, Y) + \sum_{i=0}^{\infty} (-1)^i R_i ,$$

where $R_i = \dim H_i(X, Y)$ is the ith Betti number. Then we have the following.

Theorem 11 (*Morse inequalities*). *Let M be a complete C^2 Finsler manifold modelled on a space E as above, $f: M \to R$ a C^2 function having a gradient like vector field and all of whose critical points are B-nondegenerate. Let a and b be noncritical values of f ($f^{-1}(a) \cup f^{-1}(b)$ contains no critical points). Then the pair (M^b, M^a) is admissable. If C_m denotes the number of critical points of index m in M_a^b (by Remark 2 above there are only finitely many), then*

$$R_0 \le C_0 , \qquad R_1 - R_0 \le C_1 - C_0 ,$$

$$\sum_{m=0}^{k} (-1)^{k-m} R_m \le \sum_{m=0}^{k} (-1)^{k-m} C_m ,$$

$$\chi(M^a, M^b) = \sum_{i=0}^{\infty} (-1)^i R_i = \sum_{i=0}^{\infty} (-1)^i C_i ,$$

and $R_i \le C_i$ for all i.

We conclude this section with the following important result.

Theorem 12. *Let f be a C^2 function on M which has a gradient like vector field X. Then f always assumes its infinum on any component of M, on which the infinum is greater than $-\infty$.*

Proof. Let M_0 be some component of M with $B = \inf_{x \in M_0} f$. For every positive $\varepsilon > 0$ we can find a $y \in M_0$ with $B \le f(y) < B + \varepsilon$. By following the trajectory of the gradient like vector field X through y for negative time we can find a critical point x with $B \le f(x) < B + \varepsilon$. Thus for every positive integer n we can find a critical point x_n with $B \le f(x_n) < B + 1/n$. Consequently,

$X(x_n) = 0$ and $f(x_n)$ converges to B. By (G3), $\{x_n\}$ has a subsequence converging to $z \in M_0$. Clearly $f(z) = B$ and the theorem is proved.

A central question about our theory is whether it applies to a large category of examples. Certainly it applies to complete Hilbert manifolds since the Riemannian metric provides us with a suitable vector field in the neighborhood of a zero, and nondegenerate implies B-nondegenerate. It is our contention that in most of the geometric variational problems one encounters such vector fields always exist, and moreover that they arise naturally from the variational problems themselves.

It is this fact that motivates the study of Fredholm vector fields on Banach manifolds in [20]. In the next section we give simple examples in the spirit of Palais' papers on Lusternick-Schnirelman theory and Morse theory showing how the theory applies.

5. The theory of Palais and Smale and condition (*C*)

Again M is a C^2 complete Finsler manifold, $\partial M = \emptyset$, which is modelled on a space E which has a C^1 norm.

In this section we state several theorems and propositions, but in the interest of brevity we shall omit the proofs of the theorems since the proof of Theorem 1 is especially long and technical.

Theorem 1. *Let $f: M \to R$ be a C^1 map satisfying condition (C), which is bounded below and bounded on bounded sets and has only B-nondegenerate critical points. Then there exists a gradient like vector field X for f.*

The following is quite easy to prove.

Theorem 2. *Let $f: M \to R$ be a C^2 map, satisfying condition (C) with M a sufficiently smooth Riemannian (Hilbert) manifold. Then $\nabla f: M \to TM$, the gradient of f with respect to the given Riemannian structure on TM, is gradient like for f.*

The next result will be useful in § 6.

Proposition 1. *Let $f: M \to R$ be bounded below (above) and satisfy (sequential) condition (C). Then the inverse image of bounded sets is bounded.*

Proof. Let $a < \inf_{x \in M} f(x)$ and $b > \inf_{x \in M} f(x)$ be arbitrary. It suffices to show that $f^{-1}[a, b]$ is bounded. Let $K(a, b)$ be the critical points of f in $f^{-1}[a, b]$. It follows from condition (*C*) that $K(a, b)$ is compact. Consequently there is a neighborhood N of $K(a, b)$ with diameter smaller than some $R > 0$. Let Y be a pseudo-gradient vector field for f in $M^* = M -$ (crit set f). For $p \in f^{-1}[a, b] \cap M^*$ let $\sigma_p: (\alpha, \beta) \to M$ denote the maximal integral curve of Y. Palais shows (Theorem 5.4 [12]) that as $t \to \alpha$ either $\sigma_p(t)$ drops below the level $f^{-1}(a)$ or else $\alpha = -\infty$, and $\sigma_p(t)$ has a critical point as a limit point as $t \to \alpha$. Since $f^{-1}(a) = \emptyset$, $\sigma_p(t)$ must have a critical point as a limit point as $t \to \alpha = -\infty$. Therefore for all $p \in f^{-1}[a, b] \cap M^*$ there exists a greatest $t(p) > -\infty$ with the property that $\sigma_p(t(p)) \in \bar{N}$. One can show, using condition (*C*), that

$\inf_{p \in f^{-1}[a,b]} t(p) > -\infty$ but this will not be necessary in the proof of this proposition. Note that on $f^{-}[a, b] - N$ there exists a $d > 0$ with $\|Y(p)\| \geq d$ for all $p \in f^{-1}[a, b] - N$. If this were not the case, we could find a sequence $p_n \in f^{-1}[a, b] - N$ with $\|Y(p_n)\| \to 0$ and so $\|df_{p_n}\| \to 0$. By condition (C), $\{p_n\}$ would have a convergent subsequence $\{p_{nj}\}$ converging to some $q \in N$ which is a contradiction.

We shall show that the distance of any point $p \in f^{-1}[a, b] - N$ to N is bounded by $4(b - a)/d$.

$$b - a \geq f(\sigma_p(t)) - f(p) = \int_0^t df_{\sigma_p(s)}(\sigma_p'(s))ds$$
$$\geq \int_0^t \|df_{\sigma_p(s)}\|^2 ds \geq \frac{1}{4}\int_0^t \|Y(\sigma_p(s))\|^2 ds .$$

For all t with $\sigma_p(t) \in f^{-1}[a, b] - N$ we have that this integral

$$\geq \frac{d}{4}\int_0^t \|Y(\sigma_p(s)\| ds = \frac{d}{4}\int_0^t \|\sigma_p'(s)\| ds \geq \frac{d}{4}\rho(p, \sigma_p(t)) .$$

Therefore

$$\frac{d}{4}\rho(p, \sigma_p(t(p))) \leq b - a ,$$

which implies that the distance of any which point $p \in f^{-1}[a, b] - \bar{N}$ to $\bar{N}$ is bounded by $4(b - a)/d$. Since the diameter of N is bounded by R, we can conclude that the diameter of $f^{-1}[a, b]$ is at most $8(b - a)/d + R$ and so $f^{-1}[a, b]$ is bounded.

The case for f bounded above follows immediately by setting $g = -f$ and applying what we have already proved to g.

Remark. Proposition 1 is *not* true without the assumption that f is either bounded below or above. To see this let $M = R^2$ and $f(x, y) = x^2 - y^2$. Then f satisfies condition (C) but $f^{-1}(0)$ is clearly not bounded.

In the Palais-Smale theory no assumption is made about the function $f: M \to R$ being bounded on bounded sets. Although this occurs in all examples. we know of, we have no example at hand where condition (C) is satisfied for a function f, and f is not bounded on bounded sets.

In the last proposition of this section we give a condition on the differential of f which guarantees that f is in fact bounded on bounded sets.

Proposition 2. *Let $f: M \to R$ be a smooth (C^1) function where M is a C^1 connected Finsler manifold. Suppose $\|df_p\|$ is bounded on bounded sets of M. Then f is bounded on bounded sets.*

Proof. Let S be a bounded subset of M. Let $p_0 \in S$. Then every point $p \in S$ can be joined by a path $\sigma: I \to M$ to p_0 of length less than or equal to some constant K. Thus

$$f(p) - f(p_0) = f(\sigma(1)) - f(\sigma(0))$$
$$= \int_0^1 \frac{d}{dt} f(\sigma(t))dt = \int_0^1 df_{\sigma(t)}\sigma'(t)dt ,$$

and so

$$|f(p) - f(p_0)| \leq \int_0^1 \|df_{\sigma(t)}\| \, \|\sigma'(t)\| \, dt$$
$$\leq R \int_0^1 \|\sigma'(t)\| \, dt \leq RK ,$$

where R is the bound on the norm of the differential df. Therefore f is bounded on S.

6. A return to the geodesic problem

In this section we show how on a Hilbert manifold of maps one can pose an important variational problem for which condition (C) is violated yet the theory presented in the last sections applies. We begin by reviewing the geodesic problem as studied by Palais in [10] and shall follow, in part, the notation of his § 13, and the reader is referred to that paper.

