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RESIDUES AND CHARACTERISTIC CLASSES FOR RIEMANNIAN FOLIATIONS

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Introduction

In [9] we began a study of characteristic classes for Riemannian foliations. Here we shall continue this study. The basic technique which we will use will be to start with a Killing vector field on R^{4m} with a single nondegenerate singularity at the origin and consider the resulting $(4m-2)$-codimensional Riemannian foliation on a $(4m-1)$-sphere S^{4m-1}. Given an Ad-invariant homogeneous polynomial of degree $2m$ we relate the Bott residue of the vector field determined by this polynomial and the Simons characteristic number associated to the foliation on S^{4m-1} and this polynomial. By mapping various classical groups onto S^{4m-1} and looking at the induced Riemannian foliations, we will obtain infinite classes of examples of families of foliations with trivial normal bundles for which appropriate exotic characteristic classes vary continuously.

As a consequence of these examples we obtain complete results on continuous variation in some of the possible dimensions where variation can occur. Namely, continuous variation does occur for classes in $H^{4m-1}(RW_{4m-2})$ (see § 1). A basis for $H^{4m-1}(RW_{4m-2})$ is given by $p_{j_1}\cdots p_{j_k}h_i$ where $4(j_1+\cdots+j_k)+4i_1-1=4m-1$ and $i\le j_1$ if $k>0$. Call this monomial p_Jh_i. Let $FR\Gamma_q$ denote the fiber of $BR\Gamma_q\to BGL(q)$.

Theorem (3.5). *The map $H^{4m-1}(RW_{4m-2})\to H^{4m-1}(FR\Gamma_{4m-2})$ is injective. The classes p_Jh_i all vary continuously.*

We will also conclude the uncountability of the homotopy groups $\pi_{4m-1}(FR\Gamma_{4m-2})$.

In § 1 we recall some basic facts and establish some notation. In § 2 we prove the basic results relating various connections which will enable us to deduce the relation between the residue and the Simons numbers. In § 3 we exhibit the examples of continuous families of Riemannian foliations with trivial normal bundles on various classical groups for which appropriate exotic classes vary continuously. We also examine the homotopy of $FR\Gamma_q$.

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1. Some basic notions

In [9] we introduced the complexes RW_q. For convenience, we omit mention of the Euler class in this paper. For $q = 2n$ or $q = 2n + 1$ we take the Pontryagin polynomials p_k on the Lie algebra of $GL(q)$ defined by

$$\operatorname{Det}\left(\lambda I_q - \frac{1}{2\pi}A\right) = \sum_{k=0}^{q} g_k(A)\lambda^{q-k} , \qquad p_k = g_{2k} .$$

Let

$$RW_q = R[p_1, p_2, \cdots, p_{[q/2]}]/\{P \mid \deg P > q\} \otimes \Lambda(h_1, h_2, \cdots, h_q) ,$$

where $\deg p_k = 4k$, $\deg h_k = 4k - 1$. The differential in this complex is $dp_k = 0$, $dh_k = p_k$. Let $\mathscr{F}$ be a Riemannian foliation with trivial normal bundle on a manifold M, ∇ the unique Riemannian torsion free connection on the normal bundle, γ a global frame of the normal bundle, and ∇^γ the flat connection relative to γ. This data determines a degree-preserving map

$$\delta : RW_q \to A^*(M)$$

given by $\delta(p_k) = p_k(K(\nabla))$, $\delta h_k = \Delta_{p_k}(\nabla, \nabla^\gamma)$. This map passes to a map δ^* in cohomology.

An analysis identical to that in [6] yields a basis for $H^*(RW_q)$. Namely, let $I = (i_1, \cdots, i_l)$, $J = (j_1, \cdots, j_k)$, $i_0 =$ the smallest element of I (or ∞ if I is empty), and $j_0 =$ the smallest element of J (or ∞ if J is empty). Then a basis for $H^*(RW_q)$ consists of those monomials $p_J h_I$ such that $\deg p_J h_I > q$, $4i_0 + \deg p_J > q$, and $i_0 \leq j_0$. Algebraically, we have an isomorphism

$$H^{4m}(BO(4m - 2)) \to H^{4m-1}(RW_{4m-2})$$

given by sending $p_{i_1} \cdots p_{i_k}$, with $4(i_1 + \cdots + i_k) = 4m$ and $i_1 \leq i_2 \leq \cdots \leq i_k$, to $h_{i_1} p_{i_2} \cdots p_{i_k}$. On the other hand,

$$H^{4m}(BO(4m - 2)) \xrightarrow{\cong} H^{4m}(BO(4m)) \longrightarrow H^{4m}(BT^{4m}) ,$$

where T^{4m} is the maximal torus on $O(4m)$, and the image consists of all symmetric homogeneous polynomials $\varphi(t_1^2, \cdots, t_{2m}^2)$ of degree $4m$ in two-dimensional classes $t_1, \cdots, t_{2m}$.

At this point we need a few simple facts about Simons cohomology classes. Let E be a q-dimensional vector bundle over a manifold M, ∇ a connection on E, and φ an Ad-invariant polynomial on gl (q) which determines an integral cohomology class such that $\varphi(K(\nabla)) \equiv 0$. Then we have a Simons cohomology class $S_\varphi(\nabla)$.

Lemma (1.1). *If E is trivial with global framing s, and ∇^s is globally flat* *relative to s, then*

$$\Delta_\varphi(\nabla, \nabla^s) = S_\varphi(\nabla) \qquad \bmod Z .$$

Proof. We know $\Delta_\varphi(\nabla, \nabla^s) = s^*T\varphi(\nabla)$ from [9, 1.16]. On the other hand [14, Corollary (3.18)] implies $s^*T\varphi(\nabla) = S_\varphi(\nabla)$, mod Z.

Lemma (1.2). *Let F be a trivial k-plane bundle with global framing s, and ∇^s flat relative to s. Let φ be an Ad-invariant polynomial on* gl $(q + k)$. *Then* $S_\varphi(\nabla \oplus \nabla^s) = S_\varphi(\nabla)$.

