J. DIFFERENTIAL GEOMETRY
11 (1976) 251-257

TOTALLY REAL SUBMANIFOLDS IN A KAEHLER MANIFOLD

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1. Introduction

Let $\bar{M}$ be a Kaehler manifold of dimension $2(n + p)$, $p \geq 0$, and M an n-dimensional Riemannian manifold. Let J be the complex structure of $\bar{M}$. We call M a *totally real* submanifold of $\bar{M}$ if M admits an isometric immersion into $\bar{M}$ such that

$$J(T_m(M)) \subset T_m(M)^{\perp} ,$$

where $T_m(M)$ denotes the tangent space of M at m, and $T_m(M)^{\perp}$ the normal space at m. Denote by $\bar{M}^{n+p}(c)$ a $2(n + p)$-dimensional Kaehler manifold of constant holomorphic sectional curvature c. Let h be the second fundamental form of M in $\bar{M}$, and denote by S the square of the length of the second fundamental form h. When $p = 0$, Chen-Ogiue [2] proved

Theorem A. *Let M be an n-dimensional compact totally real minimal submanifold immersed in $\bar{M}^n(c)$. If*

$$S < \frac{n(n+1)}{4(2n-1)}c ,$$

then M is totally geodesic.

Theorem B. *Let M be an n-dimensional totally real minimal submanifold immersed in $\bar{M}^n(c)$. If the sectional curvature of M is constant, then M is either totally geodesic or has nonpositive sectional curvature. Moreover, if the second fundamental form of the immersiom is parallel, then M is totally geodesic or flat.*

Theorem B is a generalization of Houh's theorem [4]. Moreover, Ludden-Okumura-Yano [5] studied an n-dimensional totally real minimal submanifold M of CP^n satisfying

$$(1.1) \qquad S = \frac{n(n+1)}{2n-1} ,$$

where CP^n denotes an n-dimensional complex projective space of constant holomorphic sectional curvature 4, and gave an example of totally real

Communicated by K. Yano, July 24, 1974, and, in revised form, October 25, 1974.

minimal surface immersed in CP^2, which just satisfies the above condition (1.1). Let S^1 be a unit sphere of dimension 1. Then $S^1 \times S^1$ is a compact minimal totally real surface immersed in CP^2 with $S = 2$. Concerning this Ludden-Okumura-Yano [5] proved

Theorem C. *If M is a compact n-dimensional* ($n > 1$) *minimal totally real submanifold of CP^n satisfying* (1.1), *then $n = 2$ and $M = S^1 \times S^1$.*

The purpose of this paper is to study a compact n-dimensional totally real submanifold M immersed in CP^n satisfying certain condition on the second fundamental form h of M, which reduces to condition (1.1) if M is minimally immersed in CP^n. Our method is based on that of Braidi-Hsiung [1].

2. Local formulas

Let $\bar{M}$ be a Kaehler manifold of dimension $2n$, and M an n-dimensional totally real submanifold immersed in $\bar{M}$. Choose a local field of orthonormal frames $e_1, \cdots, e_{2n}$ in $\bar{M}$ such that, restricted to M, the vectors $e_1, \cdots, e_n$ are tangent to M (and hence the remaining vectors $e_{n+1}, \cdots, e_{2n}$ are normal to M). Unless stated otherwise, we shall make use of the following convention on the ranges of indices:

$$1 \leq A, B, C, \cdots \leq 2n\ , \quad 1 \leq i, j, k, \cdots \leq n\ , \quad n+1 \leq a, b, c, \cdots \leq 2n\ ,$$

and when a letter appears in any term as a subscript and a superscript, it is understood that this letter is summed over its range. Denote Je_i by e_{i*} for $i = 1, \cdots, n$, and let $w^1, \cdots, w^{2n}$ be the field of dual frames with respect to the frame field of $\bar{M}$ chosen above. Then the structure equations of $\bar{M}$ are

$$dw^A = -w^A_B \wedge w^B\ , \tag{2.1}$$

$$w^A_B + w^B_A = 0\ , \quad w^i_j = w^{i*}_{j*}\ , \quad w^{i*}_j = w^{j*}_i\ , \tag{2.2}$$

$$\begin{gathered} dw^A_B = -w^A_C \wedge w^C_B + \Phi^A_B\ , \qquad \Phi^A_B = \tfrac{1}{2} K^A_{BCD} w^C \wedge w^D\ , \\ K^A_{BCD} + K^A_{BDC} = 0\ . \end{gathered} \tag{2.3}$$