Let I denote the unit interval, and $\boldsymbol{R}^n$ Euclidian n-space. By $H_0(I, \boldsymbol{R}^n)$ we mean the Hilbert space of square integrable maps from I to $\boldsymbol{R}^n$. $H_1(I, \boldsymbol{R}^n)$ is the Hilbert space of absolutely continuous maps $\sigma: I \to \boldsymbol{R}^n$ such that the derivative σ' of σ belongs to $H_0(I, \boldsymbol{R}^n)$. The inner product on $H_1(I, \boldsymbol{R}^n)$ is given by

$$\langle u, v\rangle_{H_1} = \int^1 \langle u(t), v(t)\rangle dt + \int_0^1 \langle u'(t), v'(t)\rangle dt .$$

Let $V \subset \boldsymbol{R}^n$ be a closed C^{k+4}, $k \geq 1$, Riemannian submanifold of $\boldsymbol{R}^n$, where we assume that the Riemannian structure on V comes from $\boldsymbol{R}^n$. Then the set of maps $\sigma \in H_1(I, R^n)$ with $\sigma(I) \subset V$ is a closed C^k submanifold on the Hilbert space $H_1(I, \boldsymbol{R}^n)$. If $P, Q \in V$, then the space of $\sigma \in H_1(I, \boldsymbol{R}^n)$ with $\sigma(I) \subset V$ and $\sigma(0) = P$ is also a C^k Hilbert manifold of $H_1(I, \boldsymbol{R}^n)$ which we denote by $\Omega(P)$. Similarly the space of $\sigma \in \Omega(P)$ with $\sigma(1) = Q$ is again a Hilbert submanifold of $\Omega(P)$ and consequently of $H_1(I, \boldsymbol{R}^n)$ which we denote by $\Omega(P, Q)$. In fact it can be shown (see [4]) that $\Omega(P)$ is diffeomorphic to a Hilbert space and $\Omega(P, Q) \subset \Omega(P)$ is a finite codimensional submanifold.

The tangent space $\Omega(P)_\sigma$ to $\Omega(P)$ at σ can be identified with the space of maps $h \in H_1(I, \boldsymbol{R}^n)$ with $h(0) = 0$ and $h(t) \in T_{\sigma(t)}V$. Similarly the tangent space $\Omega(P, Q)_\sigma$ to $\Omega(P, \Omega)$ at σ can be identified with the space of maps $h \in \Omega(P)_\sigma$ with $h(1) = 0$.

The Riemannian structure on $\boldsymbol{R}^n$ (and hence on V) naturally induces a

Riemannian (and hence Finsler) structure on $\Omega(P)$ and $\Omega(P, Q)$ as follows.

If $h, k \in \Omega(P)_\sigma$, define

$$\langle h, k\rangle_\sigma = \int_0^1 \left\langle \frac{Dh}{\partial t}, \frac{Dk}{\partial t} \right\rangle_{R^n} dt ,$$

where $Dh/\partial t, Dk/\partial t$ are the covariant (covariant with respect to the unique symmetric affine connection induced by the Riemannian structure on V) derivatives of h and k along σ.

Since $V \subset R^n$, there exists a smooth map $\mathscr{P}: V \to \mathscr{L}(R^n)$, the linear maps from R^n to itself, defined by $\mathscr{P}(x)$ is the orthogonal projection of R^n onto T_xV. One can show that the covariant derivative of a vector field h along σ is given by the formula

$$\frac{Dh}{\partial t} = \mathscr{P}(\sigma(t))h'(t) .$$

In [10] Palais used a different Riemannian structure on $\Omega(P, Q)$, namely he defined an "extrinsic" inner product $\langle\ ,\ \rangle_\epsilon$ on $T\Omega$ by

$$\langle h, k\rangle_{\epsilon,\sigma} = \int_0^1 \langle h', k'\rangle_{R^n} dt .$$

The next proposition, which shall be useful to us later on, shows that in one important sense there is little or no difference between these structures. Let us denote the first Riemannian structure by $\langle\ ,\ \rangle_{\mathscr{I}}$ and the two norms induced by these structures by $\|\ \ \|_\epsilon$ and $\|\ \ \|_{\mathscr{I}}$. In § 1 we saw how these norms induced metrics, say ρ_ϵ and $\rho_{\mathscr{I}}$ on $\Omega(P, Q)$.

Proposition 1. *The extrinsic and intrinsic Riemannian structures above are equivalent on bounded sets; i.e., if S is either a $\rho_{\mathscr{I}}$ or ρ_ϵ bounded set, then there exists a constant C (dependent only on the diameter of S) so that*

$$\frac{1}{C} \|v\|_{\epsilon,\sigma} \le \|v\|_{\mathscr{I},\sigma} \le C \|v\|_{\epsilon,\sigma}$$

for all $\sigma \in S$ and $v \in \Omega(P, Q)_\sigma$.

Proof. We shall show only that if S is a ρ_ϵ bounded set, then the two Riemannian structures are equivalent when restricted to S. Let $\sigma \in S$, and let V be a vector field along σ which vanishes at $t = 0$ and $t = 1$. Then

$$\frac{DV}{\partial t} = \mathscr{P}(\sigma(t))V'(t) ,$$

from which it follows that pointwise

$$\left\| \frac{DV}{\partial t} \right\|_{R^n} \leq \| V'(t) \|_{R_n} ,$$

or that $\| V \|_{\mathscr{I}} \leq \| V \|_{\epsilon}$. On the other hand

$$V'(t) = \frac{DV}{\partial t} + d\mathscr{P}_{\sigma(t)}(\sigma'(t))V(t) .$$

Consequently

$$\| V \|_{\epsilon} \leq \| V \|_{\mathscr{I}} + C_1 \| V \|_{C_0} \| \sigma' \|_{L_2} ,$$

where $\| \sigma' \|_{L_2} = \left\{ \int_0^1 \| \sigma'(t) \|_{R^n}^2 dt \right\}^{1/2}$, $\| V \|_{C_0} = \sup_t \| V(t) \|_{R^n}$,

and C_2 is a constant, which depends only on the C_0 norm of σ and hence only on the diameter of S. Since S is ρ_ϵ bounded, there is a constant C_2 with $\| \sigma' \|_{L_2} \leq C_2$ for all $\sigma \in S$. But

$$\frac{d}{dt} \| V(t) \|_{R^n}^2 = 2 \left\langle \frac{DV}{\partial t}, V \right\rangle_{R^n} .$$

This implies that

$$\| V(t) \|^2 \leq 2 \int_0^t \left\| \frac{DV}{\partial t} \right\| \| V(s) \| \, ds ,$$

or

$$\| V(t) \|^2 \leq 2 \| V \|_{C_0} \int_0^t \left\| \frac{DV}{st} \right\| ds ,$$

which by the Schwartz inequality implies that

$$\| V \|_{C_0}^2 \leq 2 \| V \|_{C_0} \| V \|_{\mathscr{I}} ,$$

or

$$\| V \|_{C_0} \leq 2 \| V \|_{\mathscr{I}} .$$

Putting things together we have that

$$\| V \|_{\mathscr{I}} \leq \| V \|_{\epsilon} \leq \| V \|_{\mathscr{I}} + 2C_1C_2 \| V \|_{\mathscr{I}} = (1 + 2C_1C_2) \| V \|_{\mathscr{I}} .$$

Setting $C = 1 + 2C_1C_2$ gives the desired result.

Proposition 2. *Define a real valued function* (*the energy function*) $J : \Omega(P, Q) \to R$ *by*

$$J(\sigma) = \frac{1}{2}\int_0^1 \|\sigma'(t)\|^2 \, dt \, .$$

J is clearly also defined on $\Omega(P)$.

In [10, § 14] it is proven that J is a C^k map satisfying condition (C). Moreover the critical points of J are the geodesics parameterized proportionally to arc length joining P and Q. These critical points are of the same differentiability class as V.

Let us study for a moment what the gradient of J looks like with respect to the Riemannian structure on $\Omega(P, Q)$ introduced above.

Let Ω^0_σ denote the Hilbert space of maps $h \in H_0(I, \boldsymbol{R}^n)$ with $h(t) \in T_{\sigma(t)}V$ for almost all t. If h is a vector field "along σ", then the map $h \to Dh/\partial t$ (covariant derivative of h along σ) establishes an isomorphism between $\Omega(P)_\sigma$ and Ω^0_σ.

Now lets compute the derivative of the energy functional. Let $h \in \Omega(P)_\sigma$. Then

$$dJ_\sigma(h) = \int_0^1 \langle \sigma'(t), h'(t)\rangle_{\boldsymbol{R}^n} dt = \int_0^1 \left\langle \sigma'(t), \frac{Dh}{\partial t}\right\rangle_{\boldsymbol{R}^n} dt \, .$$

Define $v(\sigma) \in \Omega(P)_\sigma$ to be the unique vector field along σ which solves the equation $Dv(\sigma)/\partial t = \sigma'$. Then

$$dJ_\sigma(h) = \int_0^1 \left\langle \frac{Dv(\sigma)}{\partial t}, \frac{Dh}{\partial t}\right\rangle dt \, .$$

This implies that the *gradient of the map J on $\Omega(P)$ is the vector field $\sigma \to v(\sigma)$*. Therefore the only critical points of J on $\Omega(P)$ are the zeros of V; but $v(\sigma) = 0$ if and only if σ is the constant map $\sigma(t) = P$ for all t.

