J. DIFFERENTIAL GEOMETRY
11 (1976) 133-146

KÄHLER MANIFOLDS WITH POSITIVE CURVATURE

C. J. FERRARIS

1. Introduction

This paper is concerned with the study of complex manifolds. Our main result is motivated by the following conjecture: Is a compact Kähler manifold with positive sectional curvature holomorphically equivalent to a complex projective space? The conjecture has been verified until now for complex dimension less than or equal to three, [10].

The technique used throughout this work is to consider the variational properties of the geodesic distance r of a Riemannian manifold M, where the latter is considered as a real-valued function in $M \times M$.

In § 2 we introduce the open dense submanifold of $M \times M$, denoted by $M \vee M$, which is the complement of the union of the diagonal submanifold of $M \times M$ and the set of cut pairs of M. The tangent bundle of $M \vee M$ splits into a direct sum of two subbundles V^+ and V^- both of rank dim (M). In particular, this decomposition is useful in the study of the second fundamental form of the boundary of the metric tubular neighborhoods (c-neighborhoods) of the diagonal of $M \times M$.

Since the proof of our main result requires the use of some elements on the geometry of geodesics, we include the latter in §§ 3 and 4 in the more general context of Riemannian manifolds with positive sectional curvature. Theorem 1 in § 4 gives some information about the "position" of local geodesic sprays of V^+ with respect to the metric tubular neighborhoods.

Finally, in § 5 we prove our main theorem, namely, Theorem 3: Let M be a connected compact Kähler manifold of complex dimension n with positive holomorphic bisectional curvature, then any closed n-dimensional complex analytic subvariety V (possibly singular) of $M \times M$ intersects the diagonal.

Although an alternative proof might be obtained by making use of the theory of deformations to simplify the singularities of V, we prefer a more direct method consisting of using an extended notion of the second fundamental form which applies to singular varieties.

The author wishes to acknowledge Professor E. Calabi for suggesting the problem and for his encouragement along the preparation of this work.

Communicated by E. Calabi July 17, 1974.

2. Preliminaries

Let M be a connected complete n-dimensional Riemannian manifold of class C^∞, and let us consider the product manifold $M \times M$. We distinguish in $M \times M$ the diagonal submanifold denoted by M_Δ and the set of cut pairs of M denoted by M_{cut}. The latter is a closed set of topological dimension $\leq 2n - 1$.

We consider the open submanifold

$$M \vee M = (M \times M) \backslash (M_\Delta \cup M_{cut}) .$$

The structural group of the tangent bundle of $M \vee M$, denoted by $T(M \vee M)$, can be reduced to $0(n-1)$ in a natural way as follows. Let $(p_0, q_0) \in M \vee M$, and let γ be the shortest geodesic in M between p_0 and q_0, which is unique since (p_0, q_0) is not a cut pair of M. Let us take any orthonormal frame $\{e'_i(p_0), 1 \leq i \leq n\}$ of M at p_0 such that $e'_1(p_0)$ is the unit tangent vector to γ at p_0. Then we get an orthonormal frame $\{e''_i(q_0), 1 \leq i \leq n\}$ of M at q_0 by parallel translation of the frame $\{e'_i(p_0)\}$ along γ from p_0 to q_0. After identifying $e'_i(p_0)$ with $(e'_i, 0)(p_0, q_0)$ and $e''_i(q_0)$ with $(0, e''_i)(p_0, q_0)$, we set

$$e_{\pm i}(p_0, q_0) = \frac{1}{\sqrt{2}}(e'_i(p_0) \pm e''_i(q_0)) , \qquad 1 \leq i \leq n .$$

Let us denote by $F'(M \vee M)$ the set of all frames obtained in this way. We observe that for each element in $F'(M \vee M)$ the pair (p_0, q_0) determines uniquely the vectors $e_{\pm 1}(p_0, q_0)$, and the rest of the $e_{\pm i}$'s are determined up to an element of $0(n-1)$ which acts on $F'(M \vee M)$ as follows:

$$ge_{\pm 1}(p_0, q_0) = e_{\pm 1}(p_0, q_0) ,$$
$$ge_{\pm i}(p_0, q_0) = \frac{1}{\sqrt{2}}(ge'_i(p) \pm ge''_i(q)) , \qquad i \geq 2 ,$$

for every $g \in 0(n-1)$ and $\{e_{\pm i}(p, q)\} \in F'(M \vee M)$. Accordingly the set $F'(M \vee M)$ becomes a principal bundle with $0(n-1)$ as its structural group, and is a subbundle of $F(M \vee M)$, the bundle of all orthonormal frames of $M \vee M$. The action of $0(n-1)$ as well as the product structure of $M \vee M$ as an open submanifold of $M \times M$ defines certain invariant subspaces of its tangent bundle. Among these we distinguish the following

$$V^+ = \sum_{i=1}^{n} Re_{\pm i} .$$

We shall denote by r the geodesic distance in M regarded as a real-valued function in $M \times M$. For any positive real number c, let us set

$$N_c = \{(p, q) \in M \times M / r(p, q) \leq c\} ,$$
$$W_c = \partial N_c = \{(p, q) \in M \times M / r(p, q) = c\} .$$

We call each N_c a c-neighborhood of the diagonal M_Δ in $M \times M$.

