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MINIMAL HYPERSURFACES IN SPACES OF CONSTANT CURVATURE

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Dedicated to Professor H. Behnke on his 75th birthday

In the first part of this paper we study the asymptotic cone of an immersion $M^n \to \tilde{M}^{n+1}(c)$, and prove that such an immersion is minimal if and only if there exist n orthogonal asymptotic directions u_i. This will imply that the sectional curvatures $K(u_i, u_j)$ are less than or equal to c. This is a stronger version of a theorem stating that the Ricci curvature is less than or equal to $(n-1)c$ for a minimal immersion, and gives a metric condition for M^n to be immersed minimally in $\tilde{M}^{n+1}(c)$. This can be generalized to the case of codimension p if the curvature of the normal bundle vanishes.

In the second part of this paper we classify all conformally euclidean minimal hypersurfaces of euclidean space, and show that there is only one surface of revolution, a generalized catenoid, which belongs to this class. All results in this paper are of local nature.

1. Preliminaries

We consider an n-dimensional manifold M^n immersed or imbedded in an $(n+p)$-dimensional manifold $\tilde{M}^{n+p}(c)$ of constant sectional curvature $c: M^n \to \tilde{M}^{n+p}(c)$. The metric on $\tilde{M}^{n+p}$ and its Levi-Civita connection D induce a Riemannian metric on M^n, both metrics being denoted by $\langle\ ,\ \rangle$, and its Levi-Civita connection ∇. We denote the normal bundle of the immersion by $N(M)$, tangent vectors to M^n by u, v, w, x, y, tangent vectors to $\tilde{M}^{n+p}$ by X, Y, Z, W, and normal vectors to M^n by $\xi, \eta, \cdots$.

The curvature tensor of M^n is denoted by $R(u, v)w$, the sectional curvature by $K(u, v)$, the Ricci curvature by Ric (u, v), and the scalar curvature by S. A splitting into the tangent and normal parts gives:

$$D_u\xi = -A_\xi u + \nabla^\perp_u \xi \ ,$$

where A_ξ is the second fundamental form, and $\nabla^\perp$ is a connection in the normal bundle whose curvature is denoted by $R^\perp(u, v)\xi$.

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In this paper we need two of the imbedding equations; one is the Gauss equation:

$$\langle R(u, v)w, x\rangle = c(\langle v, w\rangle\langle u, x\rangle - \langle u, w\rangle\langle v, x\rangle) \\ + \sum_i \langle A_{\xi_i}(v), w\rangle\langle A_{\xi_i}(u), x\rangle - \langle A_{\xi_i}(u), w\rangle\langle A_{\xi_i}(v), x\rangle ,$$

$\xi_1, \cdots, \xi_p$ being an orthonormal basis of the normal space, and the other one is the Ricci equation:

$$\langle R^{\perp}(u, v)\xi, \eta\rangle = \langle [A_\xi, A_\eta]u, v\rangle ,$$

where $[A_\xi, A_\eta] = A_\xi \circ A_\eta - A_\eta \circ A_\xi$. The Codazzi equation is not needed here. The use of the Ricci equation gives that if the curvature $R^\perp$ of the normal bundle vanishes, then $[A_\xi, A_\eta] = 0$ which means that all the A_ξ can be diagonalized simultaneously, independent of the normal ξ. For codimension 1 we write A instead of A_ξ where ξ is a locally chosen normal. The mean curvature normal η is defined by

$$\operatorname{trace} A_\xi = \langle \xi, \eta\rangle \qquad \text{for all normals } \xi .$$

We also need the contracted version of the Gauss equation, which gives the Ricci curvature and scalar curvature in terms of the second fundamental form. This is easily seen to be:

$$\operatorname{Ric}(u, v) = (n-1)c\langle u, v\rangle + \sum_i \langle A_{\xi_i}(u), v\rangle \operatorname{trace} A_{\xi_i} - \langle A_{\xi_i}(u), A_{\xi_i}(v)\rangle ,$$
$$S = n(n-1)c + \|\eta\|^2 - \|A\|^2 .$$

In the case of a minimal submanifold, i.e., $\eta = 0$, these equations show the following well-known theorem (see e.g. [8]).