Restriction of these frames to M gives

$$w^a = 0\ . \tag{2.4}$$

Since $0 = dw^a = -w^a_i \wedge w^i$, by Cartan's lemma we may write

$$w^a_i = h^a_{ij} w^i\ , \qquad h^a_{ij} = h^a_{ji}\ , \tag{2.5}$$

and from (2.2) it follows that

$$h^{i*}_{jk} = h^{j*}_{ik}\ . \tag{2.6}$$

Using these formulas we obtain

$$dw^i = -w^i_j \wedge w^j , \qquad w^i_j + w^j_i = 0 , \tag{2.7}$$

$$dw^i_j = -w^i_k \wedge w^k_j + \Omega^i_j , \qquad \Omega^i_j = \tfrac{1}{2} R^i_{jkl} w^k \wedge w^l , \tag{2.8}$$

$$R^i_{jkl} = K^i_{jkl} + \sum_a (h^a_{ik} h^a_{jl} - h^a_{il} h^a_{jk}) . \tag{2.9}$$

The forms (w^i_j) define the Riemannian connection of M. We call $h^a_{ij} w^i w^j e_a$ the second fundamental form of the immersion. Sometimes the second fundamental form is denoted by its components h^a_{ij}. $(\sum_i h^a_{ii} e_a)/n$ is called the mean curvature normal, and an immersion is said to be *minimal* if its mean curvature normal vanishes identically, i.e., if $\sum_i h^a_{ii} = 0$ for all a. Define the covariant derivative h^a_{ijk} of h^a_{ij}, h^a_{ijkl} and the Laplacian Δh^a_{ij} of the second fundamental form h^a_{ij} respectively by

$$h^a_{ijk} w^k = dh^a_{ij} - h^a_{il} w^l_j - h^a_{lj} w^l_i + h^b_{ij} w^a_b , \tag{2.10}$$

$$h^a_{ijkl} w^l = dh^a_{ijk} - h^a_{ljk} w^l_i - h^a_{ilk} w^l_j - h^a_{ijl} w^l_k + h^b_{ijk} w^a_b , \tag{2.11}$$

$$\Delta h^a_{ij} = \sum_k h^a_{ijkk} . \tag{2.12}$$

If $\bar{M}$ is locally symmetric, then we have the following equation (Braidi-Hsiung [1, p. 238]):

$$\begin{aligned} \sum_{a,i,j} h^a_{ij} \Delta h^a_{ij} = \sum_{a,i,j,k} & (h^a_{ij} h^a_{kkij} - K^a_{ijb} h^a_{ij} h^b_{kk} + 4K^a_{bki} h^b_{jk} h^a_{ij} \\ & - K^a_{kbk} h^a_{ij} h^b_{ij} + 2K^m_{kik} h^a_{mj} h^a_{ij} + 2K^m_{ijk} h^a_{mk} h^a_{ij}) \\ - \sum_{a,b,i,j,k,l} & [(h^a_{ik} h^b_{jk} - h^a_{jk} h^b_{ik})(h^a_{il} h^b_{jl} - h^a_{jl} h^b_{il}) \\ & + h^a_{ij} h^a_{kl} h^b_{ij} h^b_{kl} - h^a_{ij} h^a_{ki} h^b_{kj} h^b_{ll}] . \end{aligned} \tag{2.13}$$

