J. DIFFERENTIAL GEOMETRY
11 (1976) 103-128

# SOME PROPERTIES OF $\underline{k}$-FLAT MANIFOLDS

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## Introduction

In a recent paper [18], P. Molino, studying the characteristic classes of flat $G$-structures, remarked that his results more generally apply to the $G$-structures, which are locally equivalent to those defined, on a Lie group $K$, by action, under a group of automorphisms of the Lie algebra $\underline{k}$, on the left invariant parallelism of $K$. He named them "$G$-structures lisses".

Here we present some properties of this sort of structures, which we call $\underline{k}$-flat structures (structures lisses de type $\underline{k}$). Their behaviour is indeed led by a sort of flatness, where the usual abelian (i.e., torsion free) model is replaced by another one which is characterized by a canonical torsion, and is given by the local geometry of the Lie group $K$. Thus a $\underline{k}$-flat manifold admits $\underline{k}$, in a reasonable sense, as its Lie algebra (see [4]).

By defining morphisms so as to respect these Lie algebras, we obtain a category of *-flat manifolds which admits the category of Lie groups as a subcategory. Furthermore, the Lie groups are precisely the "tangent objets" in the category of *-flat manifolds. We also notice that the subcategory consisting in $\boldsymbol{R}^n$-flat manifolds is but the usual $C^\infty$ category (§ 3).

Another aspect of the theory is the $G$-structural one. In this direction, we use a special kind of connections which generalizes the Cartan-Schouten connections on Lie groups [10].

Our main results are the characterization of $\underline{k}$-flat manifolds whose structural group is of finite type (if $\underline{k}$ is a semi-simple Lie algebra, this is always the case): they are discrete quotients of some open set of the Lie group $K$ (§ 4), and the fact that if $\underline{k}$ is a reductive or a nilpotent Lie algebra, then any formal $\underline{k}$-flat structure on a manifold is $\underline{k}$-flat (§ 8).

§ 2 deals with a weak notion of $\underline{k}$-flatness which seems to us not to be lacking of interest. In §§ 5 and 6, we look for the polynomial vector fields, so useful in the study of flatness, and see them in a special kind of sub-structure (strict $\underline{k}$-flatness). Some special cohomological properties of $\underline{k}$-flatness are pointed out in § 7.

The author wishes to thank P. Molino for his helpfull suggestions while preparing this paper.

**Notation.** Let $\underline{k}$ denote a real $n$-dimensional Lie *algebra*. A basis ($e_1$,

Communicated by A. Lichnerowicz, July 8, 1974.

$\cdots, e_n)$ for its underlying vector space is given once for all, and we also use the dual basis $(e^1, \cdots, e^n)$ of the dual vector space $\underline{k}^*$. Let us denote by $\bar{\underline{k}}$ the opposite Lie algebra of $\underline{k}$, and by $u \mapsto \bar{u}$ the canonical anti-isomorphism $\underline{k} \to \bar{\underline{k}}$. One can identify with this map the spaces $\underline{gl}(\underline{k})$ and $\underline{gl}(\bar{\underline{k}})$ (resp. $GL(\underline{k})$ and $GL(\bar{\underline{k}})$). If $AUT(\underline{k})$ is the subgroup of $GL(\underline{k})$ consisting of all linear isomorphisms $a: \underline{k} \to \underline{k}$ such that $a[u, v] = [au, av]$ for all $u, v$ in $\underline{k}$, one can easily see that $AUT(\underline{k})$ agrees with $AUT(\bar{\underline{k}})$ in the former identification. We can thus write $AUT(\underline{k}) = AUT(\bar{\underline{k}}) = G_0$.

Let us introduce the structure constants of $\underline{k}$ (with respect to $(e_1, \cdots, e_n)$)

$$[e_i, e_j] = \tau_0(e_i \wedge e_j) = \tau_{ij}^k e_k ,$$

and thus $[\bar{e}_i, \bar{e}_j] = -\tau_{ij}^k \bar{e}_k$.

$K$ is the (connected and) simply connected Lie group, whose left invariant Lie algebra is $\underline{k}$, and consequently its right invariant Lie algebra is $\bar{\underline{k}}$. There is a natural isomorphism of $G_0$ onto the Lie group $AUT(K)$ ([9], [15]) given by $a(\exp u) = \exp(au)$ whenever $a \in G_0$ and $u \in k$. We will identify these two groups in this way.

## 1. Bracket manifolds

Let $M$ be an $n$-dimensional differentiable manifold. Throughout this paper this means that $M$ is a smooth, real, connected manifold which satisfies the second axiom of countability and that $\partial M = \emptyset$. A *$\underline{k}$-bracket structure* on $M$ is given by a Lie algebra bundle structure on the tangent bundle $TM$. Endowed with such a structure, $M$ is said to be a *$\underline{k}$-bracket manifold.*

A $\underline{k}$-bracket structure on $M$ is thus given by a vector valued 2-form $\tau: \bigwedge^2 TM \to TM$ satisfying Jacobi's identity $\tau \barwedge \tau = 0$, and in addition

(1) the Lie algebra structure defined by $\tau$ on each fibre of $TM$ is isomorphic to $\underline{k}$.

(2) for each $x \in M$, there exists a neighborhood $U$ of $x$ equipped with $n$ linearly independent vector fields $X_1, \cdots, X_n$ such that

$$\tau(X_i \wedge X_j) = \tau_{ij}^k X_k ,$$

$\tau_{ij}^k$ being the $\underline{k}$-structure constants; $(X_1, \cdots, X_n)$ will be called a *distinguished trivialization* of $TM$.

Frequently, one also denotes $\tau(X \wedge Y) = [\![X, Y]\!]$. Notice that a $\underline{k}$-bracket structure on $M$ is equivalently given by a *$G_0$-structure $E_\tau(M)$* on $M$. The interest of this type of structure was pointed out by K. Nomizu and K. Yano in [21].

**Example 1.1.** Let $\underline{k}$ be $\boldsymbol{R}^n$ (the abelian Lie algebra). Thus any differentiable manifold is obviously endowed with a $\boldsymbol{R}^n$-bracket structure.

**Example 1.2.** By $\mathcal{O}_K$-bilinearity we extend the left-invariant vector field bracket onto all of $TK$. This gives a $\underline{k}$-bracket structure on $K$, which will be referred to as the standard one. $(e_1, \cdots, e_n)$ is a global distinguished trivialization for it. More generally, let $H$ be a discrete subgroup of $K$. The standard $\underline{k}$-bracket structure on $K$ projects itself onto a $\underline{k}$-bracket structure on the right homogeneous space $K/H$. There is no global distinguished trivialization of $T(K/H)$, unless $H$ is a normal subgroup of $K$.

**Example 1.3.** Let $M$ be an $n$-dimensional manifold equipped with a $G$-structure $E$, where $G$ is some Lie subgroup of $G_0$. By extending the structural group of $E$ to $G_0$, one obtains a $\underline{k}$-bracket structure on $M$. So any parallelized manifold can be seen in this way as a $\underline{k}$-bracket manifold.

One can define, in an obvious sense, a *bracket morphism* $\phi: M \to M'$ from a $\underline{k}$-bracket manifold $M$ into a $\underline{k}$'-bracket manifold $M'$. If $U$ and $U'$ are some open sets in the bracket manifold $M$, they are equipped with the induced $\underline{k}$-bracket structure. A diffeomorphism $\phi$ from $U$ onto $U'$, which is a bracket morphism, is said to be a *local automorphism* of $M$. The set of such automorphisms is a pseudo-group on $M$, which obviously coincides with the pseudo-group of local automorphisms of the $G_0$-structure $E_\tau(M)$. The *infinitesimal automorphisms* of $M$ are defined in the usual manner. They are the local vector fields $Z$ such that $[Z, \tau] = 0$. Here $[\ ,\ ]$ denotes the Nijenhuis bracket of vector valued forms.

Let $\underline{h}$ be some Lie subalgebra of $\underline{k}$. An *$\underline{h}$-distribution* on a $\underline{k}$-bracket manifold is a Lie subalgebra bundle $C$ of $TM$ with fibre $\underline{h}$. If such a distribution is integrable, each leaf of the foliation thus obtained is a $\underline{h}$-bracket manifold; we shall use the word *$\underline{h}$-foliation* in such a situation.

For instance, on the $\underline{k}$-bracket manifold $K$ there is a natural $\underline{h}$-foliation, the leaves of which are the $Hz$'s, where $z \in K$ and $H$ is the connected subgroup of $K$ whose Lie algebra is $\underline{h}$.

**Proposition 1.** *Let $\underline{j}$ be a characteristic ideal of $\underline{k}$. Then on every $\underline{k}$-bracket manifold there is canonically defined a $\underline{j}$ distribution $C_{\underline{j}}$, which is invariant under the local automorphisms of $M$.*

Recall that a characteristic ideal is a subspace of $\underline{k}$ invariant under $G_0$. For every $x \in M$, choose any $r$ in the fibre over $x$ of $E_\tau(M)$ and define $C_{\underline{j}}(x) = r(\underline{j})$. This is obviously independent of the choice of $r$ and satisfies the former conditions.

Use the theory developped in [10]. From the vector valued 2-form $\tau$, one obtains two operators which act as skew-derivations on the sheaf $\bigwedge^* \mathscr{T}^*M$ of germs of alternate froms on $M$:

$$i_\tau : \bigwedge^p \mathscr{T}^*M \to \bigwedge^{p+1} \mathscr{T}^*M\ ,$$
$$\mathscr{L}_\tau : \bigwedge^p \mathscr{T}^*M \to \bigwedge^{p+2} \mathscr{T}^*M\ .$$

They are defined $i_\tau\omega = \omega \barwedge \tau$ and $\mathscr{L}_\tau\omega = [\tau, \omega]$, $\omega \in \bigwedge^p \mathscr{T}^* M$. $i_\tau$ is an $\mathscr{O}_M$-linear map. Thus we have a vector bundle complex

$$\cdots \to \bigwedge^{p-1} TM \xrightarrow{i_\tau} \bigwedge^p TM \xrightarrow{i_\tau} \bigwedge^{p+1} TM \to \cdots$$

the cohomology of which is a vector bundle with fibre $H^p(\underline{k}; \boldsymbol{R})$.

In fact, we shall use the skew-derivation of degree one

$$\delta\colon \bigwedge^p \mathscr{T}^*M \to \bigwedge^{p+1} \mathscr{T}^*M$$

defined by $\delta = d + i_\tau$. $\mathscr{L}_\tau$ is given by $\mathscr{L}_\tau = \delta^2 = d\delta + \delta d$. Also notice the naturality of these operators: for every bracket morphism $\phi\colon M \to M'$ we have $\phi^* i_{\tau'} = i_\tau \phi^*$, $\phi^*\mathscr{L}_{\tau'} = \mathscr{L}_\tau \phi^*$, $\phi^*\delta' = \delta\phi^*$.

Introduce now some terminology: let $\omega$ be a $p$-form on $M$. We say that $\omega$ is a

$$\underline{k}\text{-}basic \text{ form if } \mathscr{L}_\tau\omega = i_\tau\omega = 0\ , \quad \text{and} \quad \underline{k}\text{-}flat \text{ form if } \delta\omega = 0\ .$$

Remark that if $\omega$ is a $\underline{k}$-basic or a $\underline{k}$-flat form, so is $d\omega$. Considering global forms on $M$, we can thus define *$\underline{k}$-basic cohomology* spaces $H_b^*(M)$ and *$\underline{k}$-flat cohomology* space $H_f^*(M)$.

