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POINCARÉ'S GEOMETRIC THEOREM FOR FLOWS

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The celebrated "last geometric theorem of Poincaré" states that an area preserving homeomorphism of the annulus, which rotates the two boundary circles in opposite directions, must have at least two fixed points. Poincaré conjectured this result [3] and showed that it would imply the existence of closed orbits in the restricted three-body problem. G. D. Birkhoff gave an ingenious proof that there must be at least one fixed point [1], and attempts to establish a second fixed point have led to investigations of the fixed point indices of area preserving maps.

This note provides a very short and elementary proof of the analogous theorem for flows and also gives a generalization to systems of commuting vector fields on higher dimensional manifolds. We would like to thank Nancy Stanton for suggesting the higher dimensional generalization, and Julian Palmore for pointing out the easy way in which our proof gives a second fixed point.

Theorem. *Let A be a closed annulus, and let X be a C^1 vector field on A which is tangent to the boundary and induces an area preserving flow on A. Assume that, at each point of the boundary, X is nonzero and points in the positive direction. (That is, at each point of the boundary the pair (X, N), where N is the unit normal pointing into the annulus, is positively oriented.) Then X vanishes at at least two points of A.*

Proof. Let ω be the area form on the annulus, and let $\eta = i_X\omega$ be the interior product of ω with X. Since the flow preserves area, the Lie derivative $L_X\omega$ of ω with respect to X vanishes identically. Hence $0 = L_X\omega = i_Xd\omega + di_X\omega = d\eta$ shows that η is a closed 1-form.

Indeed, η is exact. Since every 1-cycle in A is homologous to one of the boundary circles, it suffices to show that η has period zero around a boundary circle. This holds because X is tangent to the boundary, and therefore η vanishes identically along the boundary. Hence there is a function f on the annulus such that $\eta = df$.

Since the form ω is nondegenerate, the vector fields X and grad f have exactly the same zeroes. To find a zero of X, it suffices therefore to find a critical point of f. In particular, a maximum of f in the interior of A will do. On the boundary of A the interior normal derivative of f is everywhere positive, for

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$$N(f) = df(N) = \omega(X, N) > 0 .$$

Therefore f cannot have a maximum on the boundary and so must have one in the interior.

We claim that f must have at least *two* critical points in A. Let $p \in A$ be the interior maximum. On the boundary of A, grad f is nonzero and points inward into A. Therefore, if p were the only critical point of f, we could retract A to p by following the trajectories of grad f. This proves the theorem.

Remark. The difficulty in establishing Poincaré's original conjecture lies in showing that a fixed point of an area preserving transformation has nonzero index. We avoid this difficulty by producing a maximum of the function f.

Finally, we give a higher dimensional generalization of Poincaré's theorem. Let M^{n+1} be a compact oriented manifold with boundary which admits a system of n linearly independent vector fields $X_1, \cdots, X_n$. These restrict to give a parallelization on each boundary component of M^{n+1}, and we can compare the orientation coming from this parallelization with the orientation induced from M^{n+1}. Namely, if T is a boundary component of M and N is the inward unit normal vector field, we say that $X_1, \cdots, X_n$ gives an *oriented parallelization* if at each point p of T the orientation of $\langle X_1(p), \cdots, X_n(p), N(p)\rangle$ agrees with the given orientation of M.

Theorem. *Let M^{n+1} be a compact oriented manifold such that the relative homology $H_1(M, \partial M; \boldsymbol{R})$ vanishes. Suppose M admits a system of n linearly independent commuting C^1 vector fields $X_1, \cdots, X_n$ which are volume preserving and flow tangent to the boundary. Then*

(1) *M is diffeomorphic to $T^n \times I$ where T^n is the n-torus,*

(2) *$X_1, \cdots, X_n$ restrict to an oriented parallelization on precisely one of the two boundary components of M.*

Proof. Let ω be the volume form on M, and define η to be the 1-form $i_{X_1} i_{X_2} \cdots i_{X_n} \omega$. Using the fact that the X_i preserve ω and the commutation relation $[L_X, i_Y] = i_{[X,Y]}$, one sees easily that η is closed. As before, η vanishes on vectors tangent to the boundary, so the homology assumption implies that η is exact, i.e., $\eta = df$. The function f can have no critical points, or else the X's would become linearly dependent. Therefore M must have at least two boundary components—one on which the inward normal derivative $df(N) = \eta(N) = \omega(X_1, \cdots, X_n, N)$ is positive, and one on which it is negative. Otherwise f would attain either a maximum or a minimum at some point in the interior of M. Let T be a boundary component on which the inward normal derivative is positive. Following the gradient of f shows that M is diffeomorphic to $T \times I$. Finally, T is parallelized by a system of commuting vector fields and therefore admits the structure of an abelian Lie group [2, p. 212].

Example. The two commuting vector fields on the two-torus T^2 do not extend to independent commuting volume preserving vector fields on the solid torus. If we bore out the "core" of the solid torus to obtain $T^2 \times I$, the vector fields extend, but not in such a way as to give the induced orientation on the boundary. Notice that the same argument as in the two-dimensional case implies that linear independence breaks down at at least two points.

Note. Professor G. A. Hedlund has informed us that our first theorem and its proof appeared in R. Hermann, *Some differential geometric aspects of the Legrange variational problem*, Illinois J. Math. **6** (1962) p. 641, and in R. Hermann, *Differential geometry and the calculus of variations*, Academic Press, New York, 1968, p. 180.

References

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