Proof. Let $n = q + k$. Let $G_{n,N}$ be the Grassmannian of n-planes in N-space. Let γ_n be the universal n-plane bundle, and ∇^n the universal connection on γ_n. For N large there are a map $f: M \to G_{n,N}$ and a covering map $\hat{f}: E \to \gamma_n$ such that $\hat{f}^{-1}(\nabla^n) = \nabla$, as in [14, (3.5)]. The framing s determines a bundle map $\bar{f}: E \oplus F \to \gamma_n \oplus (k)$ such that $\bar{f}^{-1}(\nabla^n \oplus \nabla') = \nabla \oplus \nabla^s$, where ∇' is globally flat relative to the canonical framing of (k). By naturality, it is enough to show $S_\varphi(\nabla^n \oplus \nabla') = S_\varphi(\nabla^n)$ for for Simons characters. Now we follow [18, (3.5)]. Let Ω^n be the curvature of ∇^n, and Ω the curvature of $\nabla^n \oplus \nabla'$. Then

$$\Omega = \begin{pmatrix} \Omega^n & 0 \\ 0 & 0 \end{pmatrix} .$$

So $\varphi(\Omega) = \varphi(\Omega^n)$. By the definition of Simons characters for $Z_{2l-1}(G_{n,N})$, if $x \in Z_{2l-1}(G_{n,N})$, then $mx = \partial y$, and

$$S_\varphi(\nabla^n \oplus \nabla')x = \overline{\frac{1}{m}(\varphi(\Omega) - \alpha)x} = \overline{\frac{1}{m}(\varphi(\Omega^n) - \alpha)x} = S_\varphi(\nabla^n)x ,$$

where α is any cochain representing the integral class associated to φ, and bar denotes reduction mod Z.

We note that $f^*S_\varphi(\nabla) = S_\varphi(f^{-1}\nabla)$.

In analogy to the definitions of [1] we need the notion of an X connection. Let X be a Killing vector field on a Riemannian manifold M with isolated nondegenerate singularities $p_1, \cdots, p_r$. An X connection ∇ is a connection on $T(M)$ with the property that there are mutually disjoint open sets $U_1, \cdots, U_r$ with $p_j \in U_j$ and such that on $M - \cup U_j$, $\nabla_Z Y = f[X, Y] + D_{\pi_2 Z} Y$, where π_1 is the orthogonal projection in the direction of X, $\pi_1 Z = fX$, π_2 the orthogonal projection perpendicular to X, and D is the Riemannian torsion free connection on $T(M)$. $\cup U_j$ will be called the support of ∇.

By way of notation, D, D' will always be used to symbolize the Riemannian connection on a manifold, whereas ∇, ∇', etc. will be other connections on $T(M)$ or on other bundles. If Y is a tangent vector, $|Y|$ will denote $\langle Y, Y\rangle^{1/2}$ where $\langle\, ,\rangle$ is the relevant metric. If X is a Killing vector field with isolated nondegenerate singularity at p, and φ an invariant polynomial, then as in [2], $\operatorname{Res}_\varphi(X, p) = \varphi(L_p)/\{\operatorname{Det}(L_p)\}^{1/2}$ where L_p is the linear transformation $(\mathscr{L}_X)_p$.

Let $\mathscr{F}$ be a foliation on a manifold M. By a flat coordinate system we will, as is customary, mean local coordinates $x_1, \cdots, x_p, y_1, \cdots, y_q$ such that the leaves are given locally by $y_1 = c_1, \cdots, y_q = c_q$.

Note. Let φ be an Ad-invariant homogeneous polynomial function on $\mathrm{gl}\,(m)$. We will say φ determines an integral class if, for any connection, $[\varphi(K(\nabla))]$ is in the image of $H^*(M; Z)$.

2. Simons classes and the residue

Let X be a Killing field on R^{2n} with a single nondegenerate singularity at the origin. The metric on R^{2n} is the standard one denoted by $\langle\, , \rangle$. Let $\mathscr{F}$ be the resulting Riemannian foliation on S^{2n-1}, and $\tilde{\nabla}$ the unique Riemannian torsion free connection on the normal bundle to $\mathscr{F}$. Let φ be an Ad-invariant homogeneous polynomial of degree n on $\mathrm{gl}\,(2n-2)$ which determines an integral class. In this section we will prove

Theorem (2.12). $S_\varphi(\tilde{\nabla})[S^{2n-1}] = c \operatorname{Res}_\varphi (X, 0)$, mod Z, *where c is a nonzero constant depending only on n and would be* 1 *if the residue were normalized.*

Note. This is of significance only when n is even. Let $(x_1, y_1, \cdots, x_n, y_n)$ be coordinates in R^{2n}. We will be primarily concerned with

$$X = \sum_{j=1}^{n} \alpha_j \left\{ y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j} \right\} .$$

Let ∇ be an X-connection on R^{2n} with support contained in the interior of B^{2n}. From the definition we observe that away from the origin, $\nabla_X Y = [X, Y]$ and $\nabla_Z Y = D_Z Y$ for $Z \perp X$. Letting ∇, D also denote ∇, D restricted to $T(R^{2n}) \,|\, S^{2n-1}$ we see immediately, from the definition of the induced connection, that the same holds on $T(R^{2n}) \,|\, S^{2n-1}$.

Let $R = \sum_{j=1}^{n} \{x_j \partial/\partial x_j + y_j \partial/\partial y_j\}$. Let V be the (trivial) vector bundle on S^{2n-1} generated by X and R. Let s' be the orthonormal framing $\{R/|R|, X/|X|\}$ of V. Let $\nabla' = \nabla \,|\, V$, that is, if q_1 is the orthogonal projection of $T(R^{2n})$ on V then $\nabla'_Z Y = q_1(\nabla_Z Y)$. Let $Y_0 = R/|R|$ and $Y_1 = X/|X|$.

Theorem (2.1). *∇' is flat relative to the framing s' of V.*

Proof. ∇ and hence ∇' are Riemannian connections. Thus $\nabla' Y_0 = a Y_1$ and $\nabla' Y_1 = -a Y_0$ where a is a 1-form on S^{2n-1}. Then $a = \langle \nabla' Y_0, Y_1 \rangle = \langle \nabla Y_0, Y_1 \rangle$. Now

$$a(X) = \langle \nabla_X Y_0, Y_1 \rangle = \langle [X, Y_0], Y_1 \rangle = X(1/|R|)\langle R, X \rangle + \langle [X, R], Y_1 \rangle / |R| .$$

Now $\langle R, X \rangle = 0$. Using $X\langle X, R \rangle = \langle [X, X], R \rangle + \langle X, [X, R] \rangle$ (since X is Killing) we find that $\langle [X, R], X \rangle = 0$ and so $\langle [X, R], Y_1 \rangle = 0$. Thus $a(X) = 0$. Now let Z be tangent to S^{2n-1} and normal to X. Then

$$a(Z) = \langle \nabla'_Z Y_0, Y_1 \rangle = \langle D_Z Y_0, Y_1 \rangle = Z(1/|R|)\langle R, Y_1 \rangle + \langle D_Z R, Y_1 \rangle / |R| .$$

Now $\langle R, Y_1\rangle = 0$, and also $Z(1/|R|) = 0$ since Z is tangent to S^{2n-1}. It is immediate that $\langle D_Z R, Y_1\rangle = \langle Z, Y_1\rangle = 0$ since Z is perpendicular to X. Thus $a(Z) = 0$. So $a \equiv 0$, and ∇' is flat relative to s'.