For instance, on the Lie group $K$ endowed with its $\underline{k}$-bracket structure, any left invariant form $\omega$ is $\underline{k}$-flat. If, moreover, we notice that the map which sends every $\underline{k}$-flat $p$-form

$$\omega = \omega_{i_1\cdots i_p} e^{i_1} \wedge \cdots \wedge e^{i_p}$$

onto the left invariant form $\omega_{i_1\cdots i_p}(\mathbf{1}) e^{i_1} \wedge \cdots \wedge e^{i_p}$ ($\mathbf{1}$ is the neutral element of $K$) commutes with $d$, then we obtain

$$H^*(\underline{k}; \boldsymbol{R}) \subset H_f^*(K)\ .$$

In particular, if $K$ is compact, we have the strict inclusion

$$H_{DR}^*(\underline{k}; \boldsymbol{R}) \subset H_f^*(K)\ .$$

The fundamental problem given by the existence of a $\underline{k}$-bracket structure on a manifold $M$ is to know if it is possible to realize this bracket by a geometric one. This actually is an integrability problem. We shall introduce two degrees of integrability:

Weak integrability for the $\underline{k}$-bracket structure means that each point of $M$ admits a neighborhood equipped with $n$ linearly independent vector fields whose vector field brackets are precisely given by $\tau$.

Strong integrability (in short, integrability) means that these vector fields further define a distinguished trivialization of $TM$.

## 2. Weak $\underline{k}$-flat manifolds

Let $M$ be a $\underline{k}$-bracket manifold, and $\sigma = (X_1, \cdots, X_n)$ a local section of the bundle of frames $B(M)$ of $M$. $\sigma$ will be called an *allowable section* if $[X_i, X_j] = [\![X_i, X_j]\!]$ for every $i, j$ in $\{1, 2, \cdots, n\}$, and $M$ a *weak $\underline{k}$-flat* *manifold* if every $x \in M$ has a neighborhood $U$ provided with an allowable section of $B(M)$.

For instance, $K$ is a weak $\underline{k}$-flat manifold, and so is the discrete homogeneous space $K/H$ with its standard bracket structure.

A very useful tool is given by some special connections: if $\sigma\colon U \to B(M)$ is an allowable section, write $\sigma = (X_1, \cdots, X_n)$ and define on $U$ three linear connections $\nabla^-, \nabla^+, \nabla^0$ by the formulas

$$\nabla^-_{X_i} X_j = 0\,, \quad \nabla^+_{X_i} X_j = [X_i, X_j]\,, \quad \nabla^0_{X_i} X_j = \tfrac{1}{2}[X_i, X_j]\,.$$

An easy computation shows that $\nabla^0$ is torsion free, and the torsion and curvature tensors of $\nabla^-$ and $\nabla^+$ are given by

$$\begin{aligned} T^- &= -\tau\,, & R^- &= 0\,, \\ T^+ &= \tau\,, & R^+ &= -\nabla^-\tau\,. \end{aligned}$$

With the use of a partition of unity, on any weak $\underline{k}$-flat manifold $M$ we can thus build linear connections the torsion of which is $-\tau, \tau$ or zero. These connections are referred to as $(-)$-*connections*, $(+)$-*connections*, $(0)$-*connections*, and generalize the so-called connections introduced on the Lie groups by Cartan-Schouten [6].

To every $(-)$-connection $\nabla$ on $M$, associate the $(+)$-connection $\bar{\nabla}$ and the $(0)$-connection $\overset{0}{\nabla}$ where

$$\bar{\nabla} = \nabla + \tau\,, \qquad \overset{0}{\nabla} = \nabla + \tfrac{1}{2}\tau\,.$$

Notice that $\nabla, \bar{\nabla}, \overset{0}{\nabla}$ admit the same geodesic curves.

This association $(\nabla, \bar{\nabla}, \overset{0}{\nabla})$ agrees with $(\nabla^-, \nabla^+, \nabla^0)$ in the case of local connections defined by an allowable section of $B(M)$. Also notice that if $\partial$ is the usual antisymmetrization map

$$\partial\colon \mathscr{T}^*M \otimes \textstyle\bigwedge^p \mathscr{T}^*M \to \textstyle\bigwedge^{p+1} \mathscr{T}^*M\,,$$

then for any $(-)$-connection $\nabla$ and $p$-form $\omega$,

$$\delta\omega = \frac{1}{p!}\,\partial\nabla\omega\,.$$

From this one can see that any allowable local section $\sigma = (X_1, \cdots, X_n)$ of $B(M)$ dualizes into a local $\underline{k}$-flat $\underline{k}$-valued 1-form $\theta = \theta^1 e_1 + \cdots + \theta^n e_n$. Hence

**Proposition 2.1.** *Let $M$ be a $\underline{k}$-bracket manifold. Then the following conditions are equivalent*:

(i) *$M$ is a weak $\underline{k}$-flat manifold,*

(ii) *each $x \in M$ has a neighborhood endowed with a flat* $(-)$*-connection,*

(iii) *each $x \in M$ has a neighborhood on which there is defined a $\underline{k}$-flat $\underline{k}$ valued 1-form $\theta$ of rank $n$.*

Thanks to condition (iii) of Proposition 2.1, it is possible to deal with the weak integrability problem with the use of some results of $H$. Goldschmidt [11], [12]. For each integer $p, 0 \leq p \leq \infty$, let $J_pT$ denote the $p$-jet bundle of $TM$. The vector field bracket induces a bracket

$$[\ ,\ ] : J_pT \otimes J_pT \to J_{p-1}T \ ,$$

while $\tau$ induces another one

$$[\![\ ,\ ]\!] : J_pT \otimes J_pT \to J_pT \ .$$

We say that a $\underline{k}$-bracket structure on a manifold $M$ is *formally weak integrable* if for every $x \in M$ there exist $Z_1, \cdots, Z_n \in (J_\infty T)_x$ such that

(a) the projections $(X_1, \cdots, X_n)$ of $(Z_1, \cdots, Z_n)$ on $T_x$ are linearly independent,

(b) $[Z_i, Z_j] = [\![Z_i, Z_j]\!]$ for any $i, j$ in $\{1, 2, \cdots, n\}$.

Thus let $\delta_p : J_pT^* \to J_{p-1} \bigwedge^2 T^*$ denote the vector bundle morphism associated to $\delta$ for every $p, 1 \leq p \leq \infty$, and $R_p = \ker \delta_p$. Condition (iii) of Proposition 2.1 implies

**Proposition 2.2.** *The $\underline{k}$-bracket structure on the manifold $M$ is formally weak integrable if and only if the natural map $R_\infty \to T^*$ is onto.*

In fact, it is easy but sometimes rather tendious to check up the following facts:

$R_1$ is a vector bundle and the natural map $R_1 \to T^*$ is onto;

the symbol of $R_1$ is involutive;

the natural map $\rho_1^2 : R_2 \to R_1$ is not onto;

let $\tilde{R}_1 = \rho_1^2(R_2)$; if $\rho_0^1 : \tilde{R}_1 \to T^*$ is of constant rank, then $R_2$ is an involutive (and thus formally integrable) differential equation. Hence the $\underline{k}$-bracket structure on $M$ is formally weak integrable if and only if $\tilde{R}_1 \to T^*$ is onto.

Introduce a local distinguished trivialization $\psi$ of $M$. It is given by the vector fields $X_1, \cdots, X_n$ on the open set $U$. Carrying by $\psi$ the vector field bracket of the $X_i's$, one obtains a tensor field $\gamma : U \to \bigwedge^2 \underline{k}^* \otimes \underline{k}$ where

$$\gamma(x)(e_i \wedge e_j) = \psi_x^{-1}[X_i, X_j](x) \ .$$

Let $\alpha : U \to \bigwedge^3 \underline{k}^* \otimes \underline{k}$ be

$$\alpha(x)(u \wedge v \wedge w) = \underset{u,v,w}{\$} ([\gamma(x)(u \wedge v), w] - \tfrac{1}{2}\gamma(x)(u \wedge [v, w])) \ ,$$

where $\underset{u,v,w}{\$}$ indicates the sum over the cyclic permutations of $(u, v, w)$. Finally, introduce the exact sequence

$$S^2\underline{k}^* \otimes \underline{k} \xrightarrow{\beta \otimes \mathrm{id}} \bigwedge^3 \underline{k}^* \otimes \underline{k} \xrightarrow{\chi} Q \longrightarrow 0 \,,$$

where $\beta : S^2\underline{k}^* \to \bigwedge^3 \underline{k}^*$ is defined as

$$(\beta\zeta)(u \wedge v \wedge w) = \underset{u,v,w}{\$} \zeta([u, v] \wedge w) \,.$$

This gives on $U$ the tensor field $\bar{\alpha}(x) = \chi(\alpha(x))$. A simple computation yields for $\theta \in T_x^*$

$$\theta \in \rho_0^1(R_1) \Longleftrightarrow \theta_\psi \circ \alpha(x) \in \operatorname{im} \beta \,,$$

where $\theta_\psi$ is the element in $\underline{k}^*$ associate with $\theta$ by $\psi_x$. Hence

**Theorem 2.1.** *The $\underline{k}$-bracket structure on $M$ is formally weak-integrable if and only if the (local) tensor field $\bar{\alpha}$ associated to each local distinguished trivialization of $TM$ is zero.*

Notice that if $\bar{\alpha}$ is zero on $U$, and $\psi'$ is another distinguished trivialization on $U$, then the tensor field $\bar{\alpha}'$ associated to $\psi'$ is necessarily zero.

On the other hand, $\bar{\alpha} = 0$ for each local distinguished trivialization means that $R_2$ is formally integrable, and thus using [11, Theorem 7.1] one obtains

**Theorem 2.2.** *Let an analytic manifold $M$ be endowed with an analytic $\underline{k}$-bracket structure. Then $M$ is a weak $\underline{k}$-flat manifold if and only if the local tensor field $\bar{\alpha}$ associated to every analytic local distinguished trivialization is zero.*

**Example 2.1.** Let $n = 3$. Then $\bar{\alpha} = 0$ (look at the dimensions) and so every analytic bracket structure of dimension 3 is weak integrable.

**Example 2.2.** Let $M$ be a Lie group, and $(X_1, \cdots, X_n)$ a left invariant parallelism on it, i.e., a basis for $\underline{m}$. According to Example 1.3 this gives on $M$ an analytic $\underline{k}$-bracket structure. Let $\gamma$ be the $\underline{m}$-bracket on $M$. Thus we have a global tensor field

$$\bar{\alpha} = \chi(\tau_0 \,\overline{\wedge}\, \gamma - \tfrac{1}{2}\gamma \,\overline{\wedge}\, \tau_0) \,,$$

which shows that the $\underline{k}$-bracket structure defined on $\boldsymbol{R}^n$ by its canonical parallelism is weak integrable ($\gamma = 0$). Take $\underline{m} = \boldsymbol{R}(e_1, e_2, e_3, e_4)$ with

$$\gamma(e_1 \wedge e_2) = e_2 \,; \; \gamma(e_4 \wedge e_1) = e_3 \,; \; \gamma(e_3 \wedge e_4) = e_1$$

(the others are zero), and take $\underline{k} = \boldsymbol{R}(e_1, e_2, e_3, e_4)$ with

$$\tau_0(e_2 \wedge e_3) = \tau_0(e_3 \wedge e_4) = \tau_0(e_4 \wedge e_2) = e_1$$

(the others are zero). The corresponding $\bar{\alpha} \neq 0$, and so we obtain a nonweak integrable $\underline{k}$-bracket structure on a manifold.