Theorem 1. *If $M^n \to \tilde{M}^{n+p}(c)$ is a minimal immersion, then the Ricci curvature and scalar curvature satisfy*

$$\operatorname{Ric}(u, v) \le (n-1)c\langle u, v\rangle , \qquad S \le n(n-1)c ,$$

and the equality holds everywhere if and only if M^n is totally geodesic in $\tilde{M}^{n+p}(c)$.

This is the only known metric restriction for M^n to have a minimal immersion in $\tilde{M}^{n+p}(c)$. In § 3 we shall prove a somewhat stronger condition on the sectional curvature.

We now recall a few other known theorems for 1-codimensional immersions, which we will need later. One defines the type number $t(p)$ by $t(p) = \operatorname{rank} A(p)$ where A is the second fundamental form. The following is a theorem of Beez (see [1, p. 368]).

Theorem 2. *For an immersion $M^n \to \tilde{M}^{n+1}(c)$ the type number is an inner geometric invariant, i.e., can be expressed in terms of the curvature operator (except that the two cases $t(p) = 0$ and $t(p) = 1$ cannot be distinguished since* *in both cases the curvature is* 0). *If* $t(p) \geq 3$ *for all* p, *then the immersion is rigid* (*already locally*), *i.e., there exist no other immersions of* M^n *into* $\tilde{M}^{n+1}(c)$ *except the given one composed with isometries of the ambient space. If* $t(p) \leq 2$ *for all* p, *then there do exist other immersions.*

Theorem 3. *Suppose that* $M^n \to \tilde{M}^{n+1}(c)$ *is a minimal immersion.*

(i) *If* $c \leq 0$, *then* M^n *cannot have constant sectional or Ricci curvature unless it is totally geodesic.*

(ii) *If* $c > 0$, *then* M^n *cannot have constant sectional curvature unless it is totally geodesic, and if the Ricci curvature is constant then* M^n *is locally a product of two spaces of constant curvature. If in addition* $\tilde{M}^{n+1}(c)$ *is the unit* $(n+1)$-*sphere* S^{n+1}, *then the immersion is the standard one of* $M^n = S^m(\sqrt{m/n}) \times S^{n-m}(\sqrt{(n-m)/n})$ *into* S^{n+1} *as a minimal hypersurface where* $S^r(k)$ *is an* r-*sphere of radius* k. (*See* [3] *or* [1, p. 386].)

From the Gauss equations follows easily (see [9, p. 200])

Theorem 4. *If* $M^n \to \tilde{M}^{n+1}(c)$ *satisfies* $\mathrm{Ric}\,(u, v) = 0$ *for all* u, v, *then* $K(u, v) = 0$ *for all* u, v.

2. The asymptotic cone

In this section $\tilde{M}^{n+p}$ need not be of constant sectional curvature. If $u \in T_pM$ with $\langle A_\xi(u), u\rangle = 0$ then u is called an asymptotic direction with respect to ξ. The set of asymptotic directions in T_pM with respect to a fixed ξ is clearly a cone, i.e., with respect to u also λu is an asymptotic direction. This set is called the asymptotic cone.

Theorem 1. *Suppose that* $M^n \to \tilde{M}^{n+p}$ *is a minimal immersion, and* ξ *a fixed normal vector.*

(i) *If* $\langle A_\xi(u), v\rangle$ *is positive or negative semidefinite, then the asymptotic cone is a linear subspace of* T_pM *of dimension equal to the dimension of the kernel of* A_ξ.

(ii) *If* $\langle A_\xi(u), v\rangle$ *is indefinite, then the asymptotic cone consists of the linear subspace* $\ker A_\xi$, *orthogonal to it a cone, which is not contained in any lower dimensional linear subspace, and all sums of both kinds of vectors. This cone is a differentiable* $(n-1)$-*dimensional submanifold of* T_pM *except at the points of* $\ker A_\xi$.

Proof. (i) is clear.

(ii) The last statement follows from the fact that the function $f\colon (T_pM - \ker A_\xi) \to \boldsymbol{R}$ given by $f(u) = \langle A_\xi(u), u\rangle$ has 0 as a regular value; 0 actually appears as a value of f since $\langle A_\xi(u), v\rangle$ is indefinite. The part of the asymptotic cone orthogonal to $\ker A_\xi$ is not contained in any linear subspace, since the asymptotic vectors would otherwise form a $(n-1)$-dimensional linear subspace V^{n-1} of T_pM and $\langle A_\xi(u), v\rangle = 0$ for $u, v \in V^{n-1}$. Therefore there exists an $(n-2)$-dimensional subspace W^{n-2} of V^{n-1} contained in $\ker A_\xi$. But this cannot be possible since then A_ξ restricted to $W^\perp$ would have only one asymptotic direction.