Let ν be the orthogonal complement to $\mathscr{F} = (X)$ on S^{2n-1}. Let q_2 be the orthogonal projection from $T(R^{2n})|S^{2n-1}$ to ν, and let $\nabla'' = \nabla|\nu = q_2 \circ \nabla$.

Theorem (2.2). *∇'' is the unique Riemannian torsion free connection on ν.*

Proof. Let $p_1: T(S^{2n-1}) \to \mathscr{F}$ and $p_2: T(S^{2n-1}) \to \nu$ be the orthogonal projections. Let D' be the Riemannian connection on S^{2n-1}. As we know, the unique torsion free Riemannian connection $\tilde{\nabla}$ on ν is given by $\tilde{\nabla}_Z Y = p_2[p_1 Z, Y] + p_2 D'_{p_2 Z} Y$. First take $Z = X$. Then $\tilde{\nabla}_X Y = p_2[X, Y]$, $q_2 \nabla_X Y = q_2\{[X, Y] + D_{\pi_2 X} Y\} = q_2[X, Y]$. (Recall, in $T(R^{2n})$, π_1 is orthogonal projection on X and π_2 is orthogonal projection perpendicular to X). Now $[X, Y]$ is tangent to S^{2n-1} and $q_2[X, Y] = \pi_2[X, Y]$. Next take Z tangent to S^{2n-1} and perpendicular to X. Then $\tilde{\nabla}_Z Y = p_2 D'_Z Y$. Since $D' = \pi \circ D$ where π is orthogonal projection from $T(R^{2n}) \to T(S^{2n-1})$, $\tilde{\nabla}_Z Y = p_2 \pi D_Z Y = q_2 D_Z Y$. Now $\nabla''_Z Y = q_2 \nabla_Z Y = q_2 D_Z Y$. Thus $\tilde{\nabla} = \nabla''$.

Remark. From now on, the unique Riemannian torsion free connection on ν will be denoted by ∇''.

We now take two connections on $T(R^{2n})|S^{2n-1}$, namely, ∇ induced from an X-connection and $\nabla' \oplus \nabla''$ corresponding to $T(R^{2n})|S^{2n-1} = V \oplus \nu$. Let φ be an Ad-invariant polynomial on gl$(2n)$ of degree n. We want to compare ∇ and $\nabla' \oplus \nabla''$, i.e., to compute $\Delta_\varphi(\nabla, \nabla' \oplus \nabla'')$.

Let $x_1, y_1, \cdots, y_{2n-2}$ be local coordinates on S^{2n-1} such that $X = \partial/\partial x_1$ (on some open set U). Let

$$f: U \to R^{2n-2}$$

be given by $f(x_1, y_1, \cdots, y_{2n-2}) = (y_1, \cdots, y_{2n-2})$. Let θ' be the connection matrix of ∇' relative to $\{Y_0, \partial/\partial x_1\}$, let θ'' be the connection matrix of ∇'' relative to $\{\partial/\partial y_1, \cdots, \partial/\partial y_{2n-2}\}$ and θ the connection matrix of ∇ relative to $\{Y_0, \partial/\partial x_1, \partial/\partial y_1, \cdots, \partial/\partial y_{2n-2}\}$.

Lemma (2.3). *$\theta' = f^*(\omega')$ for some matrix of 1-forms ω' on R^{2n-2}.*

Proof. $\nabla' Y_0 = 0$ by (2.1). $\nabla'(\partial/\partial x_1) = \nabla'(X) = \nabla'(|X| Y_0) = d|X| Y_0$ by (2.2). Thus we merely have to show that $d|X|$ is in the image of f^*. Now $d|X| = a dx_1 + \sum b_k dy_k$. Apply to X to get $X(|X|) = a$. Now $X\langle X, X\rangle = 2\langle [X, X], X\rangle = 0$. Thus $X(|X|^2) = 0$ and hence $X(|X|) = 0$. Thus $a = 0$. The fact that $X(|X|) = 0$ also shows that $|X|$ is constant along the integral curves of X. Let φ_t be the local 1-parameter group for X. Then $|X| \circ \varphi_t = |X|$. Let $m \in U$ and let t be small. Then

$$\varphi_t^*(d|X|)\{\partial/\partial y_j\}_m = \sum b_k(\varphi_t(m))\varphi_t^*(dy_k)\{\partial/\partial y_j\} .$$

Now y_k is constant on the integral curves of X, so $\varphi_t^*(dy_k) = dy_k$ and $\varphi_t^*(dy_k)\{\partial/\partial y_j\} = \delta_{jk}$. Thus $\varphi_t^*(d|X|)\{\partial/\partial y_j\}_m = b_j(\varphi_t(m))$. But $\varphi_t^*(d|X|)\cdot$

$\{\partial/\partial x_j\}_m = (\partial/\partial y_j)(|X|\circ\varphi_t)_m = (\partial/\partial y_j)(|X|)_m = d|X|\{\partial/\partial y_j\}_m = b_j(m)$. Thus $b_j(m) = b_j(\varphi_t(m))$ for all small t and all j. It follows that b_k is independent of x_1, and so $d|X| = \sum b_k(y_1, \cdots, y_{2n-2})dy_k$ is in the image of f^*.

Remark (2.4). $\theta'' = f^*(\omega'')$ for some matrix ω'' of 1-forms on R^{2n-2}. This is one of the steps in proving the vanishing theorem for Riemannian foliations [11].