We now turn our attention to the space $\Omega(P, Q)$. If $h \in \Omega(P, Q)$, then $h \to Dh/\partial t$ defines a map from $\Omega(P, Q)_\sigma \xrightarrow{D/\partial t} \Omega^0_\sigma$. The image F_σ of $\Omega(P, Q)_\sigma$ under the map $D/\partial t$ is a closed finite codimensional subspace of Ω^0_σ (in fact dim $\Omega(P)_\sigma/\Omega(P, Q)_\sigma = \dim V$). Let $\pi_\sigma\colon \Omega^0_\sigma \to F^\perp_\sigma$ be the H_0 orthogonal projection of Ω^0_σ onto the orthogonal complement of F_σ in Ω^0_σ (this is a different π_σ than used in [10, § 14]). Consider again $J\colon \Omega(P, Q) \to \boldsymbol{R}$. Then for $h \in \Omega(P, Q)_\sigma$

$$dJ_\sigma(h) = \int_0^1 \left\langle \frac{Dv(\sigma)}{\partial t}, \frac{Dh}{\partial t}\right\rangle dt = \int_0^1 \left\langle \frac{Dv(\sigma)}{\partial t} - \pi_\sigma \frac{Dv(\sigma)}{\partial t}, \frac{Dh}{\partial t}\right\rangle dt \, ,$$

since $\int_0^1 \left\langle \pi_\sigma \frac{Dv(\sigma)}{\partial t}, \frac{Dh}{\partial t}\right\rangle dt = 0$. But $\frac{Dv(\sigma)}{\partial t} - \pi_\sigma \frac{Dv(\sigma)}{\partial t} \in F_\sigma$ and thus there is a smooth vector field $\lambda(\sigma)$ along σ, $\lambda(\sigma \in \Omega(P, Q)_\sigma$ with

$$\frac{D\lambda(\sigma)}{\partial t} = \frac{Dv(\sigma)}{\partial t} - \pi_\sigma \frac{Dv(\sigma)}{\partial t} \, .$$

Therefore $dJ_\sigma(h) = \int_0^1 \left\langle \frac{D\lambda(\sigma)}{\partial t}, \frac{Dh}{\partial t} \right\rangle dt$ which implies that $\nabla J(\sigma) = \lambda(\sigma)$ and

$$\|\nabla J(\sigma)\|^2 = \int_0^1 \left\| \frac{D\lambda(\sigma)}{\partial t} \right\|_{R^n}^2 dt .$$

The vector field $\sigma \to \lambda(\sigma)$ is C^{k-1} and transverse to J, and its zeros are precisely the critical points of J.

Before moving on to another example we would like to give an alternate interpretation of $\lambda(\sigma)$ which is illuminating and very important in constructing other global vector fields transverse to a given functional. Suppose that σ and $\lambda(\sigma)$ were sufficiently smooth. Then we could integrate the following expressions for dJ_σ:

$$dJ_\sigma(h) = \int_0^1 \langle \sigma'(t), h'(t) \rangle dt , \qquad dJ_\sigma(h) = \int_0^1 \left\langle \frac{D\lambda(\sigma)}{\partial t}, \frac{Dh}{\partial t} \right\rangle dt$$

by parts to get that for all $h \in \Omega(P, Q)_\sigma$

$$-\int_0^1 \left\langle \frac{D\sigma'}{\partial t}, h \right\rangle dt = -\int_0^1 \left\langle \frac{D^2\lambda(\sigma)}{\partial t^2}, h \right\rangle , \quad \text{or} \quad \frac{D^2\lambda(\sigma)}{\partial t^2} = \frac{D\sigma'}{\partial t} .$$

Thus formally (in this case can be made precise with the selection of right spaces; e.g., $D\sigma'/\partial t \in L^2_{-1}(I, R^n)$) we should think of the gradient $\lambda(\sigma)$ as the solution to a second order *linear elliptic differential equation*. The important thing is that the equation is linear, so we can (given the boundary conditions $\lambda(\sigma)(0) = 0$, $\lambda(\sigma)(1) = 0$) uniquely solve this particular equation to give us the gradient.

Remark. It is shown in [10, § 13, Theorem 4] that the function

$$t \to \pi_\sigma(Dv/\partial)t(\sigma)$$

is absolutely continuous (our $\pi_\sigma Dv/\partial t$ is Palais's $P_\sigma h(\sigma)$) and therefore has a derivative almost everywhere which is in $L^1(I, R^n)$. We have from above that

$$\frac{D\lambda}{\partial t} = \frac{Dv}{\partial t} - \pi_\sigma \frac{Dv}{\partial t} = \sigma' - \pi_\sigma \sigma' ,$$

with $D^2\lambda/\partial t^2 = D\sigma'/\partial t$. Therefore $(D/\partial t)(\pi_\sigma Dv/\partial t) = 0$ and $\pi_\sigma Dv/\partial t$ is a parallel vector field along σ.

We would now like to duplicate the entire exposition above in a slightly different setting.

Let $L_2^2(I, R^n) = H_2(I, R^n) \subset H_1(I, R^n)$ be the Hilbert space of maps $\sigma: I \to R^n$ with $\sigma \in H_1$ and such that $\sigma': I \to R^n$ is absolutely continuous with $\sigma'' \in H_0(I, R^n)$.

The inner product on $H_2(I, R^n)$ is given by

$$\langle u, v\rangle_{H_2} = \int_0^1 \langle u(t), v(t)\rangle_{R^n} dt + \int_0^1 \langle u'(t), v'(t)\rangle_{R^n} dt + \int_0^1 \langle u''(t), v''(t)\rangle_{R^n} dt .$$

If we define $H_2(I, R^n)^0 = \{u \in H_2(I, R^n) \,|\, u(0) = u(1) = 0\}$, $H_2(I, R^n)^0$ is a closed subspace of $H_2(I, R^n)$ which admits an alternate equivalent inner product given by

$$\langle u, v\rangle = \int_0^1 \langle u'(t), v'(t)\rangle dt + \int_0^1 \langle u''(t), v''(t)\rangle dt .$$

Again let $V \subset R^n$ be a closed C^{k+4}, $k > 1$, Riemannian submanifold of R^n where the Riemannian structure on V is induced by that of R^n. The set of maps $\sigma \in H_2(I, R^n)$ with $\sigma(I) \subset V$, $\sigma(0) = P \in V$ is a closed C^k Hilbert submanifold of the Hilbert space $H_2(I, R^n)$ which we denote by $\Lambda(P)$. $\Lambda(P, Q)$ is definted similarly. $\Lambda(P)_\sigma$, the tangent space to $\Lambda(P)$ at σ, is $\{h \in H_2(I, R^n) \,|\, h(t) \in T_{\sigma(t)}V, h(0) = 0\}$ and $\Lambda(P, Q)_\sigma = \{h \in \Lambda(P)_\sigma \,|\, h(1) = 0\}$. Again the dimension of the quotient space $\dim \Lambda(P)_\sigma / \Lambda(P, Q)_\sigma = \dim V$. Let Λ_σ^1 denote those $u \in H_1(I, R^n)$ with $u(t) \in T_{\sigma(t)}V$, and let $h \in \Lambda(P)_\sigma$. Then $h \to Dh/\partial t$ defines an isomorphism between $\Lambda(P)_\sigma$ and Λ_σ^1.

The manifolds $\Lambda(P)$ and $\Lambda(P, Q)$ have natural intrinsic Riemannian (and hence Finsler) structures, given by

$$\langle h, k\rangle_\sigma = \int_0^1 \left\langle \frac{Dh}{\partial t}, \frac{Dk}{\partial t}\right\rangle_{R^n} dt + \int_0^1 \left\langle \frac{D^2h}{\partial t}, \frac{D^2k}{\partial t}\right\rangle dt$$

for $h, k \in \Lambda(P)_\sigma$ or $\Lambda(P, Q)_\sigma$. These Hilbert manifolds also admit extrinsic Riemannian structures given by

$$\langle h, k\rangle_{\epsilon,\sigma} = \int_0^1 \langle h', k'\rangle dt + \int_0^1 \langle h'', k''\rangle dt .$$

As before (cf. Prop. 1) we have

Proposition 3. *The intrinsic and extrinsic Riemannian structures on $\Lambda(P)$ and $\Lambda(P, Q)$ are equivalent on bounded sets.*

When we refer to the Riemannian manifold $\Lambda(P, Q)$ we shall always mean $\Lambda(P, Q)$ with its intrinsic Riemannian structure.