3. Integral formulas

In this section we assume that $\bar{M}$ is a Kaehler manifold of dimension $2n$ and constant holomorphic sectional curvature c. Then the curvature tensor of $\bar{M}$ is given by

$$K^A_{BCD} = \tfrac{1}{4} c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} + J_{AC}J_{BD} - J_{AD}J_{BC} + 2J_{AB}J_{CD}) , \tag{3.1}$$

where δ_{AC} denotes the Kronecker deltas. Let M be an n-dimensional totally real submanifold immersed in $\bar{M}^n(c)$. From the condition on the dimensions of M and $\bar{M}$ it follows that $e_{1*}, \cdots, e_{n*}$ is a frame for $T_m(M)^\perp$. Noticing this and using (2.6) and (3.1) we can reduce (2.13) to

$$\sum_{a,i,j} h^a_{ij} \Delta h^a_{ij} = \sum_{a,i,j,k} h^a_{ij} h^a_{kkij} + \tfrac{1}{4}(n+1)c \sum_{a,i,j} h^a_{ij} h^a_{ij} - \tfrac{1}{2} c \sum_a \left(\sum_i h^a_{ii} \right)^2$$

$$(3.2)\qquad + \sum_{a,b,i,j,k,l} (h^a_{ij}h^b_{jk}h^a_{ki}h^b_{ll} - h^a_{ij}h^b_{ij}h^a_{kl}h^b_{kl}) - \sum_{a,b,i,j,k,l} (h^a_{ik}h^b_{kj} - h^b_{ik}h^a_{kj})(h^a_{il}h^b_{lj} - h^b_{il}h^a_{lj}) \ .$$

For each a, let H_a denote the symmetric matrix (h^a_{ij}). Then (3.2) can be written as

$$(3.3)\qquad \begin{aligned}\sum_{a,i,j} h^a_{ij}\Delta h^a_{ij} = \sum_{a,i,j,k} h^a_{ij}h^a_{kkij} &+ \sum_a [\tfrac{1}{4}(n+1)c \operatorname{Tr} H^2_a - \tfrac{1}{2}c(\operatorname{Tr} H_a)^2] \\ &+ \sum_{a,b} \{\operatorname{Tr}(H_aH_b - H_bH_a)^2 - [\operatorname{Tr}(H_aH_b)]^2 \\ &+ \operatorname{Tr} H_b \operatorname{Tr}(H_aH_bH_a)\} \ ,\end{aligned}$$

where $\operatorname{Tr} H^2_a$ denotes the trace of the matrix H^2_a. (3.3) was obtained by Chen-Ogiue [2] for a totally real minimal submanifold M^n immersed in $\bar{M}^n(c)$. Now set

$$S_{ab} = \sum_{i,j} h^a_{ij}h^b_{ij} \ , \quad S_a = S_{aa} \ , \quad S = \sum_a S_a \ ,$$

so that S_{ab} is a symmetric $(n \times n)$-matrix and can be assumed to be diagonal for a suitable choice of $e_{n+1}, \cdots, e_{2n}$, and S is the square of the length of the second fundamental form h^a_{ij} of M. Since $\operatorname{Tr} A^2 = \sum_{i,j} (a_{ij})^2$ is independent of the choice of a frame, for any symmetric $A = (a_{ij})$ we can rewrite (3.3) as

$$(3.4)\qquad \begin{aligned}\sum_{a,i,j} h^a_{ij}\Delta h^a_{ij} = \sum_{a,i,j,k} h^a_{ij}h^a_{kkij} &+ \tfrac{1}{4}(n+1)cS - \sum_a S^2_a \\ &+ \sum_{a,b} \operatorname{Tr}(H_aH_b - H_bH_a)^2 - \tfrac{1}{2}c \sum_a (\operatorname{Tr} H_a)^2 \\ &+ \sum_{a,b} \operatorname{Tr} H_b \operatorname{Tr}(H_aH_bH_a) \ .\end{aligned}$$

For later development we need the following lemma (see [1] and [3]):