Remarks. 1. $M \bigvee M$ is the maximal open submanifold in $M \times M$ on which the function r is of class C^∞.

2. For each positive real number c, $W_c \cap (M \bigvee M)$ is a $(2n - 1)$-dimensional manifold.

3. $e_{-1}(p_0, q_0)$ is the "inward" unit normal vector to W_c at (p_0, q_0), where $c = r(p_0, q_0)$.

4. Since M is complete, the product manifold $M \times M$ is also complete, so that the exponential map of $M \times M$ is defined on the whole $T(M \times M)$. Let (p_0, q_0) be an element in $M \bigvee M$, and $c = r(p_0, q_0)$. Then the connected component of (p_0, q_0) in the intersection of $\bigcup_{t \in \boldsymbol{R}} \exp_{(p_0,q_0)} (te_{+1})$ with $M \bigvee M$ is contained in W_c.

5. e_{+1} is a principal direction of curvature in the tangent space of W_c, and its corresponding principal curvature is equal to zero. This follows from the fact that

$$\nabla_{e_{+1}} e_{-1} = 0 ,$$

where ∇ stands for the covariant differentiation in the Levi-Civita connection of $M \times M$.

3. Riemannian manifolds with positive sectional curvature

In the present and next sections we obtain some information about the boundary of the c-neighborhoods of M as well as the "position" of n-dimensional local geodesic sprays in $M \times M$ with respect to the c-neighborhoods by assuming that the sectional curvature of M is positive. First of all, we prove

Proposition 1. *Let $(p_0, q_0) \in M \bigvee M$ and $c = r(p_0, q_0)$. Then W_c has at (p_0, q_0) at least $n - 1$ principal directions, in which the normal curvature is positive (i.e., W_c is at least $(n - 1)$-concave at (p_0, q_0)), and at least one principal curvature equal to zero.*

Proof. The proof makes use of the variation of the length integral of a one-parameter family of curves to show that the second fundamental form of W_c is positive semi-definite on V^+. Let γ be the shortest geodesic in M between p_0 and q_0. Then we may assume that γ is parametrized by the arc-length s, $0 \leq s \leq c$. For each v in $V^+_{(p_0,q_0)} \backslash \boldsymbol{R}e_{+1}(p_0, q_0)$ we construct a one-parameter family of curves $\gamma_t(s), 0 \leq s \leq c$ and $|t| < \varepsilon$ for some positive number ε, having the foollowing properties:

(i) $\gamma_0(s) = \gamma(s)$, for all $s \in [0, c]$.

(ii) $\gamma_t(0) = \exp_{p_0}(tv') = p_t$ and $\gamma_t(c) = \exp_{q_0}(tv'')$ for all $|t| < \varepsilon$, where $v' \in T_{p_0}(M) \backslash \boldsymbol{R}e'_1, v'' \in T_{q_0}(M) \backslash \boldsymbol{R}e''_1$ and $v = v' + v''$.

(iii) For every $s \in (0, c), \gamma_t(s)$ is the geodesic tangent to $v'(s)$, where $v'(s)$ is the parallel translate of v' along γ from p_0 to $\gamma(s)$.

Let us denote by L the length integral of a curve. Then

$$(1) \qquad L(\gamma_t) \geq r(\exp_{(p_0,q_0)}(tv)) = r(p_t, q_t)$$

for every $|t| < \varepsilon$. By the construction of $\gamma_t(s)$, we have

$$L(\gamma_0) = r(p_0, q_0) , \qquad \frac{d}{dt}L(\gamma_t)|_{t=0} = 0 ,$$

and also

$$\frac{d}{dt}r(p_t, q_t)|_{t=0} = dr_{(p_0,q_0)}(v) = 0 .$$

Then from (1) we get

$$(2) \qquad \frac{d^2}{dt^2}L(\gamma_t)|_{t=0} \geq \frac{d^2}{dt^2}r(p_t, q_t)|_{t=0} .$$

Next, by computing the second variation of the arc length with respect to the family $\gamma_t(s)$ we obtain ([2], [6])

$$(3) \qquad \frac{d^2}{dt^2}L(\gamma_t)|_{t=0} < 0 .$$

On the other hand

$$(4) \qquad \frac{d^2}{dt^2}r(p_t, q_t)|_{t=0} = r_{,ij}v^i v^j = -\Pi_{W_c}[v] ,$$

where $r_{,ij}$ and Π_{W_c} stand for the second covariant differentiation in the Levi-Civita connection of M (in a coordinate system of M at (p_0, q_0)) and the second fundamental form of W_c at (p_0, q_0) respectively, and the repetition of indices indicates summation.