Define the energy functional $\hat{J}\colon \Lambda(P, Q) \to R$ by

$$\hat{J}(\sigma) = \frac{1}{2}\int_0^1 \|\sigma'(t)\|^2 dt .$$

The fact that the inclusion $i\colon \Lambda(P, Q) \to \Omega(P, Q)$ is C^∞ implies that $\hat{J} = J \circ i$ is a C^k smooth function. The critical points of $\hat{J}$ are again geodesics joining P and Q parameterized proportionally to arc length.

The functional $\hat{J}: \Lambda(P, Q) \to R$ does not satisfy condition (C). This follows directly from Proposition 1 of § 5, for if $\hat{J}$ satisfied condition (C) then the inverse image of bounded sets would be bounded. This would imply that every subset $S \subset \Lambda(P, Q)$ which is bounded in $\Omega(P, Q)$ is also bounded in $\Lambda(P, Q)$ which is clearly impossible. Although the following argument is not a complete proof it gives another indication of why $\hat{J}$ cannot satisfy condition (C).

We have already defined the linear space $H_2(I, \boldsymbol{R}^n)^0$. Let $H_1(I, \boldsymbol{R}^n)^0 = \{u \in H_1(I, \boldsymbol{R}^n) \mid u(0) = u(1) = 0\}$, and let $J: H_1(I, \boldsymbol{R}^n) \to R$ be given by $J(\sigma) = = \frac{1}{2}\int_0^1 \|\sigma'(t)\|_{\boldsymbol{R}^n}^2 dt$, and let $\hat{J} = J \mid H_2(I, \boldsymbol{R}^n)^0$. Now J satisfies condition (C) but $\hat{J}$ cannot. To see this note that

$$d\hat{J}_\sigma(h) = \int_0^1 \langle \sigma', h' \rangle dt ,$$

which implies that $\|d\hat{J}_\sigma\| \leq \|\sigma'\|_{L_2}$. But on $H_1(I, \boldsymbol{R}^n)^0$, $\sigma \to \|\sigma'\|_{L_2}$ is a norm equivalent to the H_1 norm. Consequently if $\hat{J}$ satisfied condition (C), then whenever $\sigma_n \to 0$ in H_1, σ_n would have a convergent subsequence which converged to 0 in H_2. This implies that the inclusion $i: H_2(I, \boldsymbol{R}^n)^0 \to H_1(I, \boldsymbol{R}^n)^0$ has closed range and since the range is dense it must be an isomorphism. This is clearly absurd. Therefore $\hat{J}: H_2(I, \boldsymbol{R}^n)^0 \to R$ does not satisfy condition (C). Using the Morse lemma as proved in [21] and the ideas just presented one can give another proof that $\hat{J}: \Lambda(P, Q) \to R$ does not satisfy condition (C).

However our immediate goal in this section is to produce a vector field λ which is gradient like for $\hat{J}: \Lambda(P, Q) \to R$. In fact no matter which Sobolev space H_k, $k > 1$, one chooses the energy functional restricted to H_k will always have a gradient like vector field. In fact the energy functional restricted to the Banach manifold path space $\Lambda_{L_k^p}(P, Q)$ of L_k^p maps σ, $1 \leq p < \infty$, $k \geq 2$, of the unit interval into V with $\sigma(0) = P$, $\sigma(1) = Q$ admits a gradient like vector field which for almost all P, Q would have nondegenerate zeros. In order to do Morse theory, *the choice of space does not matter*. But our purpose here is to give a simple exposition of our ideas and not to prove the most general theorem, and so we shall restrict our attention to H_2 maps.

Recall that $\Lambda(P, Q) \subset \Omega(P, Q)$. Let $\sigma \in \Lambda(P, Q)$, and let $\lambda(\sigma)$ be the vector field over σ with $(D\lambda/\partial t)(\sigma) = \sigma' - \pi_\sigma Dv/\partial t = \sigma' - \pi_\sigma \sigma'$ obtained earlier where $(D/\partial t)(\pi_\sigma \sigma') = 0$, and $\lambda \in H_1(I, \boldsymbol{R}^n)^0$. We claim that if $\sigma \in H_2(I, \boldsymbol{R}_n)$, then in fact $\lambda \in H_2(I, \boldsymbol{R}^n)^0$. This depends on the following lemmas.

Lemma 1. *Let $\sigma \in \Lambda(P, Q)$ with $\mu \in H_1(I, \boldsymbol{R}^n)$ a parallel vector field along $\sigma(D\mu/\partial t = 0)$. Then $\mu \in H_2(I, \boldsymbol{R}^n)$ with*

$$(8) \qquad \|\mu'\|_{H_1}^2 \leq const\, (\|\sigma'\|_{H_1}^2 + \|\sigma'\|_{C_0}^4 + \|\sigma'\|_{C_0}^2 \|\sigma'\|_{H_1}^2) \cdot \|\mu\|_{C_0}^2 ,$$

where $\| \quad \|_{C_0}$ denotes the supremum norm, and the constant depends only on the C_0 norm of σ.

Proof. $D\mu/\partial t = \mathscr{P}(\sigma(t))\mu'(t) = 0$, where $\mathscr{P}: V \to \mathscr{L}(\boldsymbol{R}^n)$ was the orthogonal projection map introduced earlier. Since $\mathscr{P}(\sigma(t))\mu(t) = \mu(t)$, we have that

$$\frac{D\mu}{\partial t} = \mu'(t) - d\mathscr{P}_{\sigma(t)}[\sigma'(t)]\mu(t) = 0 ,$$

or

$$\mu'(t) = d\mathscr{P}_{\sigma(t)}[\sigma'(t)]\mu(t) . \tag{9}$$

But the right hand side of (9) is clearly in $H_1(I, \boldsymbol{R}^n)$. Therefore $\mu \in H_2$. From (9) it also follows that

$$\|\mu'\|_{L_2} \leq K \|\sigma'\|_{L_2} \|\mu\|_{C_0} , \tag{10}$$

where K depends only on the C_0 norm of σ. But

$$\|\mu(t)\|^2 = \|\mu(0)\|^2 + \int_0^t \frac{d}{ds} \|\mu(s)\|^2 \, ds .$$

Since $D\mu/\partial t = 0$, the integral term vanishes and we have that $\|\mu(t)\|^2 = \|\mu(0)\|^2$, $\|\mu\|_{C_0} = \|\mu(0)\|$ and so $\|\mu'\|_{L_2} \leq K \|\sigma'\|_{L_2} \cdot \|\mu(0)\|$. Differentiating (9) again we get

$$\mu''(t) = d^2\mathscr{P}_{\sigma(t)}[\sigma'(t), \sigma'(t)]\mu(t) + d\mathscr{P}_{\sigma(t)}[\sigma''(t)]\mu(t) + d\mathscr{P}_{\sigma(t)}[\sigma'(t)]\mu'(t) .$$

Thus term by term

$$\|\mu''\|_{L_2}^2 \leq C_1 \|\sigma'\|_{C_0}^4 \|\mu\|_{C_0}^2 + C_2 \|\sigma''\|_{L_2}^2 \|\mu\|_{C_0}^2 + C_3 \|\sigma'\|_{C_0}^2 \|\mu'\|_{L_2}^2 ,$$

which applying inequality (10) gives inequality (8).

Lemma 2. *If $\sigma \in \Lambda(P, Q)$, then the function $t \to \pi_\sigma Dv/\partial t$ is in H_1 and thus by lemma 1 is in fact in H_2.*

Proof. In [10, § 14, Theorem 4] Palais showed that $(d/dt)(\pi_\sigma Dv/\partial t) = d\mathscr{P}_{\sigma(t)}[\sigma'(t)]h(\sigma)$ where $h(\sigma) \in L_2(I, \boldsymbol{R}^n)$, $\|h(\sigma)\|_{L_2} \leq \|\sigma'\|_{L_2}$ and $\mathscr{P}: V \to \mathscr{L}(\boldsymbol{R}^n)$ as before. It follows immediately from this formula that the derivatives of $\pi_\sigma Dv/\partial t$ is in L_2 or that $t \to \pi_\sigma Dv/\partial t \in H_1$.