Lemma 1. *Let A and B be symmetric $(n \times n)$-matrices. Then*

$$-\operatorname{Tr}(AB - BA)^2 \leq 2 \operatorname{Tr} A^2 \operatorname{Tr} B^2 \ ,$$

and the equality holds for nonzero matrices A and B if and only if A and B can be transformed simultaneously by an orthogonal matrix into scalar multiples of $\bar{A}$ and $\bar{B}$ respectively, where

$$\bar{A} = \left[\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & 0 \\ \hline 0 & & 0 \end{array}\right] , \qquad \bar{B} = \left[\begin{array}{cc|c} 1 & 0 & \\ 0 & -1 & 0 \\ \hline 0 & & 0 \end{array}\right] .$$

Moreover, if A_1, A_2, A_3 are symmetric $(n \times n)$-matrices such that

$$-\operatorname{Tr}(A_aA_b - A_bA_a)^2 = 2 \operatorname{Tr} A^2_a \operatorname{Tr} A^2_b \ , \quad 1 \leq a, b \leq 3 \ , \quad a \neq b \ ,$$

then at least one of the matrices A_a must be zero.

By applying Lemma 1 we obtain

$$
\begin{aligned}
&-\sum_{a,b} \operatorname{Tr}(H_aH_b - H_bH_a)^2 + \sum_a S_a^2 - \frac{1}{4}(n+1)cS \\
(3.5) \qquad &\leq 2 \sum_{a\neq b} S_aS_b + \sum_a S_a^2 - \frac{1}{4}(n+1)cS \\
&= \left[\left(2 - \frac{1}{n}\right)S - \frac{1}{4}(n+1)c\right]S - \frac{1}{n}\sum_{a>b}(S_a - S_b)^2 ,
\end{aligned}
$$

which, together with (3.4), implies

$$
(3.6) \qquad -\sum_{a,i,j} h_{ij}^a \Delta h_{ij}^a \leq W - \sum_{a,i,j,k} h_{ij}^a h_{kkij}^a ,
$$

where we have put

$$
\begin{aligned}
(3.7) \qquad W = &\left[\left(2 - \frac{1}{n}\right)S - \frac{1}{4}(n+1)c\right]S + \frac{1}{2}c\sum_a (\operatorname{Tr} H_a)^2 \\
&- \sum_{a,b} \operatorname{Tr} H_b \operatorname{Tr}(H_aH_bH_a) .
\end{aligned}
$$

Theorem 1. *Let M be an n-dimensional compact oriented totally real submanifold immersed in $\overline{M}^n(c)$. Then*

$$
(3.8) \qquad \int_M \left[W - \sum_a (\operatorname{Tr} H_a)\Delta(\operatorname{Tr} H_a)\right] *1 \geq 0 ,
$$

where $*1$ denotes the volume element of M.

Proof. First we obtain

$$
\int_M \sum_{a,i,j,k} (h_{ijk}^a)^2 *1 = -\int_M \sum_{a,i,j} h_{ij}^a \Delta h_{ij}^a *1 \geq 0 .
$$

On the other hand, we have (Braidi-Hsiung [1, p. 241])

$$
\int_M \sum_{a,i,j,k} h_{ij}^a h_{kkij}^a *1 = \int_M \sum_a (\operatorname{Tr} H_a)\Delta(\operatorname{Tr} H_a) *1 .
$$

From these equations and (3.6) follows the inequality

$$
(3.9) \qquad \int_M \left[W - \sum_a (\operatorname{Tr} H_a)\Delta(\operatorname{Tr} H_a)\right] *1 \geq \int_M \sum_{a,i,j,k} (h_{ijk}^a)^2 *1 \geq 0 ,
$$

which is just (3.8).

As a special case of Theorem 1 we have the following theorem which was proved essentially by Chen-Ogiue [2].