From (3), (4) and (2) we conclude

$$\Pi_{W_c}[v] > 0 .$$

This together with Remark 5 shows that Π_{W_c} is positive semi-definite on $V^+_{(p_0,q_0)}$. Hence W_c must have at least $n-1$ positive principal directions of curvature and at least one equal to zero (the e_{+1}) at (p_0, q_0) proving our assertion.

Now let $(p_0, q_0) \in M \vee M$, and $c = r(p_0, q_0)$. We shall prove that there exists a neighborhood U_ε of 0 in $V^+_{(p_0,q_0)}$ such that

$$r(\exp_{(p_0,q_0)}(x)) \leq r(p_0, q_0)\ , \qquad \text{for all } x \in U_\varepsilon\ .$$

Let S be a local geodesic spray of V^+ at (p_0, q_0), and let $(u_1, \cdots, u_n)$ be a coordinate system of S with center at (p_0, q_0) such that

$$u_i\left(\exp_{(p_0,q_0)}\left(\sum_{j=1}^{n} t_j e_{+j}\right)\right) = t_i\ , \qquad 1 \leq i \leq n\ ,$$

where we have chosen the e_{+i}'s, $2 \leq i \leq n$, to be the principal directions of curvature of W_c at (p, q) with principal curvatures $\mathscr{K}_i\,(> 0)$, $2 \leq i \leq n$ (see Proposition 1).

We proceed to show that for every $(p_s, p_s) \in S$ with coordinates

$$u_1((p_s, q_s)) = s\ ,$$
$$u_i((p_s, q_s)) = 0\ , \qquad 2 \leq i \leq n\ ,$$

the differential of r satisfies

$$(5) \qquad dr_{(p_s,q_s)}(v) = 0$$

for every $v \in T_{(p_s,q_s)}(S)$.

To prove (5) it will be sufficient to prove

$$(6) \qquad T_{(p_s,q_s)}(S) \subseteq (T_{(p_s,q_s)}(M \vee M)) \vdash e_{-1}\ ,$$

where $\vdash$ stands for perpendicular in the natural metric of $M \times M$. Let $v \in T_{(p_s,q_s)}(S)$. The knowledge of one of the projections of v either onto $T_{p_s}(M)$ or onto $T_{q_s}(M)$ determines the other. In fact, let us assume given v' the projection of v onto $T_{p_s}(M)$ and let us determine v'' in $T_{q_s}(M)$ such that $v' + v'' = v$. Let α be a curve in M through p_s tangent to v', i.e.,

$$\alpha(t) = \exp_{p_0}(v'(t))\ ,$$

with $v'(t) \in T_{p_0}(M)$ and $|t| < \eta$ for some positive real number η. Next, by parallel translation of $v'(t)$ along γ from p_0 to q_0 we obtain $v''(t) \in T_{q_0}(M)$, and then

$$\beta(t) = \exp_{q_0}(v''(t))\ ,$$

with $|t| < \eta$, is a curve in M through q_s whose tangent v'' belongs to $T_{q_s}(M)$ and

$$v' + v'' = v\ .$$

Remarks. (i) If v' is tangent to γ at p_s, then our construction shows that v'' is tangent to γ at q_s, and also that $v = v' + v''$ belongs to $\boldsymbol{R}e_{+1}$.

(ii) If v' belongs to $(T_{p_s}(M)) \vdash e_1'(p_s)$, then our argument together with the Gauss lemma, [2], shows that v'' belongs to $(T_{q_s}(M)) \vdash e_1''(q_s)$, and therefore $v = v' + v''$ belongs to $(T_{(p_s,q_s)}(M \vee M)) \vdash \boldsymbol{R}e_1$.

Remarks (i) and (ii) prove (6) and hence (5).

In the coordinates $(u_1, \cdots, u_n)$ the function r can be expressed as

$$r(u_1, \cdots, u_n) = r(p_0, q_0) - \tfrac{1}{2}\sum_{i=2}^{n} \mathscr{K}_i u_i^2 + \sum_{i,j=2}^{n} u_i u_j \varphi_{ij}(u) ,$$

where each $\varphi_{ij}(u_1, \cdots, u_n)$, $2 \leq i, j \leq n$, is at least linear in u_1 because of (5). Therefore we can restrict the u_i's conveniently so that

$$r(p, q) \leq r(p_0, q_0) ,$$

for every (p, q) in S, proving our assertion.