Our candidate for a gradient like vector field for $\hat{J}$ is, of course, λ. Specifically we have

Lemma 3. *If $\sigma \in \Lambda(P, Q)$ the vector field $\lambda(\sigma)$ over σ defined by $D\lambda/\partial t = \sigma' - \pi_\sigma \sigma'$ is in $H_2(I, \boldsymbol{R}^n)^0$. Moreover $\sigma \to \lambda(\sigma)$ is a C^{k-1} vector field on the H_2 Hilbert manifold $\Lambda(P, Q)$.*

Proof. Since $t \to \pi_\sigma Dv/\partial t$ is in H_2 and $D\lambda/\partial t = \sigma' - \pi_\sigma Dv/\partial t$ (or $D^2\lambda/\partial t^2 = D\sigma'/\partial t$) it follows that $D\lambda/\partial t \in H_1$ or that $\lambda \in H_2$. But $\lambda \in H_1(I, \boldsymbol{R}^n)^0$ and so $\lambda \in H_2(I, \boldsymbol{R}^n)^0$. Now $\sigma \to D\sigma'/\partial t$ is a C^k map of $\Lambda(P, Q)$ to $L_2(I, \boldsymbol{R}^n)$. Fix σ, then $D^2\lambda/\partial t^2 = L_\sigma \lambda$, where L_σ is a linear isomorphism from the H_2^0 vector fields over σ to the H_0 or L_2 vector fields over σ. The map $\sigma \to L_\sigma$ is C^{k-1} (cf. [10, Theorem 7.513]) and therefore $\sigma \to L_\sigma^{-1}D\sigma'/\partial t = \lambda(\sigma)$ is C^{k-1}.

Remark. The fact that $\lambda \in H_2(I, \boldsymbol{R}^n)^0$ if $\sigma \in \Lambda(P, Q)$ also follows directly from the theory of elliptic differential equations since we can solve uniquely the equation

$$\frac{D^2\lambda}{\partial t^2} = \frac{D\sigma'}{\partial t} \quad \text{with } \lambda(0) = \lambda(1) = 0 \quad \text{and} \quad \frac{D\sigma'}{\partial t} \in L_2 \Rightarrow \lambda \in H_2(I, \boldsymbol{R}^n)^0 .$$

Theorem 1. *The C^{k-1} vector field $\lambda \colon \Lambda \to T\Lambda$ satisfies condition (CV) and hence axiom* (G0).

Proof. $(D\lambda/\partial t)(\sigma) - \sigma' = -\pi_\sigma Dv/\partial t$.
Suppose $\lambda(\sigma_n) \to 0$ in the Riemannian structure on $T\Lambda$ (i.e., $(D\lambda/\partial t)(\sigma_n)$ and $D^2\lambda(\sigma_n)/\partial t^2$ tend to 0 in $L_2(I, \boldsymbol{R}^n)$) where σ_n is a bounded sequence in $\Lambda(P, Q)$ and hence norm bounded in $H_2(I, \boldsymbol{R}^n)$, say by a constant R_0. $\mu_n = (D\lambda/\partial t)(\sigma_n) - \sigma_n' = -\pi_{\sigma_n} Dv_n/\partial t$ $(Dv_n/\partial t = \sigma_n')$ is an H_1 parallel vector field over σ_n.

From Lemma 2 it follows that $\pi_{\sigma_n} Dv_n/\partial t \in H_2(I, \boldsymbol{R}^n)$. Now

$$\left\|\pi_{\sigma_n}\frac{Dv_n}{\partial t}\right\|_{H_1}^2 = \int_0^1 \left\|\pi_{\sigma_n}\frac{Dv_n}{\partial t}\right\|_{\boldsymbol{R}^n}^2 dt + \int_0^1 \left\|\frac{d}{dt}\left(\pi_{\sigma_n}\frac{Dv_n}{\partial t}\right)\right\|^2 dt .$$

The first term on the right is $\leq \int_0^1 \|\sigma_n'\|^2\, dt$ since π_{σ_n} is an orthogonal projection. Recall (from Lemma 2) that

$$\frac{d}{dt}\left(\pi_{\sigma_n}\frac{Dv_n}{\partial t}\right) = d\mathscr{P}_{\sigma_n(t)}(\sigma_n'(t))h(\sigma_n) ,$$

so

$$\left\|\frac{d}{dt}\left(\pi_{\sigma_n}\frac{Dv_n}{\partial t}\right)\right\|_{L_2}^2 \leq C\, \|\sigma_n'\|_{C_0}\, \|h(\sigma_n)\|_{L_2} \leq \tilde{C}\, \|\sigma_n'\|_{C_0}\, \|\sigma_n'\|_{L_2}^2 .$$

Since σ_n' is bounded in H_2, there is some constant R_1 with $\|\sigma_n'\|_{C_0} \leq R_1$ for all n. Therefore $\|\mu_n\|_{H_1}^2 \leq \text{const}\,(R_0^2 + R_1^2R_0^2)$ or μ_n is bounded in H_1. μ_n is then also bounded in C_0 and so there is an R_2 with $\|\mu_n\|_{C_0} \leq R_2$.

Applying Lemma 1 to the μ_n it follows that this sequence is bounded in $H_2(I, \boldsymbol{R}^n)$. We are assuming that $\lambda(\sigma_n) \to 0$ with respect to the Riemannian structure of $\Lambda(P, Q)$. But on bounded sets (see § 3) this implies that $\lambda(\sigma_n) \to 0$ in $H_2(I, \boldsymbol{R}^n)$. Putting everything together we see that $(D\lambda/\partial t)(\sigma_n) - \sigma_n' = \mu_n$ is a bounded sequence in H_2 and therefore (since the inclusion of H_2 into H_1 is compact) has a convergent and hence Cauchy subsequence μ_{n_j} in H_1. But $D\lambda(\sigma_n)/\partial t \to 0$ in H_1, and so σ_{n_j}' is Cauchy in H_1. Thus σ_{n_j} is Cauchy in H_2 and therefore converges to some $\sigma_0 \in H_2(I, \boldsymbol{R}^n)$. Since $\Lambda(P, Q) \subset H_2(I, \boldsymbol{R}^n)$ is closed, $\sigma_0 \in \Lambda(P, Q)$. This verifies condition (CV) for λ.

We now proceed with the completion of the proof that the vector field λ is gradient like for the energy functional $\hat{J}: \Lambda(P, Q) \to R$.

Proposition 4. *λ satisfies axiom* (G1).

Proof.

$$d\hat{J}_\sigma(\lambda) = \int_0^1 \left\langle \sigma', \frac{D\lambda}{\partial t} \right\rangle dt = -\int_0^1 \left\langle \frac{D\sigma'}{\partial t}, \lambda \right\rangle dt$$
$$= -\int \left\langle \frac{D^2\lambda}{\partial t^2}, \lambda \right\rangle dt = \int_0^1 \left\langle \frac{D\lambda}{\partial t}, \frac{D\lambda}{\partial t} \right\rangle dt > 0$$

and equals zero if and only if $\lambda = 0$ or if and only if $D\sigma'/\partial t = 0$ and σ is a geodesic parameterized by arc length. Thus the zeros of λ are precisely the critical point of $\hat{J}$.

Proposition 5. *λ satisfies* (G2).

Proof. Since $D\lambda/\partial t = \sigma' - \pi_\sigma \sigma'$, it follows easily that $\|\lambda\|_{H_2}$ is bounded on bounded sets. Let $\sigma \in \Lambda(P, Q)$ with φ_σ the trajectory of λ with maximal domain $(\alpha, \beta) \subset \boldsymbol{R}$. We consider only the behavior of the trajectory $\varphi_\sigma(t)$ as $t \to \alpha$. The situation for $t \to \beta$ is analogous and we shall omit this case.

Since $\hat{J}$ is bounded below $\hat{J}(\varphi_\sigma(s)) \not\to -\infty$ as $s \to \alpha$. Consequently we must show that $\|\lambda(\varphi_\sigma(s))\| \to 0$ as $s \to \alpha$ and that $\varphi_\sigma(\alpha, 0]$ is bounded. Let $\hat{J}(\varphi_\sigma(0)) = \hat{J}(\sigma) = b$. Our first goal is to show

$$\alpha = -\infty . \tag{11}$$

Then we shall prove

$$\|\lambda(\varphi_\sigma(s))\| \to 0 \quad \text{as} \quad s \to -\infty , \tag{12}$$

and $\varphi_\sigma(-\infty, 0]$ is bounded which will conclude the proof of axiom (G2).

Lemma 1. *$\hat{J}(\sigma) - \hat{J}(\varphi_\sigma(s)) = \int_s^0 \|D\lambda(\varphi(s))/\partial t\|_{L_2}^2 \, ds$. Consequently if $\varphi(s)$ is defined for all negative time we have that (since $\hat{J} > 0$)*

$$\int_{-\infty}^0 \left\| \frac{D}{\partial t} \lambda(\varphi(s)) \right\|_{L_2}^2 ds < \infty , \qquad \hat{J}(\varphi(s)) \leq \hat{J}(\sigma)$$

for $s \leq 0$. From this we can further concluded that there is a sequence $s_i \to -\infty$ with $\|D\lambda(\varphi(s_i))/\partial t\|_{L_2} \to 0$.

