J. DIFFERENTIAL GEOMETRY
11 (1976) 317-319

COMPACT KAEHLER MANIFOLDS WITH CONSTANT GENERALIZED SCALAR CURVATURE

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1. Introduction

Let M be an n-dimensional Kaehler manifold with fundamental 2-form Φ and Ricci 2-form γ. Following [2], we call M a *cohomological Einstein manifold* if $[\gamma] = a \cdot [\Phi]$ for some a, where $[*]$ denotes the cohomology class represented by $*$. The first Chern class of M is represented by γ.

It is well-known that *a compact cohomological Einstein Kaehler manifold is Einsteinian if the scalar curvature is constant*. The purpose of this paper is to generalize this result.

In [2], the second author introduced the notion of *generalized scalar curvatures*: Let $\omega^1, \cdots, \omega^n$ be a local field of unitary coframes, so that the Kaehler metric of M is given by $g = \frac{1}{2} \sum (\omega^\alpha \otimes \bar{\omega}^\alpha + \bar{\omega}^\alpha \otimes \omega^\alpha)$. Let $S = \frac{1}{2} \sum (R_{\alpha\bar{\beta}} \omega^\alpha \otimes \bar{\omega}^\beta + \bar{R}_{\alpha\bar{\beta}} \bar{\omega}^\alpha \otimes \omega^\beta)$ be the Ricci tensor of M. We define n scalars $\rho_1, \cdots, \rho_n$ by

$$\det (\delta_{\alpha\beta} + tR_{\alpha\bar{\beta}}) = 1 + \sum_{k=1}^{n} \rho_k t^k .$$

If we denote the scalar curvature of M by ρ, then it is easily seen that $\rho = 2\rho_1$. It is also clear that $\rho_n = \det (R_{\alpha\bar{\beta}})$.

Our main theorem is the following.

Theorem. *Let M be an n-dimensional compact Kaehler manifold ($n \geq 2$). If*

(i) *ρ_k is constant,*

(ii) *$[\gamma^k] = a \cdot [\Phi^k]$ for some a,*

(iii) *$\operatorname{rank} (R_{\alpha\bar{\beta}}) \geq k$ (or equivalently $\rho_k \neq 0$) for some $k < n$,*

then M is Einsteinian.

Assumption (iii) is redundant if $k = 1$, but it is essential if $k > 1$. Immediately from the above theorem we have

Corollary. *Let M be an n-dimensional compact Kaehler manifold ($n \geq 2$). If*

Received August 9, 1974. The first author was partially supported by NSF Grant GP-36684, and the second author by NSF Grant GP-36684 and the Matsunaga Science Foundation.

(i) ρ_k *is constant,*
(ii) *M is cohomologically Einsteinian,*
(iii) rank $(R_{\alpha\bar\beta}) \geq k$ *(or equivalently* $\rho_k \neq 0$*) for some* $k < n$,
then M is Einsteinian.

2. Proof of the theorem

Let Φ be the fnndamental 2-form of M, that is, a closed 2-form defined by

$$\Phi = \tfrac{1}{2}\sqrt{-1} \sum \omega^\alpha \wedge \bar\omega^\alpha .$$

Let γ be the Ricci 2-form of M, that is, a closed 2-form defined by

$$\gamma = \frac{\sqrt{-1}}{4\pi} \sum R_{\alpha\bar\beta}\omega^\alpha \wedge \bar\omega^\beta .$$

Let Λ be the operator of interior product by Φ. Then we have (cf. [2])

$$\Lambda^k\gamma^k = \frac{k!k!}{(2\pi)^k}\rho_k ,$$

which, together with assumption (i), implies that

$$(1) \qquad d\Lambda^k\gamma^k = 0 .$$

Let δ be the codifferential operator, and C the operator defined by $C\alpha = (\sqrt{-1})^{r-s}\alpha$, where α is a form of bidegree (r, s). Then they satisfy $d\Lambda^k - \Lambda^k d = kC^{-1}\delta C\Lambda^{k-1}$. Therefore from (1) and the fact that γ is closed we obtain

$$(2) \qquad \delta\Lambda^{k-1}\gamma^k = 0 .$$

We prove the following general lemma.

Lemma. *Let* η *be a form of bidegree* (p, q) *with* $p > 1$ *and* $q > 1$. *If* $\Lambda\delta\eta = 0$, *then* $\delta\eta = 0$.

Proof. If we denote the star isomorphism by $*$, then $\Lambda\delta\eta = 0$ is equivalent to $*\Lambda*d*\eta = 0$. If we denote the dual operator of Λ by L, then $*\Lambda*d*\eta = 0$ is equivalent to $Ld*\eta = 0$. Since L is an isomorphism (cf. for example [1]), $Ld*\eta = 0$ is equivalent to $d*\eta = 0$. Applying $*$ we obtain $*d*\eta = 0$ which is equivalent to $\delta\eta = 0$. q.e.d.

Since $\Lambda\delta = \delta\Lambda$, it follows from (2) that $\Lambda\delta\Lambda^{k-2}\gamma^k = 0$. Therefore by Lemma we have

$$\delta\Lambda^{k-2}\gamma^k = 0 .$$

Repeatedly applying this process we finally obtain

$$\delta\gamma^k = 0 ,$$

which, together with the fact that $d\gamma^k = 0$, implies that γ^k is harmonic. Therefore assumption (ii) yields that

$$\gamma^k = a\Phi^k . \tag{3}$$

At each point of M, we can choose a unitary coframe $\omega^1, \cdots, \omega^n$ with respect to which $(R_{\alpha\bar\beta})$ is of the form

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

so that $\gamma = \dfrac{\sqrt{-1}}{4\pi} \sum \lambda_\alpha \omega^\alpha \wedge \bar\omega^\alpha$. Therefore

$$\gamma^k = \left(\frac{\sqrt{-1}}{4\pi}\right)^k k! \sum_{\alpha_1 < \cdots < \alpha_k} \lambda_{\alpha_1} \cdots \lambda_{\alpha_k} \omega^{\alpha_1} \wedge \bar\omega^{\alpha_1} \wedge \cdots \wedge \omega^{\alpha_k} \wedge \bar\omega^{\alpha_k} .$$

Hence from (3) we obtain the following system of simultaneous equations

$$\begin{aligned} &\lambda_1 \cdots \lambda_{k-1}(\lambda_k - \lambda_{k+1}) = 0 , \\ &\lambda_1 \cdots \lambda_{k-1}(\lambda_k - \lambda_{k+2}) = 0 , \\ &\cdots\cdots\cdots\cdots\cdots , \end{aligned} \tag{4}$$

which consists of $\binom{n}{k-1}\binom{n-k+1}{2}$ equations. It is easily seen that, under assumption (iii), (4) implies that $\lambda_1 = \cdots = \lambda_n$. Therefore M is Einsteinian.

Remark. Assumption (iii) is essential if $k > 1$. In fact, let $M = P_{k-1}(C) \times T^{n-k+1}$, where $P_{k-1}(C)$ denotes a $(k-1)$-dimensional complex projective space with the Fubini-Study metric, and T^{n-k+1} denotes an $(n-k+1)$-dimensional complex torus with the flat metric. Then M satisfies assumptions (i) and (ii), but M is not Einsteinian.

References

[1] S. I. Goldberg, *Curvature and homology*, Academic Press, New York, 1962.
[2] K. Ogiue, *Generalized scalar curvatures of cohomological Einstein Kaehler manifolds*, J. Differential Geometry **10** (1975) 201–205.

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