4. Cut pairs

In this section by dealing with cut pairs of M we obtain a refinement (Theorem 1) of the result proved at the end of the last section. Let $(p_0, q_0) \in W_c$, $c = r(p_0, q_0)$, and let us assume that $(p_0, q_0) \in M_{cut}$. Take a shortest geodesic γ in M between p_0 to q_0, and let us consider m_0 to be any point on γ different from p_0 and q_0. Then neither (p_0, m_0) nor (m_0, q_0) is a cut pair, and therefore we can apply the result at the end of § 3 to get

$$r(p, m) \leq r(p_0, m_0) , \tag{7}$$

$$r(m, q) \leq r(m_0, q_0) \tag{8}$$

for every (p, m) and (m, q) belonging to the local geodesic sprays of V^+ through (p_0, m_0) and (m_0, q_0) respectively.

Next, by using the inequalities (7) and (8) and the fact that γ is a geodesic, we get

$$r(p, q) \leq r(p, m) + r(m, q) \leq r(p_0, m_0) + r(m_0, q_0) = r(p_0, q_0)$$

for every (p, q) in a local geodesic spray of V^+ (constructed from r) at (p_0, q_0). Finally, we can state

Theorem 1. *Let M be a connected complete n-dimensional Riemannian manifold with positive sectional curvature. Then for each (p_0, q_0) in $M \times M$ there exists a neighborhood U_ε of 0 in $V^+_{(p_0,q_0)}$ such that*

(i) $\exp_{(p_0,q_0)}(x) \in N_c$, *for every $x \in U_\varepsilon$* ,

(ii) $\dim((\exp_{(p_0,q_0)}(U_\varepsilon)) \cap W_c) \leq 1$,

where $c = r(p_0, q_0)$.

As an application we shall prove

Theorem 2. *Let M be as in Theorem* 1.

(a) *Let V^n be an n-dimensional local geodesic spray at (p_0, q_0), and assume that r attains its minimum on V^n at (p_0, q_0). Then the set C of all points in V^n*

where r is equal to c ($= r(p_0, q_0)$) *includes at least one geodesic in which the point* (p_0, q_0) *is an interior point.*

(b) *Let* V^n *be an n-dimensional submanifold of* $M \times M$ *transversal to* e_{+1}. *Then r cannot achieve its minimum on* V^n *at a point* (p_0, q_0) *which is flat relative to* e_{-1} (*i.e., the* e_{-1} *component of the second fundamental form of* V^n *at* (p_0, q_0) *is zero*).

Proof. (a_1) Let us assume that (p_0, q_0) is not a cut pair. Then we have

$$T_{(p_0,q_0)}(V^n) \subseteq T_{(p_0,q_0)}(W_c) ,$$

since r has a minimum on V^n at (p_0, q_0). Moreover,

$$V^+_{(p_0,q_0)} \subseteq T_{(p_0,q_0)}(W_c) ,$$
$$\dim (T_{(p_0,q_0)}(V^n)) = \dim (V^+_{(p_0,q_0)}) = n ,$$

and therefore

$$T_{(p_0,q_0)}(V^n) \cap V^+_{(p_0,q_0)}) \neq (0) .$$

Let $v_0 \in (T_{(p_0,q_0)}(V^n) \cap V^+_{(p_0,q_0)})\backslash\{0\}$. Then by Proposition 1

(9) $$\Pi_{W_c}[v_0] \geq 0 .$$

On the other hand

(10) $$\Pi_{W_c}|_{T_{(p_0,q_0)}(V^n)} \geq 0 .$$

From (9) and (10), we have

(11) $$\Pi_{W_c}[v_0] = 0 ,$$

and from (11),

$$T_{(p_0,q_0)}(V^n) \cap V^+_{(p_0,q_0)} = \boldsymbol{R}e_{+1} .$$

Since V^n is a local geodesic spray at (p_0, q_0), $\exp_{(p_0,q_0)} (te_{+1})$ belongs to V^n for every $t \in (a, b)$, where $a < 0$ and $b > 0$.

Set

$$C = \{(p, q) \in V^n/r(p, q) = r(p_0, q_0) = c\} .$$

It is clear that $\exp_{(p_0,q_0)} (te_{+1})$ is contained in C for every t, $a < t < b$, since (p_0, q_0) is not a cut pair.

(a_2) Now let us assume that $(p_0, q_0) \in M_{cut}$. In this case we shall introduce an auxiliar function r' which is of class C^∞ in a neighborhood of (p_0, q_0) and satisfies

(12) $$r'(p_0, q_0) = r(p_0, q_0)\ ,$$

(13) $$r'(p, q) \geq r(p, q)\ ,$$

wherever it is meaningfull.

Definition of r'. Let us take γ to be a shortest geodesic in M between p_0 and q_0, and let m_0 be any point on γ different from p_0 and q_0. Then we set

$$r'(p, q) = r'(\exp_{p_0}(x), \exp_{q_0}(y)) = r(p, m(p, q)) + r(m(p, q), q)\ ,$$

where $(p, q) = \exp_{(p_0, q_0)}(x, y)$, $x \in T_{p_0}(M)$, $y \in T_{q_0}(M)$, and

$$m(p, q) = \exp_{m_0} \frac{1}{c}[(c - s_0)x' + s_0 y']\ ,$$

where x' and y' are the parallel translations along γ of x and y from p_0 to m_0 and q_0 to m_0 respectively, and $s_0 = r(p_0, m_0)$.