## 3. $\underline{k}$-flat manifolds

A $\underline{k}$-bracket structure on $M$ is said to be *integrable* if each $x \in M$ admits a neighborhood $U$ equipped with an allowable section $\sigma$ of $E_\tau(M)$. In this case we will call $M$ a *$\underline{k}$-flat manifold*.

Thus, if $M$ is a $\underline{k}$-flat manifold, then on a neighborhood of each point of $M$ there exist $n$ linearly independent vector fields $X_1, \cdots, X_n$ such that $[X_i, X_j] = [\![X_i, X_j]\!] = \tau_{ij}^k X_k$ for any $i, j$ in $\{1, 2, \cdots, n\}$.

A map $\phi: M \to M'$, where $M$ is a $\underline{k}$-flat manifold and $M'$ a $\underline{k}'$-flat manifold, is said to be a morphism if it is a morphism of bracket manifolds. We thus obtain a category of manifolds, which we call the *$*$-flat category*.

**Example 3.1.** Every $n$-dimensional differentiable manifold is an $\boldsymbol{R}^n$-flat manifold. A morphism from an $\boldsymbol{R}^n$-flat manifold into an $\boldsymbol{R}^m$-flat manifold is but a differentiable map. In this sense, the usual category of manifolds is a subcategory of the category of $*$-flat manifolds.

**Example 3.2.** The Lie group $K$ is a $\underline{k}$-flat manifold. Let $H$ be another Lie group and therefore an $\underline{h}$-flat manifold. Any Lie homomorphism $\phi: K \to H$ is a morphism of $*$-flat manifolds, and the converse is not true. So the usual category of Lie group is a subcategory of the $*$-flat category.

Also notice that if $H$ is a discrete subgroup of $K$, then $K/H$ is a $\underline{k}$-flat manifold.

The $G_0$-structure $E_\tau(K)$ on $K$ associated with its standard $\underline{k}$-bracket structure plays a central role in the study of $\underline{k}$-flat manifolds. Its elements are the frames deduced from $(e_1, \cdots, e_n)$ by the action of an element of $G_0$.

**Theorem 3.** *Let $M$ be a $\underline{k}$-bracket manifold. Then the following properties are equivalent*:

(i) *$M$ is a $\underline{k}$-flat manifold*;

(ii) *the $G_0$-structure $E_\tau(M)$ is locally equivalent to $E_\tau(K)$*;

(iii) *there exists an open covering $(U^\alpha)$ of $M$ such that each $E_\tau(U^\alpha)$ is endowed with a flat $(-)$-connection*;

(iv) *there exists an open covering $(U^\alpha)$ of $M$ such that each $E_\tau(U^\alpha)$ is endowed with a flat $(+)$-connection*;

(v) *at each point of $M$, there exists a germ of a $\underline{k}$-flat $\underline{k}$-valued* 1-*form of rank $n$ which defines a germ of a section of $E_\tau(M)$.*

*Proof.* Obviously, condition (ii) implies any of the other ones. Moreover, it is easy to see that (i) $\Leftrightarrow$ (iii); (iii) and (v) are dual formulations of the same property. So it suffice to show that (iii) $\Leftrightarrow$ (iv) and (v) $\Rightarrow$ (ii). Let $\nabla$ be any $(-)$-connection, and $\bar{\nabla}$ the associated $(+)$-connection. The curvature tensors $R$ and $\bar{R}$ of $\nabla$ and $\bar{\nabla}$ satisfy

$$(\bar{R} - R)(X \wedge Y)Z = (\nabla_X \tau)(Y \wedge Z) - (\nabla_X \tau)(X \wedge Z)$$

for any $X, Y, Z \in \mathcal{T}M$. From this it follows that $\nabla$ is a connection on $E_\tau(M)$ if and only if $R = \bar{R}$. On the other hand, $(\nabla, \bar{\nabla}, \overset{0}{\nabla})$ are simultaneously defined on $E_\tau(M)$ if and only if one of them satisfies this property. Hence (iii) $\Leftrightarrow$ (iv).

Finally, let $\theta$ a $\underline{k}$-valued 1-form on $M$ of rank $n$. It defines a section of $E_\tau(M)$ if and only if it commutes with the brackets, i.e., $i_\tau\theta = \frac{1}{2}[\theta, \theta]$. If moreover such a form is $\underline{k}$-flat, it satisfies the Maurer-Cartan condition

$$d\theta + \tfrac{1}{2}[\theta, \theta] = 0 ,$$

and consequently (v) $\Rightarrow$ (ii).

**Corollary.** *If $M$ is a $\underline{k}$-flat manifold, then $E_\tau(M)$ is a transitive $G_0$-structure on $M$, whose structure tensor is zero.*

To obtain the first assertion, observe that $E_\tau(M)$ can be considered as the Lie group $K \dot{\times} G_0$ (semi-direct product). The left translations of $K \dot{\times} G_0$ are automorphisms of the $G_0$-structure $E_\tau(K) \to K$.

**Example 3.3.** Let $\boldsymbol{R}^n$ be equipped with the $\underline{k}$-bracket structure defined by its canonical parallelism. As it has been shown (Example 2.2) this is a weak $\underline{k}$-flat manifold. We can see that it is not a $\underline{k}$-flat manifold. In fact, the $G_0$-structure so defined on $\boldsymbol{R}^n$ is flat. It suffices to show that, without other hypothesis concerning $\underline{k}, E_\tau(K)$ is not a flat $G_0$-structure. Using the first prolongation of $E_\tau(K)$ consisting of all torsion free horizontal subspaces of $E_\tau(K) \to K$, [22], one can see that an obstruction for $E_\tau(K)$ to be flat is given by the $\partial$-Spencer cohomology class $[\Omega] \in H^{2,2}(\underline{g}_0)$, where $\Omega \in \wedge^2 \underline{k}^* \otimes \underline{g}_0$ is the value at $\mathbf{1}$ of the curvature of the (0)-connection of Cartan-Schouten on $K$. We have $\Omega(u \wedge v) = \frac{1}{4}\mathrm{ad}_{[u,v]}$. Take for instance $\underline{k} = \underline{o}(p)$; then $\underline{g}_0 = \underline{o}(\frac{1}{2}p(p-1))$ and $[\Omega] \neq 0$.

**Example 3.4.** Take $\underline{k}$ so that $\underline{Dk}$ should be an abelian Lie algebra. Thus a $\underline{k}$-bracket manifold $M$ is a $\underline{k}$-flat manifold if and only if $E_\tau(M)$ is a flat $G_0$-structure. Look at the logarithmic coordinates $(x^1, \cdots, x^n)$ associated to $(e_1, e_2, \cdots, e_n)$ on a neighborhood $U$ on $\mathbf{1}$ of $K$, and define for $x \in U$ the matrix $A(x) = (A^i_j(x))$ by

$$e^i(x) = A^i_j(x) dx^j .$$

Consider $B(x) = \mathrm{ad}_{\log x}$ on $U$. Then $B(x) \in \underline{g}_0$, and [9]

$$A(x) = \sum_{p \geq 1} \frac{1}{p!} B^{p-1}(x) .$$

It is easy too see that the condition on $\underline{k}$ implies $A(x) \in G_0$ for any $x \in U$.

In the category of $\underline{k}$-flat manifolds we have a specific notion of tangent bundle defined as follows: let $J_1(K, M)$ be the $K \dot{\times} G_0$-principal bundle over $M$ whose elements are the 1-jets at $\mathbf{1}$ of local bracket isomorphisms from $K$ into $M$. We have an isomorphism

$$J_1(K, M) = E_\tau(M) \times K ,$$

which shows that $E_\tau(M)$ is both a subbundle and a quotient bundle of $J_1(K, M)$. We define the *group tangent bundle* $T_K M$ of $M$ by

$$T_K M = J_1(K, M)/G_0 .$$

It is a nonprincipal fibre bundle over $M$ with fibre $K$, which admits $G_0$ as structural group, More precisely,

$$T_K M = E_\tau(M) \underset{G_0}{\times} K .$$

Thus $T_K M$ admits a global canonical section, the unit section, which assigns to every $x \in M$ the class $\widetilde{(r, \mathbf{1})}$, where $r \in E_\tau(M)$ projects onto $x$. Also notice that the usual tangent bundle $TM$ (which agrees with this definition when $\underline{k} = \boldsymbol{R}^n$) can be considered as

$$TM = E_\tau(M) \underset{G_0}{\times} K ,$$

and one obtains a natural exponential map $\exp : TM \to T_K M$. Thanks to this map, it is easy to see that every bracket morphism $\phi : M \to M'$, where $M$ is a $\underline{k}$-flat manifold and $M'$ a $\underline{k}'$-flat one, gives rise to a group bundle morphism

$$\begin{array}{ccc} T_K M & \xrightarrow{\tilde{\phi}} & T_{K'} M' \\ \big\downarrow & & \big\downarrow \\ M & \xrightarrow{\phi} & M' \end{array}$$

such that, for every $x \in M$, the *tangent group homomorphism* $\tilde{\phi}_x$ only depends on $j_1\phi(x)$.

Let $j : E_\tau(M) \to J_1(K, M)$ be the natural inclusion. If $\tilde{\omega}$ is a connection form on the bundle $J_1(K, M) \to M$, then $j^*\tilde{\omega}$ is a 1-form defined on $E_\tau(M)$ with values in the Lie algebra $\underline{k} \oplus \underline{g}_0$, and so decomposes into

$$j^*\tilde{\omega} = \eta + \omega .$$

The $\underline{g}_0$-part $\omega$ is a connection form on $E(M)$, and the $\underline{k}$-part $\eta$ is a tensorial form of type identity. On the other hand, choose a connection form $\omega$ on $E_\tau(M)$ and a tensorial 1-form $\eta$ of type identity with values in $\underline{k}$. Then there exists one and only one connection form $\tilde{\omega}$ on $J_1(K, M)$ such that $j^*\tilde{\omega} = \eta + \omega$. This allows us to mimic the classical construction of the development of the curves of the manifold $M$.

Let $\omega$ be a given connection form on $E_\tau(M)$, and $\tilde{\omega}$ the unique connection on $J_1(K, M)$ such that $j^*\tilde{\omega} = \theta + \omega$, where $\theta$ is the fundamental form on $E_\tau(M)$. For every $C^\infty$ curve $(x_t)$ in $M$ and every $r_0 \in E_\tau(M)$ which projects itself onto $x_0$, one can define the $\omega$-horizontal lift $(r_t)$ of $(x_t)$ in $E_\tau(M)$ from $r_0$ and the

$\tilde{\omega}$-horizontal lift $(\tilde{r}_t)$ of $(x_t)$ in $J_1(K, M)$ from $r_0$, and the $K$-development of $(x_t)$ relative to $\omega$ is $(z_t)$ such that

$$r_t = \tilde{r}_t z_t \ .$$

Thus we have

**Proposition 3.** *A curve $(x_t)$ in $M$ is a geodesic for the connection $\omega$ if and only if its development $(z_t)$ is a geodesic in $K$.*

The geodesics in $K$ are, of course, relative to the Cartan-Schouten connections, i.e., the exponential curves.