Corollary 1. *If $\langle A_\xi(u), v\rangle$ is indefinite, then there exist n linearly independent asymptotic directions with respect to ξ. This is especially the case if* trace $A_\xi = 0$.

Theorem 2. *There exist n orthogonal asymptotic directions with respect to ξ if and only if* trace $A_\xi = 0$.

Proof. If n orthogonal asymptotic directions exist, then clearly trace $A_\xi = 0$. If trace $A_\xi = 0$, we do induction on n. If $n = 2$, let u, v be the two orthogonal eigenvectors of A_ξ: $A_\xi(u) = \lambda u$, $A_\xi(v) = \mu v$, trace $A_\xi = \lambda + \mu = 0$. Then $x = u + v$, $y = u - v$ are two orthogonal asymptotic directions since $\langle A_\xi(u + v), u + v\rangle = \lambda + \mu = \langle A_\xi(u - v), u - v\rangle = 0, \langle u + v, u - v\rangle = 0$. Now we want to prove the number of orthogonal asymptotic directions to be n. By Corollary 1 there exists an asymptotic direction x: $\langle A_\xi x, x\rangle = 0$. Take the linear subspace $W \subset T_pM$ orthogonal to x. Then there exists a symmetric linear mapping $\bar{A}_\xi: W \to W$ defined by: $\langle \bar{A}_\xi(u), v\rangle = \langle A_\xi(u), v\rangle$ for all $u, v \in W$ and trace $\bar{A}_\xi = \sum_i \langle \bar{A}_\xi(u_i), u_i\rangle = \sum_i \langle A_\xi(u_i), u_i\rangle = -\langle A_\xi(x), x\rangle = 0$ ($\{u_i\}$ is an orthonormal basis of W), and by induction hypothesis there exist $n - 1$ orthogonal asymptotic directions of $\bar{A}_\xi$ which are also asymptotic directions of A_ξ.

Corollary 2. *$M^n \to \tilde{M}^{n+1}$ is minimal if and only if there exist n orthogonal asymptotic directions.* (See also [10, p. 24].)

Remark. If the codimension is greater than 1, then the asymptotic cones for different ξ are different in general and coincident in special cases as follows: If $M^2 \to \tilde{M}^{2+p}$ is a minimal immersion, and ξ, η are fixed normal vectors with $[A_\xi, A_\eta] = 0$, trace $A_\xi =$ trace $A_\eta = 0$ and $A_\xi \neq 0$, $A_\eta \neq 0$, then the asymptotic cones of A_ξ and A_η coincide, In fact, ker $A_\xi =$ ker $A_\eta = 0$ and the asymptotic cones consist of $u + v$, $u - v$ where u, v are the common eigenvectors of A_ξ and A_η. For $n > 2$ this statement becomes false, even if one assumes ker $A_\xi =$ ker A_η in addition.

Theorem 2 can be generalized in this direction as follows:

Theorem 3. *If $M^n \to \tilde{M}^{n+p}$ is a minimal immersion, $[A_\xi, A_\eta] = 0$ and* trace $A_\xi = 0$ *for all ξ, η. Then there exist n orthogonal vectors which are asymptotic with respect to all ξ.*

Proof. If $n = 2$, then the claim follows as in Theorem 2 since all A_ξ have the same eigenvectors. To do the induction, take u_i to be the common eigenvectors of all A_ξ. Then $x = \sum_i u_i$ is an asymptotic direction with respect to all ξ. Take W orthogonal to x, and define $\bar{A}_\xi$ as in Theorem 2. Then $[\bar{A}_\xi, \bar{A}_\eta] =$ trace $\bar{A}_\xi = 0$, and by assumption there exist $n - 1$ orthogonal vectors in W asymptotic with respect to $\bar{A}_\xi$ and therefore also with respect to A_ξ.