The r' just defined has the required properties (12) and (13), so finally by setting

$$N' = \{(p, q)/r'(p, q) \leq r(p_0, q_0)\}\ ,$$
$$W' = \{(p, q)/r'(p, q) = r(p_0, q_0)\}\ ,$$

we have that W' is a (2n − 1)-dimensional manifold and observe that if r attains a minimum at (p_0, q_0) it also holds for r'. Therefore the case of the cut pair (p_0, q_0) will be reduced to the non-cut pair case (a_1) by replacing r by r'.

(b) Let us assume that r achieves its minimum on V^n at (p_0, q_0) which is a flat point relative to e_{-1}. Then we get

$$T_{(p_0, q_0)}(V^n) \cap (V^+_{(p_0, q_0)} \backslash \boldsymbol{R}e_{+1}) \neq (0)$$

because of the minimality of r at (p_0, q_0) and the transversality of V^n with respect to e_{+1}.

Let $v_0 \in (T_{(p_0, q_0)}(V^n) \cap (V^+_{(p_0, q_0)} \backslash \boldsymbol{R}e_{+1})) \backslash \{0\}$. Then

(14) $$\Pi_{W_c}[v_0] > 0$$

from Proposition 1. On the other hand

(15) $$\Pi_{W_c}|_{T_{(p_0, q_0)}(V^n)} \leq 0$$

because of the minimality of r at (p_0, q_0).

The inequalities (14) and (15) lead us to a contradiction, hence r cannot achieve its minimum on V^n at a flat point with respect to e_{-1}. This concludes the proof of (b) and hence that of Theorem 2.

5. Complex manifolds

Let M be a complex manifold of complex dimension n, and let $(U; z_1, \cdots, z_n)$ be a coordinate system of M at $p \in M$ with origin at p. We proceed to define a quadratic transformation at the point p ("blowing-up"). Let $p^{n-1}(\boldsymbol{C})$ be an $(n-1)$-dimensional complex projective space with homogeneous coordinates $w_1, \cdots, w_n$, and consider the complex manifold $U \times p^{n-1}(\boldsymbol{C})$.

Let $\hat{U} \subseteq U \times p^{n-1}(\boldsymbol{C})$ be defined by

$$\hat{U} = \{(q, \zeta) \in U \times p^{n-1}(\boldsymbol{C}) / z_i(q)w_j(\zeta) = z_j(q)w_i(\zeta),\ 1 \leq i, j \leq n\} .$$

U is a complex manifold of complex dimension n. In fact, if we set V_k equal to the subset of $\boldsymbol{C}p^{n-1}$ where $w_k \neq 0$, then

$$p^{n-1}(\boldsymbol{C}) = \bigcup_{k=1}^{n} V_k ,$$

and in $(U \times V_k) \cap \hat{U}$ the defining equations give $z_j = z_k w_j / w_k$ so that $(z_k, w_1/w_k, \cdots, w_{k-1}/w_k, \cdots, w_n/w_k)$ forms a coordinate system in $(U \times V_k) \cap \hat{U}$.

We define a projection $\sigma_U : \hat{U} \rightarrow U$ by $\sigma_U(q, \zeta) = q$, which is one-to-one except in $\sigma_U^{-1}(\{p\})$ because if $q \neq p$ there exists j, $1 \leq j \leq n$, such that $z_j(q) \neq 0$, then $w_k = w_j z_k / z_j$. Hence the w's are determined up to a proportionality implying the existence of a unique $\zeta \in p^{n-1}(\boldsymbol{C})$ such that $\sigma_U(q, \zeta) = q$.

Let us denote by σ_1 the restriction of σ_U to $\hat{U} \backslash \sigma_U^{-1}(\{p\})$. Then we can define a complex manifold by setting

$$\hat{M}_p = \hat{U} \bigcup_{\sigma_1} (M \backslash \{p\}) ,$$

where the symbol $\bigcup_{\sigma_1}$ denotes the union of $\hat{U}$ with $M \backslash \{p\}$ in which the respective subsets $\hat{U} \backslash \sigma_U^{-1}(\{p\})$ and $U \backslash \{p\}$ are identified under σ_1. One gets a manifold from the fact that the graph of σ_1 in $\hat{U} \times M \backslash \{p\}$ is a closed subspace, [3].