## 4. $\underline{k}$-flat $G$-structures

Let $G$ be a Lie subgroup of $G_0$. We now study the "good" $G$-reductions of $E_\tau(M)$. Here "good" means that on the standard $\underline{k}$-flat manifold $K$ the $G$-reductions locally agree with the reduction of $E(K) = K \dot{\times} G_0$ to the subgroup $K \dot{\times} G = E_G(K)$.

First of all, notice that a manifold $M$ equipped with a $G$-structure is a $\underline{k}$-bracket manifold (cf. Example 1.3).

**Proposition 4.1.** *Let $E_G(M)$ be some $G$-structure on an $n$-dimensional manifold $M$. Then the following properties are equivalent:*

(i) *$E_G(M)$ is locally equivalent to $K \dot{\times} G$,*

(ii) *there exists an open covering $(U^\alpha)$ on $M$ such that each $E_G(U^\alpha)$ is endowed with a flat $(-)$-connection,*

(iii) *at each point of $M$, one can find a germ of a $\underline{k}$ values $\underline{k}$-flat 1-form of rank $n$ which defines a germ of a section of $E_G(M)$.*

This allows us to say that a $G$-structure $E_G(M)$ on $M$ is a *$\underline{k}$-flat $G$-structure* if it satisfies of the above properties. $E_G(K) = K \dot{\times} G$ will be referred to as the *standard $\underline{k}$-flat $G$-structure.*

**Example 4.1.** Let $\underline{k} = \boldsymbol{R}^n$ and $G \subset GL(\boldsymbol{R}^n)$. An $\boldsymbol{R}^n$ flat $G$-structure is but a flat $G$-structure. The local flat $(-)$-connections are here the local torsion free flat connections defined by local coordinates, and $E_G(\boldsymbol{R}^n)$ is the standard flat $G$-structure.

**Corollary.** *Let $E_G(M)$ be a $\underline{k}$-flat $G$-structure. Then $E_G(M)$ is transitive, and its structure tensor is $(-\tau_0) \bmod \partial(\underline{k}^* \otimes \underline{g})$.*

Remark that if $G \supset \mathrm{ad}_K$, the structure tensor of $E_G(M)$ is zero. In fact, under this hypothesis there exist $(+)$-connections and $(0)$-connections on $E_G(M)$.

We shall say that a $\underline{k}$-flat structure on a manifold $M$ is *discrete* if $E_\tau(M)$ admits a $\underline{k}$-flat $G$-reduction, where $G$ is a discrete subgroup of $G_0$. For instance, the standard $\underline{k}$-flat structure on the discrete homogeneous space $K/H$ is discrete.

**Proposition 4.2.** *The $\underline{k}$-flat structure on the manifold $M$ is discrete if and only if there exists a global flat $(-)$-connection on $E_\tau(M)$.*

The proof is straightforward.

The importance of the discrete $\underline{k}$-flatness notion is given by

**Theorem 4.1.** *If a manifold $M$ admits a discrete $\underline{k}$-flat structure, then the universal covering $\hat{M}$ of $M$ is an open set of the Lie group $K$, and the $\underline{k}$-flat structure on $M$ is a quotient of the standard $\underline{k}$-flat structure on $K$.*

*Proof.* The pull-back of $E_\tau(M)$ is a $\underline{k}$-flat $G_0$-structure on $\hat{M}$, which admits a global flat $(-)$-connection, and thus is reducible to a $\underline{k}$-flat $\{\mathbf{1}\}$-structure. This parallelism defines on $\hat{M}$ a global $\underline{k}$-valued $\underline{k}$-flat 1-form $\theta$ of rank $n$, which satisfies the Maurer-Cartan condition $d\theta + \frac{1}{2}[\theta, \theta] = 0$. Thus the theorem follows from [23, Chapter V, Theorem 2.4].

We can now state one of the main results of this section:

**Theorem 4.2.** *If $\underline{k}$ is a semi-simple Lie algebra, every $\underline{k}$-flat structure is discrete.*

*Proof.* The deep argument is that if $\underline{k}$ is a semi-simple Lie algebra, then a $\underline{k}$-flat structure on a manifold $M$ provides $M$ with a pseudo-riemannian structure (a riemannian structure if moreover $\underline{k}$ is a compact Lie algebra).

Let $M$ be equipped with a $\underline{k}$-flat $G$-structure $E_G(M)$. Look at the first prolongation $E^{(1)}_{(-)}$ of $E_G(M)$ consisting in all horizontal subspaces of $E_G(M) \to M$ with torsion tensor $-\tau_0$. (This is indeed the canonical prolongation of the $\underline{k}$-flat $G$-structure $E_G(M)$.) $G$ acts on the right on $E^{(1)}_{(-)}$. The bundle $E^{(1)}_{(-)}/G \to M$ has $G^{(1)}$ as its fibre, and thus admits global sections. Such a section is easily seen to be a $(-)$-connection on $E_G(M)$, and we obtain the

**Lemma.** *If $E_G(M)$ is a $\underline{k}$-flat structure such that $\underline{g}^{(1)} = 0$, there exists a unique $(-)$-connection on $M$.*

Also notice that this $(-)$-connection locally coincides with the local flat $(-)$-connections given by Proposition 4.1. So it is flat.

To obtain the theorem, it suffices now to use Proposition 4.2 and to remark that if $\underline{k}$ is a semi-simple Lie algebra, then $\underline{g}_0^{(1)} = 0$ because $\underline{g}_0$ keeps invariant a bilinear symmetric form on $\underline{k}$ which is nondegenerate (namely, the Killing form on $\underline{k}$).

Theorem 4.2 has an analogue without restriction on $\underline{k}$, when one supposes that the structural group $G$ is of finite type, [22], [13]. To see this, introduce for every integer $p$ the ideal of $\underline{g}$ defined by contraction

$$\underline{g}_{(p)} = \langle \underline{g}^{(p)}, S^p \underline{k} \rangle \ .$$

Write $\underline{g}_{(\infty)} = \bigcap_p \underline{g}_{(p)}$; it is the infinite ideal of $\underline{g}$, [22].

**Theorem 4.3.** *Every $\underline{k}$-flat $G$-structure admits a $\underline{k}$-flat $G_\infty$-reduction, where $G_\infty$ is a subgroup of $G$ whose Lie algebra is $\underline{g}_{(\infty)}$.*

This theorem, given in [3] in the case of flat $G$-structures, follows from Molino's theorem on finite codimensional sub-structures [18]. Indeed, $\underline{g}_{(\infty)}$ is, in the sense of [18], a finite codimensional subalgebra of $\underline{g}$. Therefore, if $E_G(M)$ is a $\underline{k}$-flat $G$-structure on $M$, and $E_G(\hat{M})$ is its pull-back over the universal covering of $M$, then $E_G(\hat{M})$ admits a global $G_{(\infty)}$-reduction, where $G_{(\infty)}$ denotes the connected Lie subgroup of $G$ whose Lie algebra is $\underline{g}_{(\infty)}$. Moreover, $E_{G_{(\infty)}}(\hat{M})$ is locally equivalent to $K \times G_{(\infty)}$, and thus is a $\underline{k}$-flat $G_{(\infty)}$-structure. From this it is not difficult to see that $E_{G(\infty)}(\hat{M})$ projects itself onto a $G_\infty$-reduction of $E_G(M), G_\infty$ being a Lie subgroup of $G$ which admits $G_{(\infty)}$ as its neutral component.

**Corollary.** *Every $\underline{k}$-flat G-structure with G of finite type is discrete.*

We have $\underline{g}_{(\infty)} = 0$ in this case. To complete this study, let us give an example of a nondiscrete $\underline{k}$-flat structure:

**Example 4.2.** Take $\underline{k} = \underline{n}(3, \boldsymbol{R})$, the Lie algebra of matrices of the type

$$\begin{pmatrix} 0 & \beta & \alpha \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{pmatrix} \qquad \alpha, \beta, \gamma \in \boldsymbol{R} .$$

Write such a matrix for $\alpha e_1 + \beta e_2 + \gamma e_3$: Since $[e_1, e_3] = [e_1, e_2] = 0$ and $[e_2, e_3] = e_1$, the Lie algebra $\underline{g}_0$ is

$$\left(\begin{array}{c|cc} \text{trace } A & \alpha & \beta \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$A$}} \\ 0 & & \end{array}\right) \qquad \alpha, \beta \in \boldsymbol{R} , \quad A \in \underline{gl}(2, \boldsymbol{R}) .$$

$\underline{Dk}$ is an abelian Lie algebra, and therefore a $\underline{k}$-flat $G_0$-structure on a manifold $M$ is flat (Example 3.4).

Choose a volume form $\bar{\omega}$ on $\$^2$, and consider on $\$^1 \times \$^2$ the structure obtained by the product of the flat parallelism on $\$^1$ and the automorphisms of $\bar{\omega}$ on $\$^2$. This is indeed a flat $H$-structure on $\$^1 \times \$^2$, where $H$ is the group of all matrices in $GL(3, \boldsymbol{R})$ of the type

$$\left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$SL(2, \boldsymbol{R})$}} \\ 0 & & \end{array}\right) .$$

$H$ is a subgroup of $G$, and (as in Example 1.3) this gives us a $\underline{k}$-flat structure on $\$^1 \times \$^2$. It is not discrete, because if it were, Theorem 4.1 would give on $\boldsymbol{R} \times \$^2$ an absolute parallelism $(X_1, X_2, X_3)$ with $X_1$ tangent to the fibers of the fibering $\boldsymbol{R} \times \$^2 \to \boldsymbol{R}$; this is obviously impossible.

## 5. Automorphisms of a $\underline{k}$-flat $G$-structure

Let $E_G(M)$ be a $\underline{k}$-flat $G$-structure on a manifold $M$, and let $\Gamma_G(M)$ denote the pseudo-group of local automorphisms of $E_G(M)$. We shall drop the $G$ if $G = G_0 \cdot \Gamma_G(M)$ is a transitive Lie pseudo-group on $M$, whose first order infinitesimal structure is $E_G(M)$. Thanks to Proposition 4.1, the local behaviors of $\Gamma_G(M)$ are the same as those of the pseudo-group $\Gamma_G(K)$ of the standard

$\underline{k}$-flat $G$-structure on $K$. Let simply denote $\Gamma_G = \Gamma_G(K)$. A local diffeomorphism $\phi$ of $K$ belongs to $\Gamma_G$ if and only if at every point $x$ in the domain of $\phi$ the tangent map $\phi_{*x}$ acts on $\underline{k}$ as some element of $G$. This means that $\phi$ is a bracket morphism whose tangent group homomorphism, at every point in its domain, belongs to $G$. (This makes sense because the existence of $E_G(M)$ implies that $T_K M$ admits $G$ as its structural group.)

$\Gamma_G(M)$ is a regular pseudo-group in the sense that it admits a Lie pseudo-algebra $\mathscr{L}_G(M)$ (this is a L.A.S. in [22]). The group structure on $E_G(K) = K \dot{\times} G$ translates into the fact that the right invariant Lie algebra $\bar{\underline{k}} \dot{\oplus} \bar{\underline{g}}$ is a subalgebra of the Lie algebra of the global sections of $\mathscr{L}_G$ over $K$. Consequently $\mathscr{L}_G(M)$ is transitive on $E_G(M)$.