Lemma (2.5). *$\theta = f^*(\omega)$ for some matrix of 1-forms ω on R^{2n-2}.*

Proof. Let $\eta = \{Y_0, \partial/\partial x_1, \partial/y_1, \cdots, \partial/\partial y_{2n-2}\}$ be the above local framing for $T(R^{2n})|S^{2n-1}$. Write $\theta = \|\theta^{ij}\|$, where

$$\theta^{ij} = a^{ij}dx_1 + \sum b_k^{ij}dy_k \ .$$

Then

$$\nabla\eta = \eta\theta \ , \quad \text{so} \quad \nabla_X\eta = \eta\theta(X) = \eta\,\|a^{ij}\| \ .$$

Now $[X, Y_0] = X(1/|R|)R + [X, R]/|R| = 0$, since $|R|$ is constant on the integral curves of a Killing field and we have seen $[X, R] = 0$. $[X, X] = 0$ and $[X, \partial/\partial y_j] = [\partial/\partial x_1, \partial/\partial y_j] = 0$. Since $\nabla_X\eta = [X, \eta]$, we have $\eta\,\|a^{ij}\| = 0$ and so $\|a^{ij}\| \equiv 0$. Next choose local vector fields $Z_1, \cdots, Z_{2n-2}$ such that $Z_j(y_k) = \delta_{jk}$ and $Z_j \perp X$ for all j. Let φ_t be the 1-parameter group for X. $\varphi_t\colon S^{2n-1} \to S^{2n-1}$ is an isometry since X is a Killing field. We want to show that each b_k^{ij} is independent of x_1, and to do this it will be sufficient to show that $\varphi_t^*(b_k^{ij}) = b_k^{ij}$ for small t. Now $y_k\circ\varphi_t = y_k$ and so $\varphi_{t,*}(Z_j)\{y_k\} = \delta_{jk}$. Since $Z_j \perp X$, we have that $\nabla_{Z_j} = D_{Z_j}$. Since φ_t is an isometry, $\varphi_t^{-1}(D) = D$. Now

$$\begin{aligned}\varphi_t^{-1}(D)_{Z_j}\eta &= \varphi_{t,*}^{-1}(D_{\varphi_{t,*}(Z_j)}\varphi_{t,*}\eta)\\ &= \varphi_{t,*}^{-1}(\varphi_{t,*}(\eta)\circ\theta(\varphi_{t,*}Z_j)) = \eta\circ\theta(\varphi_{t,*}Z_j) \ .\end{aligned}$$

Hence $\varphi_t^{-1}(D)_{Z_j}\eta = \eta\circ\theta(\varphi_{t,*}(Z_j))$, and $\varphi_t^{-1}(D)_{Z_j}\eta = D_{Z_j}\eta$. So $\eta\circ\theta(\varphi_{t,*}(Z_j)) = \eta\circ\theta(Z_j)$.

Now $\theta^{ij}(Z_l) = \sum b_k^{ij}Z_l(y_k) = b_l^{ij}$. $\theta^{ij}(\varphi_{t,*}Z_l) = \sum \varphi_t^*(b_k^{ij})\varphi_t^*(dy_k)\{Z_l\} = \varphi_t^*(b_l^{ij})$ since $\varphi_t^*(dy_k) = dy_k$. Thus

$$\eta\,\|b_k^{ij}\| = \eta\,\|\varphi_t^*(b_k^{ij})\|$$

for each i, j, k and small t. Hence $b_k^{ij} = \varphi_t^*(b_k^{ij})$.

Remark (2.6). If we consider ∇ on R^{2n} outside its support, and choose flat coordinates $x_1, y_1, \cdots, y_{2n-1}$ on R^{2n} for X, then it follows, just as in the preceeding proof, that the connection form for ∇ depends only on $y_1, \cdots, y_{2n-1}$, i.e., is pulled back from R^{2n-1}. In particular, if φ is an invariant polynomial of degree n on gl$(2n)$, then $\varphi(K(\nabla)) \equiv 0$ outside the support of ∇.

We shall now compare the connections ∇ and $\nabla'\oplus\nabla''$ on $T(R^{2n})|S^{2n-1}$. Choose local framings η' for V, η'' for ν, and $\eta = (\eta', \eta'')$ for $V\oplus\nu$ (the η's need not necessarily be as in (2.3), $\cdots$, (2.6)). Let $\theta', \theta'', \theta$ be the corresponding connection matrices. Let $\theta_0 = \begin{pmatrix} \theta' & 0 \\ 0 & \theta'' \end{pmatrix}$. Then θ_0 is the connection matrix for $\nabla' \oplus \nabla''$ relative to η. Let $\sigma = \theta - \theta_0$. Form the connection $\nabla^t = t\nabla + (1-t)\nabla' \oplus \nabla''$ on $S^{2n-1} \times I$. Let $p: S^{2n-1} \times I \to S^{2n-1}$ be the projection. The connection matrix of ∇^t relative to $p^{-1}(\eta)$ is $\theta^t = \theta_0 + t\sigma$, and the curvature matrix is $K^t = dt\sigma + td\sigma + t[\sigma, \theta_0] + t^2\sigma^2 + K_0$, where $K_0 = d\theta_0 + \theta_0^2$. Let φ be an invariant polynomial on gl$(2n)$ of degree n. Then $\Delta_\varphi(\nabla, \nabla' \oplus \nabla'') = p_*\{\varphi(K^t)\}$. Now

$$p_*\{\varphi(K^t)\} = n \sum_{i+j+k=n-1} \int t^l \varphi(\sigma \wedge (d\sigma + [\theta_0, \sigma])^i \wedge (\sigma^2)^j \wedge K_0^k) dt ,$$

where we do not care what l is.

Lemma (2.7). *$d\sigma + [\sigma, \theta_0]$, σ, σ^2, K_0 are all tensorial.*

Proof. A form Θ coming from a connection and a framing η is tensorial if, relative to ηA, Θ becomes $A\Theta A^{-1}$. Now there forms are tensorial by direct computation.

Theorem (2.8). $\Delta_\varphi(\nabla, \nabla' \oplus \nabla'') \equiv 0$.

Proof. It is enough to show that each term

$$\varphi(\sigma \wedge (d\sigma + [\theta_0, \sigma])^i \wedge (\sigma^2)^j \wedge K_0^k) \equiv 0 , \qquad i + j + k = n - 1 .$$

Since each entry σ, $d\sigma + [\theta_0, \sigma]$, K_0 are tensorial, and φ is Ad-invariant, this term remains unchanged if we use the framings η', η'', η in (2.3), $\cdots$, (2.6). In this case we have $f: U \to R^{2n-2}$, $f(x_1, y_1, \cdots, y_{2n-2}) = (y_1, \cdots, y_{2n-2})$, and each entry is in the image of f^*. Thus $\varphi(\sigma \wedge (d\sigma + [\theta_0, \sigma])^i \wedge (\sigma^2)^j \wedge K_0^k)$ is in $f^*(\Lambda^{2n-1}(R^{2n-2}))$ and so is identically zero.