There is a natural map $\sigma : \hat{M}_p \rightarrow M$, which extends the σ_U and is also onto and one-to-one except in $\sigma^{-1}(\{p\})$. The subvariety $\sigma^{-1}(\{p\})$ is an $(n-1)$-dimensional complex projective space and will be denoted by B_p. The manifold $\hat{M}_p$ is called the "blowing-up" manifold of M at the point p, and σ a quadratic transformation with respect to p. For any two coordinate systems $(U; z_1, \cdots, z_n)$ and $(U'; z'_1, \cdots, z'_n)$ in neighborhoods of p with origin at p, the natural isomorphism of $\hat{M}_p \backslash \sigma^{-1}(\{p\})$ and $M'_p \backslash \sigma'^{-1}(\{p\})$ extends naturally to a holomorphic isomorphism, [1] and [4].

We consider now the effect of the "blowing-up" of M at p on a subvariety V containing p.

Lemma 1. *Let $V \subseteq M$ be an analytic subvariety and let $p \in V$, and consider $\hat{M}_p$ with the subvariety $V_p^0 = \sigma^{-1}(V \backslash (\{p\}))$. Then the topological closure $\hat{V}_p$ of V_p^0 in $\hat{M}_p$ is an analytic subvariety of $\hat{M}_p$.*

Proof. We consider $\sigma^{-1}(V)$, which is an analytic subvariety of $\hat{M}_p$ and is a finite union of irreducible components, [8]; let us say

$$\sigma^{-1}(V) = B_p \cup A_1 \cup \cdots \cup A_m ,$$

where each A_i, $1 \leq i \leq m$, is an irreducible component. Denote by A the analytic subvariety $A_1 \cup \cdots \cup A_m$, and observe that A must contain the set V_p^0. Since σ is one-to-one in the complement of B_p, we get

$$A \backslash A \cap B_p = V_p^0 . \tag{16}$$

Next, by taking closure in $\hat{M}_p$, the identity (16) becomes

$$A = \overline{A \backslash A \cap B_p} = \overline{V_p^0} = \hat{V}_p , \tag{17}$$

since $A \backslash A \cap B_p$ is everywhere dense in A. The identity (17) proves the lemma.

The subvariety $\hat{V}_p$ is called the "blowing-up" of the variety V at the point p. Denote it by $K_p(V) = \hat{V}_p \backslash V_p^0 = \hat{V}_p \cap B_p$ and call it the *projective tangent cone* of the variety V at p. Note that if V is irreducible and d-dimensional, then the dimensions of $\hat{V}_p$ and $K_p(V)$ are d and $d - 1$ respectively.

Now let us consider M to be a Kähler manifold, and let us denote by R and J its Riemann curvature tensor and the automorphism of $T(M)$ with $J^2 = -\text{id.}$, induced by the complex structure of M, respectively.

Definition. Let M be a Kähler manifold, and let σ and σ' be two J-invariant planes in $T_p(M)$. Then the *holomorphic bisectional curvature* $H(\sigma, \sigma')$ is defined [7] by

$$H(\sigma, \sigma') = R(t, Jt, s, Js) ,$$

where t and s are unit vectors in σ and σ' respectively. By using Bianchi's identity we have

$$H(\sigma, \sigma') = R(t, s, t, s) + R(t, Js, t, Js) .$$

Finally, by taking under consideration Kähler manifolds with positive holomorphic bisectional curvature, we prove as the main result in this paper the following generalization of a result in [6].

Theorem 3. *Let M be a compact connected Kähler manifold of complex dimension n with positive holomorphic bisectional curvature. Then any closed n-dimensional complex analytic (possibly singular) subvariety V of $M \times M$ intersects M_Δ.*

Proof. We shall reach a contradiction by assuming that r achieves a positive relative minimum on V at (p_0, q_0). Since the case of the cut pair (p_0, q_0) can be reduced to the noncut pair case by introducing an auxiliary function r' (Theorem 2, § 4), we are just left with the following two cases.

(i) Let (p_0, q_0) be an element in $M \vee M$, and assume that it is a regular point of V. In this case, we have

$$T_{(p_0,q_0)}(V) \subseteq T_{(p_0,q_0)}(W_c) ,$$

$$T_{(p_0,q_0)}(W_c) = \sum_{i=1}^{n} \boldsymbol{R}e_i + \sum_{i=2}^{n} \boldsymbol{R}e_{-i} + \sum_{i=1}^{n} \boldsymbol{R}Je_i + \sum_{i=1}^{n} \boldsymbol{R}Je_{-i} ,$$

where $c = r(p_0, q_0)$, and J is the automorphism of $T_{(p_0,q_0)}(M \times M)$ with $J^2 = -\mathrm{id.}$, defined by the complex structure of $M \times M$.