The rest of this section is devoted to the study of the formal algebra $\underline{L}_G$ of $\mathscr{L}_G$.

First of all let us give an algebraic model for the jet spaces of vector fields. If $X$ is a vector field defined on some neighborhood of $\mathbf{1}$ in $K$, we can write $X = X^i e_i$. The infinite jet $j_\infty X(\mathbf{1})$ is thus determined by the sequence

$$\{e_{j_1}, \cdots, e_{j_r} X^i(\mathbf{1})\}_{r \in N, j_1, \cdots, j_r \in \{1,2,\cdots,n\}} .$$

For each $r$, define the tensor $\xi_r \in \otimes^r \underline{k}^* \otimes \underline{k}$ by

$$\xi_r(u_1 \otimes \cdots \otimes u_r) = u_1 \cdots u_r X^i(\mathbf{1}) e_i .$$

This correspondance assigns to every jet $j_\infty X(\mathbf{1})$ the tensor $\xi \in \underline{U}(\underline{k})^* \otimes \underline{k}$, where $\underline{U}(\underline{k})$ is the universal envelopping algebra of $\underline{k}$, and standard theorems on $C^\infty$ functions show that the correspondance

$$(J_\infty T)_{\mathbf{1}} \to \underline{U}(\underline{k})^* \otimes \underline{k}$$

so defined is $1:1$. We shall denote

$$\underline{J}(\underline{k}) = \underline{U}(\underline{k})^* \otimes \underline{k} . \tag{5.1}$$

Let us introduce the usual filtration $\underline{U}^{p+1}(\underline{k}) \subset \underline{U}^p(\underline{k})$ of $\underline{U}(\underline{k})$. The former correspondance gives for every integer $p$ an isomorphism

$$(J_p T)_{\mathbf{1}} \to \underline{U}^p(\underline{k})^* \otimes \underline{k} = \underline{J}_p(\underline{k}) ,$$

so that the Poincaré-Birkhoff-Witt theorem furnishes the exactness of the sequence

$$0 \to S^p \underline{k}^* \otimes \underline{k} \to \underline{J}_p(\underline{k}) \to \underline{J}_{p-1}(\underline{k}) \to 0 . \tag{5.2}$$

The bracket $[\![\ ,\ ]\!]$, defined for each $p$, $1 \le p \le \infty$, a bracket

$$[\![\ ,\ ]\!] : (J_p T)_1 \otimes (J_p T)_1 \to (J_p T)_1$$

by $[\![ j_pX(1), j_pY(1) ]\!] = j_p[\![X, Y]\!](1)$. This translates onto

$$[\![\ ,\ ]\!] : \underline{J}_p(\underline{k}) \otimes \underline{J}_p(\underline{k}) \to \underline{J}_p(\underline{k})$$

with $[\![\xi, \eta]\!] = \sum_{p,q} [\![\xi_p, \eta_q]\!]$, $\xi_p$ denoting some representant of the homogeneous component of degree $p$ of $\xi$. It is easy to show that $[\![\xi_p, \eta_q]\!] \in \otimes^{p+q}\underline{k}^* \otimes \underline{k}$ is given by

$$\begin{aligned}(5.3)\qquad & [\![\xi_p, \eta_q]\!](u_1 \otimes \cdots \otimes u_{p+q}) \\ & = \sum_{\sigma \in \sigma_{p,q}} [\xi_p(u_{\sigma_1} \otimes \cdots \otimes u_{\sigma_p}), \eta_q(u_{\sigma_{p+1}} \otimes \cdots \otimes u_{\sigma_{p+q}})] ,\end{aligned}$$

$\sigma_{p,q}$ denoting the set of shuffle permutations $(p, q)$ of $p + q$ elements. From this formula it follows that the space $\underline{J}_p(\underline{k})$ endowed with the bracket $[\![\ ,\ ]\!]$ is a Lie algebra. On the other hand, one can show that the bundle $B_p(M)$ of frames of order $p$ on any $\underline{k}$-flat manifold $M$ is canonically equipped with a $\underline{J}_p(\underline{k})$-bracket structure. Relative to these structures, the morphisms $\pi^p_q : B_p(M) \to B_p(M)$ $(0 \leq q < p)$ are bracket morphisms.

Now the vector field bracket in $\underline{J}(\underline{k})$ differs from $[\![\ ,\ ]\!]$ by a contraction part, (which is of course the only one in the flat case):

$$[\xi, \eta] = [\![\xi, \eta]\!] - \{\xi, \eta\}$$

so that $\{\xi, \eta\} = \xi \circ \eta - \eta \circ \xi$, the composition product being defined on homogeneous components by

$$(\otimes^p \underline{k}^* \otimes \underline{k}) \otimes (\otimes^q \underline{k}^* \otimes \underline{k}) \to \otimes^{p+q-1} \underline{k}^* \otimes \underline{k} ,$$

$$\begin{aligned}(5.4)\qquad & (\xi_p \circ \eta_q)(u_1 \otimes \cdots \otimes u_{p+q-1}) \\ & = \sum_{\sigma \in \sigma_{p-1,q}} \xi_p(u_{\sigma_1} \otimes \cdots \otimes u_{\sigma_{p-1}} \otimes \eta_q(u_{\sigma_p} \otimes \cdots \otimes u_{\sigma_{p+q-1}}))\end{aligned}$$

with the convention $u \circ \xi_p = 0$ for $u \in \underline{k}$.

The two Lie algebra structures on $\underline{J}(\underline{k})$ given by $[\ ,\ ]$ and $[\![\ ,\ ]\!]$ induce the same structure on the subspace $\underline{k}$.

Now, look at the map

$$d_{[} : \underline{J}(\underline{k}) \to \underline{k}^* \otimes \underline{J}(\underline{k}) = \underline{U}(\underline{k})^* \otimes \underline{k}^* \otimes \underline{k} ,$$

which assigns to every $\xi \in \underline{J}(\underline{k})$ the map $\underline{k} \to \underline{J}(\underline{k})$

$$u \mapsto [\xi, u] .$$

Then $\xi \in \underline{L}_G$ if and only if $d_{[}\xi \in \underline{U}(\underline{k})^* \otimes \underline{g}$. On a system of homogeneous components $\xi_0 + \xi_1 + \cdots + \xi_p + \cdots$ of $\xi$, this is written as:

for any integer $p$, and $u_i$'s in $\underline{k}$, the map

(5.5) $$u \mapsto [u, \xi_p(u_1 \otimes \cdots \otimes u_p)] + \xi_{p+1}(u_1 \otimes \cdots \otimes u_p \otimes u)$$
belongs to $\underline{g}$.

Introduce the usal graduation on $\underline{L}_G$. If $\underline{J}^p(\underline{k})$ denotes the kernel of the map $\underline{J}(\underline{k}) \to \underline{J}_p(\underline{k})$, define

$$\underline{L}_G^p = \underline{L}_G \cap \underline{J}^p(\underline{k}) .$$

$\underline{L}_G$ admits $\bar{\underline{k}} \oplus \bar{\underline{g}}$ as a subalgebra, and moreover

(5.6) $$\underline{L}_G = \bar{\underline{k}} \oplus \bar{\underline{g}} \oplus \underline{L}_G^1 \qquad \text{(direct sum of vector spaces)} .$$

One can see that the formal vector fields corresponding to $\bar{u} \in \bar{\underline{k}}$ or $\bar{A} \in \bar{\underline{g}}$ are not homogeneous in $\underline{J}(\underline{k})$. Systems of homogeneous components for them are $\bar{u} = (\bar{u})_0 + \cdots + (\bar{u})_p + \cdots$ and $\bar{A} = (\bar{A})_0 + \cdots + (\bar{A})_p + \cdots$ defined as

(5.7) $$\begin{aligned} &(\bar{u})_0 = u , \\ &(\bar{u})_p(v_1 \otimes \cdots \otimes v_p) = [\cdots, [u, v_1], \cdots, v_p] , \qquad p > 0 ; \end{aligned}$$

(5.8) $$\begin{aligned} &(\bar{A})_0 = 0 , \qquad (\bar{A})_1 = A , \\ &(\bar{A})_p(v_1 \otimes \cdots \otimes v_p) = [\cdots, [Av_1, v_2], \cdots, v_p] , \qquad p > 1 . \end{aligned}$$

Let $(\underline{L}_G)_p$ denote the image of $\underline{L}_G$ under the natural map $\underline{J}(\underline{k}) \to \underline{J}_p(\underline{k})$. It is a technical point to show that at least when $\underline{g} \supset \mathrm{ad}_{\underline{k}}$, the sequence

(5.9) $$0 \to \underline{g}^{(p+1)} \to (\underline{L}_G)_{p+1} \to (\underline{L}_G)_p \to 0$$

is exact. From this it follows that in this case we have

(5.10) $$\mathrm{gr}\, \underline{L}_G = \underline{k} \oplus \underline{g} \oplus \underline{g}^{(1)} \oplus \cdots \oplus \underline{g}^{(p)} \oplus \cdots$$

and therefore $\mathscr{L}_G(M)$ is transitive on all the prolongations of the $G$-structure $E_G(M)$.

The bracket $[\ ,\ ]$ on $\underline{J}(\underline{k})$ induces a bracket $[\ ,\ ]_0$ on $\mathrm{gr}\, \underline{J}(\underline{k}) = S(\underline{k}^*) \otimes \underline{k}$. Relative to $[\ ,\ ]_0$, $\mathrm{gr}\, \underline{J}(\underline{k})$ usually is not a Lie algebra, but $\mathrm{gr}\, \underline{J}^0(\underline{k})$ is. Let us say that a subalgebra $\underline{L}$ of $\underline{J}(\underline{k})$ is a *graduate Lie algebra* if $\underline{L}$ is isomorphic to the completion $\widehat{\mathrm{gr}}\, \underline{L}$ of $\mathrm{gr}\, \underline{L}$ equipped with the bracket $[\ ,\ ]_0$.