Let φ be Ad-invariant polynomial of degree n which determines an integral class. Let $s = \{Y_0, Y_1, Y_2, \cdots, Y_{2n-1}\}$ be a local orthonormal framing of $T(R^{2n})|S^{2n-1}$. Let ∇^s be the connection which is flat relative to s. Then $s'' = \{Y_2, \cdots, Y_{2n-2}\}$ is a local orthonormal framing of ν, the normal bundle to the foliation. Let $\nabla^{s''}$ be flat relative to s''.

Theorem (2.9). $\Delta_\varphi(\nabla, \nabla^s) = \Delta_\varphi(\nabla' \oplus \nabla'', \nabla^s) + dz$.

Proof. For any three connections $\Delta_\varphi(\nabla^0, \nabla^1) + \Delta_\varphi(\nabla^1, \nabla^2) + \Delta_\varphi(\nabla^2, \nabla^0) = dz$, and $\Delta_\varphi(\nabla^1, \nabla^2) = -\Delta_\varphi(\nabla^2, \nabla^1)$. Now let $\nabla^0 = \nabla$, $\nabla^1 = \nabla^s$, $\nabla^2 = \nabla' \oplus \nabla''$. Then $\Delta_\varphi(\nabla^2, \nabla^0) \equiv 0$ by (2.8) and the result follows.

Theorem (2.10). $S_\varphi(\nabla) = S_\varphi(\nabla'')$.

Note. Recall from § 1, $S_\varphi(\ )$ is the Simons cohomology class in $H^{2n-1}(S^{n-1}, R/Z)$.

Proof. Choose local framings s as in (2.9) where the first two vectors Y_0, Y_1 are always $R/|R|$, $X/|X|$. The local connection matrix for $\nabla' \oplus \nabla''$ is $\begin{pmatrix} 0 & 0 \\ 0 & \theta'' \end{pmatrix}$ ($s = (s', s'')$ where $s' = \{R/|R|, X/|X|\}$) since ∇' is flat relative to s'. Then

$\Delta_\varphi(\nabla' \oplus \nabla'', \nabla^s) = \Delta_\varphi(\nabla'', \nabla^{s''})$. From our discussion of Simons classes in § 1 it follows that $S_\varphi(\nabla) = S_\varphi(\nabla'')$.

Next we must relate the Simons characteristic number $S_\varphi(\nabla)[S^{2n-1}]$ to $\mathrm{Res}_\varphi(X, 0)$. This is essentially the Bott residue theorem. Let ∇ be an X-connection supported in an open set U with $\bar{U} \subset$ interior B^{2n}. Let φ be an invariant polynomial on gl$(2n)$ of degree n.

Theorem (2.11), (*Bott residue theorem for* B^{2n}).

$$\int_{B^{2n}} \varphi(K(\nabla)) = c\,\mathrm{Res}_\varphi(X, 0) .$$

Remark. The approach to the proof for a Killing field X on a manifold M with isolated nondegenerate singularities is to compare ∇ and D (the Riemannian connection on M) and to examine $\Delta_\varphi(\nabla, D)$ near the singularities. This is done in [2]. However for the uninitiated reader, we will carry out this analysis in the case we will need it, namely,

$$X = \sum_{i=1}^{n} \alpha_i \{y_i \partial/\partial x_i - x_i \partial/\partial y_i\} , \qquad \textit{the } \alpha\textit{'s are near } 1.$$

Proof in this special case. Let $\tilde{B}_\varepsilon = \{x \mid \sum (\alpha_i^2 x_i^2 + \alpha_i^2 y_i^2) \le \varepsilon^2\}$. We can alway choose an X-connection which is supported on U and an ε so that $\bar{U} \subsetneqq \tilde{B}_\varepsilon \subset$ interior B^{2n}. Let $\tilde{S}_\varepsilon = \partial\tilde{B}_\varepsilon$. Change coordinates. Let $u_i = \alpha_i x_i$, $v_i = \alpha_i y_i$. Then $\tilde{B}_\varepsilon = \{x \mid \sum (u_i^2 + v_i^2) \le \varepsilon^2\}$. Let us consider the global framing η of $T(R^{2n})$ given by $\eta = \{\partial/\partial x_1, \partial/\partial y_1, \cdots, \partial/\partial x_n, \partial/\partial y_n\}$. Then, relative to η and outside U, $\nabla\eta = \pi \otimes [X, \eta]$ where π is the 1-form $\pi(Y) = \langle X, Y\rangle / \langle X, X\rangle$. Notice that D is flat relative to η and $K(D) \equiv 0$. Let θ be the connection form for ∇ relative to η. Then $\theta = \pi \otimes L$ where $L = \mathrm{Diag}\left(\begin{bmatrix} 0 & \alpha_i \\ -\alpha_i & 0 \end{bmatrix}\right)$. On $R^{2n} \times I$ form $\nabla^t = t\nabla + (1-t)D$. Let $p: R^{2n} \times I \to R^{2n}$. The connection and curvature matrices of ∇^t relative to $p^{-1}(\eta)$ are $t\pi \otimes L$ and $dt\,\pi \otimes L + t\,d\pi \otimes L$. Thus

$$p_*(\varphi(K^t)) = \Delta_\varphi(\nabla, D) = \pi(d\pi)^{n-1}\varphi(L) .$$

By (2.6), $\varphi(K(\nabla)) \equiv 0$ outside U, so

$$\int_{B^{2n}} \varphi(K(\nabla)) = \int_{\tilde{B}_\varepsilon} \varphi(K(\nabla)) = \int_{\tilde{B}_\varepsilon} d\Delta_\varphi(\nabla, D) = \varphi(L) \int_{\tilde{S}_\varepsilon} \pi(d\pi)^{n-1} .$$

Now

$$\pi = \sum_{i=1}^{n} \alpha_i \{y_i dx_i - x_i dy_i\} \Big/ \sum_{i=1}^{n} \alpha_i^2 (x_i^2 + y_i^2) .$$

Thus on $\tilde{S}_\varepsilon$

$$\pi = \frac{1}{\varepsilon^2} \sum \frac{1}{\alpha^i}\{v_i du_i - u_i dv_i\} ,$$

so that on $\tilde{S}_\varepsilon$

$$\pi(d\pi)^{n-1} = (n-1)!\, \frac{2^{n-1}}{\varepsilon^{2n}} \prod_{j=1}^{n} \frac{1}{\alpha_j} \sum_{i=1}^{n} (v_i du_i - u_i dv_i) \prod_{k\neq i} dv_k du_k .$$