Remark. For $n = 2$ the conclusion also follows from [11] (if $\tilde{M}^{n+p} = \tilde{M}^{n+p}(c)$) since then M^2 lies in a 3-dimensional totally geodesic subspace of $\tilde{M}^{n+p}$. It is not true, if we assume only $[A_\xi, A_\eta] = 0$, that there exist n linearly independent vectors asymptotic with respect to all A_ξ.

3. Curvature properties of minimal submanifolds

Theorem 1. *Suppose that $M^n \to \tilde{M}^{n+1}(c)$ is an immersion.*

(i) *$\langle A(u), v\rangle$ is semidefinite if and only if $K(u, v) \geq c$ for all $u, v \in T_pM$.*

(ii) *Suppose that $\langle A(u), v\rangle$ is indefinite. Then for any u, v in the asymptotic cone, $K(u, v) \leq c$ and $K(u, v) < c$ if and only if $u + v$ is not asymptotic.*

Proof. (i) Let u_i be the eigenvectors of A, i.e., $A(u_i) = \lambda_i u_i$. Then one shows easily through the Gauss equations that $K(u, v) \geq c$ for all u, v if and only if $K(u_i, u_j) \geq c$ for all i, j. But from the Gauss equation it follows

$$K(u_i, u_j) = \langle R(u_i, u_j)u_j, u_i\rangle = c + \langle A(u_i), u_i\rangle\langle A(u_j), u_j\rangle = \lambda_i\lambda_j + c \; .$$

If $K(u_i, u_j) \geq c$, then $\lambda_i\lambda_j \geq 0$ and therefore all $\lambda_i \geq 0$ or all $\lambda_i \leq 0$. If on the other hand all $\lambda_i \geq 0$ or ≤ 0, then $K(u_i, u_j) \geq c$ and from the above remark it follows that $K(u, v) \geq c$ for all u, v.

(ii) If $\langle A(u), u\rangle = \langle A(v), v\rangle = 0$, then $K(u, v) = c - \langle A(u), v\rangle^2 \leq c$, and if $\langle A(u + v), u + v\rangle = 2\langle A(u), v\rangle \neq 0$ then $K(u, v) < c$.

From Corollary 1 of § 2 follows

Corollary 1. *If $\langle A(u), v\rangle$ is indefinite, then there exist n linearly independent u_i with $K(u_i, u_j) \leq c$. If in addition* $\ker A = 0$, *then there exist n such u_i with $K(u_i, u_j) < c$.*

From Corollary 2 of § 2 follows

Theorem 2. *If $M^n \to \tilde{M}^{n+1}(c)$ is a minimal immersion so that* trace $A = 0$, *then there exist n orthogonal vectors u_i such that $K(u_i, u_j) \leq c$.*

In view of Theorem 1 of § 1, Ric $(u_i) \leq (n-1)c$ in this case. Since Ric $(u_i) = \sum_j K(u_i, u_j)$ for u_i orthonormal, Theorem 2 is stronger than the corresponding theorem with a condition on the Ricci curvature, and gives a metric condition for M^n to have a minimal immersion. One can generalize this to the case of codimension p.

Theorem 3. *If $M^n \to \tilde{M}^{n+p}(c)$ is a minimal immersion with $R^\perp = 0$, then there exists a basis $\{u_i\}$ of T_pM with $K(u_i, u_j) \leq c$ and $\langle u_i, u_j\rangle = 0$.*

Proof. According to the Ricci equations of § 1 the conditions of Theorem 3 of § 2 are satisfied, so that there exist n vectors u_i which are asymptotic with respect to all A_ξ. Thus the Gauss equations imply $K(u_i, u_j) \leq c$.

From $K(u_i, u_j) \leq c$ for all i, j it does not follow that $K(u, v) \leq c$ for all u, v since u_i, u_j are not eigenvectors of A. In fact, $K(u, v) \leq c$ is a very strong condition:

Theorem 4. *Suppose that $M^n \to \tilde{M}^{n+1}(c)$ is a minimal immersion. If $K(u, v) \leq c$ for all $u, v \in T_pM$ then* rank $A \leq 2$.

Proof. If $A(u_i) = \lambda_i u_i$, then $K(u_i, u_j) = c + \lambda_i\lambda_j$. From $K(u_i, u_j) \leq c$ it follows that $\lambda_i\lambda_j \leq 0$, and only two of the λ_i can be distinct from 0, for otherwise the sign of the λ_i cannot be chosen properly. Therefore rank $A \leq 2$.