Proof.
$$\hat{J}(\sigma) - \hat{J}(\varphi(s)) = \int_s^0 \frac{d}{ds} \hat{J}(\varphi(s)) ds$$
$$= \tfrac{1}{2} \int_s^0 \frac{d}{ds} \left\{ \int_0^1 \left\| \frac{d}{dt} \varphi(s) \right\|_{R^n}^2 dt \right\} ds$$
$$= \int_s^0 \left\{ \int_0^1 \left\| \frac{D}{\partial t} \lambda(\varphi(s)) \right\|_{R^n}^2 dt \right\} ds$$

$$= \int_s^0 \left\| \frac{D\lambda}{\partial t}(\varphi(s)) \right\|_{L_2}^2 ds \, .$$

Lemma 2. *Suppose $\alpha > -\infty$. Then*

$$\int_\alpha^0 \left\| \frac{D\lambda}{\partial t}(\varphi(s)) \right\|_{L_2} ds < \infty \, .$$

Proof. Apply the Schwartz inequality to the integral in Lemma 1.

Lemma 3. $\alpha = -\infty$.

Proof. By Lemma 2

$$\int_\alpha^0 \| \lambda(\varphi_\sigma(s)) \|_{H_1} ds = \int_\alpha^0 \left\| \frac{D\lambda}{\partial t}(\varphi(s)) \right\|_{L_2} ds < \infty \, .$$

Therefore $\int_\alpha^0 \left\| \frac{d}{ds}\varphi(s) \right\|_{H_1} ds < \infty$ which implies that the H_1 length of $\varphi_\sigma(\alpha, 0]$ is finite and thus converges to some point in $\Omega(P, Q)$ in the H_1 topology. By Proposition 5 of § 1 (recall λ is C^1 on $\Omega(P, Q)$) this is impossible. Thus $\alpha = -\infty$.

We now proceed to (12).

Lemma 4. *For each fixed s*

$$\| \lambda(\varphi(s)) \|_{C_0} \leq 2 \left\| \frac{D}{\partial t}\lambda(\varphi(s)) \right\|_{L_2} .$$

Proof. Let ν denote an H_1 vector field (over an H_1 path σ) which vanishes at 0. Then

$$\| \nu(t) \|_{R^n}^2 = \int_0^t \left\langle \frac{D}{\partial t}\nu, \nu \right\rangle dt \, ,$$

and applying the Schwartz inequality we have

$$\| \nu(t) \|_{R^n}^2 \leq \left\| \frac{D\nu}{\partial t} \right\|_{L_2} \| \nu \|_{C_0} \, .$$

Therefore $\| \nu \|_{C_0}^2 \leq \| D\nu/\partial t \|_{L_2} \| \nu \|_{C_0}$, and dividing by $\| \nu \|_{C_0}$ gives the result of the lemma for $\nu = \lambda$ over the path $\varphi(s)$.

Lemma 5. *If $\varphi(s)$ is defined for all negative time, then $\| (D/\partial t\lambda)(\varphi(s)) \|_{L_2} \to 0$ as $s \to -\infty$. By Lemma 4 we can also conclude that $\| \lambda(\varphi(s)) \|_{C_0} \to 0$ as $s \to -\infty$.*

Proof. We present here only a sketch of the proof of this lemma since all of the details are essentially in [10] and [12]. Since the functional $J: \Omega(P, Q) \to R$ satisfies condition (C) (this is proven in [10, § 14] and in fact our proof that λ on $\Lambda(P, Q)$ satisfies condition (CV) can be modified to give a proof of this fact) it follows from Lemma 1 immediately preceding, and condition (C) for J that we can find a sequence $s_i \to -\infty$ with $\|(D\lambda/\partial t)(\varphi(s_i))\|_{L_2} \to 0$ with $\varphi(s_i)$ converging to $K(a, b)$ where the convergence is in the H_1 topology on $\Lambda(P, Q) \subset \Omega(P, Q)$.

Condition (C) for J (condition (CV) for λ) further implies that $K(a, b)$ is compact in $\Omega(P, Q)$ ((CV) implies $K(a, b)$ is compact in $\Lambda(P, Q)$). Now Theorem 5.5 in [12] can be modified to show that in fact $\varphi(s)$ converges in the H_1 topology to $K(a, b)$ as $s \to -\infty$. Since λ vanishes on $K(a, b)$ and is continuous in the H_1 topology, we can conclude that $\|D\lambda(\varphi(s))/\partial t\|_{L_1} \to 0$.

Lemma 6. *Let ψ be a nonnegative C^1 function on an interval (s_1, s_2), $-\infty \le s_1 < s_2 < \infty$, satisfying*

$$\psi(s) + \gamma(s) \ge \frac{d\psi}{ds} \ge \psi(s) - \gamma(s) ,$$

where γ is positive and bounded. Then ψ is bounded on (s_1, s_2). If $s = -\infty$, and $\gamma(s) \to 0$ as $s \to -\infty$, then $\psi(s) \to 0$ as $s \to -\infty$.

Proof. Set $\xi = \sup_s |\gamma(s)|$, so that

$$\psi(s) + \xi \ge \frac{d\psi}{ds} \ge \psi(s) - \xi .$$

Consider the functions $g, \hat{g} : (s_1, s_2) \to R$ given by

$$g(s) = e^{-s}\{\psi(s) - \xi\} , \qquad \hat{g}(s) = e^{-s}\{\psi(s) + \xi\} ,$$

$$\frac{dg}{ds} = -e^{-s}\{\psi(s) - \xi\} + e^{-s}\left\{\frac{d\psi}{ds}\right\} \ge 0 ,$$

$$\frac{d\hat{g}}{ds} = -e^{-s}\{\psi(s) + \xi\} + e^{-s}\left\{\frac{d\psi}{ds}\right\} \le 0 .$$

Therefore g is increasing on (s_1, s_2), and $\hat{g}$ is positive and decreasing. Consequently if $s_0 \in (s_1, s_2)$, then $g(s) \le g(s_0)$ for all $s, s_1 < s \le s_0$, and $\hat{g}(s) \le \hat{g}(s_0)$ for all $s, s_0 \le s < s_2$. Using this latter inequality we set

$$(13) \qquad |\psi(s) + \xi| \le e^s \hat{g}(s_0) , \qquad s \ge s_0 .$$

The function $s \to g(s)$ decreases with decreasing time, and it may be negative at some point, but if it is negative at some value $s = s_*$, then it remains negative for all $s \le s_*$. This implies that if $\psi(s_*) \le \xi$, then $0 < \psi(s) \le \xi$ for all $s \le s_*$. Thus we can conclude that on $(s_1, s_0]$ either

$$(14) \qquad 0 \le \psi(s) \le \xi ,$$

or g is positive and decreasing with decreasing s, and so

$$|\psi(s) - \xi| \leq e^s g(s_0) . \tag{15}$$

Putting inequalities (13), (14) and (15) together we see that ψ is bounded on the finite interval (s_1, s_2). Suppose now that $\gamma(s) \to 0$ as $s \to -\infty$. Let $\varepsilon > 0$ be arbitrary. Pick $s_0 < 0$ small enough so that

$$\sup_{s \in (-\infty, s_0+1]} |\gamma(s)| = \xi \leq \tfrac{1}{2}\varepsilon .$$

Let $s_2 = s_0 + 1$. Applying inequalities (7) and (8) we see that for $s \leq s_0$ either

$$0 < \psi(s) \leq \tfrac{1}{2}\varepsilon , \tag{16}$$

or

$$0 < \psi(s) \leq \tfrac{1}{2}\varepsilon + e^{s-s_0}|\psi(s_0) - \xi| . \tag{17}$$

Pick $r_0 \leq s_0$ so that $e^{s-s_0}|\psi(s_0) - \xi| < \frac{1}{2}\varepsilon$ if $s \leq r_0$. Then for all $s \leq r_0$, $0 < \psi(s) < \varepsilon$ which shows that $\psi(s) \to 0$ as $s \to -\infty$ and the proof of the lemma is completed.