Finally making the change of variables $x_i = u_i/\varepsilon$, $y_i = v_i/\varepsilon$ we get

$$\begin{aligned}\int_{\tilde{S}_\varepsilon} \pi(d\pi)^{n-1} &= (n-1)!\, 2^{n-1} \sum_{j=1}^{n} \frac{1}{\alpha_j} \int_{S_1} \sum_{i=1}^{n} \left\{(y_i dx_i - x_i dy_i) \prod_{k\neq i} dy_k dx_k\right\} \\ &= n!\, 2^n \prod_{j=1}^{n} \frac{1}{\alpha_j} \int_{B_1} \prod_{i=1}^{n} dy_i dx_i ,\end{aligned}$$

where B_1 and S_1 are the ball and sphere of radius 1. Now $\prod_{j=1}^{n} 1/\alpha_j = (\mathrm{Det}\,(L))^{-1/2}$, let $c = n!\, 2^n \int_{B_1} \prod dy_i dx_i$. Thus

$$\int_{B^{2n}} \varphi(K(\nabla)) = c \frac{\varphi(L)}{\{\mathrm{Det}\,(L)\}^{1/2}} = c \operatorname{Res}(X, 0) .$$

Theorem (2.12). $S_\varphi(\nabla'')[S^{2n-1}] = c \operatorname{Res}(X, 0)$, mod Z.

Proof. By (2.10) we know $S_\varphi(\nabla'') = S_\varphi(\nabla)$ for any X-connection ∇. Now given any local framing s of $T(R^{2n})\,|\,S^{2n-1}$, $S_\varphi(\nabla)$ is determined on the domain of this framing by $\Delta_\varphi(\nabla, \nabla^s)$. Thus we can take s to be the global framing $\{\partial/\partial x_1, \partial/\partial y_1, \cdots, \partial/\partial x_n, \partial/\partial y_n\}$. For this s, $\nabla^s = D$. Thus $\Delta_\varphi(\nabla, D)$ determines $S_\varphi(\nabla)$ on S^{2n-1}, and therefore

$$\begin{aligned}S_\varphi(\nabla)[S^{2n-1}] &= \int_{S^{2n-1}} \Delta_\varphi(\nabla, D) = \int_{B^{2n}} d\Delta_\varphi(\nabla, D) \\ &= \int_{B^{2n}} \varphi(K(\nabla)) = c \operatorname{Res}(X, 0) .\end{aligned}$$

3. Applications

In the first part of this section we will always take $n = 2m$ and work in R^{4m}. We will start with a family X_α of Killing fields, each with a singularity at the origin. The residue, with our choice of X_α, relative to suitably chosen invariant polynomial φ, will vary continuously for α_j near 1. Thus the resulting foliation $\mathscr{F}_\alpha$ on S^{4m-1} will have the property that $S_\varphi(\nabla'')[S^{4m-1}]$ will vary continuously in R/Z by Theorem (2.12). Now consider the fibrations $f\colon U(2m) \to S^{4m-1}$, $f\colon SO(4m) \to S^{4m-1}$, $f\colon Sp(m) \to S^{4m-1}$. The induced foliations $f^{-1}(\mathscr{F}_\alpha)$ will be shown to be Riemannian foliations with trivial normal bundle for α_j near 1. The continuous variation of $S_\varphi(F'')[S^{4m-1}]$ will imply the continuous variation of the associated exotic class coming from RW_{4m-2} associated to φ. The result on the variation of classes in $H^*(FR\Gamma_q)$ and the uncountability of $\pi_*(FR\Gamma_q)$ will follow from these considerations. At the end of this section we will study $\pi_*(FR\Gamma_q)$.

Let $X_\alpha = \sum_{i=1}^{2m} \alpha_i\{y_i\partial/\partial x_i - x_i\partial/\partial y_i\}$. Let φ be an Ad-invariant polynomial on gl$(4m)$ of degree $2m$. Now $L = \mathrm{Diag}\left(\begin{bmatrix} 0 & \alpha_i \\ -\alpha_i & 0 \end{bmatrix}\right)$, so $\varphi(L) = \varphi(\alpha_1^2, \cdots, \alpha_{2m}^2)$ and $\{\mathrm{Det}\,(L)\}^{1/2} = \prod_{i=1}^{2m} \alpha_i$. Thus it is easily seen that $\mathrm{Res}_\varphi\,(X_\alpha, 0)$ varies continuously with $\alpha_1, \cdots, \alpha_{2m}$.

The case $U(2m)$. Let $(z_1, \cdots, z_{2m}) \in C^{2m}$ and consider the 1-parameter group $(z_1, \cdots, z_{2m}) \to (e^{i\alpha_1 s}z_1, e^{i\alpha_2 s}z_2, \cdots, e^{i\alpha_{2m} s}z_{2m}) = z(s)$. A simple calculation shows that this is the 1-parameter group associated to

$$-X_\alpha = \sum_{j=1}^{2m} \alpha_i(y_j\partial/\partial x_j - x_j\partial/\partial y_j) .$$

Let $e = (0, 0, \cdots, 0, 1) \in C^{2m}$, and let $f\colon U(2m) \to S^{4m-1}$ be $f(A) = A(e)$.

If $f(A) = z$, then $f(e^{i\alpha s}A) = z(s)$ where $e^{i\alpha s} = \mathrm{Diag}\,(e^{i\alpha_1 s}, \cdots, e^{i\alpha_{2m} s})$. Thus $z(s) = f(AA^{-1}e^{i\alpha s}A)$ and so

$$-X_\alpha = f_* L_{A,*}(\mathrm{Ad}\,(A^{-1})x_\alpha) ,$$

where $x_\alpha = \mathrm{Diag}\,(i\alpha_1, \cdots, i\alpha_{2m})$ in $u(2m)$, and L_A is the left translation in $U(2m)$ by A.