Corollary 2. *Suppose that $M^n \to \tilde{M}^{n+1}(c)$ is a minimal immersion. Then $K(u, v) \leq c$ for all u, v if and only if* rank $A = 2$ *or* 0.

Proof. If rank $A = 1$, then trace $A \neq 0$. On the other hand, if rank $A = 0$, then $K(u, v) = c$ for all u, v; if rank $A = 2$, then the eigenvalues of A are $\lambda, -\lambda, 0$, and the sectional curvatures $K(u_i, u_j) = -\lambda^2 + c$ or c. Therefore $K(u_i, u_j) \leq c$, and $K(u, v) \leq c$ for all $u, v \in T_pM$.

This corollary characterizes the minimal immersions $M^n \to \tilde{M}^{n+1}(c)$ with sectional curvatures $\leq c$ (except the totally geodesic ones) as the ones with type number 2, and by the theorem of Beez, Theorem 2 of § 1 these are non-rigid.

To conclude this paragraph we propose a question with some partial results. In Theorem 3 of § 1 we saw that there exist no minimal immersions $M^n \to \tilde{M}^{n+1}(c)$, $c \leq 0$, of constant sectional or Ricci curvature. Do there exist minimal immersions of constant scalar curvature? The next theorem gives a partial result.

Theorem 5. *If $M^n \to \tilde{M}^{n+1}(c)$, $c \leq 0$, is a minimal immersion,* rank $A = 2$, *or $n - 1$ of the eigenvalues of A are equal, then the scalar curvature S cannot be constant, except for totally geodesic M^n.*

Proof. Since $S = n(n-1)c + \|\eta\|^2 - \|A\|^2$ and $\|A\|^2 = \sum_i \lambda_i^2$, if λ_i are the eigenvalues of A, then $S = n(n-1)c - \sum \lambda_i^2$. If rank $A = 2$, then $\lambda, -\lambda$, 0 are the eigenvalues of A. Therefore $S = n(n-1)c - 2\lambda^2$. If $S =$ constant, then $\lambda =$ constant. But according to [1] there can be only two different eigenvalues of A if the eigenvalues are constant. Thus $\lambda = 0$ and $A = 0$. If the eigenvalues of A are λ with multiplicity $n - 1$ and $-(n-1)\lambda$, then $S = n(n-1)c - n(n-1)\lambda^2$ and again $\lambda =$ constant. Therefore all eigenvalues of A are constant again, and [1, p. 374] shows that $(n-1)\lambda^2 = c \leq 0$. Hence $\lambda = 0$ and $A = 0$.

One would hope to prove that in the other cases the scalar curvature cannot be constant either. But the only thing we found is the following. From the equation for the scalar curvature it follows that $\|A\|$ is constant. Now one can use an equation of Simon's type [7, p. 372] to conclude that $\|\nabla A\|$ is constant, and also the norms of the higher covariant derivatives are constant. But this information does not seem to help very much. As Theorem 3 of § 1 shows, the situation is completely different if $c > 0$. See also [6].

4. Conformally euclidean minimal hypersurfaces

In this section we classify all conformally euclidean minimal hypersurfaces of a euclidean $(n+1)$-space R^{n+1}, $n \geq 4$. One starts with a theorem of Schouten [12]:

Theorem 1. *If $M^n \to R^{n+1}$, $n \geq 4$, is a conformally euclidean immersion, then at least $n - 1$ of the eigenvalues of A are equal.*

If one assumes now that the immersion is also minimal, then the eigenvalues of A must be λ with multiplicities $n - 1$ and $-(n-1)\lambda$. Note that, unless $A = 0$, rank $A = n$ so that such hypersurfaces are rigid. If one looks for examples of this sort, one first thinks of hypersurfaces obtained by rotating a plane curve. In more detail let $x_1 = u$, $x_2 = f(u)$ be a curve in the x_1x_2-plane lying in the halfspace $x_1 > 0$. If R^{n+1} has coordinates $x_1, \cdots, x_{n+1}$, then one can let this curve rotate around the x_2-axis, and thus obtain a hypersurface in R^{n+1}. In a chart $(u, \varphi_1, \cdots, \varphi_{n-1})$ this would look like:

$$x_1 = u \cos \varphi_1 , \quad x_2 = f(u) , \quad x_3 = u \sin \varphi_1 \cos \varphi_2, \cdots ,$$
$$x_n = u \sin \varphi_1 \cdots \cos \varphi_{n-1} , \quad x_{n+1} = u \sin \varphi_1 \cdots \sin \varphi_{n-1} .$$