Lemma 7. *For all $s \leq 0$ the L_4 norm* of $(d\varphi/dt)(s)$, *the derivative of a trajectory $\varphi(s)$ of λ, is bounded by a constant which depends only on the value of the energy $\hat{J}(\varphi(0))$ at the initial point $\varphi(0)$ of the trajectory $\varphi(s)$.*

Proof. Let $h(s) = \int_0^1 \left\| \frac{d}{dt}\varphi(s) \right\|_{R^n}^4 dt = \left\| \frac{d}{dt}\varphi(s) \right\|_{L_4}^4$. Differentiating and using the definition of trajectory we get

$$\frac{dh}{ds} = 4 \int_0^L \left\langle \frac{D\lambda}{\partial t}(\varphi(s)), \frac{d\varphi}{dt}(s) \right\rangle_{R^n} \left\| \frac{d}{dt}(\varphi(s)) \right\|_{R^n}^2 dt . \tag{18}$$

Recall from the remark on p. 71 that for all σ

$$\frac{D}{\partial t}\lambda(\sigma) = \sigma' - \pi_\sigma \sigma' = \sigma' - l(\sigma) \tag{19}$$

where (see [10, Theorem 4, § 13]) $t \to l(\sigma)(t)$ is absolutely continuous with a derivative in $L^1(I, R^n)$. What is more important is that the L^1 norm of $(d/dt)l(\sigma)$ is bounded by a constant which depends (continuously) on the value of $\hat{J}(\sigma)$. This implies that $l(\sigma)$ is in fact continuous with supremum norm bounded by a constant which depends only on $\hat{J}(\sigma)$. Applying (19) to (18) we obtain

$$\frac{dh}{ds} = 4\int_0^1 \left\| \frac{d}{dt}\varphi(s) \right\|_{R^n}^4 dt + 4\int_0^1 \left\langle l(\varphi(s)), \frac{d}{dt}\varphi(s) \right\rangle_{R^n} \left\| \frac{d}{dt}\varphi(s) \right\|_{R^n}^2 dt . \tag{20}$$

But for each fixed s, $\|l(\varphi(s))\|_{C_0} \leq \gamma_0(s)$ where $\gamma_0(s)$ is a constant depending (continuously) on the value $\hat{J}(\varphi(s))$. Since $\hat{J}(\varphi(s))$ decreases as s decreases, $\hat{J}(\varphi(s)) \leq \hat{J}(\varphi(0))$ for all $s \leq 0$ which implies that $\|l(\varphi(s))\|_{C_0} \leq \gamma$, γ a positive constant, the magnitude of which depends only on the value $\hat{J}(\varphi(0))$. Applying the Schwartz inequality to equality (20) we find that for all $s \leq 0$

$$4h(s) - 4\gamma[h(s)]^{3/4} \leq \frac{dh}{ds} \leq 4h(s) + 4\gamma[h(s)]^{3/4} . \tag{21}$$

Set $\psi(s) = h(s)^{1/4}$. Then (21) yields

$$\psi(s) - \gamma \leq \frac{d\psi}{ds} \leq \varphi(s) + \gamma .$$

Applying Lemma 6 to ψ we see that ψ and hence h is bounded on $(-\infty, 0]$.

Lemma 8. *Let* $f(s) = \left\| \frac{D^2\lambda}{\partial t^2}(\varphi(s)) \right\|_{L_2}^2 = \int_0^1 \left\| \frac{D^2}{\partial t^2}\lambda(\varphi(s)) \right\|_{R^n}^2 dt$, *for* $s \in (\alpha, 0]$, *Then* $2f(s) + \gamma(s)\sqrt{f(s)} \geq (d/ds)f(s) \geq 2f(s) - \gamma(s)\sqrt{f(s)}$ *where* γ *is a bounded nonnegative function. If* $f(s)$ *is defined for all* $s \leq 0$, *then* $\gamma(s) \to 0$ *as* $s \to -\infty$.

Proof.
$$\frac{d}{ds}f(s) = \frac{d}{ds}\int_0^1 \left\langle \frac{D^2\lambda}{\partial t^2}(\varphi(s)), \frac{D^2\lambda}{\partial t^2}(\varphi(s)) \right\rangle_{R^n} dt$$
$$= \frac{d}{ds}\int_0^1 \left\langle \frac{D}{\partial t}\frac{d}{dt}\varphi(s), \frac{D}{\partial t}\frac{d}{dt}\varphi(s) \right\rangle_{R^n} dt$$
$$= 2\int_0^1 \left\langle \frac{D}{\partial s}\frac{D}{\partial t}\frac{d}{dt}\varphi(s), \frac{D}{\partial t}\frac{d}{dt}\varphi(s) \right\rangle_{R^n} dt ,$$

and using some differential geometry (e.g., see Milnor [9, p. 43]) we get this equal to

$$2\int_0^1 \left\langle \frac{D}{\partial t}\frac{D}{\partial t}\lambda(\varphi(s)), \frac{D}{\partial t}\frac{d}{dt}\varphi(s) \right\rangle_{R^n} dt$$
$$+ 2\int_0^1 \left\langle R\left(\frac{d}{dt}\varphi(s), \frac{d}{ds}\varphi(s)\right)\frac{d}{dt}\varphi(s), \frac{D}{\partial t}\frac{d}{dt}\varphi(s) \right\rangle dt ,$$

where R is the Riemann curvator tensor. Continuing we get this equal to

$$2\left\| \frac{D^2}{\partial t^2}\lambda(\varphi(s)) \right\|_{L_2}^2 + 2\int_0^1 \left\langle R\left(\frac{d}{dt}\varphi(s), \lambda(\varphi(s))\right)\frac{d}{dt}\varphi(s), \frac{D^2}{\partial t^2}\lambda(\varphi(s)) \right\rangle dt ,$$

$$\left(\text{recall that } \frac{D^2}{\partial t^2}\lambda(\varphi(s)) = \frac{D}{\partial t}\frac{d}{dt}\varphi(s)\right) .$$

Let u, v, w be vector fields along a path σ. Then

$$\|R(u(t), v(t))w(t)\|_{R^n} \leq C \|u(t)\| \|w(t)\| \cdot \|v(t)\| ,$$

where the constant C depends only on the supremum norm of σ. Using this and the Schwartz inequality we get that

$$2\left\|\frac{D^2\lambda}{\partial t^2}(\varphi(s))\right\|_{L_2}^2 + 2C\|\lambda(\varphi(s))\|_{C_0}\cdot\left\|\frac{d}{dt}\varphi(s)\right\|_{L_4}^2\cdot\left\|\frac{D^2\lambda}{\partial t^2}(\varphi(s))\right\|_{L_2} > \frac{d}{ds}f(s)$$

$$\geq 2\left\|\frac{D^2\lambda}{\partial t^2}(\varphi(s))\right\|_{L_2}^2 - 2C\|\lambda(\varphi(s))\|_{C_0}\cdot\left\|\frac{d}{dt}\varphi(s)\right\|_{L_4}^2\cdot\left\|\frac{D^2\lambda}{\partial t^2}(\varphi(s))\right\|_{L_2} .$$

Setting $\gamma(s) = C\|\lambda(\varphi(s))\|_{C_0}\cdot\|(d/dt)\varphi(s)\|_{L_4}^2$, and noting that (i) for all $s \leq 0$, $\|(d/dt)\varphi(s)\|_{L_4}^2$ is bounded by a constant which depends only on $\hat{J}(\varphi(0))$ (Lemma 7), (ii) $\lambda(\varphi(s))$ is bounded in H_1 norm since λ is H_1 bounded on H_1 bounded sets and $\hat{J}^{-1}(0, \hat{J}(\varphi(0)))$ is bounded in the H_1 topology on $\Lambda(P, Q)$, {for all $s \leq 0$, $\varphi(s) \in \hat{J}^{-1}(0, \hat{J}(\varphi(0)))$}, (iii) $\|\lambda(\varphi(s))\|_{C_0} \leq 2\|(D\lambda/\partial t)(\varphi(s))\|_{L_2}$ (Lemma 4), we can conclude that γ is bounded.

Applying Lemma 5 we see that if $\varphi(s)$ is defined for all $s \leq 0$ then $\gamma(s) \to 0$ as $s \to -\infty$. This completes Lemma 8.

Lemma 9. *Let φ: (α, β) be a maximal trajectory for λ. Then $s \to \|(D^2/\partial t^2)\lambda(\varphi(s))\|_{L_2}$ is bounded for $s \in (\alpha, 0]$. If $\alpha = -\infty$, then $\|(D^2/\partial t^2)\lambda(\varphi(s))\|_{L_2} \to 0$ as $s \to -\infty$.*

Proof. By Lemma 8, $f(s) = \|(D^2/\partial t^2)\lambda(\varphi(s))\|_{L_2}^2$ satisfies

$$2f(s) + 2\gamma(s)\sqrt{f(s)} \geq \frac{df}{ds} \geq 2f(s) - 2\gamma(s)\sqrt{f(s)} .$$

Letting $\psi(s)^2 = f(s)$ this inequality becomes

$$\psi(s) + \gamma(s) \geq \frac{d\psi}{ds} \geq \psi(s) - \gamma(s) .$$

Note that $f(s)$ (and hence $\psi(s)$) is either strictly positive or constantly zero. This follows from the local existence and uniqueness theorem for flows of vector fields. Since $\|(D^2\lambda/\partial t^2)(\varphi(s))\|_{L_2} = 0$ implies that $\lambda(\varphi(s)) = 0$ and if $\lambda(\varphi(s)) = 0$ for any s it equals zero for all s.

Applying Lemma 6 to $\psi(s)$ finishes the proof of this lemma.