For instance, in the flat case it is a fundamental property of $\underline{L}_G$ to be a graduate Lie algebra. Let us see what happens when $\tau \neq 0$. Denote $\mathrm{gr}\, \underline{L}_G = \underline{k} \oplus \underline{g} \oplus \underline{g}^1 \oplus \cdots$ A necessary condition for $\underline{L}_G$ to be a graduate Lie algebra is that for every $A^p \in \underline{g}^p$ and $u, v \in \underline{k}$, $A^p[u, v] = 0$. This means indeed that $(\mathrm{gr}\, \underline{L}_G ; [\ ,\ ]_0)$ is a Lie algebra. This is also a sufficient condition. Introduce *the strict prolongations* of $\underline{g}$:

$$\underline{g}^{p[1]} = \{A^{p+1} \in \underline{g}^{p(1)} \mid A^{p+1}[u, v] = 0\} ,$$

and $\underline{g}^{[p]} = \underline{g}^{[p-1][1]} = \underline{g}^{[1](p-1)}$. Then we have

**Proposition 5.** *For any integer $p \geq 0$ and any sequence $(\underline{g}, \underline{g}^1, \cdots, \underline{g}^p)$ such that*

(i) $\underline{g}$ *is a subalgebra of* $\underline{g}_0$,

(ii) $\underline{g}^i \subset \underline{g}^{(i-1)[1]} \qquad (1 \leq i \leq p)$,

(iii) $[\underline{g}^i, \underline{g}^j] \subset \underline{g}^{i+j}, \qquad 0 \leq i, j,\ i + j \leq p$,

*there exists a graduate Lie subalgebra $\underline{L}_{\underline{g},\cdots,\underline{g}^p}$ of $\underline{L}_G$ which admits $\bar{\underline{k}} \oplus \underline{g}$ as a subalgebra such that*

$$\operatorname{gr} \underline{L}_{\underline{g},\cdots,\underline{g}^p} = \underline{k} \oplus \underline{g} \oplus \underline{g}^1 \oplus \cdots \oplus \underline{g}^p \oplus \underline{g}^{p[1]} \oplus \cdots \oplus \underline{g}^{p[j]} \oplus \cdots .$$

*In particular, the algebra $\underline{L}_{\underline{g}}$ so obtained from the sequence $(\underline{g})$ is the maximal graduate subalgebra of $\underline{L}_G$ which contains $\bar{\underline{k}}$.*

Let us simply give an outline of the proof. To each $A^p \in \underline{g}^{[p]}$, define the formal vector field $\overline{A^p} \in (\underline{L}_G)^p$ by the condition

$$[\bar{u}, \overline{A^p}] = \overline{A^p u} \qquad \text{for any } \bar{u} \in \bar{\underline{k}} .$$

Formulas (5.3) and (5.4), together with (5.8), show that such an $A^p$ is uniquely determined, and moreover, (5.5) holds for $\overline{A^p}$. In fact, a system of homogeneous components $\overline{A^p}$ can be found as

$$(\overline{A^p})_0 = (\overline{A^p})_1 = \cdots = (\overline{A^p})_p = 0 , \qquad (\overline{A^p})_{p+1} = A^p ,$$

$$(5.10) \quad (\overline{A^p})_q(u_1 \otimes \cdots \otimes u_q) = [\cdots, [A^p u_1 u_2 \cdots u_{p+1}, u_{p+2}], \cdots, u_q] \quad \text{if } q > p + 1 .$$

It is now a straightforward computation to show that for every $p, q \geq 0$

$$[\overline{A^p}, \overline{A^q}] = \overline{[A^p, A^q]_0} ,$$

and therefore Proposition 5 is valid.

## 6. Strict $k$-flatness

Let us consider the standard $\underline{k}$-flat $G$-structure $E_G(K)$. If $\underline{j}$ is any characteristic ideal of $\underline{k}$, there corresponds to it a bi-invariant foliation $\mathscr{F}_{\underline{j}}$ on $K$, defined by the normal subgroup $J$. $\mathscr{F}_{\underline{j}}$ is of course invariant under $\Gamma_G$. Since $J$ is also a normal subgroup of $K \mathbin{\dot{\times}} G$, $\mathscr{F}_{\underline{j}}$ lifts into a foliation $\tilde{\mathscr{F}}_{\underline{j}}$ on $E_G(K)$ with the same dimensional leaves. The elements of $\Gamma_G$ which keep $\tilde{\mathscr{F}}_{\underline{j}}$ invariant constitute a transitive Lie sub-pseudo-group $\Gamma_{G,\underline{j}}$ of $\Gamma_G$, whose equation is obtained by reduction of $E^{(1)}_{(-)} \to E_G(K)$ to the subgroup of $G^1$ whose elements vanish on $\underline{j}$. $\Gamma_{G,\underline{j}}$ is a regular pseudo-group and its Lie algebra $\underline{L}_{G,\underline{j}}$ contains $\bar{\underline{k}} \mathbin{\dot{\oplus}} \underline{g}$.

Take in particular $\underline{j} = \underline{D} = \underline{D}\underline{k}$, and call a regular pseudo-group a *graduate pseudo-group* if its formal algebra is. Thus we have

**Theorem 6.1.** *The pseudo-group $\Gamma_{G,\underline{D}}$ admits $\underline{L}_{G,\underline{D}} = \underline{L}_{\underline{g}}$ as its formal algebra. It is the maximal graduate sub-pseudo-group of $\Gamma_G$ which contains the left translations on $K$.*

*Proof.* Proposition 5 first gives $\mathrm{gr}\,\underline{L}_{G,\underline{D}} \subset \mathrm{gr}\,\underline{L}_{\underline{g}}$, so that it is enough to assign to any element $\alpha \in \mathrm{gr}\,\underline{L}_{\underline{g}}$ a local analytic vector field $X(\alpha) \in \mathscr{L}_{G,\underline{D}}$ about $\mathbf{1}$. This is already known if $\alpha \in \underline{k}$ or $\alpha \in \underline{g}$. Suppose it is done if $\alpha \in \underline{k} \oplus \underline{g} \oplus \cdots \oplus \underline{g}^{[p-1]}$, and let $A^p$ be any element in $\underline{g}^{[p]}$, $p \geq 1$. Define $X(A^p)$ by the analytic first order differential equation

$$[X(u), X(A^p)] = X(A^p u)\ , \qquad X(A^p)(\mathbf{1}) = 0.$$

Proposition 5 implies that this equation admits a unique formal solution (namely $\overline{A^p}$) and analyticity assumes the convergence of this solution. The remaining of the proof is now quite obvious.

Using the preceeding argument and the fact that $\mathscr{L}_{G,\underline{D}}$ contains infinitesimal left translations, one obtains

**Theorem 6.2.** *Let $p$ a nonnegative integer, and $(\underline{g}, \underline{g}^1, \cdots, \underline{g}^p)$ a sequence such that*

(i) *$\underline{g}$ is a subalgebra of $\underline{g}_0$,*

(ii) $\underline{g}^i \subset \underline{g}^{i-1[1]} \qquad (1 \leq i \leq p)$,

(iii) $[\underline{g}^i, \underline{g}^j] \subset \underline{g}^{i+j}, \qquad 1 \leq i, j,\ i + j \leq p.$

*Then there exists an analytic sub-pseudo-group $\Gamma^{\omega}_{\underline{g},\cdots,\underline{g}^p}$ of $\Gamma_G$ on $K$, which admits $\underline{L}_{\underline{g},\cdots,\underline{g}^p}$ as its formal algebra.*

Take now for $\Gamma_{\underline{g},\cdots,\underline{g}^p}$ the $C^\infty$ complete of $\Gamma^{\omega}_{\underline{g},\cdots,\underline{g}^p}$; on $K$ one gets a graduate Lie pseudo-group of order $p + 1$, whose first order associated infinitesimal structure is $E_G(K)$ and whose formal algebra is $\underline{L}_{\underline{g},\cdots,\underline{g}^p}$. $\Gamma_{\underline{g},\cdots,\underline{g}^p}$ is said to be the *standard strict $\underline{k}$-flat pseudo-group with model* $(\underline{g}, \underline{g}^1, \cdots, \underline{g}^p)$.

Let now $E_G(M)$ be a $\underline{k}$-flat $G$-structure on a manifold $M$. In the sense of [14] it is a $\Gamma_G$-structure on $M$. We shall call $E_G(M)$ a *strict $\underline{k}$-flat $G$-structure* if it is defined indeed by a $\Gamma_{G,\underline{D}}$-structure on $M$.

Let $\mathscr{F}_{\underline{D}}$ be the natural $\underline{D}$-foliation on $M$. If $E_G(M)$ is a strict $\underline{k}$-flat $G$-structure, $\mathscr{F}_{\underline{D}}$ lifts into a same dimensional foliation $\tilde{\mathscr{F}}_{\underline{D}}$ on $E_G(M)$, which corresponds to the foliation defined on $E_G(K)$ by the normal subgroup $DK$ of $K$.

Thus any strict $\underline{k}$-flat $G$-structure is a *foliated $G$-structure over $\mathscr{F}_{\underline{D}}$* in the sense of [17]. This fact will give some interest results in the next section.

We shall say a Lie pseudo-group $\Gamma$ of order $p + 1$ on a manifold $M$ is a *strict $\underline{k}$-flat pseudo-group modeled over* $(\underline{g}, \underline{g}^1, \cdots, \underline{g}^p)$ if

(i) there exist a Lie algebra $\underline{k}$ with the same dimension as $M$ and a sequence $(\underline{g}, \underline{g}^1, \cdots, \underline{g}^p)$ as in Theorem 6.2 such that $\Gamma$ admits a formal algebra isomorphic to $\underline{L}_{\underline{g},\cdots,\underline{g}^p}$,

(ii) the linear isotropy group $G$ of $\Gamma$ is a subgroup of $G_0$,

(iii) the $G$-structure associated to $\Gamma$ is $\underline{k}$-flat.
These conditions are redundant: condition (ii) always holds if $G$ is connected, and a consequence of Theorem 8.1 is that (iii) is true whenever (i) and (ii) are.

For such pseudo-groups, denoting $\underline{L} = \underline{k} + \underline{g} + \cdots + \underline{g}^q + \cdots$ and using the *infinite ideal of* $\underline{g}$ *with respect to* $\underline{L}$ one has an analogue of Theorem 4.3:

$$\underline{g}_{\infty,\underline{L}} = \bigcap_q \underline{g}_{q,\underline{L}} \quad \text{with} \quad \underline{g}_{q,\underline{L}} = \langle \underline{g}^q, S^q\underline{k} \rangle .$$

$\underline{g}_{\infty,\underline{L}}$ is stricty imbedded in $\underline{g}_{(\infty)}$.

**Theorem 6.3.** *If $\Gamma$ is a strict $\underline{k}$-flat pseudo-group modeled over $(\underline{g}, \cdots, \underline{g}^p)$ on the manifold $M$, there exists on $M$ a strict $\underline{k}$-flat sub-pseudo-group $\Gamma_\infty$ of $\Gamma$ modeled over $(\underline{g}_{\infty,\underline{L}})$.*

The proof is essentially the same as that of Theorem 4.3. For a quite different proof, see [1].

**Corollary.** *A $\underline{k}$-flat $G$-structure on $M$ is strict if and only if it admits a $G'$ $\underline{k}$-flat reduction, where the Lie algebra $\underline{g}'$ acts trivially on $\underline{D}\underline{k}$.*

## 7. Cohomology of $\underline{k}$-flat manifolds

If $\underline{j}$ is any characteristic ideal in $\underline{k}$, Proposition 1 gives on any $\underline{k}$-flat manifold a foliation $\mathscr{F}_{\underline{j}}$ which has the same dimension as $\underline{j}$. This yields a first obstruction to the existence of a $\underline{k}$-flat structure on a manifold $M$ with a given $\underline{k}$: let $d_{\underline{k}}$ be the subset of $\{1, 2, \cdots, n\}$ whose elements are the dimensions of all the characteristic ideals of $\underline{k}$; for any integer $p \in d_{\underline{k}}$, $M$ must admit a $p$-dimensional foliation. This leads to some topological properties of $\underline{k}$-flat manifolds. For instance, if $1 \in d_{\underline{k}}$, then $\chi(M) = 0$ for a compact $\underline{k}$-flat manifold $M$.