Let us set

$$V^+_{(p_0,q_0)} = \sum_{i=1}^{n} \boldsymbol{R}e_i = \sum_{i=1}^{n} \boldsymbol{R}Je_i .$$

Then we have

$$\dim_{\boldsymbol{R}} (T_{(p_0,q_0)}(W_c)) = 4n - 1 ,$$

$$\dim_{\boldsymbol{R}} (V^+_{(p_0,q_0)}) = \dim_{\boldsymbol{R}} (T_{(p_0,q_0)}(V)) = 2n .$$

Hence

$$\dim_{\boldsymbol{R}} ((T_{(p_0,q_0)}(V)) \cap V^+_{(p_0,q_0)}) \geq 1 .$$

Let $v_0 \in (T_{(p_0,q_0)}(V) \cap V^+_{(p_0,q_0)})\backslash\{0\}$. Then Jv_0 belongs to $(T_{(p_0,q_0)}(V) \cap V^+_{(p_0,q_0)})\backslash\{0\}$, and v_0 and Jv_0 are $\boldsymbol{R}$-linearly independent. Therefore

$$\dim_{\boldsymbol{R}} (T_{(p_0,q_0)}(V) \cap V^+_{(p_0,q_0)}) \geq 2 .$$

Now we make use of the following relations:

$$\Pi_{W_c}|_{T_{(p_0,q_0)}(V)} \leq \Pi_V , \tag{18}$$

$$\Pi_V[v] + \Pi_V[Jv] = 0 , \tag{19}$$

for all $v \in T_{(p_0,q_0)}(V)$, where Π_V stands for the component of the second fundamental form of V in the direction of the "outward" normal to W_c.

Let $v \in T_{(p_0,q_0)}(V)\backslash \boldsymbol{R}e_1 \cup \boldsymbol{R}Je_1$. Then we have

$$0 < \Pi_{W_c}[v] + \Pi_{W_c}[Jv] = -2r_{,\alpha\bar{\beta}}v^\alpha v^{\bar{\beta}} \tag{20}$$

by using the fact that M is a Kähler manifold with positive holomorphic bisectional curvature and a computation similar to that carried out in the proof of Proposition 1 in § 2. Therefore

$$\Pi_V[v_0] + \Pi_V[Jv_0] > 0 \tag{21}$$

for all v_0 in $V^+_{(p_0,q_0)} \cap T_{(p_0,q_0)}(V)\backslash \boldsymbol{R}e_1 \cup \boldsymbol{R}Je_1$, because of (20) and (18).

On the other hand

$$\Pi_V[v_0] + \Pi_V(Jv_0] = 0 \tag{22}$$

because of (19). Subtracting (21) from (22) we have

$$(\Pi_V - \Pi_{W_c})[v_0] + (\Pi_V - \Pi_{W_c})[Jv_0] < 0 ,$$

leading to a contradiction, since each summand is positive or zero because of (18).

(ii) Let (p_0, q_0) be an element of $M \bigvee M$, and assume that it is a singularity of V. In this case, we proceed as follows. Let us consider submanifolds M_1 and M_2 of $M \bigvee M$ containing (p_0, q_0) with

$$T_{(p_0,q_0)}(M_1) = T_{(p_0,q_0)}(W_c) \cap J(T_{(p_0,q_0)}(W_c)) ,$$
$$T_{(p_0,q_0)}(M_2) = V^+_{(p_0,q_0)} ,$$

respectively. Hence

$$K_{(p_0,q_0)}(M_2) \subseteq K_{(p_0,q_0)}(M_1) .$$

On the other hand

$$K_{(p_0,q_0)}(V) \subseteq K_{(p_0,q_0)}(M_1)$$

because of the minimality of the function r at (p_0, q_0). Therefore

$$K_{(p_0,q_0)}(V) \cap K_{(p_0,q_0)}(M_2) \neq \emptyset ,$$

since $K_{(p_0,q_0)}(V)$ is an algebraic variety by Chow's theorem [5], and the dimensions of $K_{(p_0,q_0)}(V)$ and $K_{(p_0,q_0)}(M_2)$ are complementary dimensional in $K_{(p_0,q_0)}(M_1)$ ($= (2n - 2)$-dimensional complex projective space).

Let A_0 be an element in $K_{(p_0,q_0)}(V) \cap K_{(p_0,q_0)}(M_2)$. We may assume also that it belongs to neither $\boldsymbol{R}e_1$ nor $\boldsymbol{R}Je_1$. Then by Lemma 1 there exists a holomorphic curve $\varphi(t)$ in $\hat{V}_p$ such that $\varphi(0) = A_0$ and with the property that it intersects B_p just at A_0 locally. The projection of $\varphi(t)$ under σ provides us with a holomorphic curve $C(t)$ in V, with $C(0) = (p_0, q_0)$ and such that if $(z_1, \cdots, z_n)$ is a coordinate system in a neighborhood of (p_0, q_0) with origin at (p_0, q_0), there exists an integer $d \geq 1$ such that

$$z_\alpha(C(t)) = (C(t))^\alpha = \sum_{j=0}^{\infty} A_j^\alpha t^{d+1}$$

with $A_0^\alpha \neq 0$ for some α, $1 \leq \alpha \leq 2n$, since the A_0's are the local homogeneous coordinates of the point A_0 in $\hat{M}_p$.