One can easily see from the geometry of the situation that the coordinate directions are all eigenvectors of A and the eigenvalues of the φ_i-directions are all the same. Therefore, according to Theorem 1, all such hypersurfaces of revolution are conformally euclidean. We will now give the eigenvalues explicitly:

$$\lambda = \frac{f''}{\sqrt{(1+f'^2)^3}} , \qquad \mu = \frac{f'}{u\sqrt{1+f'^2}} ,$$

the second one being of multiplicity $n-1$. They are just the same as those for surfaces of revolution in R^3 and can be determined in the same way. If one requires that the hypersurface be minimal, f has to satisfy the differential equation

$$\frac{1}{f'}\sqrt{1+f'^2}\cdot \operatorname{trace} A = \frac{n-1}{u} + \frac{f''}{f'} - \frac{f''\cdot f'}{1+f'^2} = 0 ,$$

integration of which gives $f' = a(u^{2n-2} - a^2)^{-\frac{1}{2}}$. For $n = 2$ this is the catenoid in R^3, which is known to be the only minimal hypersurface of revolution in R^3. For $n > 2$ this is a *generalized catenoid*, the curve f being steeper as n is larger. We have therefore proved

Theorem 2. *The only minimal hypersurface of revolution (except the hyperplane) in euclidean space is the generalized catenoid.*

The generalized catenoid is also a conformally euclidean minimal hypersurface. We now prove

Theorem 3. *The generalized catenoid is the only conformally eeuclidean minimal hypersurface (except the hyperplane) in R^{n+1}, $n > 4$.*

Proof. A classification of Kulkarni [2] states that a conformally euclidean hypersurface belongs locally to one of the following 4 types:

1. $A = 0$ or $\operatorname{rank} A = 1$, and therefore $K(u, v) = 0$ for all u, v.
2. $A = \lambda\cdot\mathrm{Id}$, and therefore $M^n = S^n$.
3. The above surfaces of revolution.
4. A tube of constant length around a curve in R^{n+1}.

Clearly a minimal hypersurface (except the hyperplane) does not belong to type 1 or 2. It does not belong to type 4 either, since there the vectors, tangent to the tube but orthogonal to the curve, are $n-1$ eigenvectors of A with the same constant eigenvalue. Therefore the last eigenvalue of A has also to be constant, and the argument used in the proof of Theorem 5 of § 3 shows that this is impossible.

But we have already shown that the only minimal hypersurface which belongs to type 3 is the generalized catenoid.

Remark. Theorems 1 and 3 are false for $n = 3$. In fact, one can show that the cotton tensor [9, p. 92] of the above described catenoid $M^3 \to R^4$ does not vanish identically, so that M^3 is not conformally euclidean, although two of the eigenvalues of A are equal.

One can given another interesting example of surfaces of revolution. In Theorem 4 of § 1 we saw that for an immersion $M^n \to R^{n+1}$ if $\mathrm{Ric} \equiv 0$ then $K \equiv 0$. One then asks if there exists a hypersurface of R^{n+1} with zero scalar curvature but nonzero sectional curvature. This is indeed the case. An example is a surface of revolution. From $S = \|\eta\|^2 - \|A\|^2 = (\sum_i \lambda_i)^2 - \sum_i \lambda_i^2 = \sum_{i \neq j} \lambda_i \lambda_j$, where λ_i are the eigenvalues of A, it follows that if λ, μ are the two eigenvalues of A as above, then

$$S = (n-1)(n-2)\mu^2 + 2(n-1)\lambda\mu = \frac{n-1}{1+f'^2}\left(\frac{(n-2)f'^2}{u^2} + \frac{2f' \cdot f''}{u(1+f'^2)}\right).$$

If one requires $S = 0$, then one has the differential equation $f'' = -\dfrac{n-2}{2u}$ $\cdot f'(1 + f'^2)$ with the solution $f' = \pm(au^{n-2} - 1)^{-\frac{1}{2}}$. This is a surface of revolution with $S = 0$ and $K \neq 0$.

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