Let $E_G(M)$ be a $\underline{k}$-flat $G$-structure on $M$, and

$$\lambda: I^p(G) \to H^{2p}(M)$$

be the Weyl homomorphism. If $\alpha \in I^p(G)$ and $\omega$ is a connection form on $M$, we shall write $\lambda_\omega(\alpha)$ the closed $2p$-form $\pi\alpha(\Omega^p)$ on $M$, which defines the class $\lambda(\alpha)$. Take a $(-)$-connection as $\omega$, whose curvature $\Omega$ is a 2-form on $E_G(M)$ with values in the subspace $\partial(\underline{k} \otimes \underline{g}^{(1)})$ of $\underline{g}$. Thus we obtain

**Theorem 7.1** (*Molino* [18]). *Let $G$ be a subgroup of $G_0$. Let $\alpha \in I^p(G)$ such that $\alpha(A^p) = 0$ for any $A \in \partial(\underline{k}^* \otimes \underline{g}^{(1)})$. Then for any $\underline{k}$-flat $G$-structure on $M$ the characteristic class $\lambda(\alpha)$ is identically zero.*

To illustrate this result, consider a vector subspace $V$ of $\underline{k}$ invariant under $G$. It defines, as in Proposition 1, a distribution $\mathscr{C}_V$ on $M$, invariant under $\Gamma_G(M)$. Notice that $\mathscr{C}_V$ is not integrable, unless $V$ is a subalgebra of $\underline{k}$. Let $E_Q$ denote the bundle of frames over $M$ transversed to $\mathscr{C}_V$ and deduced from $E_G(M)$. The natural map $E_G(M) \to E_Q$ induces an epimorphism $\rho: G \to G_Q$ on the structural groups, and so we get an injection

$$\rho^* : I^*(G_Q) \to I^*(G) \ .$$

Now, if $\beta \in I^p(G_Q)$ and $S \in \underline{k}^* \otimes \underline{g}^{(1)}$, then $\rho^*\beta(\partial S) = 0$ as long as $p > 2 \operatorname{codim}_{\underline{k}} V$, and thus the theorem can be applied. This leads to saying that a distribution $\mathscr{C}$ on a manifold $M$ is $\underline{k}$-flat if there exist a Lie algebra $\underline{k}$ and a $\underline{k}$-flat $G$-structure $E_G(M)$ on $M$ such that the infinitesimal first order structure on $M$ defining $\mathscr{C}$ can be reduced to $E_G(M)$. This means that $\mathscr{C}$ is invariant under some $\underline{k}$-flat pseuod-group. The preceeding argument gives

**Theorem 7.2** (*Bott-Molino*). *Let $Q$ be the normal vector bundle of a q-codimensional $\underline{k}$-flat distribution $\mathscr{C}$ on $M$. Then*

$$\operatorname{Pont}^l (Q) = 0 \qquad \textit{if } l > 2q \ .$$

Suppose now that moreover $\underline{g}$ consists of elements which vanish identically on $V$. Then we have Pont $(Q)$ = Pont $(M)$, and therefore

**Corollary.** *If a manifold $M$ admits a $\underline{k}$-flat $G$-structure, where $\underline{g}$ acts trivially on the subspace $V$ of $\underline{k}$, then necessarily*

$$\operatorname{Pont}^l (M) = 0 \qquad \textit{for } l > 2q \ .$$

*In particular, a necessary condition for $M$ to admit a strict $\underline{k}$-flat structure is that*

$$\operatorname{Pont}^l (M) = 0 \qquad \textit{for } l > 2 \operatorname{codim}_{\underline{k}} \underline{Dk} \ .$$

Now we shall give new cohomological invariants for $\underline{k}$-flat manifolds. This construction hinges on the following fact:

Let $M$ be a manifold equipped with a principal $G$-bundle $E \xrightarrow{\pi} M$. Let $\omega$ and $\omega'$ be two connections on $E$ which admit the same curvature $\Omega$. The difference $\eta = \omega' - \omega$ satisfies

$$D\eta + \tfrac{1}{2}[\eta, \eta] = 0 \ , \tag{7.1}$$

when $D$ is the exterior covariant differentiation relative to $\omega$.

Then for any $\alpha \in I^p(G)$ and $0 \leq j \leq p-1$

$$\mu^j_{\omega,\omega'}(\alpha) = \pi\alpha(\eta \wedge [\eta, \eta]^{p-j-1} \wedge \Omega^j) \tag{7.2}$$

is a closed $(2p-1)$-form on $M$ which defines the cohomology class $[\mu^j_{\omega,\omega'}(\alpha)] \in H^{2p-1}_{DR}(M)$.

Also notice that $\lambda_\omega(\alpha) = \lambda_{\omega'}(\alpha)$ so that the Chern formula [7]

$$\lambda_{\omega'}(\alpha) - \lambda_\omega(\alpha) = d \int_0^1 p\pi\alpha(\eta \wedge \Omega_t^{p-1}) dt \ ,$$

where $\Omega_t$ is the curvature form of $\omega_t = \omega + t\eta$, shows that

$$\nu_{\omega,\omega'}(\alpha) = \pi \int_0^1 p\alpha(\eta \wedge \Omega_t^{p-1})dt$$

defines another cohomology class $[\nu_{\omega,\omega'}(\alpha)] \in H_{DR}^{2p-1}(M)$. An easy computation gives the existence of some universal coefficients $\alpha_j(p)$ such that

$$\nu_{\omega,\omega'}(\alpha) = \sum_{0 \le j \le p-1} a_j(p)\mu^j_{\omega,\omega'}(\alpha) .$$

On the other hand, let us consider the Chern-Simons transgressed form [8]:

$$T\alpha(\omega) = \int_0^1 p\alpha(\omega \wedge \Omega_t^{p-1})dt , \quad \text{where} \quad \Omega_t = t\Omega + \tfrac{1}{2}(t^2 - t)[\omega, \omega] .$$

Using $\omega'$ for any $r \in E$ such that $\pi(r) = x$ one can define a horizontal lifting

$$\rho_\omega \colon (TM)_x \to (TE)_r$$

and a vertical one

$$\rho_\eta \colon (TM)_x \to (TE)_r$$

such that

$$(7.3) \qquad \eta\rho_\omega = \omega\rho_\eta , \quad \eta\rho_\eta = \omega\rho_\omega = 0 , \quad \pi_*\rho_\eta = 0 , \quad \pi_*\rho_\omega = \text{identity},$$

and so we obtain a projection $\rho_q \colon \bigwedge^q T^*E \to \bigwedge^q T^*M$ given by

$$\rho_q = \sum_{i+j=q} \rho_{i,j} , \quad \text{where} \quad \rho_{i,j} = \underbrace{\rho_\eta^* \wedge \cdots \wedge \rho_\eta^*}_{i} \wedge \underbrace{\rho_\omega^* \wedge \cdots \wedge \rho_\omega^*}_{j} .$$

As before one gets the existence of universal coefficients $b_j(p)$ such that

$$\rho_{2p-1}T\alpha(\omega) = \sum_{0 \le j \le p-1} b_j(p)\mu^j_{\omega,\omega'}(\alpha) .$$

Now return to the $\underline{k}$-flat case. Suppose $E$ to be the $\underline{k}$-flat $G_0$-structure $E_\tau(M)$ on $M$. It is easy to find couples of connections with the same curvature: take as $\omega$ a $(-)$-connection on $E_\tau(M)$ and as $\omega' = \bar{\omega}$ the associated $(+)$-connection. One gets $\eta = \varepsilon$, which is a canonically defined 1-form on $E_\tau(M)$. Simply denote $\mu^j_\omega(\alpha) = \mu^j_{\omega,\bar{\omega}}(\alpha)$.

**Theorem 7.3.** *If $M$ is a $\underline{k}$-flat manifold, the cohomology class defined from any $\alpha \in I^p(G_0)$ and $0 \le j \le p - 1$ by*

$$\mu^j(\alpha) = [\mu^j_\omega(\alpha)] \in H_{DR}^{2p-1}(M)$$

*is independent of the choice of the $(-)$-connection $\omega$ on $E_\tau(M)$.*

This means that the $\mu^j(\alpha)$'s can bring some information about the $\underline{k}$-flat structure on $M$. We shall call them the *secondary invariants* of $M$. Indeed, they are carried over by the $\underline{k}$-flat morphisms of rank $n$.

The proof of Theorem 7.3 goes as that given by Chern for the same property of the $\lambda(\alpha)$'s. In fact,

$$\mu^j_{\omega_1}(\alpha) - \mu^j_{\omega_0}(\alpha) = d\int_0^1 (-1)^{p-j-1} j\pi\alpha(\varepsilon \wedge [\varepsilon, \varepsilon]^{p-j-1} \wedge \omega_1 - \omega_0 \wedge \Omega_t^{j-1})dt\ ,$$

$\Omega_t$ being the curvature form of $\omega_0 + t(\omega_1 - \omega_0)$.

There are two kinds of $\mu^j(\alpha)$'s. The first one consists of those for which $0 < j < p$. The curvature $\Omega$ of some $(-)$-connection on $E_\tau(M)$ appears in their definition. So they can carry some obstructions to reduction of the structural group.

Let $\underline{j}$ be any characteristic ideal of $\underline{k}$, and $q = \operatorname{codim}_{\underline{k}} \underline{j}$. Suppose there exists over $M$ a $\underline{k}$-flat $G$-structure, where $\underline{g}$ is the ideal of $\underline{g}_0$ consisting of all derivations acting trivially on $\underline{j}$. As it has been remarked in § 6, the natural $\underline{j}$-foliation $\mathscr{F}_{\underline{j}}$ on $M$ lifts into a same dimensional foliation $\tilde{\mathscr{F}}_{\underline{j}}$ of $E_G(M)$. Moreover, $\tilde{\mathscr{F}}_{\underline{j}}$ is invariant under the right action of $G$, so that $E_G(M)$ is a foliated $G$-structure over $\mathscr{F}_{\underline{j}}$. Let $\omega$ be any $(-)$-connection on $E_G(M)$. Its torsion form $\Theta$ satisfies

$$\Theta + \tfrac{1}{2}[\theta, \theta] = 0$$

and therefore, if $\sigma$ is any allowable section of $E_G(M)$, then $\omega$ has the characteristic property

$$\sigma^*\omega \wedge \sigma^*\theta = 0\ . \tag{7.4}$$

From this it follows that the leaves of $\tilde{\mathscr{F}}_{\underline{j}}$ are $\omega$-horizontal. Moreover, Bianchi's identity implies that $i_{\tilde{X}}\Omega = 0$ for any $\tilde{X} \in \tilde{\mathscr{F}}_{\underline{j}}$. A direct consequence is

**Theorem 7.4.** *Let $E_G(M)$ be a $\underline{k}$-flat $G$-structure on $M$, $\underline{g}$ being the ideal of $\underline{g}_0$ whose elements act trivially on the characteristic ideal $\underline{j}$ of $\underline{k}$. If $q =$ $\operatorname{codim}_{\underline{k}} \underline{j}$, the characteristic class $\lambda(\alpha)$ of $E_G(M)$ defined by $\alpha \in I^p(G)$ vanishes as long as $p > [q/2]$. Moreover, if $\alpha \in I^p(G_0)$, the secondary invariants $\mu^j(\alpha)$ of $M$ vanish as long as $j > [q/2]$.*

**Corollary 1.** *If $M$ is a discrete $\underline{k}$-flat manifold, for any $\alpha \in I^p(G_0)$ and $0 < j < p$ the relations*

$$\lambda(\alpha) = 0\ , \qquad \mu^j(\alpha) = 0$$

*hold.*

**Corollary 2.** *If $M$ admits a strict $\underline{k}$-flat structure, for any $\alpha \in I^p(G_0)$ we have*

$$\lambda(\alpha) = 0 \qquad \textit{for } 2p > \operatorname{codim}_{\underline{k}} \underline{Dk} \ ,$$
$$\mu^j(\alpha) = 0 \qquad \textit{for } 2j > \operatorname{codim}_{\underline{k}} \underline{Dk} \ .$$

These corollaries are obtained by choosing $\underline{j} = \underline{k}$ and $\underline{j} = \underline{Dk}$ in the theorem.