Theorem 2. *Let M be an n-dimensional compact oriented totally real minimal submanifold immersed in $\overline{M}(c)$. Then*

(3.10) $$\int_M \left[\left(2 - \frac{1}{n}\right)S - \frac{1}{4}(n+1)c\right] S * 1 \geq 0 .$$

4. Main theorems

In this section we assume that M is an n-dimensional compact oriented totally real submanifold immersed in $\bar{M}^n(c)$, $n > 1$, and that M is not totally geodesic in $\bar{M}$ but satisfies

(4.1) $$\int_M \left[W - \sum_a (\mathrm{Tr}\, H_a)\Delta(\mathrm{Tr}\, H_a)\right] * 1 = 0 .$$

Then (3.9) implies that $h^a_{ijk} = 0$, i.e., the second fundamental form of M is covariant constant, so that $\Delta h^a_{ij} = 0$, and all terms on both sides of (3.6) vanish. It follows that inequalities (3.4) and (3.5) imply

(4.2) $$\frac{1}{n} \sum_{a>b} (S_a - S_b)^2 = 0 ,$$

(4.3) $$-\mathrm{Tr}(H_a H_b - H_b H_a)^2 = 2\, \mathrm{Tr}\, H_a^2\, \mathrm{Tr}\, H_b^2$$

for any $a \neq b$. Then by Lemma 1 we may assume that $H_a = 0$ for $a = n+3, \cdots, 2n$, which shows that $S_a = 0$ for $a = n+3, \cdots, 2n$. But by (4.2) we can see that $S_a = S_b$ for any a, b. Since M is not totally geodesic, $n = 2$ and therefore by using Lemma 1 we can assume that

(4.4) $$H_{n+1} = \lambda \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \qquad H_{n+2} = \mu \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} .$$

From this it follows that M is a minimal surface immersed in $\bar{M}^2(c)$. Since the second fundamental form h of M^2 is covariant constant, the sectional curvature of M^2 is constant and hence M^2 is flat by Theorem B. On the other hand, by using (2.10) we obtain

(4.5) $$dh^a_{ij} = h^a_{il} w^l_j + h^a_{lj} w^l_i - h^b_{ij} w^a_b .$$

Setting $a = 3$, $i = 1$, $j = 2$, we see that $d\lambda = dh^3_{12} = 0$, which means that λ is constant. Similarly, setting $a = 4$ and $i = j = 1$, we see that μ is constant. By (4.2) we get $\lambda^2 = \mu^2$, and since $S = \frac{1}{2}c$ we have $\lambda^2 + \mu^2 = \frac{1}{4}c$ so that $\lambda^2 = \frac{1}{8}c$. Since M is not totally geodesic, we may assume that $c > 0$ and $-\lambda = \mu = \frac{1}{2}\sqrt{c/2}$. Then (2.5) and (4.4) imply

$$w^3_1 = \lambda w^2 , \quad w^3_2 = \lambda w^1 , \quad w^4_1 = \mu w^1 , \quad w^4_2 = -\mu w^2 .$$

On the other hand, setting $a = 3$, $i = j = 1$ in (4.5), we have $w^3_4 = (2\lambda/\mu) w^2_1 = 2w^1_2$. Hence we obtain the following

Theorem 3. *Let M be an n-dimensional compact oriented totally real submanifold immersed in $\bar{M}^n(c)$, $n > 1$, such that M is not totally geodesic but satisfies condition* (4.1). *Then M is a flat surface minimally immersed in $\bar{M}^2(c)$, and with respect to an adapted dual orthonormal frame field w^1, w^2, w^3, w^4, the connection form (w_B^A) of $\bar{M}^2(c)$, restricted to M, is given by*

$$\begin{bmatrix} 0 & w_2^1 & -\lambda w^2 & -\mu w^1 \\ -w_2^1 & 0 & -\lambda w^1 & \mu w^2 \\ \lambda w^2 & \lambda w^1 & 0 & 2w_2^1 \\ \mu w^1 & -\mu w^2 & -2w_2^1 & 0 \end{bmatrix}, \qquad -\lambda = \mu = \frac{1}{2}\sqrt{\frac{c}{2}}\,.$$

Now we take an n-dimensional complex projective space CP^n of constant holomorphic sectional curvature 4 as an ambient space. Then Theorem 3 implies

Theorem 4. *Let M be an n-dimensional compact oriented totally real submanifold immersed in CP^n, $n > 1$, such that M is not totally geodesic but satisfies condition* (4.1). *Then $n = 2$ and $M = S^1 \times S^1$.*

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