We are going to show that for sufficiently small nonzero t

(23) $$r(C(t)) - r(C(0)) < 0 .$$

We recall that the jets of a differentiable real-valued function on $\boldsymbol{C}^m$ have bidegree (d', d'') and real (or total) degree $d = d' + d''$.

In the coordinate system $(z_1, \cdots, z_{2n})$ we can write

$$\begin{aligned} r(C(t)) - r(C(0)) = 2\mathscr{R}e(r,_{\alpha}(C(t))^{\alpha}) &+ \mathscr{R}e(r,_{\alpha\beta}(C(t))^{\alpha}(C(t))^{\beta}) \\ &+ r,_{\alpha\bar{\beta}}(C(t))^{\alpha}(C(t))^{\bar{\beta}} + \text{terms in } (t, \bar{t}) \text{ of} \\ &\quad \text{total degree greater than or equal to } 3d , \end{aligned}$$

where the barred indices $\bar{\alpha} = \alpha + 2n, \cdots$ range from $2n + 1$ to $4n$ and refer to the conjugate holomorphic coordinates $z_{\alpha+n} = z_{\bar{\alpha}} = \bar{z}_{\alpha}$.

Next, we take the average of the function $r(C(t)) - r(C(0))$, i.e.,

$$\frac{1}{2\Pi}\int_0^{2\Pi} (r(C(te^{i\theta})) - r(C(0)))d\theta .$$

On the other hand

$$\begin{aligned} &r(C(te^{i\theta})) - r(C(0)) \\ &= 2\mathscr{R}e\Big(r,_{\alpha}\Big(\sum_{j=0}^{\infty} A_j^{\alpha} t^{d+j} e^{i\theta(d+j)}\Big)\Big) \\ &\quad + \mathscr{R}e\Big[r,_{\alpha\beta}\Big(\sum_{j=0}^{\infty} A_j^{\alpha} t^{d+j} e^{i\theta(d+j)}\Big)\Big(\sum_{k=0}^{\infty} A_k^{\beta} t^{d+k} e^{i\theta(d+k)}\Big)\Big] \\ &\quad + r,_{\alpha\bar{\beta}}\Big[\Big(\sum_{j=0}^{\infty} A_j^{\alpha} t^{d+j} e^{i\theta(d+j)}\Big)\Big(\sum_{k=0}^{\infty} A_k^{\bar{\beta}} t^{d+k} e^{-i\theta(d+k)}\Big)\Big] \\ &\quad + \text{terms in } (t, \bar{t}) \text{ of total degree greater than or equal to } 3d . \end{aligned}$$

We observe that the only summand in the above expression giving any contribution when integrated from 0 to 2Π comes from

$$r,_{\alpha\bar{\beta}}\Big[\Big(\sum_{j=0}^{\infty} A_j^{\alpha} t^{d+j} e^{i\theta(d+j)}\Big)\Big(\sum_{k=0}^{\infty} A_k^{\bar{\beta}} t^{d+k} e^{-i\theta(d+k)}\Big)\Big]$$

and is given by the jet of total degree less that $3d$ of

$$r,_{\alpha\bar{\beta}}\Big(\sum_{j=k=0}^{\infty} A_j^{\alpha} A_k^{\bar{\beta}} t^{d+j} \bar{t}^{d+k}\Big) .$$

Therefore

$$\begin{aligned} &\frac{1}{2\Pi}\int_0^{2\Pi} (r(C(te^{i\theta})) - r(C(0)))d\theta \\ &\qquad = \frac{1}{2\Pi}\int_0^{2\Pi} r,_{\alpha\bar{\beta}}(A_0^{\alpha} A_0^{\bar{\beta}}\, |t|^{2d} + A_1^{\alpha} A_1^{\bar{\beta}}\, |t|^{2(d+1)} + \cdots)d\theta \end{aligned}$$

$$= r,_{\alpha\bar{\beta}}\left(\sum_{j=0}^{\infty} A_j^{\alpha} A_j^{\bar{\beta}}\ |t|^{2(d+j)}\right),$$

which is negative for small t because of (20). This proves the inequality (23) which contradicts the fact that r achieves a minimum on V at (p_0, q_0). Therefore we conclude that r cannot achieve a positive minimum on V. On the other hand, the compactness of V in $M \times M$ and the continuity of r imply the existence of some (p, q) in V, which achieves a minimum on V, and by our discussion $r(p, q)$ must be equal to zero, which shows the case (ii). Hence the proof of Theorem 3 is complete.

Ih the case where V has no singularities our Theorem 3 is [6] equivalent to Theorem 2, which was used to prove that a compact Kähler manifold of complex dimension 2 and positive sectional curvature is analytically isomorphic to $P_2(C)$, but so far our technique does not seem to be applicable to study the conjecture for greater dimensions.

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UNIVERSITY OF PENNSYLVANIA