The second kind of $\mu^j(\alpha)$'s consists of the $\mu^0(\alpha)$'s. They are only defined by the form $\varepsilon$, and so may be nontrivial on the model space $K$. Notice that on $K$ the map $\mu^0$ factorizes through the left invariant cohomology ring:

$$\begin{tikzcd} I^p(G_0) \arrow[rr, "\mu^0"] \arrow[dr] & & H^{2p-1}_{DR}(K) \\ & H^{2p-1}(\underline{k}, \boldsymbol{R}) \arrow[ur] & \end{tikzcd}$$

For instance, take $\underline{k} = \underline{o}(3)$ and therefore $K = \$^3$. An easy computation shows that for any $\alpha \in I^2(SO(3))$, $\alpha \neq 0$, the 3-form $\pi\alpha(\varepsilon \wedge [\varepsilon, \varepsilon])$ nowhere vanishes on $M$, and thus $\mu^0(\alpha) \neq 0$.

Foliations on the $\underline{k}$-flat manifolds associated to the ideals of $\underline{k}$ have some topological properties. For instance,

**Theorem 7.5.** *Let $\underline{j}$ be a characteristic ideal of $\underline{k}$. Then the $\underline{j}$-foliation $\mathscr{F}_{\underline{j}}$ on any $\underline{k}$-flat manifold $M$ admits a C.T.P. (connexion transverse projetable).*

See [19] for the definition and properties of foliations with $C.T.P.$ In fact, let $\omega$ be any $(-)$-connection on $E_\tau(M)$. Let $E^T$ denote the $G_0^T$-bundle over $M$ of the frames which are transversal to $\mathscr{F}_{\underline{j}}$ and deduced from $E_\tau(M)$. The connection $\omega$ projects onto a connection $\omega^T$ on $E^T$ which satisfies the desired conditions.

## 8. The equivalence problem for $\underline{k}$-flat $G$-structures

Let $G$ be a sub-group of $G_0$. We shall say that a $G$-structure $E_G(M)$ on a manifold $M$ is formally $\underline{k}$-flat if it is formally equivalent to the standard $\underline{k}$-flat $G$-structure $E_G(K)$ on $K$. If $G \supset \mathrm{ad}_K$, the transitivity of $\Gamma_G$ on all the prolongations of $E_G(K)$ allows us to use the construction of $V$. Guillemin [13] to give a simple criterion for formal $\underline{k}$-flatness. Using the bundle $E^{(1)}_{(-)}$ as the standard prolongation of $E_G(K)$ we see that $E_G(M)$ is formally $\underline{k}$-flat if and only if all its Guillemin structure tensors vanish.

The problem is now: are all formally $\underline{k}$-flat $G$-structures $\underline{k}$-flat structures?

More generally, a Lie pseudo-group $\Gamma$ on the manifold $K$ is said to be $\underline{k}$-flat if

(i) left translations are elements of $\Gamma$,

(ii) the linear isotropy group $G$ of $\Gamma$ is a subgroup of $G_0$,

(iii) the equation of $\Gamma$ is an analytic subbundle of $B_p(K)$.

For instance, $\Gamma_G$ is a $\underline{k}$-flat pseudo-group of order 1 on $K$. For any sequence

$(\underline{g}, \underline{g}^1, \cdots, \underline{g}^p)$ as in Theorem 6.2, the pseudo-group $\Gamma_{\underline{g},\ldots,\underline{g}^p}$ is a $\underline{k}$-flat pseudo-group of order $p+1$.

A Lie pseudo-group $\Gamma$ is said to satisfy the *equivalence theorem* if on any manifold $M$ any almost-$\Gamma$-structure is indeed a $\Gamma$-structure (with the terminology of [14]).

**Example 8.1.** If $\underline{k}$ is the abelian Lie algebra $\boldsymbol{R}^n$, then a $\boldsymbol{R}^n$-flat pseudo-group is a weak flat pseudo-group in the terminology of [20]. [5] and [20] lead thus to the result: any $\boldsymbol{R}^n$-flat pseudo-group satisfies the equivalence theorem.

**Example 8.2.** If $\underline{k}$ is a semi-simple Lie algebra, Theorem 4.2 implies that any $k$-flat pseudo-group is of finite type. [13] leads then to: any $\underline{k}$-flat pseudo-group satisfies the equivalence theorem.

**Example 8.3.** If $\underline{Dk}$ is an abelian Lie algebra, Example 3.4 implies that the pseudo-group $\Gamma_{G_0}(K)$ is a flat pseudo-group. Theorefore $\Gamma_{G_0}(K)$ satisfies the equivalence theorem.

It seems reasonable to think that any $\underline{k}$-flat pseudo-group satisfies the equivalence theorem. Our purpose is now to show how relative flatness theorem of P. Molino can be used to give an affirmative answer to this question for some types of $\underline{k}$-flat pseudo-groups.

If $\Gamma$ is a $\underline{k}$-flat pseudo-group on $K$ with linear isotropy group $G$, Examples 8.1 and 2 allow us to suppose there exist ideals of $\underline{k}$ which are invariant under $G$. Let $\underline{j}$ be such an ideal, and $\mathscr{F}_{\underline{j}}$ the $\underline{j}$-foliation on $K$ associated to it. The leaf of $\mathscr{F}_{\underline{j}}$ through $\mathbf{1}$ is the connected subgroup of $K$ with algebra $\underline{j}$, and thus it is closed. Let $\Gamma'_{\underline{j}}$ be the pseudo-group on $J$ obtained by restriction of the actions of elements of $\Gamma$. The formal completion $\Gamma_{\underline{j}}$ of $\Gamma'_{\underline{j}}$ is a $\underline{j}$-flat pseudo-group on $J$.

In the same way, $\mathscr{F}_{\underline{j}}$ defines a fibering $K \rightarrow K/J$, and the analycity of the equation of $\Gamma$ allows us to consider the quotient pseudo-group $\Gamma^T_{\underline{j}}$ of $\Gamma$ by this fibering (see [16]). Let $q: \Gamma \rightarrow \Gamma^T_{\underline{j}}$ be the natural map. $\Gamma^T_{\underline{j}}$ is a $\underline{k}/_{\underline{j}}$-flat pseudo-group on the Lie group $K/J$. So it admits a sub-pseudo-group $\gamma^T_{\underline{j}}$ which is simply transitive on $K/J$, namely, the local restrictions of left translations of $K/J$. Thus $\gamma_{\underline{j}} = \bar{q}^1(\gamma^T_{\underline{j}})$ is a $\underline{k}$-flat sub-pseudo-group of $\Gamma$ whose linear isotropy algebra is $\underline{g} \cap \operatorname{der}(\underline{k}, \underline{j})$.

**Lemma.** *If the pseudo-groups $\Gamma^T_{\underline{j}}$ and $\gamma_{\underline{j}}$ satisfy the equivalence theorem, so does $\Gamma$.*

*Proof.* Consider an almost-$\Gamma$-structure on a manifold $M$. Let $\mathscr{F}_{\underline{j}}$ be the induced $\underline{j}$-foliation on $M$. Since restriction to a leaf and projection along the leaves of $\mathscr{F}_{\underline{j}}$ are formal properties (they are done on jet spaces), we see that any local submanifold $W$ transverse to $\mathscr{F}_{\underline{j}}$ (resp. any leaf of $\mathscr{F}_{\underline{j}}$) is endowed with an almost-$\Gamma^T_{\underline{j}}$-structure (resp. an almost $\Gamma_{\underline{j}}$-structure). Let $x$ be a point of $W$. The hypothesis on $\Gamma^T_{\underline{j}}$ implies that in some neighborhood of $x$ in $M$ the almost-$\Gamma$-structure reduces to an almost-$\gamma_{\underline{j}}$-structure. The lemma thus follows.

Suppose now $\Gamma$ is a strict $\underline{k}$-flat pseudo-group. *If* $p+1$ denotes its order, there exists a sequence $(\underline{g}, \underline{g}^1, \cdots, \underline{g}^p)$ as in Theorem 6.2 such that the formal algebra $\underline{L}$ of $\Gamma$ is $\underline{L}_{g,g^1,\dots,g^p}$. Take $\underline{j} = \underline{Dk}$. Since $g^1 \subset g^{[1]}$, $\underline{Dk}$ is an ideal of $\underline{L}$, and thus $\bar{\underline{k}}$ is a finite dimensional subalgebra of $\underline{L}$ which is flat relative to $\underline{L}$. Molino's theorem on relative flatness [20] thus gives

**Theorem 8.1.** *Any strict $\underline{k}$-flat pseudo-group $\Gamma$ satisfies the equivalence theorem.*

Now, if $\underline{k}$ is a reductive Lie algebra, its radical $\underline{r}$ equals its center. So the quotient pseudo-group $\Gamma_{\underline{r}}^T$ is $\underline{k}/_{\underline{r}}$-flat with $\underline{k}/_{\underline{r}}$ semi-simple. On the other hand, $\gamma_{\underline{r}}$ is a strict $\underline{k}$-flat pseudo-group. From the above lemma and Theorem 8.1, we obtain

**Theorem 8.2.** *If $\underline{k}$ is a reductive Lie algebra, any $\underline{k}$-flat pseudo-group satisfies the equivalence theorem.*

Notice that this theorem contains Examples 8.1 and 8.2.

Another case to which these technics apply is given by the hypothesis: $\underline{k}$ is a nilpotent algebra. Look in this case at the central sequence

$$\underline{k} \supset \underline{Ck} \supset \cdots \supset \underline{C}_q\underline{k} \supset 0 \ .$$

Take $\underline{j} = \underline{Ck}$. $\underline{k}/_{\underline{Ck}}$ is an abelian Lie algebra. Therefore the proof of the above lemma shows that any almost-$\Gamma$-structure locally reduces to an almost $\Gamma_1$-structure with linear isotropy algebra contained in der $(\underline{k}, \underline{Ck})$. Notice that any $A \subset$ der $(\underline{k}; \underline{Ck})$ satisfies $A(\underline{Dk}) \subset \underline{C}_2\underline{k}$. We can thus apply once more the lemma with $\underline{j} = \underline{C}_2\underline{k}$, and so on. Finally, the relations

$$A \in \text{der}\,(\underline{k}, C_r\underline{k}) \Rightarrow A(D\underline{k}) \subset \underline{C}_{r+1}\underline{k} \ , \qquad 1 \leq r \leq q \ ,$$

lead us to a $\underline{k}$-flat parallelims in some neighborhood of any point of $M$. We thus have

**Theorem 8.3.** *If $\underline{k}$ is a nilpotent Lie algebra, any $\underline{k}$-flat pseudo-group satisfies the equivalence theorem.*

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