In the Lie algebra $u(2m)$, let x_j be the matrix with an i in the jj position and all other zero, y_{jk}, $j < k$, have an i in the jk and kj position and all other zeros, and z_{jk}, $j < k$, have a 1 in jk, -1 in kj and all other zeros. Then $\{x_j, y_{jk}, z_{jk}\}$ form a basis for $u(2m)$, $\{x_j, y_{jk}, z_{jk}\}_{k<n}$ form a basis for $u(2m-1)$ and for each A

$$f_* \circ L_{A,*} \text{ maps } u(2m-1) \oplus [\mathrm{Ad}\,(A^{-1})x_\alpha] \text{ onto } [X_\alpha] .$$

Thus

$$T(f^{-1}(X_\alpha))_A = L_{A,*}(u(2m-1)) \oplus [L_{A,*} \circ \mathrm{Ad}\,(A^{-1})x_\alpha] .$$

Lemma (3.1). *$\{L_{A,*}(y_{j\,2m}), L_{A,*}(z_{j\,2m})\}$ project to a basis for*

$$T(U(2m))_A / T(f^{-1}(X_\alpha))_A$$

for α_j near 1.

Proof. x_j for $j < n$, y_{jk} for $k < n$, z_{jk} for $k < n$, y_{jn}, z_{jn}, and $\mathrm{Ad}\,(A^{-1})x_\alpha$ give us a basis for $u(2m)$ when $\alpha_j = 1$, since $\mathrm{Ad}\,(A^{-1})x_\alpha = x_1 + \cdots + x_n$ for $\alpha_j = 1$. Call this set of vectors $\beta_{\alpha,A}$. Let $g\colon R^{2m} \times U(2m) \to R$ be given by $g(\alpha, A) = \mathrm{Det}\,(\beta_{\alpha,A})$. For each A, $g(1, A) \neq 0$. By compactness there is a neighborhood W of 1 in R^{2m} such that $g(\alpha, A) \neq 0$ for all α in W and all A in $U(2m)$. Thus for α in W, $\{L_{A,*}(y_{j\,2m}), L_{A,*}(z_{j\,2m})\}$ give us a basis for

$$T(U(2m))/T(f^{-1}X_\alpha) .$$

Now $f^{-1}(\mathscr{F}_\alpha)$ is a Riemannian foliation. Let s_α be the framing of the normal bundle arising from (3.1). This is a $(4m-2)$-codimensional foliation of $U(2m)$. Let

$$\delta^* : H_*(RW_{4m-2}) \to H^*(U(2m), R) .$$

Theorem (3.2). *Let $p_J h_i$ be an element of $H^{4m-1}(RW_{4m-2})$. Then $\delta^*(p_J h_i)$ varies continuously in $H^{4m-1}(U(2m), R)$.*

Proof. $f^{-1}(\nabla'')$ is the Riemannian connection on the normal bundle to $f^{-1}(\mathscr{F}_\alpha)$. $\delta^*(p_J h_i) = [p_J(f^{-1}\nabla'')\Delta_{p_i}(f^{-1}\nabla'', \nabla^s)]$.

Let $\varphi = p_J p_i$, a polynomial of degree $4m$. Then by [9, 1.6]

$$p_J(f^{-1}\nabla'')\Delta_{p_i}(f^{-1}\nabla'', \nabla^s) = \Delta_\varphi(f^{-1}\nabla'', \nabla^s) .$$

Now $[\Delta_\varphi(f^{-1}\nabla'', \nabla^s)] = S_\varphi(f^{-1}\nabla'')$, mod Z. Since

$$H^{4m-1}(S^{4m-1}, R/Z) \to H^{4m-1}(U(2m), R/Z)$$

is an injection, and $S_\varphi(\nabla'')$ varies continuously, so does $f^* S_\varphi(\nabla'') = S_\varphi(f^{-1}\nabla'')$ and hence also $\delta^*(p_J h_i)$.

The case $SO(4m)$. As in the case of $U(2m)$, let $f(A) = A(e)$ where $e = (0, 0, \cdots, 0, 1)$ in R^{4m}. The integral curve for $-X_\alpha$ through $z = (x_1, y_1, \cdots, x_{2m}, y_{2m})$ is

$$(\cdots, x_j \cos \alpha_j s - y_j \sin \alpha_j s, x_j \sin \alpha_j s + y_j \cos \alpha_j s, \cdots) = z(s) .$$

Now let

$$D = \mathrm{Diag}\left(\begin{bmatrix} \cos \alpha_j s & -\sin \alpha_j s \\ \sin \alpha_j s & \cos \alpha_j s \end{bmatrix}\right) .$$

Then $z(s) = Dz$, so if $f(A) = z$, then $f(DA) = z(s)$.

Now in the Lie slgebra $so(4m)$, let x_{jk} for $j < k$ be the matrix with -1 in jk and $+1$ in kj position and all other zero. Let $v_j = x_{2j-1,2j}$. Then $\{x_{jk}\}$ form a basis for $so(4m)$ and

$$D = \exp s(\alpha_1 v_1 + \cdots + \alpha_{2m} v_{2m}) .$$

Let $v_\alpha = \alpha_1 v_1 + \cdots + \alpha_{2m} v_{2m}$ and write $D = e^{sv_\alpha}$. Then $f(AA^{-1}e^{sv_\alpha}A) = z(s)$ and so

$$f_* L_{A,*}(so(4m-1) \oplus [\mathrm{Ad}\,(A^{-1})v_\alpha]) = [X_\alpha]_{f(A)} .$$

Now just as in the $U(2m)$ case we have

Lemma (3.3). *$\{L_{A,*}(x_{j,4m}), j \neq 4m-1\}$ project to a basis for*

$$T(SO(4m))/f^{-1}(\mathscr{F}_\alpha) \qquad \text{for } \alpha_j \text{ near } 1 .$$

As in the case of $U(2m)$, $f^{-1}(\mathscr{F}_\alpha)$ is a family of Riemannian foliations, $f^{-1}(\nabla'')$ is the Riemannian connection on the normal bundle, δ^* is determined by $f^{-1}(\nabla'')$ and ∇^s where s is given by the framing of (3.3). Then just as in the proof of (3.2) we have

Theorem (3.4). *$\delta^*(p_J h_i)$ varies continuously.*

Note that the remark following (3.2) applies equally well to $SO(4m)$.