Lemma 10. *Let φ: $(\alpha, 0] \to \Lambda(P, Q)$ be as above. Then $s \to \|\lambda(\varphi(s))\|_{H_2}$ is bounded and if $\alpha = -\infty$, $\|\lambda(\varphi(s))\|_{H_2} \to 0$ as $s \to -\infty$. In addition $\varphi(\alpha, 0]$ is bounded in the H_2 metric on $\Lambda(P, Q)$.*

Proof. By Lemma 9, $s \to \|(D^2\lambda/\partial t^2)(\varphi(s))\|_{L_2}$ is bounded, and if $\alpha > -\infty$, it tends to zero as $s \to -\infty$. From Lemma 5 we know that $\|(D\lambda/\partial t)(\varphi(s))\|_{L_2} \to 0$ as $s \to -\infty$. Thus $\|\lambda(\varphi(s))|_{H_2} \to 0$ as $s \to -\infty$. In either case $\|(D^2\lambda/\partial t^2)(\varphi(s))\|_{L_2} = \|(D/\partial t)(d/dt)\varphi(s)\|_{L_2}$ is bounded. $\|(d/dt)\varphi(s)\|_{L_2}^2 = \hat{J}(\varphi(s))$ is bounded by $\hat{J}(\varphi(0))$. But $\varphi(s) \in \Lambda(P, Q)$ whence the boundedness of the first two derivatives of $\varphi(s)$ in L_2 implies that $\varphi(s)$ is bounded in $H_2(I, \boldsymbol{R}^n)$ and so $\varphi(\alpha, 0]$ is bounded in $\Lambda(P, Q)$. This concludes Lemma 10 and also the proof of Proposition 5.

Let us push onto

Proposition 6. *λ satisfies axiom* (G3).

Proof. Let σ be a critical point of $\hat{J}$ (and therefore a zero of λ) in $\hat{J}^{-1}(a, b)$. It follows it a straightforward way as in Palais [10] that σ is in fact C^∞, but we must show that the set of all such σ in $\hat{J}^{-1}(a, b)$ is bounded in $\Lambda(P, Q)$.

If σ is critical, $D\sigma'/\partial t = 0$. Thus

$$\frac{D\sigma'}{\partial t} = \mathscr{P}(\sigma(t))\sigma''(t) = 0 \ .$$

Since

$$\begin{aligned}\frac{d}{dt}\sigma'(t) &= \frac{d}{dt}\mathscr{P}(\sigma(t))\sigma'(t) \\ &= \mathscr{P}(\sigma(t))\sigma''(t) + d\mathscr{P}_{\sigma(t)}(\sigma'(t))\cdot\sigma'(t) \ ,\end{aligned}$$

we have

$$\sigma''(t) = d\mathscr{P}_{\sigma(t)}(\sigma'(t))\cdot\sigma'(t) \ . \tag{18}$$

This implies that

$$\|\sigma''\|_{C_0} \leq C\,\|\sigma'\|_{C_0}^2 \ , \tag{19}$$

where the constant C depends only on the C_0 (supremum) norm of σ. Differentiating (18) again we get that

$$\begin{aligned}\sigma'''(t) = {}& d^2P_{\sigma(t)}(\sigma'(t), \sigma'(t))(\sigma'(t)) \\ &+ dP_{\sigma(t)}(\sigma''(t))\cdot\sigma'(t) + dP_{\sigma(t)}(\sigma'(t))\cdot\sigma''(t) \ ,\end{aligned}$$

which yields

$$\|\sigma'''\|_{C_0} \leq K\{\|\sigma'\|_{C_0}^3 + \|\sigma''\|_{C_0}\|\sigma'\|_{C_0}\} \ .$$

Using (19) we see that

$$\|\sigma'''\|_{C_0} \leq \tilde{K}\,\|\sigma'\|_{C_0}^3 \ , \tag{20}$$

where the constant $\tilde{K}$ depends only on the C_0 norm of σ. But $D\sigma'/\partial t = 0$ implies that $\|\sigma'(t)\|$ is constant in t ($(d/dt)\,\|\sigma'(t)\|^2 = 2\langle D\sigma'/\partial t, \sigma'\rangle = 0$).

Therefore $\|\sigma'(t)\| = c$ some constant, and $\|\sigma'\|_{L_2} = \|\sigma'\|_{C_0}$, whence from (19) we get $\|\sigma''\|_{C_0} \leq C\,\|\sigma'\|_{L_2}^2$ and from (20) we get $\|\sigma'''\|_{C_0} \leq \tilde{K}\,\|\sigma'\|_{L_2}^3$. This implies that the H_3 norm of σ satisfies

$$\|\sigma\|^2_{H_3} \leq \text{const}\,\{\|P\| + \|\sigma'\|_{L_2} + \|\sigma'\|^2_{L_2} + \|\sigma'\|^3_{L_2}\}\,, \tag{21}$$

where $\|P\|$ is the norm of $P \in V \subset \boldsymbol{R}^n$. Thus

$$\|\sigma\|^2_{H_3} \leq \text{const}\,\{\|P\| + \sqrt{b} + b + b^{3/2}\}\,.$$

But the inclusion of H_3 into H_2 is compact. Thus (21) shows that $K(a, b)$ is bounded in $\Lambda(P, Q)$ with the bound depending on b and is also compact. This establishes (G3), and concludes the proof that λ is gradient like, which we state formally as

Theorem 2. *The vector field λ on $\Lambda(P, Q)$ defined by the differential equation $D^2\lambda/\partial t^2 = D\sigma'/\partial t$ is gradient like for the function $\hat{J}$.*

Thus we have a full Morse theory for the geodesic problem on H_2 if we can show that there exists nondegenerate critical points in our sense for almost all P, Q.

Theorem 3. *A critical point for $\hat{J}$ is B-nondegenerate if and only if it is a nondegenerate critical point for J. Therefore by classical theorem of Marston Morse, $\hat{J}$ has nondegenerate critical points for almost all P, Q.*

Proof. This follow from the fact that if σ is critical we have the commutative diagram

$$\begin{array}{ccc} \Omega(P, Q)_\sigma & \xrightarrow{\lambda_*(\sigma)} & \Omega(P, Q)_\sigma \\ \Big\downarrow \text{inc} & & \Big\downarrow \text{inc} \\ \Lambda(P, Q)_\sigma & \xrightarrow{\lambda_*(\sigma)} & \Lambda(P, Q)_\sigma \end{array}$$

where $\lambda_*(\sigma)$ denotes the Frechét derivative of the vector field λ at σ, on both the tangent spaces $\Omega(P, Q)_\sigma$ and $\Lambda(P, Q)_\sigma$ of $\Omega(P, Q)$ and $\Lambda(P, Q)$ at σ. It is shown in [20] and [6] that $\lambda_*(\sigma)$ is of the form identity plus completely continuous. Consequently by the Fredholm alternative theorem and the fact that $\Lambda(P, Q)_\sigma$ is dense $\Omega(P, Q)$ we see that the top arrow is an isomorphism if and only if the bottom arrow is. Therefore it follows that σ is nondegenerate for $J: \Omega(P, Q) \to R$ if and only if it is B-nondegenerate for $\hat{J}: \Lambda(P, Q) \to R$ (see the definition of B-nondegeneracy at the end of § 3).

Remark. Let us repeat that we could have done the complete Morse theory for the energy functional

$$J(\sigma) = \frac{1}{2}\int_0^1 \|\sigma'(t)\|^2\,dt$$

where σ belongs to any Banach manifold path space $\Lambda^p_k(P, Q)$ of the L^p_k maps σ of the unit interval into V with $\sigma(0) = P$, $\sigma(1) = Q$ with $k \geq 2$, $1 < p < \infty$. For almost all P, Q the associated gradient like vector field will have B-nondegenerate zeros.

Our purpose in this section was to again emphasize our point of view *that it is not the space which is important for Morse theory*; *that the functional under consideration need not determine the space one must use. We intend to make this point clearer in future papers.*

In an addendum to this paper (which will remain unpublished) the author shows that the functional

$$E\colon \Lambda_1^4(P, Q) \to R$$

defined on the Sobolev space of L_1^4 maps of I into V taking 0 to P and 1 to Q given by

$$E(\sigma) = \frac{1}{2}\int \|\sigma'(t)\|^2 + \frac{1}{4}\int_0^1 \|\sigma'(t)\|^4\,dt$$

is smooth, satisfies condition (C) and has a gradient like vector field. What is more surprising is that the critical points of E are also the geodesics joining P and Q parameterized by arc length, and for almost all P, Q the critical points of E will be B-nondegenerate. Hence our Morse theory applies to E. Finally by remarks in § 1 we know that a Morse lemma does not hold about the critical points of E.

Therefore *the Morse lemma is not necessary for Morse theory*. On the other hand, $\hat{J}\colon \Lambda(P, Q) \to R$ considered earlier also had geodesics as critical points; for almost all P, Q the critical points of $\hat{J}$ are B-nondegenerate and a Morse lemma holds about these critical points (e.g., see [21]). However, condition (C) does not hold for E. *Thus condition* (C) *is also not essential for Morse theory*.

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