The case Sp (m). We will think of Sp $(m) \subset U(2m)$ and so sp $(m) \subset u(2m)$. sp (m) consists of matrices $\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$ with $A_4 = -A_1^T, A_3 = A_3^T, A_2 = A_2^T$. Let r_j have an i in jj and $-i$ in $m+j, m+j$ for $j \leq m$. Let v_{jk} have an i in jk and kj and $-i$ in $m+j, m+k$ and $m+k, m+j$ for $j < k \leq m$. Let z_{jk} have a $+1$ in jk, -1 in kj, $+1$ in $j+m, k+m$ and -1 in $k+m, j+m$ for $j < k \leq m$. Let w_{jk} have i in $j, m+k, k, m+j, j+m, k$, and $k+m, j$ for $j < k \leq m$. Let u_{jk} have $+1$ in $j, k+m$, $+1$ in $k, j+m$, -1 in $j+m, k$, -1 in $k+m, k$. Then $\{r_j, v_{jk}, z_{jk}, w_{jk}, u_{jk}\}$ form a basis for sp (m). Let $r_\alpha = \alpha_1 r_1 + \cdots + \alpha_m r_m$, and $e^{r_\alpha s} = \exp s r_\alpha$. Let $z \in C^{2m}$. Then $e^{r_\alpha s} z = z(s)$ is the integral curve for

$$X_\alpha = \sum_{j=1}^{2m} \alpha_j \left\{ -y_j \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial y_j} + y_{j+2m} \frac{\partial}{\partial x_{j+2m}} - x_{j+2m} \frac{\partial}{\partial y_{j+2m}} \right\} .$$

An easy computation shows $\mathrm{Res}_\varphi\,(X, 0)$ is as for the previous X_α. A lemma identical to Lemmas (3.1) and (3.3) shows that

$$L_{A,*}\{v_{j\,2m} \text{ for } j < 2m,\ z_{j\,2m} \text{ for } j < 2m,\ w_{j\,2m} j \leq 2m,\ u_{j\,2m} j \leq 2m\}$$

projects to a basis for $T(\mathrm{Sp}\,(m))/T(f^{-1}X_\alpha)$. Using this framing and $f^{-1}(\nabla'')$ we get, as for $U(2m)$ and $SO(4m)$ that $\delta^*(p_J h_i)$ varies continuously.

Theorem (3.5). *The map $H^{4m-1}(RW_{4m-2}) \to H^{4m-1}(FR\Gamma_{4m-2})$ is injective. The classes $\{p_J h_i\}$, with constitute a basis for $H^{4m-1}(RW_{4m-2})$ all vary continuously.*

Proof. In § 1 we have described a basis $\{p_J h_i\}$ for $H^{4m-1}(RW_{4m-2})$. Consider the polynomials $\{p_J p_i\}$ and label them $\varphi_{(J,i)}$. The foliation $f^{-1}(\mathscr{F}_\alpha)$ in Lemma (3.1) gives rise to a commutative diagram:

$$\begin{array}{ccc} H^{4m-1}(RW_{4m-2}) & \xrightarrow{\delta^*} & H^{4m-1}(U(2m)) \\ & \searrow \quad \swarrow & \\ & H^{4m-1}(FR\Gamma_{4m-2}) & \end{array}$$

Now, if the map $H^{4m-1}(RW_{4m-2}) \to H^{4m-1}(FR\Gamma_{4m-2})$ were not injective, then some linear combination $\sum c_{(J,i)} p_J h_i$ would be identically zero in $H^{4m-1}(FR\Gamma_{4m-2})$.

Let $\varphi_I = \varphi_{(J,i)}$. Then for all $\mathscr{F}_\alpha$,

$$\delta^*(\sum c_{(J,i)} p_J h_i) = 0 , \qquad \mathrm{mod}\, Z .$$

But $S_{\varphi_I}(\nabla'')[S^{4m-1}] = c\varphi_I(\alpha_1^2, \cdots, \alpha_{2m}^2)/\pi\alpha_i$, mod Z and so

$$c \sum (c_I \varphi_I(\alpha_1^2, \cdots, \alpha_{2m}^2)/\pi\alpha_i) \in Z$$

for all choices of $\alpha_1, \cdots, \alpha_{2m}$. This is impossible. Now we have already seen that all of the cohomology classes $\delta^*(p_J h_i)$ vary continuously.

The homotopy of $BR\Gamma_q$, $FR\Gamma_q$. Let X_α be the family of Killing fileds given in the beginning of § 3. Let $\mathscr{F}_\alpha$ be the resulting Riemannian foliation on S^{4m-1} with ∇'' the Riemannian connection on the normal bnndle. Consider the fibration

$$FR\Gamma_{4m-2} \to BR\Gamma_{4m-2} \to BO(4m-2) .$$

Let $f_\alpha \colon S^{4m-1} \to BR\Gamma_{4m-2}$ be the map which classifies $\mathscr{F}_\alpha$.

As in the previous theorem, let $\varphi_1, \cdots, \varphi_r$ be the polynomials $p_J p_i$ where $\{p_J h_i\}$ form a basis for $H^{4m-1}(RW_{4m-2})$. Let $f_\alpha \colon S^{4m-1} \to BR\Gamma_{4m-2}$ be the map which classifies $\mathscr{F}_\alpha$. Let $\varphi_j(\alpha)$ be the element in $H^{4m-1}(BR\Gamma_{4m-2}, R/Z)$ corresponding to $S_\varphi(\nabla'')$. Define

$$\Phi \colon \pi_{4m-1}(BR\Gamma_{4m-2}) \to \underbrace{R/Z + \cdots + R/Z}_{r}$$

by $\Phi([f]) = (f^*(\varphi_1(\alpha))[S^{4m-1}], \cdots, f^*(\varphi_r(\alpha))[S^{4m-1}])$. Choosing $f = f_\alpha$ we get

$$\begin{aligned} \Phi([f_\alpha]) &= (S_{\varphi_1}(\nabla'')[S^{4m-1}], \cdots, S_{\varphi_r}(\nabla'')[S^{4m-1}]) \\ &= \left(\frac{(\varphi_1(\alpha_1^2, \cdots, \alpha_{2m}^2)}{\pi\alpha_i}, \cdots, \frac{\varphi_r(\alpha_1^2, \cdots, \alpha_{2m}^2)}{\pi\alpha_j}\right) . \end{aligned}$$

For various choices of $\alpha = (\alpha_1, \cdots, \alpha_r)$ we get a surjection onto a neighborhood of the identity. Thus

Theorem (3.6.) *$\pi_{4m-1}(BR\Gamma_{4m-2})$ and $\pi_{4m-1}(FR\Gamma_{4m-2})$ map surjectively onto $\bigoplus_r R/Z$, where $r = \dim H^{4m-1}(RW_{4m-2})$. Thus these homotopy groups are not countably generated.*

Note that these homotopy groups were first studied in [12] using methods of [1].

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